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SPECTRAL MULTIPLIERS FOR SCHRÖDINGER OPERATORS

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ABSTRACT. We prove a sharp Hörmander multiplier theorem for Schrödinger operators $H=-\Delta+V$ on $\mathbb{R}^n$. The result is obtained under certain condition on a weighted L^∞ estimate, coupled with a weighted L^2 estimate for H, which is a weaker condition than that for nonnegative operators via the heat kernel approach. Our approach is elaborated in one dimension with potential V belonging to certain critical weighted L^1 class. Namely, we assume that $\int(1+|x|)|V(x)|\,dx$ is finite and H has no resonance at zero. In the resonance case, we assume $\int(1+|x|^2)|V(x)|\,dx$ is finite.

1. Introduction

Let $H=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^n$, where $\Delta=\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and V is real-valued. In this paper, we are concerned with proving a spectral multiplier theorem on L^p spaces for H and we then consider potentials in some critical class L_1^1 in one dimension, where V may *not* be positive. As is well known, spectral multiplier theorem plays a significant role in harmonic analysis and PDEs [1, 2, 4, 5, 8, 10, 15, 16, 20, 25, 28].

For a Borel measurable function $\phi:\mathbb{R}\to\mathbb{C}$ we define $\phi(H)=\int\phi(\lambda)\,dE_\lambda$ by functional calculus, where $H=\int\lambda\,dE_\lambda$ is the spectral resolution of the selfadjoint operator H acting in $L^2(\mathbb{R}^n)$. The spectral multiplier problem is to find sufficient condition on a bounded function μ on $\mathbb{R}$ (with minimal smoothness) so that $\mu(H)$ is bounded on $L^p(\mathbb{R}^n)$, $1<p<\infty$.

In the Fourier case, i.e., $V=0$, Hörmander [21] essentially proved (for radial multipliers) the multiplier theorem on $L^p(\mathbb{R}^n)$, under the condition that the scaling-invariant local Sobolev norm on μ is finite for $s>n/2$,

$$\|\mu\|_{W^s_{2,sloc}} := \sup_{t>0}\|\mu(t\cdot)\chi\|_{W^s_2(\mathbb{R})}<\infty. \qquad (1)$$

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Here $\chi \in C_0^\infty(\mathbb{R}\setminus\{0\})$ is a fixed C^∞-smooth function with compact support away from zero and W_2^s denotes the usual Sobolev space endowed with the norm $\|f\|_{W_2^s} = \|(1-\Delta)^{s/2} f\|_2$. The proof in [21] mainly requires that the kernel $K_\mu(x,y)$ of $\mu(-\Delta)$ satisfy

$$\int_{|x-\bar{y}|>2|y-\bar{y}|} |K_\mu(x,y) - K_\mu(x,\bar{y})|\, dx \le C \tag{2}$$

for all y, $\bar{y}$ (for the weak $(1,1)$ estimate). However, the regularity condition in (2) is invalid for H when $V \neq 0$.

For $V \ge 0$, Hebisch [20] proved a multiplier theorem with $s > \frac{n+1}{2}$ based on heat kernel estimates. His approach was essentially to control the low energy part of $\mu(H)$ by a pointwise decay of the kernel, see (5). This heat kernel approach has been recently developed in proving sharp multiplier theorems (with $s > n/2$) in various settings for positive elliptic operators on manifolds or metric spaces [1, 7, 14], see [15] for a comprehensive survey and the references therein.

The question remains open for general V where the heat kernel estimates may not hold. In this paper, we formulate a Hörmander type spectral multiplier theorem (Theorem 1.2) for general H on $\mathbb{R}^n$. We show that Theorem 1.2 is true if the two weighted estimates in Assumption 1.1, namely a weighted L^2 estimate (in high energy) and an integral form of pointwise decay estimate (in low energy), are satisfied for H. In Sections 3–5, we elaborate the approach in one dimension by considering potentials in the class $L^1_\gamma := \{f : \int (1+|x|)^\gamma |f(x)|\, dx < \infty\}$, $\gamma = 1, 2$.

For a (continuous) function ϕ, let $\phi(H)(x,y)$ denote the kernel of $\phi(H)$, $x, y \in \mathbb{R}^n$ and let $\lambda_j = 2^{-j/2}$, $j \in \mathbb{Z}$. By $\phi \in X(\Omega)$, where $\Omega \subset \mathbb{R}$ and X is a function space on $\mathbb{R}$, we mean that $\phi \in X$ and has support in Ω. Throughout this paper, c or C will denote an absolute constant and χ_Ω the characteristic function on the set Ω.

Assumption 1.1. *Assume that H satisfies the following two estimates.*

(a) (*Weighted L^2 estimate*) *There exists some $s > n/2$ so that for all j and $\phi \in W_2^s([\frac{1}{4}, 1] \cup [-1, -\frac{1}{4}])$,*

$$\sup_y \big\| |x-y|^s \phi(\lambda_j^2 H)(x,y) \big\|_{L_x^2} \le c \lambda_j^{s-n/2} \|\phi\|_{W_2^s}. \tag{3}$$

(b) (*Weighted L^∞ estimate*) *There exist a finite measure $d\zeta$ and $0 < \varepsilon \le 1$ so that for all x, y, j and $\phi \in W_2^{n+\varepsilon}([-1,1])$,*

$$|\phi(\lambda_j^2 H)(x,y)| \le c \lambda_j^{-n} \int_{\mathbb{R}^n} (1 + \lambda_j^{-1}|x-y-u|)^{-n-\varepsilon}\, d\zeta(u), \tag{4}$$

where $c = c(\|\phi\|_{W_2^{n+\varepsilon}})$.

The assumption is intrinsic in the sense that it only depends on H and does not depend on the multiplier μ. Note that when $V \geq 0$, Hebisch [20] essentially used in the proof the following pointwise decay

$$|\phi(\lambda_j^2 H)(x,y)| \leq c\lambda_j^{-n}(1+\lambda_j^{-1}|x-y|)^{-n-\varepsilon}, \tag{5}$$

which is implied by the upper Gaussian bound for $e^{-tH}(x,y)$. Assumption 1.1(b) is a much weaker condition than (5). When V is negative, the decay in (5) does not hold, not even for V being a Schwartz function, cf. [24, 37].

Theorem 1.2. *Suppose H satisfies Assumption 1.1 for some $s > n/2$. If $\|\mu\|_{W^s_{2,sloc}} < \infty$, then $\mu(H)$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and has weak type $(1,1)$. Moreover,*

$$\|\mu(H)\|_{L^1 \to weak\text{-}L^1} \leq c\|\mu\|_{W^s_{2,sloc}}. \tag{6}$$

That the critical exponent $\frac{n}{2}$ is sharp is well-known in the literature [6, 15, 29]. Note that the condition in (1) implies $\mu \in L^\infty$ by Sobolev embedding

$$\|\mu\|_\infty \leq c\|\mu\|_{W^s_{2,sloc}} \tag{7}$$

whenever $s > 1/2$. Also, note that one has an equivalent norm for $\|\cdot\|_{W^s_{2,sloc}}$ if in (1) χ is replaced with any other φ in $C_0^\infty(\mathbb{R}\setminus\{0\})$.

Remark 1.3. From the proof given in Section 2, we easily observe that Theorem 1.2 actually holds for any self-adjoint operator L in place of H that satisfies Assumption 1.1(a) and

(b′) There exist $d\zeta_k \in M$, $k \in \mathbb{Z}$, M the set of finite measures, with $0 < \varepsilon \leq 1$ and $\sum_k \|\zeta_k\|_M < \infty$, so that for all x, y, j

$$|\Phi(\lambda_j^2 L)(x,y)| \leq c \sum_{k,\pm} \lambda_k^{-n}(1+{\lambda_k}^{-1}|\cdot|)^{-n-\varepsilon} * d\zeta_k(\pm x \pm y), \tag{8}$$

where $\Phi \in C^\infty([-1,1])$ is given as in (10), $f * d\zeta(x) = \int f(x-u)\,d\zeta(u)$ is the usual convolution.

Applying Theorem 1.2 to the one dimensional $H_V := -d^2/dx^2 + V$, we obtain the following theorem.

Theorem 1.4. *Suppose V is in $L^1_1(\mathbb{R})$ and assume that there is no resonance at zero. If for some $s > 1/2$, $\|\mu\|_{W^s_{2,sloc}}$ is finite, then the conclusions of Theorem 1.2 hold. Furthermore, the conclusions also hold true for all $V \in L^1_2(\mathbb{R})$.*

A typical example for μ is $\mu_\gamma(\xi) = |\xi|^{i\gamma}$, $\gamma \in \mathbb{R}$. Hence, $H_V^{i\gamma}$ is bounded on L^p, $1 < p < \infty$, and maps L^1 to weak-L^1.

Let $\{\Phi, \varphi_j\} \in C_0^\infty(\mathbb{R})$ be a dyadic system satisfying $\operatorname{supp}\Phi \subset \{x : |x| \le 1\}$, $\operatorname{supp}\varphi \subset \{x : \frac{1}{4} \le |x| \le 1\}$ and

$$\sum_{j=-\infty}^{\infty} \varphi_j(x) = 1 \quad \forall x \neq 0, \tag{9}$$

$$\Phi(x) + \sum_{j=1}^{\infty} \varphi_j(x) = 1 \quad \forall x, \tag{10}$$

where $\varphi_j(x) = \varphi(2^{-j}x)$, and note that $\Phi(x) \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Using the dyadic system above, we will make the high and low energy cutoffs of $\mu(H)$ in the proof of Theorem 1.2. As in [12, 22, 35], we can also define $B_p^{\alpha,q}(H)$ and $F_p^{\alpha,q}(H)$, the Besov spaces and Triebel–Lizorkin spaces associated with H. We can show that the sharp spectral multiplier theorem also hold on these spaces, see the statement in Theorem 2.3.

