Illinois Journal of Mathematics
Volume 54, Number 2, Summer 2010, Pages 601–620
S 0019-2082

THE PATH-CONNECTIVITY OF MRA WAVELETS IN $L^2(\mathbb{R}^d)$

ZHONGYAN LI, XINGDE DAI, YUANAN DIAO AND WEI HUANG

ABSTRACT. We show that for any $d \times d$ expansive matrix A with integer entries and $|\det(A)| = 2$, the set of all A-dilation MRA wavelets is path-connected under the $L^2(\mathbb{R}^d)$ norm topology. We do this through the application of A-dilation wavelet multipliers, namely measurable functions f with the property that the inverse Fourier transform of $(f\widehat{\psi})$ is an A-dilation wavelet for any A-dilation wavelet ψ (where $\widehat{\psi}$ is the Fourier transform of ψ). In this process, we have completely characterized all A-dilation wavelet multipliers for any integral expansive matrix A with $|\det(A)| = 2$.

1. Introduction

Let A be a $d \times d$ real expansive matrix, i.e., a matrix with real entries whose eigenvalues are all of modules greater than one. An *A-dilation wavelet* is a function $\psi \in L^2(\mathbb{R}^d)$ such that the set

$$\{|\det A|^{\frac{n}{2}}\psi(A^n\mathbf{t} - \ell) : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$. For any function $f(\mathbf{t}) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, its Fourier transform is defined by

$$\mathcal{F}(f(\mathbf{t})) = \widehat{f}(\mathbf{s}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(\mathbf{t}) e^{-i\mathbf{t}\circ\mathbf{s}}\, d\mathbf{t}, \tag{1.1}$$

where $\mathbf{t} \circ \mathbf{s}$ is the standard inner product of the vectors $\mathbf{s},\ \mathbf{t} \in \mathbb{R}^d$. The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$.

Received November 6, 2008; received in final form January 18, 2010.

Zhongyan Li is partially supported by the grant of Young Teachers Study Abroad of China Scholarship Council (2005) and the Fundamental Research Funds for the Central Universities and the Project sponsored by SRF for ROCS, SEM. She thanks the Department of Mathematics and Statistics at UNC Charlotte for its hospitality during 2007–2008.

2010 *Mathematics Subject Classification.* Primary 42-XX, 46-XX.

One of the many problems in wavelet theory concerns the construction of different wavelets. Naturally, one may attempt to construct new wavelets from an existing one. This approach leads to the concept of *wavelet multipliers* [3]. A measurable function f is called an A-dilation wavelet multiplier if the inverse Fourier transform of $(f\widehat{\psi})$ is an A-dilation wavelet for any A-dilation wavelet ψ. Our study in this paper concerns the case where the dilation matrix A is an expansive matrix with integer entries such that $|\det(A)|=2$. For such matrices, the wavelet multipliers have been studied extensively and are completely characterized for dimension 1 [11, 14] and dimension 2 [9, 10]. These results then lead to the characterization of the phases of A-dilation MRA wavelets and the establishment of the path-connectedness of the set of all A-dilation MRA wavelets under the $L^2(\mathbb{R}^1)$ or $L^2(\mathbb{R}^2)$ norm topology for dimension $d=1$ and dimension $d=2$. Here in this paper, we will generalize the above mentioned results to all $d\times d$ expansive matrices with integer entries and determinant ± 2. It is important for us to point out that the approach used for the case $d=2$ depends on certain special properties a 2×2 integral expansive matrix A (with $|\det(A)|=2$) possesses [9, 10]. This is no longer the case for $d\geq 3$ so a different approach has to be used.

The rest of the paper is organized as follows. In the next section, we introduce the notations and terms needed for this paper, with some preliminary results needed in the later sections. In Section 3, we state and prove our results on wavelet multipliers on $L^2(\mathbb{R}^d)$. In Section 4, we prove the path-connectivity of the set of all A-dilation MRA wavelets. In the last section, we compare our approach in this paper with the ones used in lower dimensions ($d=1$ and $d=2$). We show that in the higher dimensions $d\geq 3$ there exist integral expansive matrices that would prohibit the direct generalizations of the approaches used to solve the path-connectivity of A-dilation MRA wavelets in $d=1$ and $d=2$ ([9]–[11], [14]).

2. Notations, definitions and preliminary results

Let $M_d^{(2)}(\mathbb{Z})$ be the set of all $d\times d$ expansive integral matrices (i.e., matrices with integer entries) whose determinants are ± 2. Throughout this paper, we will limit our discussion to matrices $A\in M_d^{(2)}(\mathbb{Z})$. We will use T, D_A as the translation and dilation unitary operators acting on $L^2(\mathbb{R}^d)$ defined by $(T^\ell f)(\mathbf{t}) = f(\mathbf{t}-\ell)$, $(D_A f)(\mathbf{t}) = |\det(A)|^{\frac{1}{2}} f(A\mathbf{t})$, $\forall f\in L^2(\mathbb{R}^d)$, $\mathbf{t}\in\mathbb{R}^d$ and $\ell\in\mathbb{Z}^d$. A measurable function $\psi\in L^2(\mathbb{R}^d)$ is called an A-dilation wavelet if $\{D_A^n T^\ell\psi : n\in\mathbb{Z},\ \ell\in\mathbb{Z}^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. Whenever we state that two functions $f,\ g\in L^2(\mathbb{R}^d)$ are equal, it is understood that $f(\mathbf{s}) = g(\mathbf{s})$ for almost all $\mathbf{s}\in\mathbb{R}^d$. Furthermore, we say that $E=F$ for two measurable sets F and E in $\mathbb{R}^d$ if $(F\setminus F)\cup(E\setminus F)$ is a measure zero set.

DEFINITION 2.1. A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ is called an A-dilation multi-resolution analysis (or A-dilation MRA for short) if the following hold:

(i) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;
(ii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$;
(iii) $f(\mathbf{t}) \in V_j$ if and only if $f(A^{-j}\mathbf{t}) \in V_0$ for $j \in \mathbb{Z}$; and
(iv) There exists $\phi(\mathbf{t})$ in V_0 such that $\{\phi(\mathbf{t}-\ell) : \ell \in \mathbb{Z}^d\}$ is an orthonormal basis for V_0.

The function $\phi(\mathbf{t})$ defined in (iv) above is called an A-dilation scaling function for the MRA. In our case, it is known that a single A-dilation wavelet can be derived from the above A-dilation MRA [12] (due to the fact that $|\det(A)| - 1 = 1$). An A-dilation wavelet $\psi \in V_1 \cap V_0^{\perp}$ so obtained is called an MRA wavelet. For any $f \in V_1$, $f(A^{-1}\mathbf{t}) \in V_0$, hence we have

$$f(\mathbf{t}) = |\det(A)| \sum_{\ell\in\mathbb{Z}^d} c_\ell \phi(A\mathbf{t}-\ell). \tag{2.1}$$

If we define $m_f(\mathbf{s}) = \sum_{\ell\in\mathbb{Z}^d} c_\ell e^{-i\ell\circ\mathbf{s}}$, then by taking Fourier transform on both sides of (2.1) we obtain $\widehat{f}(A^\tau\mathbf{s}) = m_f(\mathbf{s})\widehat{\phi}(\mathbf{s})$, where A^τ is the transpose of A. In particular, we have

$$\widehat{\phi}(A^\tau\mathbf{s}) = m(\mathbf{s})\widehat{\phi}(\mathbf{s}) \tag{2.2}$$

for some function $m(s)$ of the form similar to (2.1). $m(s)$ is called the *low pass A-dilation filter* of the MRA.

A measurable function $f(\mathbf{t}) \in L^2(\mathbb{R}^d)$ is called a $2\pi\mathbb{Z}^d$ periodic if $f(\mathbf{t}+2\pi\ell) = f(\mathbf{t})$ on $\mathbb{R}^d$ for any $\ell \in \mathbb{Z}^d$.

The following lemmas are well-known results and can be easily obtained by standard textbook arguments [2, 4, 7].

LEMMA 2.1. *ψ is an A-dilation wavelet iff the following conditions hold*:

(i) $\|\psi\|_2 = 1$;
(ii) $\sum_{j\in\mathbb{Z}} |\widehat{\psi}((A^\tau)^j\mathbf{s})|^2 = 1/(2\pi)^d$ *and*
(iii) $\sum_{j=0}^{\infty} \widehat{\psi}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi}((A^\tau)^j(\mathbf{s}+2\pi\ell))} = 0 \ \forall \ell \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$.

LEMMA 2.2. *An A-dilation wavelet ψ is an A-dilation MRA wavelet iff*

$$D_\psi(\mathbf{s}) = \sum_{n=1}^{\infty} \sum_{\ell\in\mathbb{Z}^d} \left|\widehat{\psi}\big((A^\tau)^n(\mathbf{s}+2\pi\ell)\big)\right|^2 = \frac{1}{(2\pi)^d}. \tag{2.3}$$

LEMMA 2.3. *ϕ is an A-dilation scaling function for an MRA iff the following conditions hold*:

(i) $\sum_{\ell\in\mathbb{Z}^d} |\widehat{\phi}(\mathbf{s}+2\pi\ell)|^2 = 1/(2\pi)^d$;
(ii) $\lim_{j\to\infty} |\widehat{\phi}((A^\tau)^{-j}\mathbf{s})| = 1/(2\pi)^{\frac{d}{2}}$ *and*

(iii) *there exists a* $2\pi\mathbb{Z}^d$ *periodic function* $m(\mathbf{s}) \in L^2([-\pi,\pi)^d)$ *such that* $\widehat{\phi}(A^\tau\mathbf{s}) = m(\mathbf{s})\widehat{\phi}(\mathbf{s})$.

Lemma 2.4 below describes some special properties a matrix in the set $M_d^{(2)}(\mathbb{Z})$ possesses. Its proof is elementary and our reader can also refer to [5].

LEMMA 2.4. *Let* $A \in M_d^{(2)}(\mathbb{Z})$. *Then the group* $\mathbb{Z}^d/A^\tau\mathbb{Z}^d$ *is isomorphic to* $(A^\tau)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ *and the order of* $\mathbb{Z}^d/A^\tau\mathbb{Z}^d$ *is equal to* 2. *In particular, for any* $\mathbf{h}_1 \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$, $\mathbb{Z}^d = A^\tau\mathbb{Z}^d \cup (A^\tau\mathbb{Z}^d + \mathbf{h}_1)$ *and* $(A^\tau)^{-1}\mathbb{Z}^d = \mathbb{Z}^d \cup (\mathbb{Z}^d + \mathbf{h}_2)$ *where* $\mathbf{h}_2 = (A^\tau)^{-1}\mathbf{h}_1$.

