

ERGODICITY AND MIXING OF W^* -DYNAMICAL SYSTEMS IN TERMS OF JOININGS

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ABSTRACT. We study characterizations of ergodicity, weak mixing and strong mixing of W^* -dynamical systems in terms of joinings and subsystems of such systems. Ergodic joinings and Ornstein’s criterion for strong mixing are also discussed in this context.

1. Introduction

In [6], we studied joinings of W^* -dynamical systems, and in particular gave a characterization of ergodicity in terms of joinings, similar to the measure theoretic case. In this paper, we continue to extend certain results regarding joinings of measure theoretic dynamical systems to the noncommutative setting of W^* -dynamical systems. First, we generalize the necessary condition for ergodicity to arbitrary group actions, and also prove a similar set of sufficient and necessary conditions for weak mixing in terms of ergodic compact systems and discrete spectra (see Section 2). Section 3 is devoted to an interesting (and known) class of examples of W^* -dynamical systems obtained from group von Neumann algebras of discrete groups and their automorphisms, however we express our results in the language of locally compact quantum groups. Next, we study ergodic joinings in Section 4. In Sections 2 and 4, we also consider simple applications for the case where the group action is that of a countable discrete amenable group, namely a weak ergodic theorem and a Halmos–von Neumann type theorem, respectively. In the latter, we make the rather strong assumption of asymptotic Abelianness “in density.” The focus in this paper is on building some general aspects of the theory of joinings of W^* -dynamical systems, and these applications are more for illustration of how joinings can potentially be used rather than being important results in themselves. In Section 5, we present a joining characterization of strong mixing (for the special

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case where the acting group is \mathbb{Z}), and use it to obtain a version of Ornstein's criterion for strong mixing in the case of W^* -dynamical systems. Sections 2 and 3 differ from Sections 4 and 5 in the sense that in the former subsystems of W^* -dynamical systems play a central role while in the latter they do not. At the same time Sections 4 and 5 just take initial steps in the respective topics, while the topics in Sections 2 and 3 are more fully developed. Along the way we give indications of further work that might be done.

We use the same basic definitions as in [6], and will again refer to a W^* -dynamical system simply as a “dynamical system,” or even just a “system.” For convenience, we summarize the essential definitions used in [6]: A *dynamical system* $\mathbf{A} = (A, \mu, \alpha)$ consists of a faithful normal state μ on a σ -finite von Neumann algebra A , and a representation $\alpha : G \rightarrow \text{Aut}(A) : g \mapsto \alpha_g$ of an arbitrary group G as $*$ -automorphisms of A , such that $\mu \circ \alpha_g = \mu$ for all g . We will call \mathbf{A} an *identity system* if $\alpha_g = \iota_A$ for all g where $\iota_A : A \rightarrow A$ is the identity mapping, while we call it *trivial* if $A = \mathbb{C}1_A$ where 1_A (often denoted simply as 1) is the unit of A . In the rest of the paper, the symbols \mathbf{A} , \mathbf{B} and \mathbf{F} will denote dynamical systems (A, μ, α) , (B, ν, β) and (F, κ, φ) respectively, all making use of actions of the same group G . A *joining* of \mathbf{A} and \mathbf{B} is a state ω (i.e., a positive linear functional with $\omega(1) = 1$) on the algebraic tensor product $A \odot B$ such that $\omega(a \otimes 1_B) = \mu(a)$, $\omega(1_A \otimes b) = \nu(b)$ and $\omega \circ (\alpha_g \odot \beta_g) = \omega$ for all $a \in A$, $b \in B$ and $g \in G$. The set of all joinings of \mathbf{A} and \mathbf{B} is denoted by $J(\mathbf{A}, \mathbf{B})$. We call \mathbf{A} *disjoint* from \mathbf{B} when $J(\mathbf{A}, \mathbf{B}) = \{\mu \odot \nu\}$. A dynamical system \mathbf{A} is called *ergodic* if its *fixed point algebra* $A_\alpha := \{a \in A : \alpha_g(a) = a \text{ for all } g \in G\}$ is trivial, that is, $A_\alpha = \mathbb{C}1_A$. We call \mathbf{F} a *subsystem* of \mathbf{A} if there exists an injective unital $*$ -homomorphism h of F onto a von Neumann subalgebra of A such that $\mu \circ h = \kappa$ and $\alpha_g \circ h = h \circ \varphi_g$ for all $g \in G$. (In [6], the terminology “factor” instead of “subsystem” was used.) If furthermore $h : F \rightarrow A$ is surjective, then we say that h is an *isomorphism* of dynamical systems, and the systems \mathbf{A} and \mathbf{F} are *isomorphic*.

Unlike [6], in this paper, we will have occasion to use completions of the algebraic tensor product. Even though A and B are von Neumann algebras, we will encounter the maximal C^* -algebraic tensor product $A \otimes_m B$ in Sections 2, 4 and 5. In Section 3, we do use the von Neumann algebraic tensor product, however in this case it is to handle locally compact quantum groups and not directly related to joinings.

The work in this paper is of course strongly influenced by previous work on joinings in measure theoretic ergodic theory which originates in Furstenberg's work [9]. In this regard, we mention that [5] and [10], as well as unpublished lecture notes by A. del Junco, served as very useful sources.

For example, the joining obtained in [6, Construction 3.4], and which we will again use here, can be viewed (ignoring dynamics) as a generalization of a diagonal measure $\Delta(Y \times Z) := \rho(Y \cap Z)$ defined in terms of some measure ρ on a measurable space X and where $Y, Z \subset X$. A noncommutative version

of a diagonal measure using essentially the same idea as our construction of a joining appeared in [8, Section 4].

Also keep in mind that the use of joinings in noncommutative dynamical systems is not without precedent, as a special case of this idea (under the name “stationary couplings”) is used in work on entropy [22].

2. Ergodicity and weak mixing

We start by improving on the characterization of ergodicity given in [6]. In particular, we prove a stronger version of [6, Theorem 3.7] using a simpler proof. We do this by using an approach given in unpublished lecture notes by A. del Junco for the measure theoretic case.

THEOREM 2.1. *A dynamical system \mathbf{A} is ergodic if and only if it is disjoint from all identity systems.*

Proof. Suppose \mathbf{A} is ergodic, and let \mathbf{B} be any identity system. Consider any $\omega \in J(\mathbf{A}, \mathbf{B})$. From this joining, we obtain (see [6, Construction 2.3 and Proposition 2.4]) a conditional expectation operator $P_\omega : H_\mu \rightarrow H_\nu$ (i.e. $\langle P_\omega x, y \rangle = \langle x, y \rangle$) such that $U_g P_\omega^* = P_\omega^* V_g$, where $\gamma_\mu : A \rightarrow H_\mu$ and $\gamma_\nu : B \rightarrow H_\nu$ are the GNS constructions for (A, μ) and (B, ν) respectively, U and V the corresponding unitary representations of α and β on the Hilbert spaces H_μ and H_ν respectively, and we denote by Ω_ω their common unit cyclic vector (in the GNS Hilbert space obtained from ω , which contains H_μ and H_ν). Therefore for any $b \in B$, we have $U_g P_\omega^* \gamma_\nu(b) = P_\omega^* \gamma_\nu(b)$, since \mathbf{B} is an identity system. But \mathbf{A} is ergodic, hence by [3, Theorem 4.3.20] the fixed point space of U is $\mathbb{C}\Omega_\omega$, so $P_\omega^* \gamma_\nu(b) = \langle \Omega_\omega, P_\omega^* \gamma_\nu(b) \rangle \Omega_\omega = \nu(b)\Omega_\omega$. For any $a \in A$, it follows that

$$\omega(a \otimes b) = \langle \gamma_\mu(a^*), \gamma_\nu(b) \rangle = \langle \gamma_\mu(a^*), P_\omega^* \gamma_\nu(b) \rangle = \mu(a)\nu(b)$$

hence $\omega = \mu \otimes \nu$, which means that \mathbf{A} is disjoint from \mathbf{B} . The converse was proven in [6, Theorem 3.3] using a subsystem of \mathbf{A} . □

Before we move on to weak mixing, we give a simple application of Theorem 2.1, namely we prove a weak ergodic theorem. The result itself is not that interesting, but we do this to illustrate how joinings can in principle be used to prove results that don’t refer to joinings in their formulation (see in particular Corollary 2.4). Again, we follow the basic plan for the measure theoretic case given in the unpublished lecture notes by del Junco.