1.1. Weighted estimates for the kernel of $\phi(2^{-j}H_V)$. Let $V \in L_1^1(\mathbb{R})$ and assume 0 is not a resonance or let $V \in L_2^1(\mathbb{R})$ in general. From [13] or Section 3, H_V has resonance at 0 means that the Wronskian vanishes at 0, that is, $\nu := W(0) = 0$. From Theorem 1.2 and the remark that follows, we know that the main technical difficulty in proving Theorem 1.4 is to verify the two weighted estimates in Assumption 1.1(a), (b′).

The proofs of (3) and (8) for H_V require some new and refined formulas and asymptotic estimates for $m_\pm(x,k)$, the modified Jost functions, and $t(k)$, $r_\pm(k)$, the associated transmission and reflection coefficients. The main tools are Volterra integral equations for $m_\pm(x,k)$ as well as its Fourier transforms. These are motivated by and developed from the treatment in [13].

For the L^∞ estimates in (8) for the low energy, we use Wiener's lemma in order to prove the existence of finite measures $d\zeta_k$, which are actually L^1 functions up to a delta measure, see [19] for a similar treatment when considering the dispersive estimates for H_V.

For the L^2 estimates in (3) for the high energy, we prove (3) for $1/2 < s < 1$ by interpolating between the cases $s = 0$ and $s = 1$, which can be viewed as Plancherel formula for $D^s\phi_j$ with respect to the Fourier transform associated to H_V.

The remaining of the paper is organized as follows. In Section 2, we prove the weak $(1,1)$ estimate for general H under the hypothesis in Assumption 1.1. Sections 3–5 are devoted to the proof of Theorem 1.4, which is quite long verification of the estimates in (3) and (8) in one dimension. In certain cases, it involves delicate and subtle technicalities.

2. Proof of weak-(1,1) boundedness

In this section, we mainly give the proof of Theorem 1.2. Since $\mu \in L^\infty$, $\mu(H)$ is bounded on L^2. Hence by interpolation and duality it is sufficient to show that $\mu(H)$ has weak type (1,1), which will follow from Lemma 2.1, Lemma 2.2 and Calderón–Zygmund decomposition. The proof is a modification of the arguments in [20] and [14]. Let $\{\Phi, \varphi_j\}$ be as in (9), (10). Write $\mu_j = \mu\varphi_j$, $\Phi_j(x) = \Phi(2^{-j}x)$, $j \in \mathbb{Z}$.

LEMMA 2.1. *Let H satisfy Assumption 1.1(a) with $s > n/2$. Let $y \in I$, $I \subset \mathbb{R}^n$ a cube with length $t = \ell(I) = 2^{-j_I/2}$, $j_I \in \mathbb{Z}$. Then*

(a) *For all $j \geq j_I$,*

$$\int_{|x-y|\geq 2t} \left|\big(\mu_j(1-\Phi_{j_I})\big)(H)(x,y)\right| dx \leq c(2^{j/2}t)^{\frac{n}{2}-s}\|\mu\|_{W^s_{2,sloc}}.$$

(b)

$$\int_{|x-y|\geq 2t} \sum_{j=-\infty}^{\infty} \left|\big(\mu_j(1-\Phi_{j_I})\big)(H)(x,y)\right| dx \leq c\|\mu\|_{W^s_{2,sloc}}.$$

Proof. Inequality (a) is consequence of Assumption 1.1(a) and Schwarz inequality. Let $\tilde{\mu}_j = \mu_j(1-\Phi_{j_I})$. We have for $s > n/2$, $j \geq j_I$,

$$\begin{aligned}
&\int_{|x-y|\geq 2t} |\tilde{\mu}_j(H)(x,y)|\,dx \\
&\quad = \int_{|x-y|\geq 2t} |x-y|^{-s}|x-y|^{s}|\tilde{\mu}_j(H)(x,y)|\,dx \\
&\quad \leq c_{n,s}(2^{j/2}t)^{n/2-s}\|\mu\|_{W^s_{2,sloc}}.
\end{aligned}$$

(b) is an easy consequence of (a). Note that since $\operatorname{supp}\varphi_j \subset \{2^{j-2} \leq |\xi| \leq 2^j\}$ and $\operatorname{supp}(1-\Phi_{j_I}) \subset \{|\xi| \geq 2^{j_I-1}\}$, it follows that $\tilde{\mu}_j(\xi) = 0$ if $j \leq j_I - 1$. □

LEMMA 2.2. *Let H satisfy Assumption 1.1(b) with some finite measure $d\zeta$ and $\varepsilon \in (0,1]$. Let $y \in I$, I a cube with length $t = \ell(I) = 2^{-j_I/2}$, $j_I \in \mathbb{Z}$ and volume $|I|$. Then for all x and all $y \in I$*

$$|\Phi_{j_I}(H)(x,y)| \leq c|I|^{-1}\int_{u\in\mathbb{R}^n}\int_{z\in I} 2^{j_I n/2}(1+2^{j_I/2}|x-z-u|)^{-n-\varepsilon}\,dz\,d\zeta(u).$$

Proof. Since $\Phi \in C^\infty([-1,1]) \subset W_2^{n+\varepsilon}([-1,1])$, according to (4), $\Phi_{j_I}(H)(x,y)$ is dominated by

$$\begin{aligned}
&c\int_{\mathbb{R}^n} \lambda_j^{-n}(1+\lambda_j^{-1}|x-y-u|)^{-n-\varepsilon}\,d\zeta(u) \\
&\quad \leq c|I|^{-1}\int_{\mathbb{R}^n}\int_{z\in I} 2^{j_I n/2}(1+2^{j_I/2}|x-z-u|)^{-n-\varepsilon}\,dz\,d\zeta(u), \\
&\quad \forall x \in \mathbb{R}^n, y \in I,
\end{aligned}$$

where $\lambda_j = 2^{-j_I/2}$ and we observed that for all x and $t = \ell(I)$

$$\begin{aligned}\sup_{y\in I}(1+|x-y|/t)^{-n-\varepsilon} &\le c\min_{y\in I}(1+|x-y|/t)^{-n-\varepsilon}\\ &\le \frac{c}{|I|}\int_I(1+|x-z|/t)^{-n-\varepsilon}\,dz.\end{aligned}$$

□

2.1. Proof of the weak-$(1,1)$. Let $f\in L^1\cap L^2$. For any given $\alpha>0$, apply the C–Z decomposition to obtain that $f=g+b$ for some $g\in L^1\cap L^2$, and $b\in L^1$ with $b=\sum_k b_k$, where $\operatorname{supp} b_k\subset I_k$, I_k being disjoint cubes in $\mathbb{R}^n$ with lengths $\ell(I_k)$ equal to integer powers of $\sqrt{2}$ and:

(i) $|g(x)|\le c\alpha$ a.e. x,

(ii) $|I_k|^{-1}\int_{I_k}|f(x)|\,dx\le c\alpha$,

(iii) $\sum_k|I_k|\le c\alpha^{-1}\|f\|_1$.

We will prove that there exists a constant C such that $\forall f\in L^1\cap L^2$,

$$\big|\{x:|\mu(H)f(x)|>\alpha\}\big|\le C\alpha^{-1}\|f\|_1(\|\mu\|_{W^s_{2,sloc}}+\|\mu\|_\infty^2+1). \tag{11}$$

Since $\mu\in L^\infty$, Chebeshev inequality gives

$$\begin{aligned}\big|\{x:|\mu(H)g(x)|>\alpha/2\}\big| &\le(\alpha/2)^{-2}\|\mu(H)g\|_2^2\\ &\le c\|\mu\|_\infty^2\alpha^{-1}\|f\|_1.\end{aligned}$$

The main task is to deal with the “bad” function b. Let Φ be as in (10), $\Phi_j(x)=\Phi(2^{-j}x)$. Write

$$\mu(H)b(x)=\sum_k\mu(H)\big(1-\Phi_{j_k}(H)\big)b_k(x)+\sum_k\mu(H)\Phi_{j_k}(H)b_k(x),$$

where $2^{-j_k}=\ell(I_k)^2$. Denote by I_k^* the cube having length $5\sqrt{n}$ times the length of I_k with the same center as I_k. We need to show

$$\begin{aligned}&\Big|\Big\{x\in\mathbb{R}^n\Big\backslash\bigcup_k I_k^*:|\mu(H)b(x)|>\alpha/2\Big\}\Big|\\ &\le\Big|\Big\{x\in\mathbb{R}^n\Big\backslash\bigcup_k I_k^*:\ \sum_k\big|\mu(H)\big(1-\Phi_{j_k}(H)\big)b_k(x)\big|>\alpha/4\Big\}\Big|\\ &\quad+\Big|\Big\{x\in\mathbb{R}^n\Big\backslash\bigcup_k I_k^*:\ \Big|\sum_k\mu(H)\Phi_{j_k}(H)b_k(x)\Big|>\alpha/4\Big\}\Big|\\ &\le c\|\mu\|_{W^s_{2,sloc}}\alpha^{-1}\|f\|_1.\end{aligned}$$

(a) *High energy cut-off.* If $x\notin\bigcup_k I_k^*$, then $I_k\subset\{y:|y-x|>2\sqrt{n}t_k\}$, $t_k=2^{-j_k/2}$. We have

$$\mu(H)\big(1-\Phi_{j_k}(H)\big)b_k(x)=\int_{|y-x|>2t_k}\big(\mu(1-\Phi_{j_k})\big)(H)(x,y)b_k(y)\,dy.$$

Applying Lemma 2.1(b) for $s > n/2$, we obtain

$$
\begin{aligned}
&\left|\left\{x \notin \bigcup I_k^* : \left|\sum_k \big(\mu(1-\Phi_{j_k})\big)(H) b_k(x)\right| > \alpha/4\right\}\right| \\
&\quad \le c(\alpha/4)^{-1} \int_{\mathbb{R}^n \setminus \bigcup I_k^*} \left|\sum_k \big(\mu(1-\Phi_{j_k})\big)(H) b_k(x)\right| dx \\
&\quad \le c\alpha^{-1} \int \sum_k |b_k(y)|\, dy \int_{|y-x|>2t_k} \big|\big(\mu(1-\Phi_{j_k})\big)(H)(x,y)\big|\, dx \\
&\quad \le c\|\mu\|_{W^s_{2,sloc}} \alpha^{-1} \|f\|_1,
\end{aligned}
$$

where we note that

$$
\begin{aligned}
&\int_{|x-y|>2t_k} \big|\big(\mu(1-\Phi_{j_k})\big)(H)(x,y)\big|\, dx \\
&\quad \le \int_{|x-y|>2t_k} \sum_j \big|\big(\mu_j(1-\Phi_{j_k})\big)(H)(x,y)\big|\, dx \le c\|\mu\|_{W^s_{2,sloc}}.
\end{aligned}
$$