REMARK 2.1. Since $A \in M_d^{(2)}(\mathbb{Z})$, any noninteger entry in $(A^\tau)^{-1}$ is a rational number with denominator 2 (namely a number of the form $\frac{1}{2}(2r+1)$ with $r \in \mathbb{Z}$. It follows that $\mathbf{h}_2 = (A^\tau)^{-1}\mathbf{h}_1 \notin \mathbb{Z}^d$ has at least one noninteger entry and all noninteger entries are rational numbers with denominator 2. We will use $\rho(\mathbf{h}_2)$ to denote the index of the first such noninteger entry in $\mathbf{h}_2$.

REMARK 2.2. Notice that for $\mathbf{h}_1, \mathbf{h}_1' \in \mathbb{Z}^d \setminus A^\tau(\mathbb{Z}^d)$ and $\mathbf{h}_2 = (A^\tau)^{-1}\mathbf{h}_1$, $\mathbf{h}_2' = (A^\tau)^{-1}\mathbf{h}_1'$, we have $\rho(\mathbf{h}_2) = \rho(\mathbf{h}_2')$ since $\mathbf{h}_2 - \mathbf{h}_2' \in \mathbb{Z}^d$. Thus the index $\rho(\mathbf{h}_2)$ only depends on A. Hence, it is appropriate to denote such an index by $\rho(A)$. Let $\mathbf{u} \in \mathbb{R}^d$ be the vector with all of its entries being zero except that at its $\rho(A)$th coordinate, where it has 1 as its entry. Then $e^{\pm i2\pi\mathbf{h}_2\circ\mathbf{u}} = -1$. We leave it to our reader to verify that there is a unique element $\mathbf{h}_1 \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$ such that $(A^\tau)^{-1}\mathbf{h}_1 = \mathbf{h}_2$ is a nonzero vector whose entries are either 1/2 or 0. In this case $\mathbf{h}_2 \circ \mathbf{u} = 1/2$. From now on, $\mathbf{h}_1$ and $\mathbf{h}_2$ will be understood to be these two uniquely determined vectors to avoid any confusion.

LEMMA 2.5. *Suppose that* ψ *is an* A*-dilation MRA wavelet with scaling function* ϕ *and low pass filter* $m(\mathbf{s})$, *then*

$$|\widehat{\phi}(\mathbf{s})|^2 = \sum_{j=1}^{\infty} |\widehat{\psi}((A^\tau)^j\mathbf{s})|^2, \tag{2.4}$$

$$|m(\mathbf{s})|^2 + |m(\mathbf{s} + 2\pi\mathbf{h}_2)|^2 = 1. \tag{2.5}$$

A measurable set $F \in \mathbb{R}^d$ is called a $2\pi\mathbb{Z}^d$ *translation tiling domain* (or just a tiling domain for short) if $\{F + 2\pi\ell : \ell \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. It is shown in [6] that there exist special translation tiling domains F with the property that $\mathbf{0} \in F$, $F \subset A^\tau F$ and $\bigcup_{n\geq 0}(A^\tau)^n F = \mathbb{R}^d$. The set $E = A^\tau F \setminus F$ is called a generalized Shannon-type wavelet set (from which an A-dilation MRA wavelet can be obtained). A function f with the property $|f| = 1$ is called a *unimodular function.*

PROPOSITION 2.1. *Let* $\phi \in L^2(\mathbb{R}^d)$ *be a scaling function for an* A*-dilation MRA* $\{V_j\}$ *and let* m *be its associated low pass filter. Let* $\psi \in W_0 = V_1 \cap V_0^\perp$, *then* $\{\psi(\mathbf{t}-\ell):\ell\in\mathbb{Z}^d\}$ *is an orthonormal basis for* W_0 *iff there exists a* $2\pi\mathbb{Z}^d$ *periodic, measurable and unimodular function* $v:\mathbb{R}^d\to\mathbb{C}$ *such that*

$$\widehat{\psi}(A^\tau\mathbf{s})=e^{i(\mathbf{s}\circ\mathbf{u})}v(A^\tau\mathbf{s})\overline{m(\mathbf{s}+2\pi\mathbf{h}_2)}\widehat{\phi}(\mathbf{s}), \tag{2.6}$$

where $\mathbf{u}$ *is the vector defined in Remark* 2.2.

Let us give an outline of the proof for Proposition 2.1. From the discussion following (2.1), we have $\widehat{\psi}(A^\tau\mathbf{s})=m_\psi(\mathbf{s})\widehat{\phi}(\mathbf{s})$ for some $2\pi\mathbb{Z}^d$ periodic function m_ψ. Again, standard arguments show that $\{\psi(\mathbf{t}-\ell):\ell\in\mathbb{Z}^d\}$ is an orthonormal basis for W_0 iff the equations $|m(\mathbf{s})|^2+|m(\mathbf{s}+2\pi\mathbf{h}_2)|^2=1$, $|m_\psi(\mathbf{s})|^2+|m_\psi(\mathbf{s}+2\pi\mathbf{h}_2)|^2=1$ and $m(\mathbf{s})\overline{m_\psi(\mathbf{s})}+m(\mathbf{s}+2\pi\mathbf{h}_2)\overline{m_\psi(\mathbf{s}+2\pi\mathbf{h}_2)}=0$ hold. The reader can verify that the solution for $m_\psi(\mathbf{s})$ (in terms of $m(\mathbf{s})$) is of the form $e^{i(\mathbf{s}\circ\mathbf{u})}v(A^\tau\mathbf{s})\overline{m(\mathbf{s}+2\pi\mathbf{h}_2)}$.

PROPOSITION 2.2. *Let* ψ *be an* A*-dilation MRA wavelet, then* $e^{i(\mathbf{s}\circ A^{-1}\mathbf{u})}|\widehat{\psi}(\mathbf{s})|$ *is the Fourier transform of an* A*-dilation MRA wavelet.*

Proof. Let ϕ be the corresponding scaling function with low pass filter m, then $\mathcal{F}^{-1}(|\widehat{\phi}|)$ is also an A-dilation scaling function whose associated low pass filter is $|m|$ by Lemma 2.3. Thus, the function ψ_1 defined by $\widehat{\psi_1}(A^\tau\mathbf{s})=e^{i(\mathbf{s}\circ\mathbf{u})}|\overline{m(\mathbf{s}+2\pi\mathbf{h}_2)}\widehat{\phi}(\mathbf{s})|=e^{i(\mathbf{s}\circ\mathbf{u})}|\psi(A^\tau\mathbf{s})|$ is an A-dilation MRA wavelet by Proposition 2.1. ☐

A measurable subset E of $\mathbb{R}^d$ is *simple* if $E(\Omega)=\bigcup_{\ell\in\mathbb{Z}^d}(E\cap(\Omega+2\pi\ell)-2\pi\ell)\subset\Omega$ is a disjoint union (where $\Omega=[-\pi,\pi)^d$). Two simple sets $E,F\subset\mathbb{R}^d$ are said to be $2\pi\mathbb{Z}^d$ translation congruent (or just translation congruent for short) to each other if $E(\Omega)=F(\Omega)$.

PROPOSITION 2.3. *Let* F *be a* $2\pi\mathbb{Z}^d$ *translation tiling domain of* $\mathbb{R}^d$, *then there exist disjoint subsets* F_0 *and* F_1 *of* F *such that* (1) $F=F_0\cup F_1$; (2) *each of* F_0, F_1 *is a* $2\pi(A^\tau)^{-1}\mathbb{Z}^d$ *translation tiling domain of* $\mathbb{R}^d$; (3) $F_1-2\pi\mathbf{h}_2$ *is* $2\pi\mathbb{Z}^d$ *translation congruent to* F_0.

Proposition 2.3 is key to our proof for the path-connectivity of the A-dilation MRA wavelets so we provide a detailed proof here.

Proof of Proposition 2.3. Since F is a $2\pi\mathbb{Z}^d$ translation tiling domain of $\mathbb{R}^d$, $(A^\tau)^{-1}F$ is a $2\pi(A^\tau)^{-1}\mathbb{Z}^d$ translation tiling domain of $\mathbb{R}^d$. For each $\ell\in\mathbb{Z}^d$, define $F_\ell=F\cap(((A^\tau)^{-1})F+2\pi\ell)$. F_ℓ is measurable since it is the intersection of two measurable sets. Now define $F_1=\bigcup_{\ell\in\mathbb{Z}^d}F_\ell$. F_1 is measurable since it is the union of countably many measurable sets. It follows that $F_0=F\setminus F_1$ is also measurable and conditions (1) is met. We now proceed to prove (2) and (3).

Claim 1. For any two points, $\mathbf{x},\mathbf{y}\in F_1$, $\mathbf{x}-\mathbf{y}\notin 2\pi(A^\tau)^{-1}\mathbb{Z}^d$. For if not, then $\exists\mathbf{x},\mathbf{y}\in F_1$ such that $\mathbf{x}\neq\mathbf{y}$ but $\mathbf{x}-\mathbf{y}=2\pi(A^\tau)^{-1}\ell$ for some $\ell\in\mathbb{Z}^d$. By the definition of F_1, $\mathbf{x}=(A^\tau)^{-1}\mathbf{x}_0+2\pi\ell_1$ for some $\mathbf{x}_0\in F$ and $\ell_1\in\mathbb{Z}^d$,

$\mathbf{y} = (A^\tau)^{-1}\mathbf{y}_0 + 2\pi\ell_2$ for some $\mathbf{y}_0 \in F$ and $\ell_2 \in \mathbb{Z}^d$. It follows that $\mathbf{x} - \mathbf{y} = (A^\tau)^{-1}(\mathbf{x}_0 - \mathbf{y}_0) + 2\pi(\ell_1 - \ell_2) = 2\pi(A^\tau)^{-1}\ell$ hence $\mathbf{x}_0 - \mathbf{y}_0 \in 2\pi\mathbb{Z}^d$. Thus, $\mathbf{x}_0 = \mathbf{y}_0$ since F is a tiling domain. This in turn implies that $\mathbf{x} - \mathbf{y} \in 2\pi\mathbb{Z}^d$ so $\mathbf{x} = \mathbf{y}$ (again because F is a tiling domain). This contradicts to our assumption that $\mathbf{x} \neq \mathbf{y}$.