DEFINITION 2.2. For a dynamical system \mathbf{A} , consider the cyclic representation (H, π, Ω) of (A, μ) obtained by the GNS construction. Set $\tilde{A} := \pi(A)'$, define the state $\tilde{\mu}$ on \tilde{A} by $\tilde{\mu}(b) := \langle \Omega, b\Omega \rangle$, and let the unital $*$ -homomorphism $\delta : A \odot \tilde{A} \rightarrow B(H)$ be defined by $\delta(a \otimes b) := \pi(a)b$. The state μ_Δ on the unital $*$ -algebra $A \odot \tilde{A}$ defined by $\mu_\Delta(t) := \langle \Omega, \delta(t)\Omega \rangle$ will be called the *diagonal state* for (A, μ) .

The state μ_{Δ} is in fact a joining of \mathbf{A} and its “mirror image” $\tilde{\mathbf{A}}$ constructed on $(\tilde{A}, \tilde{\mu})$ defined above by carrying α to \tilde{A} using the natural $*$ -anti-isomorphism $a \mapsto Ja^*J$ where J is the modular conjugation associated with $(\pi(A), \Omega)$ (see [6, Construction 3.4]). But it is not this aspect of μ_{Δ} that will be used in the next proposition (see Section 5 for further elaboration on the joining aspect).

PROPOSITION 2.3. *Let \mathbf{A} be ergodic, with G countable, discrete and amenable, and consider any right Følner sequence (Λ_n) in G . We can extend the diagonal state for (A, μ) to a state μ_{Δ} on the maximal C^* -algebraic tensor product $A \otimes_m \tilde{A}$, and then*

$$w * \text{-}\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \mu_{\Delta} \circ (\alpha_g \otimes_m \iota_{\tilde{A}}) = \mu \otimes_m \tilde{\mu},$$

where $w * \text{-}\lim$ denotes the weak* limit and $\iota_{\tilde{A}}$ is the identity mapping on \tilde{A} .

Proof. We will make use of the identity system $\mathbf{B} := (\tilde{A}, \tilde{\mu}, \iota_{\tilde{A}})$. The maximal tensor product has the property that δ in Definition 2.2 can be extended to a $*$ -homomorphism $A \otimes_m \tilde{A} \rightarrow B(H)$, and hence we can easily extend the diagonal state to a state μ_{Δ} on $A \otimes_m \tilde{A}$. (The general case of such extensions is discussed in Section 4.) Then

$$\omega_n := \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \mu_{\Delta} \circ (\alpha_g \otimes_m \iota_{\tilde{A}})$$

is also a state on $A \otimes_m \tilde{A}$. The set S of states of the unital C^* -algebra $A \otimes_m \tilde{A}$ is weakly* compact (see for example [3, Theorem 2.3.15]), hence the sequence (ω_n) has a cluster point ρ in S in the weak* topology.

We now show that $\rho|_{A \otimes \tilde{A}}$ is a joining of \mathbf{A} and \mathbf{B} . For each $\varepsilon > 0$, $a \in A$, $b \in \tilde{A}$ and $N \in \mathbb{N}$, there is an $n > N$ such that $|\rho(a \otimes b) - \omega_n(a \otimes b)| < \varepsilon$. Furthermore, $\omega_n(a \otimes 1_{\tilde{A}}) = \mu(a)$ and $\omega_n(1_A \otimes b) = \tilde{\mu}(b)$. Therefore, $|\rho(a \otimes 1_{\tilde{A}}) - \mu(a)| < \varepsilon$ and $|\rho(1_A \otimes b) - \tilde{\mu}(b)| < \varepsilon$ for all $\varepsilon > 0$, and so $\rho(a \otimes 1_{\tilde{A}}) = \mu(a)$ and $\rho(1_A \otimes b) = \tilde{\mu}(b)$. Next, note that for all $h \in G$

$$\begin{aligned} & |\omega_n \circ (\alpha_h \otimes_m \iota_{\tilde{A}})(a \otimes b) - \omega_n(a \otimes b)| \\ &= \frac{1}{|\Lambda_n|} \left| \sum_{g \in (\Lambda_n h) \setminus \Lambda_n} \mu_{\Delta} \circ (\alpha_g \otimes_m \iota_{\tilde{A}})(a \otimes b) \right. \\ &\quad \left. - \sum_{g \in \Lambda_n \setminus (\Lambda_n h)} \mu_{\Delta} \circ (\alpha_g \otimes_m \iota_{\tilde{A}})(a \otimes b) \right| \\ &\leq \frac{|\Lambda_n \Delta (\Lambda_n h)|}{|\Lambda_n|} \|a \otimes b\| \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Since ρ is a cluster point of (ω_n) , we conclude that $\rho \circ (\alpha_g \otimes_m \iota_{\tilde{A}}) = \rho$, and therefore $\rho|_{A \otimes \tilde{A}} \in J(\mathbf{A}, \mathbf{B})$.

By Theorem 2.1 and continuity, it follows that $\rho = \mu \otimes_m \tilde{\mu}$. In particular, this means that $\mu \otimes_m \tilde{\mu}$ is the unique weak* cluster point of (ω_n) , which implies that (ω_n) converges to $\mu \otimes_m \tilde{\mu}$, as required. \square

To clarify the meaning of Proposition 2.3, we include the following weak mean ergodic theorem in terms of a Hilbert space (the conventional proof of the mean ergodic theorem is both more elementary, and delivers a stronger result than the current approach, but again, our motivation here is to illustrate that results regarding joinings can have nontrivial consequences). The proof of this result is essentially a partial converse to the proof of [6, Theorem 3.7].

COROLLARY 2.4. *Consider the situation in Definition 2.2 and Proposition 2.3, and let U be the unitary representation of α on H , in other words $\pi(\alpha_g(a)) = U_g \pi(a) U_g^*$ and $U_g \Omega = \Omega$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \langle U_g x, y \rangle = \langle (\Omega \otimes \Omega)x, y \rangle$$

for all $x, y \in H$, where $\Omega \otimes \Omega$ is the projection of H onto $\mathbb{C}\Omega$, i.e. $(\Omega \otimes \Omega)x = \Omega \langle \Omega, x \rangle$.