(b) *Low energy cut-off.* Since $\mu(H)$ is bounded on L^2, the proof is complete if we can show

$$
\int \left|\sum_k \Phi_{j_k}(H) b_k(x)\right|^2 dx \le c\alpha \|f\|_1. \tag{12}
$$

To show this, let $\rho_j = 2^{jn/2}(1 + 2^{j/2}|\cdot|)^{-n-\varepsilon}$. According to Lemma 2.2, $\forall h \in L^2$,

$$
\begin{aligned}
&\left|\left\langle \sum_k \Phi_{j_k}(H) b_k, h\right\rangle\right| \\
&\quad = \left|\sum_k \int h(x)\, dx \int_{y\in I_k} \Phi_{j_k}(H)(x,y) b_k(y)\, dy\right| \\
&\quad \le \sum_k |I_k|^{-1} \int |b_k(y)|\, dy \int |h(x)|\, dx \int_{z\in I_k} \int_u \rho_j(x-z-u)\, d\zeta(u)\, dz \\
&\quad \le c\alpha \int \sum_k \chi_{I_k}(z)\, dz \int (M_{HL}h)(z+u)\, d\zeta(u) \\
&\quad \le c\alpha \left\|\sum_k \chi_{I_k}\right\|_2 \|M_{HL}h * d\tilde{\zeta}\|_2 \quad \big(d\tilde{\zeta} = d\zeta(-\cdot) \text{ a finite measure}\big) \\
&\quad \le c\alpha \left(\sum_k |I_k|\right)^{1/2} \|h\|_2 \le c\alpha^{1/2} \|f\|_1^{1/2} \|h\|_2,
\end{aligned}
$$

which proves (12) by duality. We have used the fact that if $\rho_t = t^{-n}\rho(x/t)$ is any approximation to the identity so that $\rho \in L^1(\mathbb{R}^n)$ is positive and decreasing, then

$$\sup_{t>0} |\rho_t * f(x)| \le M_{HL} f(x),$$

where M_{HL} denotes the Hardy–Littlewood maximal function on $\mathbb{R}^n$.

Therefore, (11) is established. In view of (7), the weak-(1,1) bound in (6) follows via the same argument above if, for given $\alpha > 0$, instead of decomposing f at height α, one decomposes f at height $\alpha/\max(\|\mu\|_{W^s_{2,sloc}}, \|\mu\|_\infty)$, see for example, [9] for details.

2.2. Besov and Triebel–Lizorkin spaces. For a general selfadjoint operator acting on $L^2(\mathbb{R}^n)$, one can define the associated Besov and Triebel–Lizorkin spaces [17, 22, 25]. Let $H = -\Delta + V$ on $\mathbb{R}^n$. Under the same conditions for H and μ as in Theorem 1.2, we can show that $\mu(H)$ is bounded on these generalized spaces (cf. Theorem 2.3, where W^s_2 is replaced with an abstract space). Like in the Fourier case [3, 32], the spectral multiplier theorems on them are closely related to some of the main results in Littlewood–Paley theory for H (interpolation, embedding and identification) [12, 18, 22, 24, 36].

Let $\alpha \in \mathbb{R}$ and $1 \le p, q \le \infty$. The homogeneous *Besov space* associated with H, denoted by $\dot{B}^{\alpha,q}_p(H)$, is defined to be the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, where the norm $\|\cdot\|_{\dot{B}^{\alpha,q}_p(H)}$ is given by

$$\|f\|_{\dot{B}^{\alpha,q}_p(H)} = \left(\sum_{j=-\infty}^{\infty} 2^{j\alpha q} \|\varphi_j(H) f\|_p^q \right)^{1/q}. \tag{13}$$

Similarly, the homogeneous *Triebel–Lizorkin space* $\dot{F}^{\alpha,q}_p(H)$ is defined by the norm

$$\|f\|_{\dot{F}^{\alpha,q}_p(H)} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{j\alpha q} |\varphi_j(H) f|^q \right)^{1/q} \right\|_p.$$

For $s \in \mathbb{R}$ let $X^s \subset \mathcal{S}'(\mathbb{R})$ be a Banach space endowed with a norm $\|\cdot\|_{X^s}$, where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions on $\mathbb{R}$. Further assume that $\{X^s\}_{s\in\mathbb{R}}$ satisfies the following properties.

(a) $C_0^\infty(\mathbb{R}) \subset X^s$, $\forall s$,
(b) $X^{1/2+\varepsilon} \subset L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, $\forall \varepsilon > 0$,
(c) $\|uv\|_{X^s} \le c\|u\|_{X^s}\|v\|_{X^s}$, $\forall u, v \in X^s$, $s > n/2$.

Examples of X^s include $W^s_p(\mathbb{R})$, $p \in (1,\infty)$, and $B^{s,q}_p(\mathbb{R})$, $p, q \in (1,\infty)$, the classical Sobolev and Besov spaces, see [3, Section 6.8] or [32].

THEOREM 2.3. *Suppose $H = -\Delta + V$ verifies Assumption 1.1*(a), (b) *with $X^s, X^{n+\varepsilon}$ replacing W^s_2, $W^{n+\varepsilon}$ respectively. Let μ satisfy for some $s > n/2$*

$$\|\mu\|_{X^s_*} := \sup_{t>0} \|\mu(t\cdot)\chi\|_{X^s} < \infty,$$

where χ *is a fixed function in* $C_0^\infty(\mathbb{R}\setminus\{0\})$. *Then* $\mu(H)$ *extends to a bounded operator on* $\dot{B}_p^{\alpha,q}(H)$ *for* $1<p<\infty$, $1\le q\le\infty$, $\alpha\in\mathbb{R}$ *and* $\dot{F}_p^{\alpha,q}(H)$ *for* $1<p<\infty$, $1<q<\infty$, $\alpha\in\mathbb{R}$.

Note that Theorem 1.2 holds under the same hypothesis in Theorem 2.3 with the same proof given in this section. The statement for $B_p^{\alpha,q}(H)$ follows immediately from (13). To show the statement for $F_p^{\alpha,q}(H)$, we need to prove, as a key step, that the operator $T_\mu := \{\mu_j(H)\}$ maps $L^1(\ell^q)$ continuously to weak-$L^1(\ell^q)$, where $\mu_j=\mu\varphi_j$ and T_μ is given by $\{f_j\}\mapsto\{\mu_j(H)f_j\}$. This can be achieved by a vector-valued version of the proof in Section 2.1. The details are presented in [26].

Under additional smoothness condition on V, one can identify $F_p^{\alpha,q}(H)=F_p^{2\alpha,q}(\mathbb{R}^n)$, which allows us to obtain the boundedness of $\mu(H)$ on $F_p^{\alpha,q}$ and $B_p^{\alpha,q}$ spaces on $\mathbb{R}^n$ according to Theorem 2.3, cf. [25, 32].

REMARK 2.4. We would like to mention that the boundedness of $\mu(H)$ on L^p, $1<p<\infty$, can also be obtained from wave operator method [11, 33, 34]. However our results give the endpoint estimate $L^1\to$ weak-L^1 and also the boundedness for $F_p^{\alpha,q}$ spaces (including Sobolev space), which consequently lead to interpolation and embedding results. The reason is that wave operator method can transfer the integrability but somehow lose the pointwise information.

3. Weighted L^∞ estimates: High energy

Let $V\in L^1(\mathbb{R})$. Then H_V has the form domain $W_2^1(\mathbb{R})$, whose absolute continuous spectrum $\sigma_{\rm ac}(H_V)=[0,\infty)$ and singular continuous spectrum is empty. The pure point spectrum $\sigma_{\rm pp}(H_V)$ is finite provided that $\int(1+|x|)|V|\,dx<\infty$. Let $H_{\rm pp}$ and $H_{\rm ac}$ denote the projections of H_V onto the pure point and absolute continuous subspaces of $L^2(\mathbb{R})$, respectively. From [30, Section C.3], we know that the eigenfunctions have exponential decay $\lesssim e^{-c|x|}$, $c>0$.[1] It follows that $\|\mu(H_{\rm pp})f\|\le c\|f\|_p$, $1\le p\le\infty$. Hence, in view of the remark following Theorem 1.2, it suffices to verify (3) and (8) for $H_{\rm ac}$ in place of H. As we will show, (8) is a result of Lemma 3.5 and Lemma 4.3, and (3) is a result of interpolation between Lemmas 5.1 and 5.2.

3.1. Kernel formula. Let $R_V(z)=(H_V-z)^{-1}$ be the resolvent of H_V, $z\in\mathbb{C}\setminus[0,\infty)$. For $\phi\in C(\mathbb{R})$, $\phi(H)$ has the resolvent expression [27, Section XIII.6]

$$\phi(H_{\rm ac})f(x)=\frac{1}{\pi}\int_0^\infty\phi(\lambda)\Im R_V(\lambda+i0)f\,d\lambda =\frac{1}{2\pi i}\int_0^\infty\phi(\lambda)[R_V(\lambda+i0)-R_V(\lambda-i0)]f\,d\lambda. \tag{14}$$

[1] $A\lesssim B$ stands for the usual notion $A\le cB$ for some absolute constant c.