Claim 2. F_1 is $2\pi\mathbb{Z}^d$ translation congruent to $(A^\tau)^{-1}F$. $\forall \mathbf{x} \in F_1$, then $\mathbf{x} = (A^\tau)^{-1}\mathbf{y} + 2\pi\ell$ for some $\mathbf{y} \in F$ and $\ell \in 2\pi\mathbb{Z}^d$. Thus, $\mathbf{x} - (A^\tau)^{-1}\mathbf{y} \in 2\pi\mathbb{Z}^d$ and it follows that F_1 is $2\pi\mathbb{Z}^d$ translation congruent to a subset of $(A^\tau)^{-1}F$. Similarly, for any $\mathbf{y} \in (A^\tau)^{-1}F$, $\mathbf{y} = (A^\tau)^{-1}\mathbf{x}$ for some $\mathbf{x} \in F$. Since F is a tiling of $\mathbb{R}^d$, $\mathbf{y} = \mathbf{z} + 2\pi\ell$ for some $\mathbf{z} \in F$ and $\ell \in \mathbb{Z}^d$. It follows that $\mathbf{z} = \mathbf{y} - 2\pi\ell = (A^\tau)^{-1}\mathbf{x} - 2\pi\ell \in F_{-\ell_1} \subset F_1$. Thus, $(A^\tau)^{-1}F$ is $2\pi\mathbb{Z}^d$ translation congruent to a subset of F_1. Therefore, F_1 is translation congruent to $(A^\tau)^{-1}F$.

Since $(A^\tau)^{-1}F$ is a tiling of $\mathbb{R}^d$ under the $2\pi(A^\tau)^{-1}\mathbb{Z}^d$ translation, it follows that F_1 is a tiling of $\mathbb{R}^d$ under the $2\pi(A^\tau)^{-1}\mathbb{Z}^d$ translation. This proves half of (2).

Claim 3. $F_1 - 2\pi\mathbf{h}_2$ is translation congruent to F_0. $\forall\ \mathbf{x} \in F_1$, then $\mathbf{x} - 2\pi\mathbf{h}_2 = \mathbf{y} + 2\pi\ell_1$ for some $\mathbf{y} \in F, \ell_1 \in \mathbb{Z}^d$ since F is a tiling domain. If $\mathbf{y} \in F_1$, then $\mathbf{y} = (A^\tau)^{-1}\mathbf{y}_1 + 2\pi\ell_2$, and $\mathbf{x} = (A^\tau)^{-1}\mathbf{x}_1 + 2\pi\ell_3$, where $\mathbf{y}_1, \mathbf{x}_1 \in F$. So $(A^\tau)^{-1}\mathbf{x}_1 + 2\pi\ell_3 - 2\pi\mathbf{h}_2 = (A^\tau)^{-1}\mathbf{y}_1 + 2\pi\ell_2 + 2\pi\ell_1$. Multiplying both sides of this by A^τ shows that $\mathbf{x}_1 - \mathbf{y}_1 \in 2\pi\mathbb{Z}^d$ hence $\mathbf{x}_1 = \mathbf{y}_1$. But then we have $\mathbf{h}_2 = \ell_3 - \ell_1 - \ell_2 \in \mathbb{Z}^d$, which contradicts to $\mathbf{h}_2 \notin \mathbb{Z}^d$. So we must have $\mathbf{y} \in F_0$ and this shows that $F_1 - 2\pi\mathbf{h}_2$ is translation congruent to a subset of F_0. For any $\mathbf{x}_0 \in F_0$, $\mathbf{x}_0 = \mathbf{x}_1 + 2\pi(A^\tau)^{-1})\ell_1$ for some $\mathbf{x}_1 \in F_1$ and $\ell_1 \in \mathbb{Z}^d$. It is necessary $(A^\tau)^{-1})\ell_1 \notin \mathbb{Z}^d$, otherwise $\mathbf{x}_0 = \mathbf{x}_1$ (again because F is a tiling domain), which contradicts to $\mathbf{x}_0 \notin F_1$. So $2\pi(A^\tau)^{-1})\ell_1 = 2\pi\mathbf{h}_2 + 2\pi\ell_2$ for some $\ell_2 \in \mathbb{Z}^d$. It follows that $\mathbf{x}_0 + 2\pi\mathbf{h}_2$ is congruent equivalent to $\mathbf{x}_1$ modulo $2\pi\mathbb{Z}^d$. Thus, F_0 is translation congruent to a subset of $F_1 - 2\pi\mathbf{h}_2$ (3) is proven. The other half of (2) now follows easily: for any $\mathbf{z} \in \mathbb{R}^d$, $\mathbf{z} = \mathbf{x} + 2\pi(A^\tau)^{-1}\ell$ for some $\mathbf{x} \in F_0$. But $\mathbf{x} = \mathbf{x}_1 - 2\pi\mathbf{h}_2 + 2\pi\ell_1$ for some $\ell_1 \in \mathbb{Z}^d$, so $\mathbf{z} = \mathbf{x}_1 - 2\pi\mathbf{h}_2 + 2\pi\ell_1 + 2\pi(A^\tau)^{-1}\ell = \mathbf{x}_1 + 2\pi(A^\tau)^{-1}(\ell + \ell_1 - A^\tau\mathbf{h}_2) = \mathbf{x}_1 + 2\pi(A^\tau)^{-1}\ell_2$ where $\ell_2 = \ell + \ell_1 - A^\tau\mathbf{h}_2 \in \mathbb{Z}^d$ and since $A^\tau\mathbf{h}_2 \in \mathbb{Z}^d$. □

A direct consequence of Proposition 2.3 is the following corollary. Its proof is elementary and is left to the reader.

Corollary 2.1. *Let $A \in M_d^{(2)}(\mathbb{Z})$, F a tiling domain and $\mathbf{h}_1$, $\mathbf{h}_2$ be as defined in Remark 2.2. Then for any $m \in L^1(\mathbb{R}^d)$ that is $2\pi\mathbb{Z}^d$ periodic, we have:*

(i) $\int_F m((A^\tau)^{-1}\mathbf{s})\,d\mathbf{s} = 2\int_{F_1} m(\mathbf{s})\,d\mathbf{s} = 2\int_{F_1(\Omega)} m(\mathbf{s})\,d\mathbf{s}$ *and*
(ii) $\int_{F_j} m(\mathbf{s})\,d\mathbf{s} = \int_{F_j(\Omega)} m(\mathbf{s})\,d\mathbf{s} = \int_{F_{1-j}(\Omega)} m(\mathbf{s} + 2\pi\mathbf{h}_2)\,d\mathbf{s} = \int_{F_{1-j}} m(\mathbf{s} + 2\pi\mathbf{h}_2)\,d\mathbf{s}$ *where* $j = 0, 1$.

3. A-dilation wavelet multipliers

A measurable function f is called an A-dilation wavelet multiplier (or wavelet multiplier for short) if the inverse Fourier transform of $f\widehat{\psi}$ is an A-dilation wavelet whenever ψ is an A-dilation wavelet. In this section, we characterize the A-dilation wavelet multipliers. By an argument similar to the one used in [9], it can be shown that any wavelet multiplier f has to be unimodular. Thus, in the following we will limit our discussion to such functions. Instead of trying to characterize the scaling function multiplier or the low pass filter multiplier (which is the approach used in [9]), we will use a different approach.

THEOREM 3.1. *A unimodular function $f \in L^\infty(\mathbb{R}^d)$ is an A-dilation wavelet multiplier iff the function $k(\mathbf{s}) = f(A^\tau \mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic.*

Proof. "$\Longleftarrow$" Assume that $f \in L^\infty(\mathbb{R}^d)$ is a unimodular function and that $k(\mathbf{s}) = f(A^\tau \mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic. To show that f is a wavelet multiplier, we need to show that for any A-dilation wavelet ψ, $\eta = \mathcal{F}^{-1}(f\widehat{\psi})$ is also a wavelet. It suffices to verify that $\widehat{\eta}$ satisfies conditions (ii) and (iii) in Lemma 2.1. It is easy to see that (ii) holds for $\widehat{\eta}$ since $|\widehat{\eta}| = |\widehat{\psi}|$ and (ii) holds for $\widehat{\psi}$. Applying the relation $f(A^\tau \mathbf{s}) = k(\mathbf{s})f(\mathbf{s})$ repeatedly, for any $j \geq 1$ and $\ell \in \mathbb{Z}^d$, we obtain

$$f((A^\tau)^j \mathbf{s}) = k((A^\tau)^{j-1}\mathbf{s}) \cdots k(A^\tau \mathbf{s})k(\mathbf{s})f(\mathbf{s}), \tag{3.1}$$

and

$$\begin{aligned} f\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big) &= k\big((A^\tau)^{j-1}(\mathbf{s}+2\pi\ell)\big)k\big((A^\tau)^{j-2}(\mathbf{s}+2\pi\ell)\big) \\ &\quad \cdots k\big(A^\tau(\mathbf{s}+2\pi\ell)\big)k(\mathbf{s}+2\pi\ell)f(\mathbf{s}+2\pi\ell) \\ &= k((A^\tau)^{j-1}\mathbf{s}) \cdots k(A^\tau \mathbf{s})k(\mathbf{s})f(\mathbf{s}+2\pi\ell). \end{aligned}$$

Since $k(\mathbf{s})$ is unimodular, this leads to

$$\begin{aligned} & f((A^\tau)^j \mathbf{s}) \cdot \overline{f\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)} \\ &\quad = k((A^\tau)^{j-1}\mathbf{s}) \cdots k(A^\tau \mathbf{s})k(\mathbf{s})f(\mathbf{s}) \cdot \overline{k((A^\tau)^{j-1}\mathbf{s}) \cdots k(A^\tau \mathbf{s})k(\mathbf{s})f(\mathbf{s}+2\pi\ell)} \\ &\quad = f(\mathbf{s})\overline{f(\mathbf{s}+2\pi\ell)} \end{aligned}$$

for any $j \geq 0$ and $\ell \in \mathbb{Z}^d$. Thus,

$$\begin{aligned} &\sum_{j=0}^{\infty} \widehat{\eta}((A^\tau)^j \mathbf{s})\overline{\widehat{\eta}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)} \\ &\qquad = f(\mathbf{s})\overline{f(\mathbf{s}+2\pi\ell)} \sum_{j=0}^{\infty} \widehat{\psi}((A^\tau)^j \mathbf{s})\overline{\widehat{\psi}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)} = 0 \end{aligned}$$

for any $\ell \in \mathbb{Z}^d \backslash A^\tau \mathbb{Z}^d$. So condition (iii) of Lemma 2.1 holds for $\widehat{\eta}$ as well.