Proof. For $x := \pi(a)\Omega$ and $y := b\Omega$ where $a \in A$ and $b \in \tilde{A}$, it follows from Proposition 2.3 that

$$\begin{aligned} \langle (\Omega \otimes \Omega)x, y \rangle &= \mu \otimes_m \tilde{\mu}(a^* \otimes b) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \mu_{\Delta}(\alpha_g(a^*) \otimes b) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \langle U_g x, y \rangle \end{aligned}$$

but $\pi(A)\Omega$ and $\tilde{A}\Omega$ are both dense in H , since μ is faithful and normal. \square

We now proceed to weak mixing, our goal being an analogue of Theorem 2.1.

DEFINITION 2.5. Consider a dynamical system \mathbf{A} and let (H, π, Ω) be the cyclic representation of (A, μ) obtained from the GNS construction, and let U be the corresponding unitary representation of α on H , that is, $U_g \pi(a)\Omega = \pi(\alpha_g(a))\Omega$. An *eigenvector* of U is an $x \in H \setminus \{0\}$ such that there is a function, called its *eigenvalue*, $\chi_x : G \rightarrow \mathbb{C}$ such that $U_g x = \chi_x(g)x$ for all $g \in G$. The eigenvalue $g \mapsto 1$ will be denoted as 1. Denote by H_0 the Hilbert subspace spanned by the eigenvectors of U . The set of all eigenvalues is denoted by $\sigma_{\mathbf{A}}$ and is called the *point spectrum* of \mathbf{A} . We call \mathbf{A} *weakly mixing* if $\dim H_0 = 1$. We say \mathbf{A} *has discrete spectrum* if $H_0 = H$. We call \mathbf{A} *compact* if the orbit

U_Gx is totally bounded in H for every $x \in H$, or, equivalently, if $\alpha_G(a)$ is totally bounded in $(A, \|\cdot\|_\mu)$ for every $a \in A$, where $\|a\|_\mu := \sqrt{\mu(a^*a)}$.

We have the following equivalence when G is Abelian.

PROPOSITION 2.6. *Let G be Abelian. Then \mathbf{A} has discrete spectrum if and only if it is compact.*

Proof. By [12, Section 2.4] (or see [2, Lemma 6.6] for the special case that we are using here), H_0 is the set of all $x \in H$ whose orbits U_Gx are totally bounded in H . □

It is not clear if Proposition 2.6 can be extended to non-Abelian G . Therefore, we are going to give the sufficient and necessary conditions for weak mixing separately in terms of compactness and discrete spectra, respectively.

THEOREM 2.7. *Let \mathbf{A} be ergodic. If \mathbf{A} is disjoint from all ergodic compact systems, then it is weakly mixing.*

Proof. The plan is essentially the same as for the proof of the corresponding direction in Theorem 2.1 (see [6, Theorem 3.3]). Suppose \mathbf{A} is not weakly mixing, then by [2, Propositions 6.5 and 6.7(1)] it has a nontrivial compact subsystem, say \mathbf{F} . Since \mathbf{A} is ergodic, so is \mathbf{F} . So by [6, Construction 3.4 and Lemma 3.5] we are finished. □

THEOREM 2.8. *If \mathbf{A} and \mathbf{B} are ergodic, and \mathbf{B} has discrete spectrum with $\sigma_{\mathbf{A}} \cap \sigma_{\mathbf{B}} = \{1\}$, then \mathbf{A} is disjoint from \mathbf{B} . In particular, if \mathbf{A} is weakly mixing, then it is disjoint from all ergodic systems with discrete spectrum.*

Proof. As in the proof of Theorem 2.1, we employ a conditional expectation operator. So consider any $\omega \in J(\mathbf{A}, \mathbf{B})$, and then use the same notation as in Theorem 2.1’s proof. Let $y \in H_\nu$ be any eigenvector of V with eigenvalue χ , then $y = \gamma_\nu(e)$ for some $e \in B$ by [23, Theorem 2.5], while $U_g P_\omega^* y = \chi(g) P_\omega^* y$. So either $P_\omega^* y = 0$ or $\chi \in \sigma_{\mathbf{A}} \cap \sigma_{\mathbf{B}}$. In the latter case $P_\omega^* y \in \mathbb{C}\Omega_\omega$, since \mathbf{A} is ergodic, hence in either case we have $P_\omega^* y \in \mathbb{C}\Omega_\omega$. Therefore,

$$\begin{aligned} \langle \gamma_\mu(a^*), \gamma_\nu(e) \rangle &= \langle \gamma_\mu(a^*), P_\omega^* \gamma_\nu(e) \rangle = \langle \gamma_\mu(a^*), \Omega_\omega \rangle \langle \Omega_\omega, P_\omega^* \gamma_\nu(e) \rangle \\ &= \mu(a) \langle \Omega_\omega, \gamma_\nu(e) \rangle \end{aligned}$$

for all $a \in A$. For an arbitrary $b \in B$ one has a sequence (b_n) of linear combinations of such *eigenoperators* e , such that $\gamma_\nu(b_n) \rightarrow \gamma_\nu(b)$, since \mathbf{B} has discrete

spectrum. Hence,

$$\begin{aligned} \omega(a \otimes b) &= \langle \gamma_\mu(a^*), \gamma_\nu(b) \rangle = \lim_{n \rightarrow \infty} \langle \gamma_\mu(a^*), \gamma_\nu(b_n) \rangle \\ &= \lim_{n \rightarrow \infty} \mu(a) \langle \Omega, \gamma_\nu(b_n) \rangle \\ &= \mu(a) \nu(b) \end{aligned}$$

which means that $J(\mathbf{A}, \mathbf{B}) = \{\mu \odot \nu\}$. □

3. The quantum group duals of discrete groups

Halmos [11] studied dynamical systems consisting of an automorphism of a compact Abelian group, with the automorphism providing an action of \mathbb{Z} on the group by iteration. In particular, he characterized ergodicity (which turns out to be equivalent to strong mixing in this case) in terms of the orbits of the induced action in the dual group (or character group). Here, we study a generalization of this type of system, where the compact group is replaced by a compact quantum group obtained as the dual of a discrete group Γ which need not be Abelian. For simplicity, we also only consider actions of $G = \mathbb{Z}$ in this section.

We use the von Neumann algebra setting for locally compact quantum groups (which include both our discrete group and its compact quantum group dual as special cases), as developed by Kustermans and Vaes [14] (also see [27] and [13]). Below we briefly review the definitions of this theory to fix the conventions and notations that we will use. Other useful sources regarding this material is [26], and [24, Section 18] which focusses on Hopf-von Neumann algebras and Kac algebras.

We should mention that since we are ultimately only interested in discrete groups and their dual quantum groups, we could in principle work in the setting of Kac algebras or even in terms of group von Neumann algebras. However, the framework set up in [14] is simple and powerful, and very convenient to work in, while the language of quantum groups also makes the generalization from Abelian to general discrete groups clearer, and opens the window to possible further generalization when replacing the discrete group by a discrete quantum group (which we will not do in this paper).