Let $W(\lambda)$ be the Wronskian of f_+, f_-, then for $\lambda \neq 0$

$$R_V(\lambda^2 \pm i0)(x,y) = \begin{cases} \frac{f_+(x,\pm\lambda)f_-(y,\pm\lambda)}{W(\pm\lambda)}, & x > y, \\ \frac{f_+(y,\pm\lambda)f_-(x,\pm\lambda)}{W(\pm\lambda)}, & x < y, \end{cases}$$

where $f_\pm(x,z)$ are the Jost functions that solve for $\Im z \geq 0$

$$-f''_\pm(x,z) + V(x)f_\pm(x,z) = z^2 f_\pm(x,z) \tag{15}$$

and satisfy the asymptotics

$$f_\pm(x,z) \to \begin{cases} e^{\pm izx}, & x \to \pm\infty, \\ \frac{1}{t(z)}e^{\pm izx} + \frac{r_\mp(z)}{t(z)}e^{\mp izx}, & x \to \mp\infty, \end{cases}$$

$t(z), r_\pm(z)$ being the transmission and reflection coefficients respectively, see [13, 19].

Let $m_\pm(x,z) = e^{\mp izx} f_\pm(x,z)$ be the *modified Jost functions*. We obtain, from formula (14) of the spectral measure of H_{ac}, that

$$\phi(H_{\mathrm{ac}})(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\lambda^2) m_+(x,\lambda) m_-(y,\lambda) t(\lambda) e^{i\lambda(x-y)}\, d\lambda, \tag{16}$$

where $t(\lambda) = -2i\lambda/W(\lambda)$, see e.g. [19, 25].[2]

3.2. Fourier transforms of $m_\pm(x,k)$. The following lemma for $m_\pm$, t, $r_\pm$ are basically recorded from [13], see also [25]. Let $B_\pm(x,y)$ be the pair of functions satisfying the Marchenko equations in (28), (29).

LEMMA 3.1. *Let $V \in L^1_1$. Then*

$$\begin{aligned} m_+(x,k) &= 1 + \int_0^\infty B_+(x,y) e^{2iky}\, dy, \\ m_-(x,k) &= 1 + \int_{-\infty}^0 B_-(x,y) e^{-2iky}\, dy, \\ t(k)^{-1} &= 1 - \frac{1}{2ik}\int_{-\infty}^\infty V(t) m_\pm(t,k)\, dt \\ &= 1 - \frac{\nu}{2ik} - \frac{1}{2ik}\int_{-\infty}^\infty V(t)\, dt \int_0^\infty B_+(t,y)(e^{2iky} - 1)\, dy \\ &= 1 - \frac{\nu_0}{2ik} - \frac{1}{2ik}\int_{-\infty}^\infty V(t)\, dt \int_0^\infty B_+(t,y) e^{2iky}\, dy, \\ r_\pm(k) t(k)^{-1} &= \frac{1}{2ik}\int_{-\infty}^\infty e^{\mp 2ikt} V(t) m_\mp(t,k)\, dt, \end{aligned}$$

[2] Since the kernel formula coincides with the one using Lippmann–Schwinger scattering eigenfunctions, (16) is valid for both $x > y$ and $x < y$.

where $\nu_0 = \int_{-\infty}^{\infty} V(t)\,dt$ *and*

$$\nu := W(0) = \int_{-\infty}^{\infty} V(t) m_+(t,0)\,dt = \int_{-\infty}^{\infty} V(t)\,dt\biggl(1 + \int_0^{\infty} B_+(t,y)\,dy\biggr), \tag{17}$$

see [13, Remark 9, p. 152].

Let $\hat{f}(k) = \int f(x) e^{-ikx}\,dx$ and $g^\vee(x) = \int g(k) e^{ikx}\,dk$. The following lemma gives estimates on the Fourier transforms of $m_\pm$, which is an easy consequence of Lemmas 3.1 and 4.6(c).

LEMMA 3.2. *Let* $V \in L^1_1$. *Let* $x > 0, y < 0$. *Then there exists a constant* $c = c(\|V\|_{L^1_1})$ *independent of* x, y *such that* $\forall u$

$$|m_+(x,\pm\cdot)^\vee(u)| \le 2\pi\delta + c\chi_{\{\pm u<0\}}\rho^+(\mp u/2) \in \mathbb{R}_+\delta + L^1(\mathbb{R}_\mp),$$
$$|m_-(y,\pm\cdot)^\vee(u)| \le 2\pi\delta + c\chi_{\{\pm u<0\}}\rho^-(\pm u/2) \in \mathbb{R}_+\delta + L^1(\mathbb{R}_\mp),$$

where δ *is the Dirac measure at zero,* $\rho^+(u) = \int_u^\infty |V(t)|\,dt$, $\rho^-(u) = \int_{-\infty}^u |V(t)|\,dt$, $\mathbb{R}_+ = (0,\infty)$ *and* $\mathbb{R}_- = (-\infty, 0)$.

The next lemma provides series expansions for $t(k)$, $r_\pm(k)$ in the high energy, whose proofs will be postponed till the end of this section.

LEMMA 3.3. *Let* $V \in L^1_1$, *then there are* $a_\pm(\mathbb{R})$, $b \in L^1(\mathbb{R}_-)$ *such that for* $|k| > k_0 := k_0(\|V\|_{L^1_1}) > 1$

$$t(k) = 1 + \sum_{n=1}^{\infty} (2ik)^{-n}\bigl(\nu_0 + \hat{b}(k)\bigr)^n,$$

$$r_\pm(k) = \bigl(-\nu_0 + \hat{a}_\pm(k)\bigr)\sum_{n=1}^{\infty}(2ik)^{-n}\bigl(\nu_0 + \hat{b}(k)\bigr)^{n-1},$$

where $\nu_0 = \int V(t)\,dt$ *and* k_0 *is a fixed constant depending on* $\|V\|_{L^1_1}$ *and* $\|a_\pm\|_1$, $\|b\|_1 \le c(\|V\|_{L^1_1})$.

We will also need the relations between m_+ and m_- [13, Ch. 2, p. 144].

LEMMA 3.4. *Let* $V \in L^1_1$.

$$t(k) m_-(x,k) = e^{2ikx} r_+(k) m_+(x,k) + m_+(x,-k),$$
$$t(k) m_+(x,k) = e^{-2ikx} r_-(k) m_-(x,k) + m_-(x,-k).$$

3.3. High energy cutoff for $\Phi_j(H_{\mathrm{ac}})(x,y)$. We are ready to prove (8) for the high energy.

LEMMA 3.5. *Let* $V \in L^1_1$ *and* $\Phi \in C^\infty([-1,1])$ *as in* (10). *Then there exists a finite measure* $d\zeta_{high}$ *in* $\mathbb{R}_+\delta + L^1$ *such that for all* x, y *and* $j \ge j_0 := j_0(\|V\|_{L^1_1})$,

$$\bigl|\bigl((1-\Phi_{j_0})\Phi_j\bigr)(H_{\mathrm{ac}})(x,y)\bigr| \le \sum_{\pm} (\rho_0 + \rho_j) * d\zeta_{high}(\pm x \pm y), \tag{18}$$

where j_0 *is a fixed number depending on* $\|V\|_{L^1_1}$ *only,* $0 \le \rho_0(x) \le c_N(1+|x|)^{-N}$, $0 \le \rho_j(x) \le c_N 2^{j/2}(1+2^{j/2}|x|)^{-N}$, $\forall N$.

Proof. In the following we always assume $x > y$. The estimates for $x < y$ follow by symmetry. We divide the discussions into three cases. (a) $x>0, y<0$, (b) $x>0, y>0$, and (c) $x<0, y<0$.

Let $\tilde{\psi}_j(k) = (1-\Psi_{j_0}(k))\Psi_j(k)$, $\Psi_j(k) = \Phi_j(k^2)$. Let $j_0 := \max(2 + [2\log_2 k_0], 2\log_2 \|d\sigma\|_M)$, $d\sigma = |\nu_0|\delta + |b|$, where k_0, b are the same as in Lemma 3.3.

Case (a). $x>0$, $y<0$. According to (16) and Lemma 3.3, we have for $j \ge j_0$,

$$\begin{aligned} &2\pi\big((1-\Phi_{j_0})\Phi_j\big)(H_{\text{ac}})(x,y) \\ &\quad = \sum_{n=0}^{\infty}(1/2i)^n \int \tilde{\psi}_j(k)k^{-n}\big(\nu_0+\hat{b}(k)\big)^n m_+(x,k)m_-(y,k)e^{i(x-y)k}\,dk \\ &\quad := \sum_{n=0}^{\infty} I_n(x,y). \end{aligned}$$

By Lemma 3.2, if $x>0$, $y<0$,

$$|m_+(x,\cdot)^\vee * m_-(y,\cdot)^\vee(u)| \le d\zeta_0 := c\delta + \rho_1 \in \mathbb{R}_+\delta + L^1(\mathbb{R}_-).$$

If $n=0$,

$$\begin{aligned} |I_0(x,y)| &= \frac{1}{4\pi^2}\left|\int \tilde{\psi}_j^\vee(x-y-u)m_+(x,\cdot)^\vee * m_-(y,\cdot)^\vee(u)\,du\right| \\ &\le \int |\tilde{\psi}_j^\vee(x-y-u)|\,d\zeta_0(u), \end{aligned}$$

where since $\Psi \in C_0^\infty$, we have

$$|\tilde{\psi}_j^\vee(x)| \le 2^{j_0/2}(1+2^{j_0/2}|x|)^{-N} + 2^{j/2}(1+2^{j/2}|x|)^{-N}$$

by writing

$$\begin{aligned} \tilde{\psi}_j^\vee(\eta) &= \Psi_j^\vee(\eta) - \Psi_{j_0}^\vee(\eta) \\ &= 2^{j/2}\Psi^\vee(2^{j/2}\eta) - 2^{j_0/2}\Psi^\vee(2^{j_0/2}\eta). \end{aligned} \tag{19}$$

For $n=1$, observe that

$$\begin{aligned} (\tilde{\psi}_j(k)k^{-1})^\vee(\xi) &= \frac{1}{2i}\left(\int_\xi^\infty \tilde{\psi}_j^\vee(u)\,du - \int_{-\infty}^\xi \tilde{\psi}_j^\vee(u)\,du\right) \\ &= -i\int_\xi^\infty \tilde{\psi}_j^\vee(u)\,du = i\int_{-\infty}^\xi \tilde{\psi}_j^\vee(u)\,du, \end{aligned} \tag{20}$$

where $\int \tilde{\psi}_j^\vee(u)\,du = 2\pi\tilde{\psi}_j(0) = 0$. It is easy to see from (20) and (19) that for each $N \in \mathbb{N}$ there exists a constant $c_N > 0$ such that for all $j \geq j_0$,

$$\int_\xi^\infty |\tilde{\psi}_j^\vee(\eta)|\,d\eta \leq c_N(1+|\xi|)^{-N} \quad \forall \xi.$$

Thus,

$$\begin{aligned}
|I_1(x,y)| &= \left|\int \tilde{\psi}_j(k)k^{-1}\big(\nu_0 + \hat{b}(k)\big)m_+(x,k)m_-(y,k)e^{i(x-y)k}\,dk\right| \\
&\leq \int |(\tilde{\psi}_j k^{-1})^\vee(x-y-u)|(|\nu_0|\delta + |b|) * d\zeta_0(u) \\
&\leq c_N \int (1+|x-y-u|)^{-N}\,d\zeta_1(u),
\end{aligned}$$

where $d\zeta_1 = (|\nu_0|\delta + |b|) * d\zeta_0$ is in $\mathbb{R}_+\delta + L^1(\mathbb{R}_-)$.