"$\Longrightarrow$" We need to show that $k(\mathbf{s}) = f(A^\tau \mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic. Let ψ be any A-dilation MRA wavelet such that $\mathrm{supp}(\widehat{\psi}) = \mathbb{R}^d$ (the existence of such ψ is proven in [5, Example 5.14]). By Proposition 2.2, the function $\psi_1(t)$ defined by

$$\widehat{\psi_1} = e^{i(A^\tau)^{-1}\mathbf{s}\circ\mathbf{u}}|\widehat{\psi}(\mathbf{s})| = e^{i\mathbf{s}\circ A^{-1}\mathbf{u}}|\widehat{\psi_1}(\mathbf{s})| \tag{3.2}$$

is an A-dilation wavelet. Since $\mathcal{F}^{-1}(f\widehat{\psi_1})$ is also an A-dilation wavelet, $\widehat{\psi_1}$ and $f\widehat{\psi_1}$ both satisfy condition (iii) of Lemma 2.1, that is,

$$\sum_{j=0}^{\infty} \widehat{\psi_1}((A^\tau)^j\mathbf{s}) \cdot \overline{\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)} = 0 \quad \text{and} \tag{3.3}$$

$$\begin{aligned} &\sum_{j=0}^{\infty} f((A^\tau)^j\mathbf{s})\widehat{\psi_1}((A^\tau)^j\mathbf{s}) \\ &\qquad \times \overline{f\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)}\overline{\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)} = 0 \end{aligned} \tag{3.4}$$

for any $\ell \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$. Since $\ell \in \mathbb{Z}^d \setminus A^\tau\mathbb{Z}^d$, there exists $\ell_1 \in \mathbb{Z}^d$ such that $\ell = A^\tau\ell_1 + \mathbf{h}_1 = A^\tau(\ell_1 + \mathbf{h}_2)$ by Lemma 2.4. Thus,

$$\begin{aligned} \widehat{\psi_1}(\mathbf{s})\overline{\widehat{\psi_1}(\mathbf{s}+2\pi\ell)} &= e^{i\mathbf{s}\circ A^{-1}\mathbf{u}}|\widehat{\psi_1}(\mathbf{s})| \cdot e^{-i(\mathbf{s}+2\pi\ell)\circ A^{-1}\mathbf{u}}|\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &= e^{-i2\pi(\ell_1+\mathbf{h}_2)\circ\mathbf{u}}|\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &= e^{-i2\pi\mathbf{h}_2\circ\mathbf{u}}|\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &= -|\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)|, \end{aligned}$$

since $\ell_1 \circ \mathbf{u}$ is an integer and $e^{-i2\pi\mathbf{h}_2\circ\mathbf{u}} = -1$ (see Remark 2.2). On the other hand, for any $j > 0$,

$$\begin{aligned} &\widehat{\psi_1}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)} \\ &\quad = e^{i(A^\tau)^j\mathbf{s})\circ A^{-1}\mathbf{u}}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot e^{-i(A^\tau)^j(\mathbf{s}+2\pi\ell)\circ A^{-1}\mathbf{u}}\big|\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)\big| \\ &\quad = \big|\widehat{\psi_1}((A^\tau)^j\mathbf{s})\big| \cdot \big|\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)\big|. \end{aligned}$$

Thus, (3.3) and (3.4) can be rewritten as

$$\begin{aligned} &|\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &\quad = \sum_{j=1}^{\infty} |\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot \big|\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)\big| \quad \text{and} \end{aligned} \tag{3.5}$$

$$\begin{aligned} &f(\mathbf{s})\overline{f(\mathbf{s}+2\pi\ell)} \cdot |\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &\quad = \sum_{j=1}^{\infty} f((A^\tau)^j\mathbf{s})\overline{f\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \\ &\qquad \times \big|\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)\big|. \end{aligned} \tag{3.6}$$

Since f is unimodular, $\overline{f} = 1/f$. Hence, (3.6) can be rewritten as

$$\begin{aligned}(3.7)\qquad &\frac{f(\mathbf{s})}{f(\mathbf{s}+2\pi\ell)}|\widehat{\psi_1}(\mathbf{s})|\cdot|\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &= \sum_{j=1}^{\infty} \frac{f((A^\tau)^j\mathbf{s})}{f((A^\tau)^j(\mathbf{s}+2\pi\ell))}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})|\cdot\left|\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)\right|.\end{aligned}$$

Combining this with (3.5) then leads to

$$\begin{aligned}(3.8)\qquad &\sum_{j=1}^{\infty}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})|\cdot\left|\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)\right| \\ &= \sum_{j=1}^{\infty}\frac{f(\mathbf{s}+2\pi\ell)}{f(\mathbf{s})}\frac{f((A^\tau)^j\mathbf{s})}{f((A^\tau)^j(\mathbf{s}+2\pi\ell))}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \\ &\quad\times\left|\widehat{\psi_1}\big((A^\tau)^j(\mathbf{s}+2\pi\ell)\big)\right|.\end{aligned}$$

Finally, since $|\widehat{\psi_1}((A^\tau)^j\mathbf{s})|\cdot|\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))| > 0$ by the choice of ψ_1 and $|\frac{f(\mathbf{s}+2\pi\ell)}{f(\mathbf{s})}\frac{f((A^\tau)^j\mathbf{s})}{f((A^\tau)^j(\mathbf{s}+2\pi\ell))}| = 1$, it follows that $\frac{f(\mathbf{s}+2\pi\ell)}{f(\mathbf{s})}\frac{f((A^\tau)^j\mathbf{s})}{f((A^\tau)^j(\mathbf{s}+2\pi\ell))} = 1$. In particular, $k(\mathbf{s}) = \frac{f(A^\tau\mathbf{s})}{f(\mathbf{s})} = \frac{f(A^\tau(\mathbf{s}+2\pi\ell))}{f(\mathbf{s}+2\pi\ell)} = k(\mathbf{s}+2\pi\ell)\ \forall\ell\in\mathbb{Z}^d\setminus A^\tau\mathbb{Z}^d$. If $\ell\in A^\tau\mathbb{Z}^d$, then $\ell-\mathbf{h}_1\notin A^\tau\mathbb{Z}^d$, and $k(\mathbf{s}+2\pi\ell) = k(\mathbf{s}+2\pi\mathbf{h}_1+2\pi(\ell-\mathbf{h}_1)) = k(\mathbf{s}+2\pi\mathbf{h}_1) = k(\mathbf{s})$ as well. Therefore, $k(\mathbf{s}) = f(A^\tau\mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic. $\square$

Combining Lemma 2.2 and Theorem 3.1 then leads to the following corollary.

Corollary 3.1. *A unimodular function $f\in L^\infty(\mathbb{R}^d)$ is an A-dilation MRA wavelet multiplier iff the function $k(\mathbf{s}) = f(A^\tau\mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic.*

Next, we show that all A-dilation wavelet multipliers can be constructed in the way described in the following theorem. A measurable set $E\subset\mathbb{R}^d$ is called an A-dilation wavelet set if $\mathcal{F}^{-1}((2\pi)^{-\frac{d}{2}}\chi_E)$ is an A-dilation wavelet. It is known that E is an A-dilation wavelet set iff both the sets $\{A^nE : n\in\mathbb{Z}\}$ and $\{E+2\pi\ell : \ell\in\mathbb{Z}^d\}$ are partitions of $\mathbb{R}^d$ [1].

Theorem 3.2. *Let E be an A-dilation wavelet set, $k(\mathbf{s})$ be a measurable unimodular $2\pi\mathbb{Z}^d$ periodic function and $g(\mathbf{s})$ be a measurable unimodular function defined on E. Define*

$$f(\mathbf{s}) = \begin{cases} g(\mathbf{s}), & \mathbf{s}\in E,\\ k((A^\tau)^{-1}\mathbf{s})\cdots k((A^\tau)^{-n}\mathbf{s})\cdot g((A^\tau)^{-n}\mathbf{s}), & \mathbf{s}\in(A^\tau)^nE, n\ge 1,\\ \overline{k(\mathbf{s})k(A^\tau\mathbf{s})\cdots k((A^\tau)^{n-1}\mathbf{s})}\cdot g((A^\tau)^n\mathbf{s}), & \mathbf{s}\in(A^\tau)^{-n}E, n\ge 1,\\ 1, & \mathbf{s}=0.\end{cases}$$

Then f is an A-dilation wavelet multiplier. Moreover, any A-dilation wavelet multiplier can be constructed this way.

Proof. Since $k(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic, it suffices to show that $f(A^\tau\mathbf{s}) = k(\mathbf{s})\cdot f(\mathbf{s})$ in order to show that f is an A-dilation wavelet multiplier by Theorem 3.1.

Case 1. $\mathbf{s}\in E$. Then $A^\tau\mathbf{s}\in A^\tau E$ and

$$f(A^\tau\mathbf{s}) = k((A^\tau)^{-1}A^\tau\mathbf{s})g((A^\tau)^{-1}A^\tau\mathbf{s}) = k(\mathbf{s})g(\mathbf{s}) = k(\mathbf{s})f(\mathbf{s}).$$

Case 2. $\mathbf{s}\in(A^\tau)^nE$ where $n\geq 1$. Then $A^\tau\mathbf{s}\in(A^\tau)^{n+1}E$ and

$$f(A^\tau\mathbf{s}) = k(\mathbf{s})k((A^\tau)^{-1}\mathbf{s})\cdots k((A^\tau)^{-n}\mathbf{s})g((A^\tau)^{-n}\mathbf{s}) = k(\mathbf{s})f(\mathbf{s}).$$

Case 3. $\mathbf{s}\in(A^\tau)^{-1}E$. Then $A^\tau\mathbf{s}\in E$ and $f(\mathbf{s}) = \overline{k(\mathbf{s})}g(A^\tau\mathbf{s})$, so $f(A^\tau\mathbf{s}) = g(A^\tau\mathbf{s}) = k(\mathbf{s})f(\mathbf{s})$.

Case 4. $\mathbf{s}\in(A^\tau)^{-n}E$ where $n>1$. Then $A^\tau\mathbf{s}\in(A^\tau)^{-(n-1)}E$ and

$$f(A^\tau\mathbf{s}) = k(\mathbf{s})\overline{k(\mathbf{s})k(A^\tau\mathbf{s})\cdots k((A^\tau)^{n-1}\mathbf{s})}g((A^\tau)^n\mathbf{s}) = k(\mathbf{s})f(\mathbf{s}).$$

Since $\{(A^\tau)^nE : n\in\mathbb{Z}\}$ is a partition of $\mathbb{R}^d$ modulo a null set, the above four cases have exhausted all possibilities for $\mathbf{s}\in\mathbb{R}^d$ in the sense.

Now suppose that $f(\mathbf{s})$ is an A-dilation wavelet multiplier. Let $g(\mathbf{s}) = f(\mathbf{s})$ for $\mathbf{s}\in E$, and $k(\mathbf{s}) = f(A^\tau\mathbf{s})/f(\mathbf{s})$. Then $k(\mathbf{s})$ is $2\pi\mathbb{Z}^d$ periodic and is unimodular. We leave it to our reader to verify that $f(\mathbf{s})$ has the form given in the theorem. □

4. Path-connectivity of the set of A-dilation MRA wavelets

In this section, we prove the main result of this paper, namely that the set of all A-dilation MRA wavelets is path-connected under the $L^2(\mathbb{R}^d)$ norm topology. For more discussions and related results on this topic, interested reader may refer to [11, 13, 14].