A *locally compact quantum group* is defined to be a von Neumann algebra M with a unital normal $*$ -homomorphism $\Delta : M \rightarrow M \otimes M$ (where $M \otimes N$ denotes the von Neumann algebraic tensor product of two von Neumann algebras), such that $(\Delta \otimes \iota_M) \circ \Delta = (\iota_M \otimes \Delta) \circ \Delta$ (where ι_M denotes the identity map on M), and on which there exist normal semi-finite faithful (n.s.f.) weights φ and ψ which are left and right invariant respectively, namely $\varphi((\theta \otimes \iota_M) \circ \Delta(a)) = \varphi(a)\theta(1)$ for all $a \in \mathcal{M}_\varphi^+$ and $\psi((\iota_M \otimes \theta) \circ \Delta(a)) = \psi(a)\theta(1)$ for all $a \in \mathcal{M}_\psi^+$, for all $\theta \in M_*^+$, where M_*^+ is the positive normal linear functionals on M , and $\mathcal{M}_\varphi^+ = \{a \in M^+ : \varphi(a) < \infty\}$. This quantum

group is denoted as (M, Δ) . We will call (M, Δ) a *compact quantum group* if we can take $\varphi = \psi$ as a state, which we will call the *Haar state*. Note that the Haar state is faithful and normal.

The dual $(\hat{M}, \hat{\Delta})$ of (M, Δ) is again a locally compact quantum group and is defined as follows (also see [27, Definition 3.1]), where we assume M is in standard form with respect to the Hilbert space H : Denote by $W \in M \otimes B(H)$ the so-called multiplicative unitary of (M, Δ) with respect to the GNS construction on H obtained from some φ as above; see [14, Theorem 1.2]. Let $\hat{\lambda} : M_* \rightarrow B(H) : \theta \mapsto (\theta \otimes \iota)(W)$ with ι the identity on $B(H)$. Then \hat{M} is defined to be the σ -weak closure of $\{\hat{\lambda}(\theta) : \theta \in M_*\}$ and $\hat{\Delta}$ is defined by $\hat{\Delta}(a) := \Sigma W(a \otimes 1)W^* \Sigma$ where $\Sigma : H \otimes H \rightarrow H \otimes H$ is the “flip map” and $1 \in M$ is the identity operator on H . The symbol ι will always denote the identity map on some von Neumann algebra which will be clear from context.

Next, we give the basic definitions and results which we use to build our dynamical systems.

DEFINITION 3.1. An *automorphism* of (M, Δ) is a $*$ -automorphism $\alpha : M \rightarrow M$ such that $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta$.

PROPOSITION 3.2. Let α be an automorphism of a compact quantum group (A, Δ) with Haar state μ . Then $\mu \circ \alpha = \mu$.

Proof. From the strong form of left invariance [14, Proposition 3.1], we have $\mu \circ \alpha(a)1 = \alpha^{-1}[\mu(\alpha(a))1] = \alpha^{-1}[(\iota \otimes \mu) \circ \Delta \circ \alpha(a)] = \alpha^{-1} \circ (\iota \otimes \mu) \circ (\alpha \otimes \alpha) \circ \Delta(a) = [\iota \otimes (\mu \circ \alpha)] \circ \Delta(a)$ but this says that $\mu \circ \alpha$ is also left invariant, hence by uniqueness of left invariant states (which also explains the terminology *the* Haar state) we have $\mu \circ \alpha = \mu$. □

Hence, $\mathbf{A} = (A, \mu, \alpha)$ is a dynamical system with $G = \mathbb{Z}$ by simply setting $\alpha_n := \alpha^n$ for $n \in \mathbb{Z}$. Let us now look at the specific case that will interest us throughout the rest of this section, and also fix the notation that we will use:

Let Γ be any group and assign to it the discrete topology and counting measure. We set

$$(3.1) \quad (A, \Delta) := (\widehat{L^\infty(\Gamma)}, \hat{\Delta}_\Gamma),$$

where $[\Delta_\Gamma(f)](g, h) := f(gh)$ for all $f \in L^\infty(\Gamma)$ and $g, h \in \Gamma$, and where of course we view the elements of $L^\infty(\Gamma)$ as linear operators on $H := L^2(\Gamma)$ by multiplication. In this situation, we in fact have that A is generated by $\{\lambda(g) : g \in \Gamma\}$ where we write $\lambda(g) \equiv \lambda(\theta_g)$ with $\lambda : L^\infty(\Gamma)_* \rightarrow B(H)$ defined in the same way as $\hat{\lambda}$ above, and $\theta_g(f) := f(g)$, which translates into $\lambda : \Gamma \rightarrow B(H)$ being a unitary representation of Γ with $[\lambda(g)f](h) = f(g^{-1}h)$ for all $f \in H$ and $g, h \in \Gamma$, and having the property $\Delta(\lambda(g)) = \lambda(g) \otimes \lambda(g)$. In this case we also have that the Haar state μ is tracial, that is, $\mu(ab) = \mu(ba)$ for all $a, b \in M$, however this doesn't play a direct role in our further work.

Furthermore, let

$$T : \Gamma \rightarrow \Gamma$$

be any automorphism of the group Γ . From T , we now obtain an automorphism of (A, Δ) as follows: Define a unitary operator $U : H \rightarrow H$ by $Uf := f \circ T$. Since $[U^*\lambda(g)Uf](h) = [\lambda(g)(f \circ T)](T^{-1}(h)) = (f \circ T)(g^{-1}T^{-1}(h)) = f(T(g)^{-1}h) = [\lambda(T(g))f](h)$, we have

$$U^*\lambda(g)U = \lambda(T(g))$$

which in particular means that the set of generators of A is invariant under $U^*(\cdot)U$ and hence A itself as well. So we have a well-defined mapping

$$\alpha : A \rightarrow A : a \mapsto U^*aU$$

which we call the *dual* of T . It remains to show that α is an automorphism of (A, Δ) . Note that $\Delta \circ \alpha(\lambda(g)) = \Delta(\lambda(T(g))) = \lambda(T(g)) \otimes \lambda(T(g)) = (\alpha \otimes \alpha)(\lambda(g) \otimes \lambda(g)) = (\alpha \otimes \alpha) \circ \Delta(\lambda(g))$, and by linearity and σ -weak continuity this extends to all of A , that is to say $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta$ as required.

We will refer to the dynamical system $\mathbf{A} = (A, \mu, \alpha)$ as the *dual system* of (Γ, T) , and this notation will be fixed throughout the rest of this section. Our eventual goal in this section is a refinement of Theorems 2.1 (one direction) and 2.7 for dual systems, however we first develop some general theory regarding dual systems.

As we show next, every automorphism of (A, Δ) is the dual of some automorphism of Γ , hence assuming the automorphism T of Γ to be given places no restriction on the dynamics obtained as automorphisms of (A, Δ) .

We will use the following additional notation: By δ_g with $g \in \Gamma$, we denote the element of H defined by $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $g \neq h \in \Gamma$. In particular we set $\Omega := \delta_1$ where 1 here denotes the identity of Γ . Then Ω is cyclic and separating for A , and $\mu(a) = \langle \Omega, a\Omega \rangle$ so (H, ι_A, Ω) is the cyclic representation of (A, μ) obtained in the GNS construction. Also note that $\lambda(g)\Omega = \delta_g$. We will use the notation $\chi_g := \delta_g$ when we want to view this function as an element of $L^\infty(\Gamma)$ rather than $H = L^2(\Gamma)$; this makes some the arguments slightly easier to read. Using the notation $\gamma : A \rightarrow H : a \mapsto a\Omega$, the multiplicative unitary W of (A, Δ) has the following defining property (see [14, Theorem 1.2]): $W^*[\gamma(a) \otimes \gamma(b)] = (\gamma \otimes \gamma)[\Delta(b)(a \otimes 1)]$.