If $n \geq 2$, we have by integration by parts: for $j \geq j_0$, $N \geq 1$,

$$\begin{aligned}
&\left|(1+\xi^N)\int \tilde{\psi}_j(k)k^{-n}e^{ik\xi}\,dk\right| \\
&\quad = \left|\int_{2^{(j_0-1)/2} \leq |k| \leq 2^{j/2}} e^{ik\xi}(1+i^N\partial_k^N)[\tilde{\psi}_j(k)k^{-n}]\,dk\right| \\
&\quad \leq c_{j_0,N}n^{N-1}2^{-(j_0-1)n/2} \quad \forall \xi.
\end{aligned}$$

Hence, with $d\sigma = |\nu_0|\delta + |b(u)|$,

$$\begin{aligned}
&\sum_{n=2}^\infty |I_n(x,y)| \\
&\quad \leq c_N \sum_{n=2}^\infty n^{N-1}2^{-j_0 n/2}2^{-n/2}\int (1+|x-y-u|)^{-N}\overbrace{d\sigma * \cdots * d\sigma}^{n} * d\zeta_0(u) \\
&\quad \leq c_N \int (1+|x-y-u|)^{-N}\,d\tilde{\zeta}(u),
\end{aligned}$$

where by our choice $j_0 > 2\log_2 \|d\sigma\|_M - 1$ so that

$$d\tilde{\zeta} := \sum_{n=2}^\infty n^{N-1}2^{-j_0 n/2}2^{-n/2}\overbrace{d\sigma * \cdots * d\sigma}^{n} * d\zeta_0(u)$$

is a finite measure in $\mathbb{R}_+\delta + L^1$. Combining the the above estimates for $I_n(x,y)$, $n = 0, 1$ and ≥ 2, we thus establish (18) in Case (a).

Case (b). $x > y > 0$. By (16) and Lemma 3.4

$$\begin{aligned} &2\pi\big((1-\Phi_{j_0})\Phi_j\big)(H_{\mathrm{ac}})(x,y) \\ &= \int \tilde{\psi}_j^{\vee}(x+y-u)(r_+(\cdot)m_+(x,\cdot)m_+(y,\cdot))^{\vee}(u)\,du \\ &\qquad + \int \tilde{\psi}_j^{\vee}(x-y-u)(m_+(x,\cdot)m_+(y,-\cdot))^{\vee}(u)\,du. \end{aligned}$$

Similar to Case (a), using Lemma 3.2 and the formula for $r_+(k)$ in Lemma 3.3 we obtain that there exists some finite measure $d\zeta_2 \in \mathbb{R}_+\delta + L^1$ so that for all $x > 0, y > 0$ and $j \geq j_0$,

$$\begin{aligned} &\big|\big((1-\Phi_{j_0})\Phi_j\big)(H_{\mathrm{ac}})(x,y)\big| \\ &\leq \int |\tilde{\psi}_j^{\vee}(x-y-u)|\,d\zeta_2(u) + c_N \sum_{\pm} \int (1+|x\pm y-u|)^{-N}\,d\zeta_2(u). \end{aligned}$$

Case c. $0 > x > y$. Similar to Case (b), we obtain that there exists some finite measure $d\zeta_3 \in \mathbb{R}_+\delta + L^1$ so that for all $0 > x > y$ and $j \geq j_0$,

$$\begin{aligned} &\big|\big((1-\Phi_{j_0})\Phi_j\big)(H_{\mathrm{ac}})(x,y)\big| \\ &\leq c_N \int (1+|x+y+u|)^{-N}\,d\zeta_3(u) + \int |\tilde{\psi}_j^{\vee}(x-y-u)|\,d\zeta_3(u). \qquad \square \end{aligned}$$

3.4. Proof of Lemma 3.3. By Lemma 3.1, if $|k| > k_0 = k_0(\|V\|_{L^1_1})$ large enough, we have a geometric series expansion

$$\begin{aligned} t(k) &= \left(1 - \frac{\nu_0}{2ik} - \frac{1}{2ik}\int V(t)\,dt \int_0^{\infty} B_+(t,y)e^{2iky}\,dy\right)^{-1} \\ &= \sum_{n=0}^{\infty} (2ik)^{-n}\left(\nu_0 + \int V(t)\,dt\int_0^{\infty} B_+(t,y)e^{2iky}\,dy\right)^n \\ &= \sum_{n=0}^{\infty} (2ik)^{-n}\big(\nu_0 + \hat{b}(k)\big)^n, \end{aligned}$$

where $\hat{b}(k) = \hat{\beta}(-2k)$ and $\beta(y) = \int V(t)\chi_{(0,\infty)}(y)B_+(t,y)\,dt$, which is in $L^1(\mathbb{R}_+)$ by Lemma 4.6(a).

Let

$$\alpha_{\pm}(k) = \big(1 + r_{\pm}(k)\big)t(k)^{-1}, \tag{21}$$

then there exist $a_{\pm} \in L^1(\mathbb{R})$ such that

$$\alpha_{\pm}(k) = 1 - \frac{\nu_0}{2ik} + \frac{1}{2ik}\hat{a}_{\pm}(k) \quad \forall k \neq 0. \tag{22}$$

Indeed, similar to the way we deal with $t(k)$, write

$$\begin{aligned}
\alpha_+(k) &= 1 + \frac{1}{2ik}\int (e^{-2ikt}-1)V(t)m_-(t,k)\,dt \\
&= 1 + \frac{1}{2ik}\int (e^{-2ikt}-1)V(t)\,dt \\
&\quad + \frac{1}{2ik}\int (e^{-2ikt}-1)V(t)\,dt\int_{-\infty}^{0} B_-(t,y)e^{-2iky}\,dy \\
&= 1 - \frac{\nu_0}{2ik} + \frac{1}{2ik}\hat{V}(2k) \\
&\quad + \frac{1}{2ik}\Bigg[\bigg(\chi_{(-\infty,\infty)}(y)\int_y^{\infty} V(t)B_-(t,y-t)\,dt\bigg)^{\wedge}(2k) \\
&\quad - \bigg(\chi_{(-\infty,0)}(y)\int V(t)B_-(t,y)\,dt\bigg)^{\wedge}(2k)\Bigg].
\end{aligned}$$

It is easy to see from Lemma 4.6(a) that the last two functions of y in the parentheses are in L^1 if $V\in L^1_1$. Thus, (22) holds for $\alpha_+(k)$ with some $a_+\in L^1$, and so

$$\begin{aligned}
r_+(k) &= \alpha_+(k)t(k) - 1 \\
&= \big(-\nu_0 + \hat{a}_+(k)\big)\sum_{n=1}^{\infty}(2ik)^{-n}\big(\nu_0+\hat{b}(k)\big)^{n-1}.
\end{aligned}$$

Similarly, we obtain the formulas for $\alpha_-(k)$ and $r_-(k)$.

4. Weighted L^∞ estimates: Low energy

In this section, we prove (8) for $\Phi_j(H_{\mathrm{ac}})(x,y)$ for $j<j_0$, where j_0 is taken to be the same number as in Lemma 3.5. Recall that $\Psi_j(k)=\Phi_j(k^2)=\Phi(2^{-j}k^2)$. The following lemma gives Fourier transform formulas of $t, r_\pm$ for the low energy.

LEMMA 4.1. (a) *Let $V\in L^1_1$ and $\nu\neq 0$. Then there exist $f_1, g_{1,\pm}\in L^1$ such that for all $j<j_0$,*

$$\begin{aligned}
(\Psi_j(k)t(k))^\vee(u) &= \Psi_j^\vee * f_1(u), \\
(\Psi_j(k)r_\pm(k))^\vee(u) &= \Psi_j^\vee * (g_{1,\pm}-\delta)(u),
\end{aligned}$$

equivalently,

$$\begin{aligned}
\Psi_j(k)t(k) &= \Psi_j(k)\hat{f}_1(k), \\
\Psi_j(k)r_\pm(k) &= \Psi_j(k)\big(\hat{g}_{1,\pm}(k)-1\big).
\end{aligned}$$

(b) *Let $V\in L^1_2$ and $\nu=0$. Then there exist $f, g_\pm$ in L^1 such that*

$$\begin{aligned}
t(k) &= 1+\hat{f}(k), \\
r_\pm(k) &= \hat{g_\pm}(k).
\end{aligned}$$

We postponed the proof till Sections 4.2 and 4.3. Combining Lemma 4.1 and Lemma 3.2, we readily obtain the following lemma.