THEOREM 4.1. *For any two A-dilation MRA wavelets ψ_0 and ψ_1, there exists a continuous map $\gamma : [0,1] \longrightarrow L^2(\mathbb{R}^d)$ such that $\gamma(0) = \psi_0$, $\gamma(1) = \psi_1$ and $\gamma(t)$ is an A-dilation MRA wavelet for $\forall t\in[0,1]$.*

We will prove the theorem by directly constructing a continuous path connecting the two MRA wavelets. Since the proof is of constructive nature and is fairly long, we will break it into several lemmas. For a given A-dilation MRA wavelet ψ_0, we will associate with it three special subsets of A-dilation MRA wavelets denoted by $\mathcal{M}_{\psi_0}$, $\mathcal{W}_{\psi_0}$ and $\mathcal{S}_{\psi_0}$: $\mathcal{M}_{\psi_0}$ contains all A-dilation MRA wavelets ψ such that $\widehat{\psi} = v\widehat{\psi_0}$ for some A-dilation wavelet multiplier v, $\mathcal{W}_{\psi_0}$ contains all A-dilation MRA wavelets ψ such that $|\widehat{\psi}| = |\widehat{\psi_0}|$ and $\mathcal{S}_{\psi_0}$ contains all A-dilation MRA wavelets ψ such that $|\widehat{\phi}| = |\widehat{\phi_0}|$, where ϕ and ϕ_0 are the corresponding scaling functions of ψ and ψ_0.

LEMMA 4.1. *$\mathcal{S}_{\psi_0} = \mathcal{M}_{\psi_0} = \mathcal{W}_{\psi_0}$ for any A-dilation MRA wavelet ψ_0.*

Proof. Let $\psi \in \mathcal{W}_{\psi_0}$, then $|\widehat{\psi}| = |\widehat{\psi_0}|$. By (2.4), we have

$$|\widehat{\phi}(\mathbf{s})|^2 = \sum_{j=1}^{\infty} |\widehat{\psi}((A^\tau)^j \mathbf{s})|^2 = \sum_{j=1}^{\infty} |\widehat{\psi_0}((A^\tau)^j \mathbf{s})|^2 = |\widehat{\phi_0}(\mathbf{s})|^2,$$

so $\psi \in \mathcal{S}_{\psi_0}$. This shows that $\mathcal{W}_{\psi_0} \subseteq \mathcal{S}_{\psi_0}$. $\mathcal{M}_{\psi_0} \subseteq \mathcal{W}_{\psi_0}$ by definition. $\mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0}$ follows from an argument similar to the one used in the proof of [9, Theorem 1.2] and Proposition 2.1. Thus, $\mathcal{M}_{\psi_0} \subseteq \mathcal{W}_{\psi_0} \subseteq \mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0}$. Therefore, $\mathcal{S}_{\psi_0} = \mathcal{M}_{\psi_0} = \mathcal{W}_{\psi_0}$. □

LEMMA 4.2. *Let ψ_0 be an A-dilation MRA wavelet. Then $\mathcal{M}_{\psi_0}$ is path-connected.*

Proof. This is proved in [9, Theorem 1.3] for two special cases of A. However, that proof can be easily modified for the general case and is left to the reader. □

Let F be a tiling domain with the property that $\mathbf{0} \in F$, $F \subset A^\tau F$ and $\bigcup_{n\geq 0} (A^\tau)^n F = \mathbb{R}^d$ (the existence of such sets is shown in [6] as we mentioned in Section 2). In this case, the function ψ_0 defined by

$$\widehat{\psi_0}(\mathbf{s}) = (2\pi)^{-\frac{d}{2}} e^{i\mathbf{s}\circ A^{-1}\mathbf{u}} \chi_E(\mathbf{s}) \tag{4.1}$$

is an A-dilation MRA wavelet (where $E = A^\tau F \setminus F$) and is called a generalized Shannon-type wavelet. To show that any two A-dilation MRA wavelets are connected by a continuous path, it suffices to show any A-dilation MRA wavelet ψ can be path-connected to ψ_0. We will do this by showing that there exists a $\psi_1 \in \mathcal{S}_\psi$ such that ψ_1 is path-connected to ψ_0. By Lemmas 4.1 and 4.2 above, ψ_1 is path-connected to ψ hence ψ_0 is path-connected to ψ as well.

Note that the corresponding scaling function and low pass filter of ψ_0 are given by is $\hat{\phi}_0(\mathbf{s}) = (2\pi)^{-\frac{d}{2}} \chi_F$ and $m_0(\mathbf{s}) = \chi_{(A^\tau)^{-1}F}$. Keep in mind that F is a $2\pi\mathbb{Z}^d$ tiling domain so it is $2\pi\mathbb{Z}^d$ translation congruent to $\Omega = [-\pi, \pi]^d$.

By Lemma 2.3 and Proposition 2.2, there exists $\psi_1 \in \mathcal{S}_\psi$ such that its corresponding scaling function ϕ_1 and low pass filter m_1 satisfy $\widehat{\phi_1} \geq 0$, $m_1 \geq 0$ and

$$\widehat{\psi_1}(\mathbf{s}) = e^{i\mathbf{s}\circ A^{-1}\mathbf{u}} m_1\big((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_2\big) \widehat{\phi_1}\big((A^\tau)^{-1}\mathbf{s}\big). \tag{4.2}$$

We will now show that this particular choice of ψ_1 suffices for our purpose. In the following, we will first build a path that connects the low-pass filters, then use this filter path to construct the path for the scaling functions and ultimately the path that connects wavelet functions ψ_1 and ψ_0.

Keep in mind that m_0 and m_1 are $2\pi\mathbb{Z}^d$ periodic functions. Let $F_1 = (A^\tau)^{-1}F \subset F$ and $F_0 = F \setminus F_1$ (thus, the support of m_0 on F is simply F_1). Recall from Proposition 2.3 (and its proof) that F_0 and F_1 are both $2\pi(A^\tau)^{-1}\mathbb{Z}^d$ tiling domains of $\mathbb{R}^d$ and $F_1 \pm 2\pi\mathbf{h}_2$ is $2\pi\mathbb{Z}^d$ translation congruent to F_0. So the measure of F_0 and F_1 are both $(2\pi)^d/2$ and the measure of

$(A^\tau)^{-k}F_1$ is $(2\pi)^d/2^{k+1}$ for any $k \geq 0$. For any $t \in (0,1)$, there exists an integer $k_0(t) \geq 0$ such that $2^{-(k_0(t)+1)} < 1-t \leq 2^{-k_0(t)}$. If $1-t = 2^{-k_0(t)}$, define $H_t = (A^\tau)^{-k_0(t)}F_1$. If $2^{-(k_0(t)+1)} < 1-t < 2^{-k_0(t)}$, then there exists a positive real number $r_t > 0$ such that the set $G_t = B(r_t) \cap (A^\tau)^{-k_0(t)}(F_1 \setminus (A^\tau)^{-1}F_1)$ has measure $(1-t-2^{-(k_0(t)+1)})(2\pi)^d/2 = (1-t-2^{-(k_0(t)+1)})\mu(F_1)$, where $B(r_t) = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}|^2 \leq r_t^2\}$ and μ denotes the Lebesgue measure in $\mathbb{R}^d$. And we will define $H_t = G_t \cup (A^\tau)^{-(k_0(t)+1)}F_1$. Under this definition, observe that the measure of H_t is precisely $(1-t)\mu(F_1)$. Furthermore, for any $\mathbf{s} \in F$, it is obvious that $(A^\tau)^{-k}\mathbf{s} \in H_t$ for any $k \geq k_0(t)+2$. For $t=0$ and $t=1$, H_t is defined as F_1 and the empty set respectively. Since H_t is a subset of F_1 and $F_1 \pm 2\pi\mathbf{h}_2$ is $2\pi\mathbb{Z}^d$ translation congruent to F_0, $H_t \pm 2\pi\mathbf{h}_2$ is $2\pi\mathbb{Z}^d$ translation congruent to a subset (uniquely determined by H_t modular a zero measure set) $J_t \subset F_0$. Equivalently, $J_t \pm 2\pi\mathbf{h}_2$ is $2\pi\mathbb{Z}^d$ translation congruent to H_t. The low-pass filter path $m_t(\mathbf{s})$ is then defined for any $\mathbf{s} \in F$ first as below and then extended to $\mathbb{R}^d$ as a $2\pi\mathbb{Z}^d$ periodic function (this is possible since F is a tiling domain).

$$m_t(\mathbf{s}) = \begin{cases} \sqrt{1-t+tm_1^2(\mathbf{s})}, & \mathbf{s} \in F_1 \setminus H_t, \\ 1, & \mathbf{s} \in H_t, \\ \sqrt{t}m_1(\mathbf{s}), & \mathbf{s} \in F_0 \setminus J_t, \\ 0, & \mathbf{s} \in J_t. \end{cases} \tag{4.3}$$

For $t=0$ and $t=1$, $m_t(\mathbf{s})$ is just $m_0(\mathbf{s})$ and $m_1(\mathbf{s})$. Furthermore, $m_t(\mathbf{s})$ satisfies the equation $|m_t(\mathbf{s})|^2 + |m_t(\mathbf{s}+2\pi\mathbf{h}_2)|^2 = 1$. These are left to our reader to verify.

LEMMA 4.3. *For each $t \in [0,1]$, the function ϕ_t defined by*

$$\widehat{\phi}_t(\mathbf{s}) = (2\pi)^{-\frac{d}{2}} \prod_{j=1}^{\infty} m_t((A^\tau)^{-j}\mathbf{s}) \tag{4.4}$$

is an MRA A-dilation scaling function hence ψ_t defined by

$$\widehat{\psi_t}(\mathbf{s}) = e^{i\mathbf{s}\circ A^{-1}\mathbf{u}} m_t\big((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_2\big)\widehat{\phi}_t((A^\tau)^{-1}\mathbf{s}) \tag{4.5}$$

is an MRA A-dilation wavelet.