THEOREM 3.3. *Every automorphism α of (A, Δ) in (3.1) is the dual of some automorphism T of the discrete group Γ .*

Proof. Using the notation above, we define a unitary operator $U : H \rightarrow H$ by $U^*a\Omega := \alpha(a)\Omega$. We first show that

$$(3.2) \quad (U \otimes U)W = W(U \otimes U)$$

to enable us to define an automorphism of $(\hat{A}, \hat{\Delta})$. Using the defining property of W , we have

$$\begin{aligned} (U^* \otimes U^*)W^*(\delta_g \otimes \delta_h) &= (U^* \otimes U^*)(\gamma \otimes \gamma)[\Delta(\lambda(h))(\lambda(g) \otimes 1)] \\ &= (U^* \otimes U^*)(\gamma \otimes \gamma)[\lambda(hg) \otimes \lambda(h)] \\ &= [\gamma \circ \alpha \circ \lambda(hg)] \otimes [\gamma \circ \alpha \circ \lambda(h)] \\ &= (\gamma \otimes \gamma)[[(\alpha \otimes \alpha) \circ \Delta(\lambda(h))][(\alpha \circ \lambda(g)) \otimes 1]] \\ &= (\gamma \otimes \gamma)[[\Delta(\alpha(\lambda(h)))]([\alpha(\lambda(g)) \otimes 1])] \\ &= W^*(U^* \otimes U^*)(\delta_g \otimes \delta_h) \end{aligned}$$

which proves (3.2). For any $\theta \in A_*$ it follows from the definition of $\hat{\lambda}$ and from (3.2) that

$$\begin{aligned} U\hat{\lambda}(\theta)U^* &= (\theta \otimes \iota)[(1 \otimes U)W(1 \otimes U^*)] \\ &= (\theta \otimes \iota)[(U^* \otimes 1)W(U \otimes 1)] \\ &= [(\theta \circ \alpha) \otimes \iota](W) \\ &= \hat{\lambda}(\theta \circ \alpha) \end{aligned}$$

and therefore $U\hat{A}U^* = \hat{A}$ so

$$\hat{\alpha} : \hat{A} \rightarrow \hat{A} : a \mapsto UaU^*$$

is a well-defined $*$ -automorphism of \hat{A} . Next observe from the definition of $\hat{\Delta}$ and again using (3.2) that

$$\begin{aligned} (\hat{\alpha} \otimes \hat{\alpha}) \circ \hat{\Delta}(a) &= (U \otimes U)\Sigma W(a \otimes 1)W^*\Sigma(U^* \otimes U^*) \\ &= \Sigma W(\hat{\alpha}(a) \otimes 1)W^*\Sigma \\ &= \hat{\Delta} \circ \hat{\alpha}(a) \end{aligned}$$

in other words $\hat{\alpha}$ is an automorphism of $(\hat{A}, \hat{\Delta})$. However, by Pontryagin duality (see, for example, [14, p. 75]) $(\hat{A}, \hat{\Delta}) = (L^\infty(\Gamma), \Delta_\Gamma)$. With somewhat tedious but fairly elementary arguments, one can then show that $\hat{\alpha}(f) = f \circ T$ for all $f \in L^\infty(\Gamma)$ for some automorphism T of the group Γ . Lastly we show that α is the dual of T in the sense defined earlier. Define $U_T : H \rightarrow H : f \mapsto f \circ T$, then $U_T^*fU_T\delta_g = f(T^{-1}(g))\delta_g = U^*fU\delta_g$ for all $f \in L^\infty(\Gamma)$, so $U_TU^*f = fU_TU^*$, in particular for $f = \chi_h$. Hence, for all $g \neq h$ in Γ we have $\chi_hU_TU^*\delta_g = 0$ and so $U^*\delta_g = k(g)U_T^*\delta_g = k(g)\delta_{T(g)}$ for some complex number $k(g)$ of modulus 1. But then $\alpha(\lambda(g))\Omega = U^*\delta_g = k(g)\lambda(T(g))\Omega$ and therefore $k(g)\lambda(T(g)) \otimes \lambda(T(g)) = \Delta(k(g)\lambda(T(g))) = (\alpha \otimes \alpha) \circ \Delta(\lambda(g)) = [k(g)]^2\lambda(T(g)) \otimes \lambda(T(g))$ since Ω is separating for A , so $k(g) = 1$ which means that $U_T = U$ as required. \square

Having set up the framework, we can now start doing ergodic theory. We first discuss the theorem of Halmos in the current setting. Recall that \mathbf{A} is

strongly mixing if

$$\lim_{n \rightarrow \infty} \mu(\alpha^n(a)b) = \mu(a)\mu(b)$$

for all $a, b \in A$. We will say that $g \in \Gamma$ has a finite orbit under T if the orbit $T^{\mathbb{N}}(g) := \{T^n(g) : n \in \mathbb{N}\}$ is a finite set, where $\mathbb{N} = \{1, 2, 3, \dots\}$.

The following result is fairly standard, but often expressed in terms of group C^* -algebras, or in the language of group von Neumann algebras (see [1, Proposition 2.12] for an example of this type of result). For completeness, and since we use this result later, we include a proof based on that of the Abelian case in for example, [19, Section 2.5].

THEOREM 3.4. *If the dual system \mathbf{A} is ergodic, then the only element of Γ with a finite orbit under T is its identity 1. Conversely, if 1 is the only element of Γ with finite orbit under T , then \mathbf{A} is strongly mixing. It follows that \mathbf{A} is strongly mixing if and only if it is ergodic.*

Proof. Suppose $g \in \Gamma \setminus \{1\}$ has a finite orbit under T . Then there is a smallest $n \in \mathbb{N}$ such that $T^n g = g$. Hence, this is the smallest n in \mathbb{N} for which $(U^*)^n \delta_g = \delta_g$. Set

$$x := \delta_g + U^* \delta_g + (U^*)^2 \delta_g + \dots + (U^*)^{n-1} \delta_g$$

then $U^* x = x$. It is also easily seen that $U^* \Omega = \Omega$, but as we now show, $x \notin \mathbb{C}\Omega$. Since $x = \delta_g + \delta_{T(g)} + \dots + \delta_{T^{n-1}(g)}$, while $g \neq 1$ and hence $T^j(g) \neq 1$, we have $x(1) = 0 \neq 1 = \Omega(1)$. At the same time $x(g) = 1$ so $x \neq 0$, so $x \notin \mathbb{C}\Omega$. This means the fixed point space of U^* has dimension larger than 1, and therefore \mathbf{A} is not ergodic.

Conversely, suppose 1 is the only element of Γ with finite orbit under T . Consider $g, h \in G$. If $g = h = 1$, then it is easily seen that $\lim_{n \rightarrow \infty} \langle (U^*)^n \delta_g, \delta_h \rangle = 1 = \langle \delta_g, \Omega \rangle \langle \Omega, \delta_h \rangle$. Otherwise, if at least one of g or h is not 1, then from our supposition, $T^n(h) \neq h$ for n large enough, and therefore it is again easily seen that $\lim_{n \rightarrow \infty} \langle (U^*)^n \delta_g, \delta_h \rangle = 0 = \langle \delta_g, \Omega \rangle \langle \Omega, \delta_h \rangle$. From this, we deduce that

$$\lim_{n \rightarrow \infty} \langle (U^*)^n x, y \rangle = \langle x, \Omega \rangle \langle \Omega, y \rangle$$

for all $x, y \in H$, but this means that \mathbf{A} is strongly mixing. □

Weak mixing is an intermediate condition between ergodicity and strong mixing and is therefore also equivalent to these two conditions. Another simple corollary of Theorem 3.4 (which can also be seen directly) is that if $1 < |\Gamma| < \infty$, then \mathbf{A} cannot be ergodic.