LEMMA 4.2. (a) *Let* $V \in L_1^1$ *and* $\nu \neq 0$. *Then there exist positive functions* h_1, h_2 *and* h_3 *in* L^1 *independent of* x, y *such that*:

(i) $\forall x > 0, y < 0$,

$$|(\Psi_j(k)t)^\vee * m_+(x,\cdot)^\vee * m_-(y,\cdot)^\vee(u)| \lesssim |\Psi_j^\vee| * (\delta + h_1)(u),$$

(ii) $\forall x > 0, y > 0$,

$$|(\Psi_j(k)r_+)^\vee * m_+(x,\cdot)^\vee * m_+(y,\cdot)^\vee(u)| \lesssim |\Psi_j^\vee| * (\delta + h_2)(u),$$

(iii) $\forall x < 0, y < 0$,

$$|(\Psi_j(k)r_-)^\vee * m_-(x,\cdot)^\vee * m_-(y,\cdot)^\vee(u)| \lesssim |\Psi_j^\vee| * (\delta + h_3)(u).$$

(b) *Let* $V \in L_2^1$ *and* $\nu = 0$. *Then there exist positive functions* f_1, f_2 *and* f_3 *in* L^1, *independent of* x, y, *such that*:

(i) $\forall x > 0, y < 0$,

$$|t^\vee * m_+(x,\cdot)^\vee * m_-(y,\cdot)^\vee(u)| \lesssim \delta + f_1(u),$$

(ii) $\forall x > 0, y > 0$,

$$|r_+^\vee * m_+(x,\cdot)^\vee * m_+(y,\cdot)^\vee(u)| \lesssim \delta + f_2(u),$$

(iii) $\forall x < 0, y < 0$,

$$|r_-^\vee * m_-(x,\cdot)^\vee * m_-(y,\cdot)^\vee(u)| \lesssim \delta + f_3(u).$$

Thus, the estimate in (8) for the low energy cutoff follows from Lemma 4.2 by proceeding the way similar to (but much simpler than) the high energy case in Section 3.3.

LEMMA 4.3. *Let* $V \in L_1^1$ *and* H_V *has no resonance at zero or* $V \in L_2^1$. *Then there exist a finite measure* $d\zeta_{low} \in \mathbb{R}_+\delta + L^1$ *such that for all* $j < j_0$

$$|\Phi_j(H_{\mathrm{ac}})(x,y)| \leq c \sum_{\pm} \int |\Psi_j^\vee(\pm x \pm y - u)| \, d\zeta_{low}(u).$$

The detail of the proof is straightforward and hence omitted.

4.1. Fourier transforms of $t(k)$**,** $r_\pm(k)$**.** Lemma 4.1 tells that in the cases of $V \in L_1^1$, $\nu \neq 0$ and $V \in L_2^1$, low energy cut-offs of $t(k), r_\pm(k)$ are the Fourier transforms of L^1 functions up to $c\delta$. We will show that this is true by Wiener's lemma [23, Lemma 6.3].

LEMMA 4.4 (Wiener). *Let* $f, h \in L^1(\mathbb{R})$. *Suppose* $\operatorname{supp} \hat{f}$ *is compact and* $\hat{h}$ *is nonzero on* $\operatorname{supp} f$. *Then there exists some* $g \in L^1(\mathbb{R})$ *such that* $\hat{f} = \hat{h}\hat{g}$ *or* $\hat{f}/\hat{h} = \hat{g}$.

The following variant of Wiener's lemma can be found in, for example, [31, Ch. V, $S3$].

LEMMA 4.5. *Let $g \in L^1(\mathbb{R})$ such that $\hat{g}(x)+1$ is nonzero for all x. Then there exists a function $f \in L^1(\mathbb{R})$ such that*

$$\hat{f}(x)+1=\frac{1}{\hat{g}(x)+1}.$$

Recall from [13, Theorem 1] that (i) if $\nu=0$, then $t(k)\neq 0$, $\forall k$. (ii) if $\nu \neq 0$, then $t(0)=0$ but $t(k)\neq 0$, $\forall k \neq 0$ (cf. also Lemma 4.8).

Since $W(k)=-2ik/t(k)$, by Lemma 3.1

$$W(k)=-2ik\left(1-\frac{\nu}{2ik}-\int V(t)\,dt\int_0^\infty B_+(t,y)\frac{e^{2iky}-1}{2ik}dy\right) \tag{23}$$

$$=-2ik\left(1-\frac{\nu_0}{2ik}-\int V(t)\,dt\int_0^\infty B_+(t,y)\frac{e^{2iky}}{2ik}\,dy\right). \tag{24}$$

4.2. Proof of Lemma 4.1(a). In this case $\nu=W(0)\neq 0$, hence $W(k)\neq 0$, $\forall k$. Write

$$\Psi_{j_0}(k)t(k)=\frac{-2ik\Psi_{j_0}(k)}{\chi(k)W(k)}, \tag{25}$$

where we take $\chi \in C_0^\infty$ with $\chi(x)=1$ on $\operatorname{supp}\Psi_{j_0}$. From (24), we have

$$W(k)=-2ik+\nu_0+\left(\chi_{(0,\infty)}(\cdot)\int V(t)B_+(t,\cdot)\,dt\right)^\vee(2k),$$

where we note that in terms of Lemma 4.6(a), the function $y\mapsto \chi_{(0,\infty)}(y)\times \int V(t)B_+(t,y)\,dt$ is in L^1 provided $V\in L^1_1$. Thus, we find that χW, which is nonzero on the support of Ψ_{j_0}, is the Fourier transform of an L^1 function. According to Wiener's lemma (Lemma 4.4),

$$\Psi_{j_0}(k)t(k)=\hat{f}_1(k) \tag{26}$$

for some $f_1\in L^1$. Hence, for $j<j_0$

$$(\Psi_j(k)t(k))^\vee=c\Psi_j^\vee*(\Psi_{j_0}t(k))^\vee=\Psi_j^\vee*f_1,$$

where note that $\Phi_{j_0}(k)\equiv 1$ on support of $\Phi_j(k)$.

Let $\alpha_\pm(k)=(1+r_\pm(k))t(k)^{-1}$, then

$$\Psi_j(k)r_\pm(k)=\Psi_j(k)\alpha_\pm(k)t(k)-\Psi_j(k).$$

It is sufficient to deal with the first term. By (22), there exist $a_\pm\in L^1$ such that

$$2ik\alpha_\pm(k)=2ik-\nu_0+\hat{a}_\pm(k).$$

Thus, $\Psi_{j_0}(k)(-2ik)\alpha_\pm(k) = \hat{g}_{0,\pm}(k)$ for some $g_{0,\pm} \in L^1$. We have for $j < j_0$,

$$
\begin{aligned}
(\Psi_j(k)\alpha_\pm(k)t(k))^\vee &= c\Psi_j^\vee * \left(\Psi_{j_0}(k)\frac{t(k)}{-2ik}\right)^\vee * (\Psi_{j_0}(k)(-2ik)\alpha_\pm(k))^\vee \\
&= c\Psi_j^\vee * \left(\frac{\Psi_{j_0}(k)}{W(k)}\right)^\vee * g_{0,\pm} \\
&= \Psi_j^\vee * g_{1,\pm}, \quad g_{1,\pm} = cf_0 * g_{0,\pm},
\end{aligned}
$$

where in view of (25), the same way as showing (26) we see that $\frac{\Psi_{j_0}(k)}{W(k)} = \hat{f}_0$ for some $f_0 \in L^1$. This proves that for $j < j_0$

$$\Psi_j(k)r_\pm(k) = \Psi_j(k)\big(\hat{g}_{1,\pm}(k) - 1\big).$$

4.3. Proof of Lemma 4.1(b). First, we observe the following formula when $\nu = 0$,

$$
\begin{aligned}
t(k)^{-1} - 1 &= -\int V(t)\,dt \int_0^\infty \left(\int_\xi^\infty B_+(t,y)\,dy\right) e^{2ik\xi}\,d\xi \\
&= -\left(\chi_{(0,\infty)}(\xi)\int V(t)\,dt\int_\xi^\infty B_+(t,y)\,dy\right)^\vee (2k).
\end{aligned}
\tag{27}
$$

Indeed, since $\nu = 0$, we have by (23)

$$t(k)^{-1} = 1 - \int V(t)\,dt \int_0^\infty B_+(t,y)\frac{e^{2iky}-1}{2ik}\,dy.$$

Then (27) follows by using $\frac{e^{2iky}-1}{2ik} = \int_0^y e^{2ik\xi}\,d\xi$ and Fubini theorem.

Since $V \in L_2^1$, Lemma 4.6(b) implies that the function given by $\xi \mapsto \chi_{(0,\infty)}(\xi)\int V(t)\,dt\int_\xi^\infty B_+(t,y)\,dy$ belongs to L^1. Hence, $t(k)^{-1} - 1 = \hat{g}_0(k)$ for some $g_0 \in L^1$.

Now by Lemma 4.5 there exists $h \in L^1$ (evidently $1 + \hat{g}_0(k) = t(k)^{-1} \neq 0$, $\forall k$) so that

$$t(k) = \frac{1}{1 + \hat{g}_0(k)} = 1 + \hat{h}(k).$$

A similar argument shows that there exists some $\omega_\pm \in L^1$ so that $\alpha_\pm(k) = (1 + r_\pm(k))t(k)^{-1} = 1 + \hat{\omega}_\pm(k)$ by applying Lemma 4.6. Therefore, $r_\pm = \alpha_\pm t - 1 = \hat{\omega}_\pm + \hat{h} + \hat{\omega}_\pm\hat{h}$ are in $(L^1)^\wedge$.

4.4. Marchenko equation. From Lemma 3.1, we know that for each x, $B_\pm(x,y)$ are the Fourier transforms of $m_\pm(x,\pm k) - 1$. They are real-valued, supported in $\mathbb{R}_\pm$ and belong to $L^2(\mathbb{R}_\pm)$ [13, 33]. Moreover, $B_\pm(x,y)$ satisfy the *Marchenko equations*

$$B_+(x,y) = \int_{x+y}^{\infty} V(t)\,dt + \int_0^y dz \int_{t=x+y-z}^{\infty} V(t)B_+(t,z)\,dt, \tag{28}$$

$$B_-(x,y) = \int_{-\infty}^{x+y} V(t)\,dt + \int_y^0 dz \int_{-\infty}^{t=x+y-z} V(t)B_-(t,z)\,dt. \tag{29}$$

The following lemma is mainly on certain weighted L^1 inequalities for $B_\pm$, which contributes to several kernel estimates as we have seen.