Proof. The statement holds trivially for $t=0$ and 1, so we only need to consider the case $0<t<1$. $\widehat{\phi}_t$ is well defined since $0 \leq m_t(\mathbf{s}) \leq 1$, so is $\widehat{\psi_t}$. From the definition of $\widehat{\phi}_t$, we have $\widehat{\phi}_t(A^\tau\mathbf{s}) = m_t(\mathbf{s})\widehat{\phi}_t(\mathbf{s})$, So $\widehat{\phi}_t$ satisfies condition (iii) of Lemma 2.3. On the other hand, $\widehat{\phi}_t(\mathbf{s}) = (2\pi)^{-\frac{d}{2}}, \forall \mathbf{s} \in H_t$, also by the definition of $\widehat{\phi}_t$. Since $0<t<1$, $(A^\tau)^{-(k_0(t)+1)}F_1 \subset H_t$ so H_t contains a neighborhood of $\mathbf{0}$ since F (hence F_1) contains a neighborhood of $\mathbf{0}$. From this condition (ii) of Lemma 2.3 follows. We now prove that ϕ_t satisfies condition (i) of Lemma 2.3 as well, which then implies that ϕ_t is a scaling function and ψ_t is an MRA A-dilation wavelet.

As we observed earlier, $\forall \mathbf{s} \in F$, $(A^\tau)^{-k}\mathbf{s} \in H_t$, $\forall k > k_0(t)+1 = k_0'$. So by the definition of $m_t(\mathbf{s})$, we have $m_t((A^\tau)^{-k}\mathbf{s}) = 1$ for all such k. On the other hand, if $1 \le k \le k_0'$, we have $(A^\tau)^{-k}\mathbf{s} \in F_1$ hence $m_t((A^\tau)^{-k}\mathbf{s}) \ge \sqrt{1-t}$. It follows that for any $\mathbf{s} \in F$ we have

$$\begin{aligned}\widehat{\phi}_t(\mathbf{s}) &= (2\pi)^{-\frac{d}{2}} \prod_{k=1}^{k_0'} m_t((A^\tau)^{-k}\mathbf{s}) \prod_{k=k_0'+1}^{\infty} m_t((A^\tau)^{-k}\mathbf{s}) \\ &= (2\pi)^{-\frac{d}{2}} \prod_{k=1}^{k_0'} m_t((A^\tau)^{-k}\mathbf{s}) \ge (2\pi)^{-\frac{d}{2}} (1-t)^{k_0'/2}.\end{aligned}$$

This implies that $\chi_F(\mathbf{s}) \le (2\pi)^{\frac{d}{2}} \widehat{\phi}_t(\mathbf{s})/(1-t)^{k_0'/2}$. For each $k \ge 1$, define

$$p_{t,k}(\mathbf{s}) = (2\pi)^{-\frac{d}{2}} \chi_F((A^\tau)^{-k}\mathbf{s}) \cdot \prod_{j=1}^{k} m_t((A^\tau)^{-j}\mathbf{s}).$$

Then

$$p_{t,k}(\mathbf{s}) \le \frac{\widehat{\phi}_t((A^\tau)^{-k}\mathbf{s})}{(1-t)^{k_0'/2}} \prod_{j=1}^{k} m_t((A^\tau)^{-j}\mathbf{s}) = \frac{\widehat{\phi}_t(\mathbf{s})}{(1-t)^{k_0'/2}}.$$

For $k \ge 2$, $\int_{\mathbb{R}^d} |p_{t,k}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s}$ can be rewritten as (substituting $\mathbf{s}$ for $(A^\tau)^{-k}\mathbf{s}$)

$$\begin{aligned}&\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\chi_F((A^\tau)^{-k}\mathbf{s})|^2 \cdot \prod_{j=1}^{k} |m_t((A^\tau)^{-j}\mathbf{s})|^2 \cdot e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s} \\ &= \frac{2^k}{(2\pi)^d} \int_F \prod_{j=0}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})}\, d\mathbf{s} \\ &= \frac{2^k}{(2\pi)^d} \Bigg(\int_{F_0} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})}\, d\mathbf{s} \\ &\qquad + \int_{F_1} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})}\, d\mathbf{s} \Bigg).\end{aligned}$$

Since $F_0 \pm 2\pi\mathbf{h}_2$ is $2\pi\mathbb{Z}^d$ translation congruent to F_1 and e^{-ins}, $m_t(\mathbf{s})$ are $2\pi\mathbb{Z}^d$ periodic, the above equality becomes (keep in mind that $F_1 = (A^\tau)^{-1}F$)

$$\begin{aligned}&\frac{2^k}{(2\pi)^d} \Bigg(\int_{F_1} |m_t(\mathbf{s}+2\pi\mathbf{h}_2)|^2 \prod_{j=1}^{k-1} \left|m_t\big((A^\tau)^j(\mathbf{s}+2\pi\mathbf{h}_2)\big)\right|^2 e^{-i\mathbf{n}\circ((A^\tau)^k(\mathbf{s}+2\pi\mathbf{h}_2))}\, d\mathbf{s} \\ &\qquad + \int_{F_1} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})}\, d\mathbf{s} \Bigg)\end{aligned}$$

$$= \frac{2^k}{(2\pi)^d} \int_{F_1} \left(|m_t(\mathbf{s})|^2 + |m_t(\mathbf{s} + 2\pi\mathbf{h}_2)|^2\right) \prod_{j=1}^{k-1} |m_t((A^\tau)^j \mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k \mathbf{s})}\, d\mathbf{s}$$

$$= \frac{2^k}{(2\pi)^d} \int_{F_1} \prod_{j=1}^{k-1} |m_t((A^\tau)^j \mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k \mathbf{s})}\, d\mathbf{s}$$

$$= \frac{2^{k-1}}{(2\pi)^d} \int_{F} \prod_{j=0}^{k-2} |m_t((A^\tau)^j \mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^{k-1} \mathbf{s})}\, d\mathbf{s} = \int_{\mathbb{R}^d} |p_{t,k-1}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s}.$$

Repeating the above procedure then leads to

$$\begin{aligned}
&\int_{\mathbb{R}^d} |p_{t,k}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s} \\
&\quad = \int_{\mathbb{R}^d} |p_{t,1}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s} \\
&\quad = \frac{2}{(2\pi)^d} \int_{F} |m_t(\mathbf{s})|^2 e^{-i\mathbf{n}\circ(A^\tau\mathbf{s})}\, d\mathbf{s} = \frac{2}{(2\pi)^d} \int_{F_1} e^{-i\mathbf{n}\circ(A^\tau\mathbf{s})}\, d\mathbf{s} \\
&\quad = \frac{1}{(2\pi)^d} \int_{F} e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s} = \frac{1}{(2\pi)^d} \int_{\Omega} e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s} = \delta_{\mathbf{n},\mathbf{0}}.
\end{aligned}$$

So $\|p_{t,k}\|^2 = 1$. Clearly, $\lim_{k\to\infty} p_{t,k}(\mathbf{s}) = \widehat{\phi}_t(\mathbf{s})$ for all $\mathbf{s} \in \mathbb{R}^d$. Thus, $\phi_t \in L^2(\mathbb{R}^d)$ by Fatou's lemma. Since $p_{t,k}(\mathbf{s})$ is dominated by $\widehat{\phi}_t(\mathbf{s})/(1-t)^{k_0'/2}$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |p_{t,k}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s} = \int_{\mathbb{R}^d} |\widehat{\phi}_t(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}}\, d\mathbf{s} = \delta_{\mathbf{n},\mathbf{0}}$$

by Lebesgue's dominated convergence theorem. This is equivalent to

$$\sum_{\ell\in\mathbb{Z}^d} |\widehat{\phi}_t(\mathbf{s} + 2\pi\ell)|^2 = 1/(2\pi)^d.$$

By Lemma 2.3, ϕ_t is an MRA scaling function. $\square$

LEMMA 4.4. *For each fixed $t_0 \in [0,1]$, the mapping $t \mapsto m_t(\mathbf{s})$ is continuous at t_0 for almost all $\mathbf{s} \in \mathbb{R}^d$. That is, there exists a measure zero set $N_{t_0} \subset \mathbb{R}^d$ such that for each $\mathbf{s} \in \mathbb{R}^d \setminus N_{t_0}$, $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $|m_t(\mathbf{s}) - m_{t_0}(\mathbf{s})| < \varepsilon$ for all $t \in [0,1] \cap (t_0 - \delta, t_0 + \delta)$.*

Proof. We shall only prove the case for $0 < t_0 < 1$ and leave the cases $t_0 = 0$ and $t_0 = 1$ (which are simpler than the case of $0 < t_0 < 1$) to the reader. Recall from the definition of the set H_t that $H_{t_2} \subset H_{t_1}$ (hence $F_1 \setminus H_{t_1} \subset F_1 \setminus H_{t_2}$) whenever $t_2 \geq t_1$ and that $\mu(H_t) = (1-t)\mu(F_1)$.

For any $t_0 \in (0,1)$, define

$$M_{t_0} = \left(\bigcap_{t>t_0} \left((F_1 \setminus H_t) \cap H_{t_0}\right)\right) \cup \left(\bigcap_{t<t_0} (H_t) \cap (F_1 \setminus H_{t_0})\right).$$

It is easy to check that for any $t > t_0$, we have $\mu((F_1 \setminus H_t) \cap H_{t_0}) = \mu(H_{t_0} \setminus H_t) = (t - t_0)\mu(F_1)$. Similarly, for any $t < t_0$, $\mu(H_t \cap (F_1 \setminus H_{t_0})) = \mu(H_t \setminus H_{t_0}) = (t_0 - t)\mu(F_1)$. Hence, $\mu(M_{t_0}) \leq 2|t - t_0|\mu(F_1)$ for any $t \neq t_0$. Since t is arbitrary and M_{t_0} is a fixed measurable set, we must have $\mu(M_{t_0}) = 0$. Now define $N_{t_0} = \bigcup_{\ell \in \mathbb{Z}^d}(M_{t_0} + \pi\ell)$, which is also a measure zero set.

For any $\mathbf{s} \in \mathbb{R}^d \setminus N_{t_0}$, there exists a (unique) $\mathbf{s}_0 \in F$ such that $\mathbf{s} - \mathbf{s}_0 \in 2\pi\mathbb{Z}^d$, $\mathbf{s}_0 \in F_1 \setminus M_{t_0}$ or $\mathbf{s}_0 \in F_0 \setminus M'_{t_0}$ where M'_{t_0} is the subset in F_0 that is $2\pi\mathbb{Z}^d$ translation congruent to $M_{t_0} + 2\pi\mathbf{h}_2$. Since m_t is $2\pi\mathbb{Z}^d$ translation periodic, we have $m_t(\mathbf{s}) = m_t(\mathbf{s}_0)$. Consider first the case that $\mathbf{s}_0 \in F_1 \setminus M_{t_0}$. By the definition of M_{t_0}, there exist $t_1 < t_0 < t_2$ such that $\mathbf{s}_0 \notin H_{t_1} \setminus H_{t_2}$. Choose $\delta > 0$ small enough so that $\delta < \min\{t_0 - t_1, t_2 - t_0\}$ and $|\sqrt{1 - t + bt} - \sqrt{1 - t_0 + bt_0}| < \varepsilon$ for any $t \in (0,1) \cap (t_0 - \delta, t_0 + \delta)$ and any $0 \leq b \leq 1$. Then for any such t we have $\mathbf{s}_0 \in H_{t_2} \subset H_{t_0}$ or $\mathbf{s}_0 \in F_1 \setminus H_{t_1} \subset F_1 \setminus H_{t_0}$. In the first case, $|m_t(\mathbf{s}_0) - m_{t_0}(\mathbf{s}_0)| = 0$ and in the second case $|m_t(\mathbf{s}_0) - m_{t_0}(\mathbf{s}_0)| = |\sqrt{1 - t + tm_1^2(\mathbf{s}_0)} - \sqrt{1 - t_0 + t_0 m_1^2(\mathbf{s}_0)}| < \varepsilon$ by the choice of δ. The case of $\mathbf{s}_0 \in F_0 \setminus M'_{t_0}$ can be similarly proven. $\square$

The result of Lemma 4.4 can be easily extended to the following corollary. We leave the proof to the reader.