We now move on to subsystems and compactness. In our current situation, if \mathbf{A} is not weakly mixing then it is not ergodic, and so one can obtain a non-trivial compact subsystem by considering the fixed point algebra of α . But a result purely in terms of dual systems would be preferable, and from the point of view of weak mixing we want a result in terms of a compact subsystem that need not be an identity system. Hence, we consider the following:

Let $E := \{g \in \Gamma : T^{\mathbb{N}}(g) \text{ is finite}\}$ and let F denote the von Neumann algebra generated by $\{\lambda(g) : g \in E\}$. Then

THEOREM 3.5. *The system $\mathbf{F} = (F, \kappa, \varphi) := (F, \mu|_F, \alpha|_F)$ is isomorphic to the dual system of $(E, T|_E)$ and it is a compact subsystem of \mathbf{A} . Furthermore, if \mathbf{F} is trivial then \mathbf{A} is ergodic.*

Proof. One easily sees that $T|_E$ is an automorphism of the subgroup E of Γ , hence $\alpha(F) = F$. So \mathbf{F} is indeed a subsystem of \mathbf{A} . It is also readily seen that if K is the closure of $F\Omega$ in H , then $\pi : F \rightarrow B(K) : a \mapsto a|_K$ is well-defined and (K, π, Ω) is the cyclic representation of (F, κ) obtained in the GNS construction. Also note that K is the closure of $D := \text{span}\{\delta_g : g \in E\}$.

Note that π is injective since Ω is separating for F , and then one can verify that $\pi(F)$ is generated by $\lambda_E : E \rightarrow B(K)$ where $[\lambda_E(g)f](h) := f(\widehat{g^{-1}h})$ for $f \in K = L^2(E)$ in terms of the counting measure on E . So $\pi(F) = \widehat{L^\infty(E)}$. It is readily verified that π is an isomorphism (as defined in Section 1) of the dynamical system \mathbf{F} and the dual system of $(E, T|_E)$.

Consider any $v = \sum_{j=1}^r c_j \delta_{g_j}$ in D and let $n_g = |T^{\mathbb{N}}(g)|$ denote the length of g 's orbit, that is, it is the smallest element of \mathbb{N} such that $T^{n_g}(g) = g$. Then $(U^*)^{n_{g_1} \cdots n_{g_r}} v = v$, in other words v also has a finite orbit. For arbitrary $x \in K$ and $\varepsilon > 0$, there will be a $v \in D$ with $\|x - v\| < \varepsilon$ and therefore $\|(U^*)^n x - (U^*)^n v\| < \varepsilon$ for all $n \in \mathbb{N}$, but since $(U^*)^{\mathbb{N}}v$ is finite, $(U^*)^{\mathbb{N}}x$ is totally bounded. We conclude that \mathbf{F} is compact.

When \mathbf{A} is not ergodic, $E \neq \{1\}$ by Theorem 3.4, and therefore $K \neq \mathbb{C}\Omega$ so $F \neq \mathbb{C}1$ by the definition of K . In other words, \mathbf{F} is nontrivial. □

We will refer to \mathbf{F} defined above as the *finite orbit subsystem* of \mathbf{A} .

Before we proceed with subsystems and joinings, we give the following analogue of Theorem 3.4 as an application of (part of) Theorem 3.5.

THEOREM 3.6. *The dual system \mathbf{A} is compact if and only if all the orbits in (Γ, T) are finite.*

Proof. Suppose \mathbf{A} is compact. Then in particular for any $g \in \Gamma$ the orbit $\delta_{T^{\mathbb{N}}(g)} := \{\delta_{T^n(g)} : n \in \mathbb{N}\} = (U^*)^{\mathbb{N}}\delta_g$ is totally bounded in H . Hence, there is a finite set $N \subset H$ such that for every $\delta_{T^n(g)}$ there is an $x \in N$ with $\|\delta_{T^n(g)} - x\| < 1/\sqrt{2}$. However, for any pair $h \neq j$ in Γ we have $\|\delta_h - \delta_j\| = \sqrt{2}$, so for any $x \in N$ the ball $\{y \in H : \|y - x\| < 1/\sqrt{2}\}$ contains at most one point in the orbit $\delta_{T^{\mathbb{N}}(g)}$. Since N is finite it follows that $T^{\mathbb{N}}(g)$ is finite.

Conversely, assume that all the orbits $T^{\mathbb{N}}(g)$ in Γ are finite, that is, $E = \Gamma$, so $F = A$ and therefore \mathbf{A} is compact by Theorem 3.5. □

Using Theorems 3.4 and 3.6, one can now in a standard way easily construct concrete examples of dynamical systems which are either ergodic or compact or neither. For example if Γ is a free group generated by an alphabet S and

$T : S \rightarrow S$ is a bijection which we extend to a automorphism $T : \Gamma \rightarrow \Gamma$, then we get a dual system which is ergodic or compact or neither depending on whether the orbits of T on S are all infinite or all finite or neither. Another example is to consider the group Γ of finite permutations of a possibly infinite set S with automorphisms given by $g \mapsto h^{-1}gh$ where h is an element of the group of all bijections of S , in which case one can again obtain ergodicity or compactness or neither by choosing h appropriately.

Next, we mention a simple converse for Theorem 3.5.

PROPOSITION 3.7. *If \mathbf{A} is ergodic, then it has no nontrivial compact subsystems.*

Proof. Note that \mathbf{A} is weakly mixing by Theorem 3.4 and hence does not have a nontrivial compact subsystem by [2, Theorem 6.8]. □

We now reach our final goal for this section, namely to make a connection with joinings.

THEOREM 3.8. *If the dual system \mathbf{A} is disjoint from all compact dual systems, then it is ergodic.*

Proof. Define a group $\tilde{\Gamma}$ that consists of the same elements as Γ but with the product $g \cdot h := hg$. Then T is an automorphism of $\tilde{\Gamma}$. Let \mathbf{B} be the dual system of $(\tilde{\Gamma}, T)$. It is easily verified that $B \subset A'$. Let \mathbf{F} be the finite orbit subsystem of \mathbf{B} , so in particular \mathbf{F} is compact and isomorphic to a dual system by Theorem 3.5. Then we see that $\omega : A \odot F \rightarrow \mathbb{C} : t \mapsto \langle \Omega, \delta(t)\Omega \rangle$ is a joining of \mathbf{A} and \mathbf{F} where $\delta : A \odot F \rightarrow B(H)$ is defined through $\delta(a \otimes b) = ab$. (This is again the “diagonal measure” idea.) Note that as in [6, Lemma 3.5], ω is trivial (i.e., equal to $\mu \odot \nu$) if and only if \mathbf{F} is trivial. But since $(\tilde{\Gamma}, T)$ has the same orbits as (Γ, T) , we know from Theorem 3.4 that \mathbf{B} is ergodic if and only if \mathbf{A} is. Hence, if we assume that \mathbf{A} is not ergodic, then \mathbf{F} is nontrivial by Theorem 3.5. □

Proposition 3.7 suggests that the converse of Theorem 3.8 might be true, however I don’t have a proof or a counter example.