LEMMA 4.6. (a) *If $V \in L^1_1$, then there exists $c = c(\|V\|_{L^1_1})$ so that for all $x \in \mathbb{R}$*

$$\int_{-\infty}^{\infty} |B_\pm(x,y)|\,dy \le c\big(1 + \max(0, \mp x)\big). \tag{30}$$

(b) *If $V \in L^1_2$, then there exists $c = c(\|V\|_{L^1_2})$ so that for all $x \in \mathbb{R}$*

$$\int_{-\infty}^{\infty} |y||B_\pm(x,y)|\,dy \le c\big(1 + \max(0, \mp x)\big)^2. \tag{31}$$

(c) *Let $V \in L^1_1$, then for all $x, y \in \mathbb{R}$*

$$|B_\pm(x,y)| \le e^{\gamma^\pm(x)}\rho^\pm(x+y), \tag{32}$$

where $\gamma^+(x) = \int_x^\infty (t-x)|V(t)|\,dt$, $\gamma^-(x) = \int_{-\infty}^x (x-t)|V(t)|\,dt$, $\rho^\pm$ are as in Lemma 3.2.

Estimate (c) is known, see, for example, [13] or [19]. The estimates in (a), (b) were obtained in [11, Lemma 3.2, Lemma 3.3] using Gronwall's inequality. See also [25, Lemma 4.5] for a generalized version of these inequalities for $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$

$$\int_{-\infty}^{\infty} |y|^n |B_\pm(x,y)|\,dy \le c\big(1 + \max(0, \mp x)\big)^{n+1}, \tag{33}$$

$c = c(\|V\|_{L^1_{n+1}})$, which follows from direct iterations of (28) and (29).

4.5. Modified Jost functions. Let $h(x,k) = \frac{e^{2ikx}-1}{2ik}$. It is well known that $m_\pm(x,k)$ satisfy the equations

$$m_+(x,k) = 1 + \int_x^\infty h(t-x,k)V(t)m_+(t,k)\,dt, \tag{34}$$

$$m_-(x,k) = 1 + \int_{-\infty}^x h(x-t,k)V(t)m_-(t,k)\,dt. \tag{35}$$

LEMMA 4.7. *Let $V \in L^1_1$. Then*

$$|m_\pm(x,k)| \le c\big(1 + \max(0, \mp x)\big),$$

$$|\dot{m}_\pm(x,k)| \le c\frac{1 + \max(0, \mp x)}{|k|} \quad \forall k \neq 0,$$

where $c = c(\|V\|_{L^1_1})$.

LEMMA 4.8. (a) *Let* $V \in L^1_1$. *Then*: (i) $|t(k)| \le 1, |r_\pm(k)| \le 1$. (ii) $\dot t(k) = O(1/k), \dot r_\pm(k) = O(1/k)$ *as* $|k| \to \infty$. (iii) *If* $\nu \ne 0$, *then* $t(k) = O(k)$ *as* $k \to 0$.

(b) *Let* $V \in L^1_1$ *and* $\nu \ne 0$ *or* $V \in L^1_2$. *Then*

$$\dot t(k) = O(1/k), \qquad \dot r_\pm(k) = O(1/k) \quad \text{as } k \to 0.$$

The asymptotics in Lemmas 4.7 and 4.8(a) are known, see [13, Lemma 1, p. 130] or [33]. We will give the proof of Lemma 4.8(b) below.

4.6. Proof of Lemma 4.8(b). (A) Let $V \in L^1_1$ and $\nu \ne 0$. By Lemma 3.1, we have

$$t(k)^{-1} = 1 - \frac{1}{2ik} \int V(t) m_+(t,k)\, dt.$$

Taking derivative in k and applying Lemma 4.7 give

$$|\partial_k(t(k)^{-1})| \le c/k^2 \quad \forall k \ne 0,$$

thus,

$$\dot t(k) = -t(k)^2 \partial_k (t(k)^{-1}) = \begin{cases} O(1), & |k| < 1, \\ O(1/k^2), & |k| \ge 1, \end{cases} \tag{36}$$

where we used $t(k) = O(k)$, $k \to 0$ if $\nu \ne 0$, by Lemma 4.8(a). From (21) and Lemma 3.1, we see

$$r_\pm(k) = t(k)\left(1 + \int h(\mp t, k) V(t) m_\pm(t,k)\, dt\right) - 1.$$

Now the estimate

$$|\dot r_\pm(k)| \le c/|k|$$

can be established by using (36), the estimates

$$|h(t,k)| \le \min(|t|, 1/|k|),$$
$$|\dot h(t,k)| \le 2\frac{|t|}{|k|},$$

Lemma 4.7 and Lemma 3.4.

(B) Let $V \in L^1_2$ and $\nu = 0$. By (27), integrating by parts we have

$$\begin{aligned} k\partial_k(t(k)^{-1}) &= -\int V(t)\, dt \int_0^\infty \left(\int_\xi^\infty B_+(t,\eta)\, d\eta\right) \xi\, d_\xi(e^{2ik\xi}) \\ &= \int V(t)\, dt \int_0^\infty e^{2ik\xi} \left(\int_\xi^\infty B_+(t,\eta)\, d\eta - \xi B_+(t,\xi)\right) d\xi, \end{aligned}$$

where note that $|\xi \int_\xi^\infty B_+(t,\eta)\,d\eta| \le \int_\xi^\infty |\eta B_+(t,\eta)|\,d\eta \to 0$ as $\xi\to\infty$ by virtue of Lemma 4.6(b). Then Fubini theorem and again Lemma 4.6(b) give

$$\begin{aligned}|k\partial_k(t(k)^{-1})| &\le \int |V(t)|\,dt \left(\int_0^\infty \left(\int_\xi^\infty |B_+(t,\eta)|\,d\eta + \xi|B_+(t,\xi)|\right) d\xi\right)\\ &\le c\int (1+|t|)^2|V(t)|\,dt.\end{aligned}$$

Thus,

$$\dot t(k) = -t(k)^2\partial_k(t(k)^{-1}) = O(1/k). \tag{37}$$

Finally, the estimate $\dot r_\pm(k) = O(1/k)$ follows from those routine asymptotics for $\dot h(t,k), \dot m_\pm(x,k)$ and (37).

Remark 4.9. The proof in Part A actually has shown the asymptotics for $t(k), r_\pm(k)$ for both $k\to 0$ and $|k|\to\infty$, for the latter we only require $V\in L_1^1$.

5. Weighted L^2 estimates

We will prove (3) for H_{ac} with $s=1$ and $s=0$ (Lemmas 5.1 and 5.2). Then the case $1/2<s<1$ follows via interpolation. The case $s=1$ requires certain improved asymptotics for k-derivatives of $m_\pm(x,k)$ $t(k), r_\pm(k)$ than [13, p. 134]. In the most difficult (and subtle) case (Case c), we use the Volterra type expansions for $\dot m_\pm(x,k)$[3] in order to deal with the inconsistency of the weight and distorted phase.

In the following, we use the abbreviations $H^s = W_2^s$, $\dot H^s = \dot W_2^s$, and so,

$$\begin{aligned}\|f\|_{H^s(\mathbb{R})} &= \|(1+|\xi|^2)^{s/2}\hat f\|_2,\\ \|f\|_{\dot H^s(\mathbb{R})} &= \||\xi|^s\hat f\|_2 \approx \|(-\Delta)^s f\|_2.\end{aligned} \tag{38}$$

Lemma 5.1. *Let $V\in L_1^1$, $\nu\ne 0$ or $V\in L_2^1$. Then for all y, $j\in\mathbb{Z}$ and $\phi\in H^1([\frac14,1])$,*

$$\|(x-y)\phi_j(H_{\text{ac}})(x,y)\|_{L_x^2} \le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])}. \tag{39}$$

Proof. Let $\psi_j(k) := \phi_j(k^2) = \phi(2^{-j}k^2)$. By symmetry, we will only need to show (39) in the following three cases:

$$\forall y<0,\quad \left\|\chi_{\{x>0\}}(x-y)\phi_j(H)(x,y)\right\|_{L_x^2} \le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])}, \tag{39a}$$

$$\forall y>0,\quad \left\|\chi_{\{x>y\}}(x-y)\phi_j(H)(x,y)\right\|_{L_x^2} \le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])}, \tag{39b}$$

$$\forall y<0,\quad \left\|\chi_{\{y<x<0\}}(x-y)\phi_j(H)(x,y)\right\|_{L_x^2} \le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])}. \tag{39c}$$

Case (a) $x>0$, $y<0$. Using the formula in Lemma 3.1

$$m_+(x,k) = 1 + \int_0^\infty B_+(x,u)e^{2iku}\,du, \tag{40}$$

[3] We will follow the convention that $\dot f(k) = \partial_k f(k)$.

we write

$$\begin{aligned} 2\pi\phi_j(H_{\rm ac})(x,y) &= \int \psi_j(k)t(k)m_-(y,k)e^{i(x-y)k}\,dk \\ &\quad + \int \psi_j(k)t(k)m_-(y,k)e^{i(x-y)k}\,dk \int_0^\infty B_+(x,u)e^{2iku}\,du \\ &:= I_1^a(x,y) + I_2^a(x,y). \end{aligned}$$

In view of (38), we have

$$\|(x-y)I_1^a(x,y)\|_{L^2_x} = \|\psi_j(k)t(k)m_-(y,k)\|_{\dot H^1_k} \le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])},$$

where we have used the following estimates by Lemmas 4.7 and 4.8: For $i=0,1$,

$$\begin{cases} \|\partial_k^i\psi_j\|_{L^2} \le c2^{-\frac{j}{2}(i-\frac12)}\|\phi\|_{H^1([\frac14,1])}, \\ \partial_k^i t(k) = O(1/k^i), \\ \partial_k^i m_-(y,k) = O(1/k^i), \quad \forall y<0. \end{cases}$$

Applying Minkowski inequality and Lemma 4.6(c) for $x>0$, we obtain by (38) that for each $y<0$,

$$\begin{aligned} &\|\chi_{\{x>0\}}(x-y)I_2^a(x,y)\|_{L^2_x} \\ &= \left\| \int_0^\infty B_+(x,u)\,du(x-y)\int \psi_j(k)t(k)m_-(y,k)e^{i(x-y+2u)k}\,dk \right\|_{L^2_{\{x>0\}}} \\ &\le \int_0^\infty \rho^+(u)\,du \left\| (x-y+2u)\int \psi_j(k)t(k)m_-(y,k)e^{i(x-y+2u)k}\,dk \right\|_{L^2_{\{x>0\}}} \\ &\le \int_0^\infty \rho^+(u)\,du\, \|\psi_j(k)t(k)m_-(y,k)\|_{\dot H^1_k} \le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])}. \end{aligned}$$

So combining the estimates for I_1^a and I_2^a gives (39a).