Corollary 4.1. *For each fixed $t_0 \in [0,1]$ and each fixed $k \geq 1$, the mapping $t \mapsto \prod_{j=1}^k m_t((A^\tau)^{-j}\mathbf{s})$ is continuous at t_0 for almost all $\mathbf{s} \in \mathbb{R}^d$. That is, there exists a measure zero set $N_{t_0}^k \subset \mathbb{R}^d$ such that for each $\mathbf{s} \in \mathbb{R}^d \setminus N_{t_0}^k$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|\prod_{j=1}^k m_t((A^\tau)^{-j}\mathbf{s}) - \prod_{j=1}^k m_{t_0}((A^\tau)^{-j}\mathbf{s})| < \varepsilon$ for all $t \in [0,1] \cap (t_0 - \delta, t_0 + \delta)$.*

Lemma 4.5. *For any $t_0 \in [0,1]$, $\lim_{t\to t_0} \widehat{\phi_t}(\mathbf{s}) = \widehat{\phi_{t_0}}(\mathbf{s})$ for $\mathbf{s} \in \mathbb{R}^d$. More precisely, for each fixed $t_0 \in [0,1]$, there exists a measure zero set $N'_{t_0} \subset \mathbb{R}^d$ such that for any $\mathbf{s} \in \mathbb{R}^d \setminus N'_{t_0}$, we have $\lim_{t\to t_0} \widehat{\phi_t}(\mathbf{s}) = \widehat{\phi_{t_0}}(\mathbf{s})$.*

Proof. Since $\widehat{\phi_1} \geq 0$, $\lim_{j\to\infty} \widehat{\phi_1}((A^\tau)^{-j}\mathbf{s}) = 1/(2\pi)^{\frac{d}{2}}$. For any given $\varepsilon > 0$ and $\mathbf{s} \in \mathbb{R}^d$ (modular a zero measure set), there exists a positive integer n_0 such that $\widehat{\phi_1}((A^\tau)^{-n}\mathbf{s}) > 1/(2\pi)^{\frac{d}{2}} - \varepsilon/2$ and $(A^\tau)^{-n}\mathbf{s} \subset (A^\tau)^{-1}F$ for any $n \geq n_0$. It follows that $m_t((A^\tau)^{-n}\mathbf{s})$ is either 1 or $\sqrt{(1-t) + tm_1^2((A^\tau)^{-n}\mathbf{s})}$ for any $t \in [0,1]$. In either case, $m_t((A^\tau)^{-n}\mathbf{s}) \geq m_1((A^\tau)^{-n}\mathbf{s})$, thus $\widehat{\phi_t}((A^\tau)^{-n}\mathbf{s}) \geq \widehat{\phi_1}((A^\tau)^{-n}\mathbf{s})$ for any $t \in [0,1]$. Since $\widehat{\phi_t}(\mathbf{s}') \leq 1/(2\pi)^{\frac{d}{2}}$ for any $\mathbf{s}' \in \mathbb{R}^d$ by its definition, it follows that for any t_1, $t_2 \in [0,1]$, we have

$$|\widehat{\phi_{t_1}}((A^\tau)^{-n}\mathbf{s}) - \widehat{\phi_{t_2}}((A^\tau)^{-n}\mathbf{s})| < \varepsilon/2. \tag{4.6}$$

On the other hand, by Corollary 4.1, for each $t_0 \in [0,1]$, there exists a zero measure set $N_{t_0}^{n_0} \subset \mathbb{R}^d$ such that for each $\mathbf{s} \in \mathbb{R}^d \setminus N_{t_0}^{n_0}$, $\exists \delta > 0$ such that $|\prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s}) - \prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})| < \varepsilon/2$ for all $t \in [0,1] \cap (t_0 - \delta,$

$t_0+\delta)$. Hence, $|\widehat{\phi}_t(\mathbf{s})-\widehat{\phi_{t_0}}(\mathbf{s})|$ is bounded by

$$\begin{aligned}
&\left|\prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s})\widehat{\phi}_t((A^\tau)^{-n_0}\mathbf{s})-\prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\widehat{\phi_{t_0}}((A^\tau)^{-n_0}\mathbf{s})\right|\\
&\quad=\left|\prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s})\cdot\widehat{\phi}_t((A^\tau)^{-n_0}\mathbf{s})-\prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\widehat{\phi}_t((A^\tau)^{-n_0}\mathbf{s})\right.\\
&\qquad\left.+\prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\widehat{\phi}_t((A^\tau)^{-n_0}\mathbf{s})-\prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\widehat{\phi_{t_0}}((A^\tau)^{-n_0}\mathbf{s})\right|\\
&\quad\le(2\pi)^{-\frac{d}{2}}\left|\prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s})-\prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\right|\\
&\qquad+|\widehat{\phi}_t((A^\tau)^{-n_0}\mathbf{s})-\widehat{\phi_{t_0}}((A^\tau)^{-n_0}\mathbf{s})|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

So $\lim_{t\to t_0}\widehat{\phi}_t(\mathbf{s})=\widehat{\phi_{t_0}}(\mathbf{s})$. □

By the continuity of $m_t(\mathbf{s})$ and $\widehat{\phi}_t$, we now have $\lim_{t\to t_0}\widehat{\psi_t}(\mathbf{s})=\widehat{\psi_{t_0}}(\mathbf{s})$ for almost every $\mathbf{s}\in\mathbb{R}^d$.

LEMMA 4.6. *For $t_0,t\in[0,1]$, $\lim_{t\to t_0}\|\widehat{\psi_t}-\widehat{\psi_{t_0}}\|^2=0$.*

Proof. Since $\|\widehat{\psi_t}\|^2=\|\widehat{\psi_{t_0}}\|^2=1$, $\|\widehat{\psi_t}-\widehat{\psi_{t_0}}\|^2=\langle\widehat{\psi_t}-\widehat{\psi_{t_0}},\widehat{\psi_t}-\widehat{\psi_{t_0}}\rangle=2-\langle\widehat{\psi_t},\widehat{\psi_{t_0}}\rangle-\langle\widehat{\psi_{t_0}},\widehat{\psi_t}\rangle$. Thus, it suffices to show that $\lim_{t\to t_0}\langle\widehat{\psi_t},\widehat{\psi_{t_0}}\rangle=1$.

Since $\widehat{\psi_{t_0}}\in L^2(\mathbb{R}^d)$, for any $\varepsilon>0$, there exists a sufficiently large number $r>0$ such that $(\int_{|\mathbf{s}|>r}|\widehat{\psi_{t_0}}(\mathbf{s})|^2\,d\mathbf{s})^{\frac{1}{2}}<\varepsilon/4$. By Hölder's Inequality, we then have $\int_{|\mathbf{s}|>r}|\widehat{\psi_t}(\mathbf{s})-\widehat{\psi_{t_0}}(\mathbf{s})|\cdot|\overline{\widehat{\psi_{t_0}}(\mathbf{s})}|\,d\mathbf{s}\le\|\widehat{\psi_t}(\mathbf{s})-\widehat{\psi_{t_0}}(\mathbf{s})\|(\int_{|\mathbf{s}|>r}|\widehat{\psi_{t_0}}(\mathbf{s})|^2\,d\mathbf{s})^{\frac{1}{2}}<\varepsilon/2$ since $\|\widehat{\psi_t}(\mathbf{s})-\widehat{\psi_{t_0}}(\mathbf{s})\|\le\|\widehat{\psi_t}(\mathbf{s})\|+\|\widehat{\psi_{t_0}}(\mathbf{s})\|=2$. On the other hand, $|\widehat{\psi_t}(\mathbf{s})-\overline{\widehat{\psi_{t_0}}(\mathbf{s})}|\le 1/(\pi)^{d/2}$ since $|\widehat{\psi_t}(\mathbf{s})|\le 1/(2\pi)^{\frac{d}{2}}$ and $|\overline{\widehat{\psi_{t_0}}(\mathbf{s})}|\le 1/(2\pi)^{\frac{d}{2}}$ by (4.4), (4.5) and the fact that $|m_t|\le 1$ for any t. Thus, by the dominated convergence theorem, we have $\lim_{t\to t_0}\int_{|\mathbf{s}|\le r}|\widehat{\psi_t}(\mathbf{s})-\widehat{\psi_{t_0}}(\mathbf{s})|\,d\mathbf{s}=0$. Therefore, there exists a number $\delta>0$ such that $\int_{|\mathbf{s}|\le r}|\widehat{\psi_t}(\mathbf{s})-\widehat{\psi_{t_0}}(\mathbf{s})|\,d\mathbf{s}<\varepsilon/2$ whenever $|t-t_0|<\delta$. Combining the above leads to

$$\begin{aligned}
|\langle\widehat{\psi_t},\widehat{\psi_{t_0}}\rangle-1|&=|\langle\widehat{\psi_t},\widehat{\psi_{t_0}}\rangle-\langle\widehat{\psi_{t_0}},\widehat{\psi_{t_0}}\rangle|=\left|\int_{\mathbb{R}^d}\big(\widehat{\psi_t}(\mathbf{s})-\widehat{\psi_{t_0}}(\mathbf{s})\big)\cdot\overline{\widehat{\psi_{t_0}}(\mathbf{s})}\,d\mathbf{s}\right|\\
&\le\int_{|\mathbf{s}|\le r}\left|\big(\widehat{\psi_t}(\mathbf{s})-\widehat{\psi_{t_0}}(\mathbf{s})\big)\overline{\widehat{\psi_{t_0}}(\mathbf{s})}\right|d\mathbf{s}\\
&\quad+\int_{|\mathbf{s}|>r}\left|\big(\widehat{\psi_t}(\mathbf{s})-\widehat{\psi_{t_0}}(\mathbf{s})\big)\overline{\widehat{\psi_{t_0}}(\mathbf{s})}\right|d\mathbf{s}<\varepsilon.
\end{aligned}$$

So $\lim_{t\to t_0}\|\widehat{\psi_t}-\widehat{\psi_{t_0}}\|^2=0$. □

Since the inverse Fourier transform preserves norm, it follows that the mapping $t \mapsto \psi_t$ is continuous in the $L^2(\mathbb{R}^d)$ norm topology. This completes the proof of the connectedness theorem.