4. Ergodic joinings

In this section, we briefly motivate and study ergodic joinings. We begin by noting that for our systems \mathbf{A} and \mathbf{B} from Section 1, every state on the (unital) $*$ -algebra $A \odot B$ can in fact be extended to a state on the maximal C^* -algebraic tensor product $A \otimes_m B$. This is a consequence of the following proposition pointed out to me by the referee, which is certainly known, but for which I have no reference.

PROPOSITION 4.1. *Let A and B be unital C^* -algebras, and ω any state on their algebraic tensor product $A \odot B$. Then ω is bounded with respect to the maximal C^* -norm on $A \odot B$.*

Proof. In this proof, the notation $s \leq t$ for $s, t \in A \odot B$ means that $t - s$ is a finite sum of terms of the form u^*u with $u \in A \odot B$. The proof has two steps.

Firstly, we assume the so-called *Axiom A₁* of F. Combes [4, p. 38], namely that for every $t \in A \odot B$ there exists a scalar $\lambda_t \geq 0$ such that

$$s^*t^*ts \leq \lambda_t s^*s$$

for all $s \in A \odot B$. We now show that from this assumption it follows that ω is bounded with respect to the maximal C*-norm $\|\cdot\|_m$ on $A \odot B$. Let $(H_\omega, \pi_\omega, \Omega_\omega)$ be the cyclic representation of $(A \odot B, \omega)$ obtained from the GNS construction, so we have a linear $\gamma_\omega : A \odot B \rightarrow H_\omega$ such that $\gamma_\omega(A \odot B)$ is dense in H_ω , $\langle \gamma_\omega(s), \gamma_\omega(t) \rangle = \omega(s^*t)$, and

$$\pi_\omega(t)\gamma_\omega(s) := \gamma_\omega(ts)$$

for all $s, t \in A \odot B$. Then

$$\|\pi_\omega(t)\gamma_\omega(s)\|^2 = \omega((ts)^*ts) \leq \lambda_t \omega(s^*s) = \lambda_t \|\gamma_\omega(s)\|^2$$

for all $s, t \in A \odot B$, and since $\gamma_\omega(A \odot B)$ is dense in H_ω , it follows that $\pi_\omega(t)$ can be extended to a bounded linear operator on H_ω . As for the cyclic representation of a state on a C*-algebra, $\pi_\omega : A \odot B \rightarrow B(H_\omega)$ is a *-homomorphism, for example $\langle \pi_\omega(t)^*\gamma_\omega(u), \gamma_\omega(s) \rangle = \langle \gamma_\omega(u), \gamma_\omega(ts) \rangle = \omega(u^*ts) = \overline{\omega(s^*t^*u)} = \overline{\langle \gamma_\omega(s), \pi_\omega(t^*)\gamma_\omega(u) \rangle}$ from which $\pi_\omega(t^*) = \pi_\omega(t)^*$ follows. This implies that $t \mapsto \|\pi_\omega(t)\|$ is a C*-seminorm on $A \odot B$, and therefore $\|\pi_\omega(t)\| \leq \|t\|_m$ for $t \in A \odot B$; see [16, p. 193], for example. But using the cyclic representation, we then have

$$|\omega(t)| = |\langle \Omega_\omega, \pi_\omega(t)\Omega_\omega \rangle| \leq \|\pi_\omega(t)\| \leq \|t\|_m$$

and hence ω is bounded with respect to the maximal C*-norm as required.

Secondly we show that Combes' axiom is indeed satisfied, and this will complete the proof. Note that for $0 \leq a \in A$ and $0 \leq b \in B$ we have $a \otimes b = (a^{1/2} \otimes b^{1/2})^*(a^{1/2} \otimes b^{1/2}) \geq 0$. For $a_2 \geq a_1$ in A and $b_2 \geq b_1$ in B it follows that $a_2 \otimes b_2 - a_1 \otimes b_1 = (a_2 - a_1) \otimes (b_2 - b_1) + a_1 \otimes (b_2 - b_1) + (a_2 - a_1) \otimes b_1 \geq 0$, hence

$$a_2 \otimes b_2 \geq a_1 \otimes b_1.$$

For an arbitrary $t = \sum_{k=1}^n a_k \otimes b_k \in A \odot B$, it follows from [17, Inequality 8.5], the inequality above, and the fact that for $c \geq 0$ in a unital C*-algebra one has $c \leq \|c\|$, that

$$t^*t \leq n \sum_{k=1}^n (a_k^*a_k) \otimes (b_k^*b_k) \leq \left(n \sum_{k=1}^n \|a_k\|^2 \|b_k\|^2 \right) 1_A \otimes 1_B$$

hence Combes' axiom holds with $\lambda_t = n \sum_{k=1}^n \|a_k\|^2 \|b_k\|^2$. □

In particular for any systems \mathbf{A} and \mathbf{B} , it follows that in effect every element ω of $J(\mathbf{A}, \mathbf{B})$ is a state on $A \otimes_m B$, and by continuity we have $\omega \circ (\alpha_g \otimes_m \beta_g) = \omega$ for all $g \in G$. In the rest of this section, we work in terms of this setting.

PROPOSITION 4.2. *The set $J(\mathbf{A}, \mathbf{B})$ is weakly* compact, and it is the closed convex hull of its extreme points. In particular, this set of extreme points, which we will denote by $J_e(\mathbf{A}, \mathbf{B})$, is not empty.*

Proof. Let S be the set of states on $A \otimes_m B$. Since S is weakly* compact, and it is readily verified that $J(\mathbf{A}, \mathbf{B})$ is weakly* closed in S , it follows that $J(\mathbf{A}, \mathbf{B})$ is weakly* compact. It is easy to see $J(\mathbf{A}, \mathbf{B})$ is convex. Since $\mu \otimes_m \nu \in J(\mathbf{A}, \mathbf{B})$, it follows from the Krein–Milman theorem that $J_e(\mathbf{A}, \mathbf{B})$ is not empty and that $J(\mathbf{A}, \mathbf{B})$ is the closed convex hull of $J_e(\mathbf{A}, \mathbf{B})$. \square

DEFINITION 4.3. A C^* -dynamical system (C, τ) consists of a unital C^* -algebra C and a representation $\tau : G \rightarrow \text{Aut}(C) : g \mapsto \tau_g$ of a group G . Let E_τ denote the extreme points of the set of τ -invariant states on C .

This set of extreme points connects to ergodicity (in the sense that we have been using the term) in the following way: by [3, Theorem 4.3.20] a W^* -dynamical system is ergodic if and only if the unitary representation of its dynamics on its GNS Hilbert space has a one-dimensional fixed point space. On the other hand, according to [3, Theorem 4.3.17], if the triple (C, ρ, τ) is G -Abelian (see [3, Definition 4.3.6]) for some τ -invariant state ρ on C , then $\rho \in E_\tau$ if and only if the unitary representation of τ on the GNS Hilbert space of (C, ρ) has a one-dimensional fixed point space. So in this case, we can say that every element ρ of E_τ gives us an ergodic C^* -dynamical system of the form (C, ρ, τ) . The latter will appear again in Section 5, but without the assumption that it is G -Abelian.