Case (b) $x>y>0$. By Lemma 3.4, we write

$$\begin{aligned} 2\pi\phi_j(H_{\rm ac})(x,y) &= \int \psi_j(k)r_+(k)m_+(x,k)m_+(y,k)e^{i(x+y)k}\,dk \\ &\quad + \int \psi_j(k)m_+(x,k)m_+(y,-k)e^{i(x-y)k}\,dk. \end{aligned}$$

By (40), we see that (39b) can be proved as in Case a by applying Lemmas 4.6(c), 4.7 and 4.8.

Case (c) $y<x<0$. Using Lemma 3.4, we write

$$\begin{aligned} 2\pi\phi_j(H_{\rm ac})(x,y) &= \int \psi_j(k)r_-(k)m_-(x,k)m_-(y,k)e^{-i(x+y)k}\,dk \\ &\quad + \int \psi_j(k)m_-(x,-k)m_-(y,k)e^{i(x-y)k}\,dk \\ &:= I_1^c(x,y) + I_2^c(x,y). \end{aligned}$$

The term I_1^c can be dealt with in a way similar to Case (a) or (b). We can estimate $\|\chi_{\{y<x<0\}}(x-y)I_1^c(x,y)\|_2$ by writing

$$m_-(x,k) = 1 + \int_{-\infty}^0 B_-(x,u)e^{-2iku}\,du$$

(cf. Lemma 3.1, [13, p. 137] or [33]) and using Lemma 4.6(c) and the estimates for $m_-(y,k)$, $r_-(k)$ in Lemmas 4.7 and 4.8, where we have observed if $y < x < 0$, then $|x-y| \le |x+y|$ and

$$|x-y| \le |x+y+2u| \quad \forall u<0.$$

For I_2^c, if following the same line one would have to require for all $y<x<0$ and $u<0$

$$|x-y| \le |x+y-2u|,$$

which is unfortunately not valid. Here, we proceed by exploiting the expansion of $m_-(x,-k)$ as follows. Iterating (35), we write with $t_0 = x$

$$\begin{aligned} m_-(x,-k) \;&=\; 1 + \sum_{n=1}^{\infty} \int_{-\infty}^{t_0} h(t_0-t_1,-k)V(t_1)\,dt_1 \\ &\qquad \times \int_{-\infty}^{t_1} h(t_1-t_2,-k)V(t_2)\,dt_2 \cdots \\ &\qquad \times \int_{-\infty}^{t_{n-1}} h(t_{n-1}-t_n,-k)V(t_n)\,dt_n \\ &:= \sum_{n=0}^{\infty} M_n^-(x,k). \end{aligned}$$

Observe that

$$h(x-t,-k) = \int_0^{x-t} e^{-2iku}\,du, \tag{41}$$

$$\partial_k h(x-t,-k) = \frac{1}{k}\left((x-t)e^{-2ik(x-t)} - \int_0^{x-t} e^{-2iku}\,du\right). \tag{42}$$

We have by integration by parts

$$\begin{aligned} -i(x-y)I_2^c(x,y) &= \sum_{n=0}^{\infty} \int \partial_k[\psi_j(k)m_-(y,k)M_n^-(x,k)]e^{i(x-y)k}\,dk \\ &:= \sum_{n=0}^{\infty} A_n^-(x,y). \end{aligned}$$

For $y<0$, $j\in\mathbb{Z}$ it is easy to see from Lemma 4.7 that

$$\|A_0^-(x,y)\|_{L_x^2}=\|\psi_j(k)m_-(y,k)\|_{\dot H_k^1}\le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])}.$$

For $n\ge 1$ using (41) and exchanging order of integration give that

$$\begin{aligned}A_n^-(x,y)&=\int_{-\infty}^{t_0}V(t_1)\,dt_1\int_0^{t_0-t_1}du_1\cdots\int_{-\infty}^{t_{n-1}}V(t_n)\,dt_n\int_0^{t_{n-1}-t_n}du_n\\&\quad\times\int\partial_k[\psi_j m_-(y,k)]e^{i(x-y-2u_1-\cdots-2u_n)k}\,dk\\&\quad+\int\psi_j(k)m_-(y,k)\partial_k M_n^-(x,k)e^{i(x-y)k}\,dk:=\Pi_1(x,y)+\Pi_2(x,y),\end{aligned}$$

where

$$\begin{aligned}&\partial_k M_n^-(x,k)\\&\quad=\int_{t_0>t_1>t_2>\cdots>t_n}\partial_k\big(h(t_0-t_1,-k)\cdots h(t_{n-1}-t_n,-k)\big)\\&\qquad\times V(t_1)\cdots V(t_n)\,dt_1\cdots dt_n:=J_{n,1}^-+J_{n,2}^-+\cdots+J_{n,n}^-,\end{aligned}$$

$J_{n,i}^-$ denoting the integral involving $\partial_k h(t_{i-1}-t_i,-k)$, $i=1,\dots,n$.

We estimate by Minkowski inequality

$$\begin{aligned}&\big\|\chi_{\{y<x<0\}}\Pi_1(x,y)\big\|_{L_x^2}\\&\quad\le\int_{-\infty}^0|V(t_1)|\,dt_1\int_0^{-t_1}du_1\cdots\int_{-\infty}^{t_{n-1}}|V(t_n)|\,dt_n\int_0^{-t_n}du_n\\&\qquad\times\left\|\int\partial_k[\psi_j m_-(y,k)]e^{i(x-y-2u_1-\cdots-2u_n)k}\,dk\right\|_{L_x^2}\\&\quad\le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])}\int_{-\infty}^0(-t_1)|V(t_1)|\,dt_1\cdots\int_{-\infty}^{t_{n-1}}(-t_n)|V(t_n)|\,dt_n\\&\quad\le c2^{-j/4}\|\phi\|_{H^1([\frac14,1])}\frac{(\|tV\|_1)^n}{n!}.\end{aligned}$$

For Π_2 we estimate the first term by using (42), Minkowski inequality and Plancherel theorem to obtain

$$\begin{aligned}&\left\|\chi_{\{y<x<0\}}\int\psi_j(k)m_-(y,k)J_{n,1}^-(x,k)e^{i(x-y)k}\,dk\right\|_{L_x^2}\\&\quad\le\int_{-\infty}^0(-t_1)|V(t_1)|\,dt_1\int_{-\infty}^{t_1}|V(t_2)|\,dt_2\int_0^{t_1-t_2}du_2\cdots\\&\qquad\times\int_{-\infty}^{t_{n-1}}|V(t_n)|\,dt_n\int_0^{t_{n-1}-t_n}du_n\\&\qquad\times\left\|\int\frac{\psi_j(k)}{k}m_-(y,k)e^{-i(x+y-2t_1+2u_2+\cdots+2u_n)k}\,dk\right\|_{L_x^2}\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{0}|V(t_1)|\,dt_1\int_0^{-t_1}du_1\int_{-\infty}^{t_1}|V(t_2)|\,dt_2\int_0^{t_1-t_2}du_2\cdots\\
&\times\int_{-\infty}^{t_{n-1}}|V(t_n)|\,dt_n\int_0^{t_{n-1}-t_n}du_n\\
&\times\left\|\int\frac{\psi_j(k)}{k}m_-(y,k)e^{i(x-y-2u_1-2u_2-\cdots-2u_n)k}\,dk\right\|_{L^2_x}\\
\le{}& c2^{-j/4}\|\phi\|_{L^2([\frac14,1])}\frac{(\|tV\|_1)^n}{n!}.
\end{aligned}
$$

The same estimate holds for other terms involving $J^-_{n,i}$, $i=2,\dots,n$. And so,

$$
\left\|\chi_{\{y<x<0\}}A_n^-(x,y)\right\|_2\le c2^{-j/4}(1+n)\frac{(\|tV\|_1)^n}{n!}\|\phi\|_{H^1([\frac14,1])}.
$$

It follows that for all $j\in\mathbb{Z}$

$$
\left\|\chi_{\{y<x<0\}}(x-y)I_2^c(x,y)\right\|_2\le c2^{-j/4}e^{\|tV\|_1}\|\phi\|_{H^1([\frac14,1])},
$$

which proves (39c). □

LEMMA 5.2. *Let $V\in L^1_1$. Then for all y, $j\in\mathbb{Z}$ and $\phi\in L^2([\frac14,1])$*

$$
\|\phi_j(H_{\mathrm{ac}})(x,y)\|_{L^2_x}\le c2^{j/4}\|\phi\|_{L^2([\frac14,1])}.
$$

The proof is straightforward and follows the same line as in the weighted case $s=1$ but much simpler, where we only need Lemma 4.6(a) and the following asymptotics in Lemma 4.7 and Lemma 4.8: If $V\in L^1_1$, then

$$
\begin{cases}
|m_\pm(y,k)|\le c\big(1+\max(0,\mp y)\big),\\
t(k)=O(1),\\
r_+(k)=O(1).
\end{cases}
$$

We omit the details.

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