5. Further discussions

The path-connectivity for the set of all 2-dilation MRA wavelets was first established in [11, 14] for the one dimensional case. In [10], the authors solved the path-connectivity problem for all matrices in $M_2^{(2)}(\mathbb{Z})$. The approach used in [10] depends on a special property that a matrix in $M_2^{(2)}(\mathbb{Z})$ possesses. The purpose of this section is to show that a matrix in $M_3^{(2)}(\mathbb{Z})$ may no longer possess this property. Consequently, the argument used in this paper to establish the path-connectivity of all A-dilation MRA wavelets for any $A \in M_d^{(2)}(\mathbb{Z})$ is not a simple generalization of the earlier approaches.

Two $d \times d$ integral matrices B and C are said to be *integrally similar* if there exists an integral $d \times d$ matrix P such that $|\det(P)| = 1$ and $P^{-1}BP = C$. The integral similarity then defines an equivalent relation among matrices of $M_d^2(\mathbb{Z})$. For $d = 2$, there are exactly six integrally similar classes in $M_2^{(2)}(\mathbb{Z})$ [8]. A representative from each of these classes is listed below.

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}.$$

An important point about the above representatives is that each of them has the property that $(A^\tau)^{-1}\Omega \subset \Omega$ where $\Omega = [-\pi, \pi)^2$. For an expansive matrix with this property, one could then employ the nice geometric structures of Ω and $(A^\tau)^{-1}\Omega$ in the construction of m_t (and $\widehat{\phi_t}(\mathbf{s})$, $\widehat{\psi_t}(\mathbf{s})$). In particular, a general Shannon-type wavelet can be visualized in these cases. For a matrix in $M_2^{(2)}(\mathbb{Z})$ that does not have this property, one can use its representative in the above list (in its equivalent class) and use the following useful theorem (proved in [10]).

THEOREM 5.1 ([10]). *For any 2×2 integral matrix P with $|\det P| = 1$, let $\Phi_P : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$ be the operator defined by $\Phi_P(g(\mathbf{t})) = g(P\mathbf{t})$. If B and C are two 2×2 integral, expansive matrices such that $P^{-1}BP = C$, then the following statements hold*:

(i) *ψ is a B-dilation wavelet iff $\Phi_P(\psi)$ is a C-dilation wavelet*;
(ii) *A function $f \in L^2(\mathbb{R}^2)$ is a B-dilation wavelet multiplier iff the function $\Phi_{(P^\tau)^{-1}}(f)$ is a C-dilation wavelet multiplier.*

The linear operator $\Phi_P : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$ defined above is obviously continuous and unitary (since $|\det P| = 1$). In the case that P is also integral and $P^{-1}BP = C$, then Theorem 5.1 asserts that $\Phi_P : \mathcal{W}_B \longrightarrow \mathcal{W}_C$ is a continuous and bijective mapping, where $\mathcal{W}_B$ is the set of all B-dilation wavelets and $\mathcal{W}_C$ is the set of all C-dilation wavelets. Φ_P is also a bijection between the set of all B-dilation MRA wavelets and the set of all C-dilation MRA wavelets. These observations imply that at least for the two dimensional case, the path-connectivity problem of all B-dilation MRA wavelets for any $B \in M_2^{(2)}(\mathbb{Z})$ is equivalent to the path-connectivity problem of all A-dilation MRA wavelets where A is one of the six matrices listed above. This is precisely what these authors did in [10]: they established the path-connectivity of all A-dilation MRA wavelets where A is one of the six matrices listed above, which then implied that all B-dilation MRA wavelets are path-connected for any $B \in M_2^{(2)}(\mathbb{Z})$. A different way to interpret the above theorem (or the classifications of $M_2^{(2)}(\mathbb{Z})$ matrices into the 6 integrally similar equivalent classes) is that for any matrix A in $M_2^{(2)}(\mathbb{Z})$, a general Shannon-type A-dilation MRA wavelet can be constructed using a set F where F has the form $P\Omega$ for some $P \in M_2^{(1)}(\mathbb{Z})$. Of course, such explicit expression for an MRA wavelet would be desirable. In some sense, the approach in [10] made the maximum use of this property of $M_2^{(2)}(\mathbb{Z})$.

Let us now consider the possibility of extending that approach in the case of $d = 3$. In other words, one would like to establish the following:

(1) Identify all integrally similar equivalent classes of matrices in $M_3^{(2)}(\mathbb{Z})$;

(2) Show that in each such class there is a representative matrix A with the property that $(A^\tau)^{-1}\Omega \subset \Omega$ where $\Omega = [-\pi, \pi)^3$;

(3) Explore the structure of $(A^\tau)^{-1}\Omega$ and the possibility of using this structure in the definition of $m_t(\mathbf{s})$ (and ϕ_t, ψ_t) that would lead to the establishment of path-connectivity of all A-dilation MRA wavelets for such matrix A.

Unfortunately, the first task seems to be a very difficult problem. The authors failed to find an answer to this question in the literature. The third task, even when successfully carried out, can only solve the problem for some matrices in $M_3^{(2)}(\mathbb{Z})$. The biggest problem turned out to be task 2: there are integrally similar equivalent classes in $M_3^{(2)}(\mathbb{Z})$ which do not have any representative A with the property $(A^\tau)^{-1}\Omega \subset \Omega$. The authors do not intend to elaborate the details here. Instead, we will just list some of our findings about $M_3^{(2)}(\mathbb{Z})$.

Firstly, there are 14 similar equivalent classes (but we are not sure if they are all the integrally similar classes) in $M_3^{(2)}(\mathbb{Z})$. A representative from each class is listed below.

$$\begin{pmatrix}0&-1&-1\\0&-1&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&-1\\2&-1&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&-1&1\\0&-1&-1\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\2&0&0\\0&1&0\end{pmatrix},$$
$$\begin{pmatrix}0&1&0\\0&0&1\\2&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&-1\\2&-1&0\end{pmatrix},\begin{pmatrix}1&-1&0\\0&0&-1\\1&1&0\end{pmatrix},\begin{pmatrix}1&0&1\\1&0&-1\\0&-1&0\end{pmatrix},$$
$$\begin{pmatrix}0&0&1\\2&1&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&2&0\\0&0&-1\\1&0&0\end{pmatrix},\begin{pmatrix}-1&-1&-1\\0&-1&1\\1&0&0\end{pmatrix},\begin{pmatrix}-1&-1&-1\\-1&0&0\\0&-1&1\end{pmatrix},$$
$$\begin{pmatrix}-1&-1&0\\0&0&-1\\1&-1&1\end{pmatrix},\begin{pmatrix}0&-1&-1\\0&1&-1\\1&0&1\end{pmatrix}.$$

Secondly, of these 14 classes, 4 of them do not have any representative A with the property $(A^\tau)^{-1}\Omega\subset\Omega$. The last four matrices listed above are the representatives of these four classes. It would be interesting to know if one can find a bounded set F (as used in the definition of a general Shannon type wavelet) that is also geometrically nice and simple.

Thirdly, the 10 other representatives in the above list all have the property $(A^\tau)^{-1}\Omega\subset\Omega$. It is interesting to note that within each of these 10 classes, there are exactly 24 matrices with this property. For any other matrix in one of these classes, it is not clear whether it is integrally similar to one of these 24 representatives. We suspect that is the case.

Finally, for $d\geq 4$, the situation will be even more complicated and it is plausible that similar equivalent classes like the last 4 in the above list exist. Since our findings above are obtained through exhaustive search, the method cannot be easily generalized to handle the more general cases in higher dimension. This would be a problem for future study.

References

[1] X. Dai, D. R. Larson and D. M. Speegle, *Wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **3**(2) (1997), 451–456. MR 1468374

[2] M. Bownik, *A characterization of affine dual frames in* $L^2(\mathbb{R}^n)$, Appl. Comput. Harmon. Anal. **8** (2000), 203–221. MR 1743536

[3] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Memoirs Amer. Math. Soc. **134**(640) (1998). MR 1432142

[4] I. Daubechies, *Ten lecture on wavelets*, CBMS Lecture Notes **61**, SIAM, Philadelphia, PA, 1992. MR 1162107

[5] D. Bakic, I. Krishtal and E. N. Wilson, *Parseval frame wavelets with* $E_n^{(2)}$*-dialtions*, Appl. Comput. Harmon. Anal. **19** (2005), 386–431. MR 2186451

[6] Q. Gu and D. Han, *On multiresolution analysis (MRA) wavelets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **6**(4) (2000), 437–447. MR 1776974

[7] E. Hernándes and G. Weiss, *A first course on wavelets*, CRC Press, Boca Raton, FL, 1996. MR 1408902

[8] J. C. Lagarias and Y. Wang, *Haar type orthonormal wavelet basis in* $\mathbb{R}^2$, J. Fourier Anal. Appl. **2**(1) (1995), 1–14. MR 1361539

[9] Y. Li, *On a class of bidimensional nonseparable wavelet multipliers*, J. Math. Anal. Appl. **270** (2002), 543–560. MR 1916596

[10] Z. Li, X. Dai, Y. Diao and J. Xin, *Multipliers, phases and connectivity of MRA wavelets in* $L^2(\mathbb{R}^2)$, J. Fourier Anal. Appl. **16**(2) (2010), 155–176.

[11] R. Liang, *Some properties of wavelets*, Ph.D. Dissertation, University of North Carolina at Charlotte, 1998. MR 2698134

[12] Y. Meyer, *Wavelets and operators*, Cambridge Studies in Advanced Mathematics **37**, Cambridge Univ. Press, Cambridge, 1992. MR 1228209

[13] D. Speegle, *The s-elementary wavelets are path-connected*, Proc. Amer. Math. Soc. **127**(1) (1999), 223–233. MR 1468204

[14] The Wutam Consortium, *Basic properties of wavelets*, J. Fourier Anal. Appl. **4**(4) (1998), 575–594. MR 1658652

Zhongyan Li, Department of Mathematics and Physics, North China Electric Power University, Beijing, 102206, China

E-mail address: lzhongy@ncepu.edu.cn

Xingde Dai, Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA

E-mail address: xdai@uncc.edu

Yuanan Diao, Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA

E-mail address: ydiao@uncc.edu

Wei Huang, Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA

E-mail address: whuang7@uncc.edu