PROPOSITION 4.4. *If \mathbf{A} and \mathbf{B} are ergodic, then $J_e(\mathbf{A}, \mathbf{B}) \subset E_{\alpha \otimes_m \beta}$ where $(\alpha \otimes_m \beta)_g := \alpha_g \otimes_m \beta_g$.*

Proof. Since \mathbf{A} and \mathbf{B} are ergodic, we have $\mu \in E_\alpha$ and $\nu \in E_\beta$; see for example [3, Theorem 4.3.17]. Now consider any $\omega \in J_e(\mathbf{A}, \mathbf{B})$ and write $\omega = r\omega_1 + (1 - r)\omega_2$ where ω_1 and ω_2 are states invariant under $\alpha \otimes_m \beta$, and $0 < r < 1$. Then $\mu = \omega(\cdot \otimes 1_B) = r\omega_1(\cdot \otimes 1_B) + (1 - r)\omega_2(\cdot \otimes 1_B)$, but $\mu \in E_\alpha$, hence $\mu = \omega_j(\cdot \otimes 1_B)$ and likewise $\nu = \omega_j(1_A \otimes \cdot)$. Thus, $\omega_j \in J(\mathbf{A}, \mathbf{B})$, but ω is extremal in the latter set, therefore $\omega = \omega_j$. This shows that $\omega \in E_{\alpha \otimes_m \beta}$. \square

This proposition motivates the term *ergodic joining* (of \mathbf{A} and \mathbf{B}) for each element of $J_e(\mathbf{A}, \mathbf{B})$ when \mathbf{A} and \mathbf{B} are both ergodic.

We end this section with another illustration of how joinings can in principle be used, by proving a Halmos–von Neumann type theorem for W^* -dynamical systems in terms of Hilbert space. Unfortunately, we require a form of asymptotic Abelianness defined as follows.

DEFINITION 4.5. Consider a C^* -dynamical system (C, τ) whose group G is countable, discrete and amenable. Let (Λ_n) be any Følner sequence in G . If

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} \|[a, \tau_g(b)]\| = 0$$

for all $a, b \in C$ where $[\cdot, \cdot]$ is the commutator, then we say (C, τ) is (Λ_n) -asymptotically Abelian.

This type of asymptotic Abelianness was also used in [17] for the case $G = \mathbb{Z}$. We will not in fact need any properties of Følner sequences; we will only use (4.1), for example, it does not matter if (Λ_n) is a right or left Følner sequence.

PROPOSITION 4.6. *Let \mathbf{A} and \mathbf{B} be ergodic, (Λ_n) -asymptotically Abelian and have the same point spectrum, i.e. $\sigma_{\mathbf{A}} = \sigma_{\mathbf{B}}$. Then the unitary representations U and V of α and β , respectively (as in Definition 2.5) can be done on Hilbert subspaces of some Hilbert space, such that the eigenvectors of U and V span the same Hilbert subspace, say H_0 , and such that $U_g x = V_g x$ for all $x \in H_0$ and $g \in G$.*

Proof. We follow the basic plan due to Lemańczyk [15] (also see [10, Theorem 7.1]) for the measure theoretic case. By Proposition 4.2, there exists an $\omega \in J_e(\mathbf{A}, \mathbf{B})$. Note furthermore that $(A \otimes_m B, \alpha \otimes_m \beta)$ is (Λ_n) -asymptotically Abelian, and hence it is easy to see that the pair $(A \otimes_m B, \omega)$ is G -Abelian (see [3, Definition 4.3.6]). Now consider the “combined” GNS construction for $(A \otimes_m B, \omega)$, (A, μ) and (B, ν) as given by [6, Construction 2.3], namely $(H_\omega, \gamma_\omega)$, (H_μ, γ_μ) and (H_ν, γ_ν) , and the corresponding unitary representations W , U and V of $\alpha \otimes_m \beta$, α and β , respectively. From $(H_\omega, \gamma_\omega)$ and (H_μ, γ_μ) , we of course also obtain the respective cyclic representations with common cyclic vector: $(H_\omega, \pi_\omega, \Omega_\omega)$ and $(H_\mu, \pi_\mu, \Omega_\omega)$.

Take any $\chi \in \sigma_{\mathbf{A}} = \sigma_{\mathbf{B}}$ then by [23, Theorem 2.5] the corresponding eigenvectors of U and V are of the form $\gamma_\mu(a)$ and $\gamma_\nu(b)$ for some $a \in A$ and $b \in B$, and furthermore $\alpha_g(a) = \chi(g)a$ and $\beta_g(b) = \chi(g)b$. Hence,

$$W_g \gamma_\omega(a^* \otimes b) = \gamma_\omega(\alpha_g(a)^* \otimes \beta_g(b)) = \gamma_\omega(a^* \otimes b)$$

since $|\chi| = 1$. Therefore, $\gamma_\omega(a^* \otimes b) = c\Omega_\omega$ for some $c \in \mathbb{C}$ by [3, Theorem 4.3.17] (which uses above mentioned G -Abelianness). So

$$c\gamma_\mu(a) = c\pi_\mu(a)\Omega_\omega = \pi_\omega(a \otimes 1_B)\pi_\omega(a^* \otimes b)\Omega_\omega = \gamma_\omega((aa^*) \otimes b) = d\gamma_\nu(b)$$

for some $d \in \mathbb{C} \setminus \{0\}$, since $\alpha_g(aa^*) = |\chi(g)|^2 aa^* = aa^* \neq 0$ and \mathbf{A} is ergodic. We conclude that $\gamma_\mu(a)$ and $\gamma_\nu(b)$ are proportional, and therefore the eigenvectors of U and V span the same Hilbert subspace H_0 of H_ω . Lastly, for any $x \in H_0$, we have $U_g x = W_g x = V_g x$ by [6, Construction 2.3]. \square

That some form of asymptotic Abelianness should be necessary is perhaps not surprising (see [23, Remark 2.7]), however it would probably be desirable to rather have a version of Proposition 4.6 for C*-dynamical systems (with an invariant state).

5. Strong mixing

Throughout this section, we consider the situation in Definition 2.2, but with $G = \mathbb{Z}$. Remember that as in the special case in Section 3, \mathbf{A} is *strongly mixing* when

$$\lim_{n \rightarrow \infty} \mu(\alpha_n(a)b) = \mu(a)\mu(b)$$

for all $a, b \in A$. Let $\tilde{\mathbf{A}} = (\tilde{A}, \tilde{\mu}, \tilde{\alpha})$ be the “mirror image” of \mathbf{A} which we referred to after Definition 2.2. It turns out that $\tilde{\alpha}_n(b) = U_n b U_n^*$ for all $b \in \tilde{A}$ with U as in Definition 2.5; see [6, Construction 3.4] for more details. Then one can define a joining Δ_n of \mathbf{A} and $\tilde{\mathbf{A}}$ for every n by

$$\Delta_n(a \otimes b) := \mu_\Delta(\alpha_n(a) \otimes b)$$

for all $a \in A$ and $b \in \tilde{A}$. It is easy to verify that Δ_n is indeed a joining, and in particular $\mu_\Delta = \Delta_0$ is a joining. This joining is an example of what in measure theoretic ergodic theory is called a *graph joining* (see, for example, [10, Examples 6.3] or [5, Section 2.2]). We then have the following simple joining characterization of strong mixing.

PROPOSITION 5.1. *The system \mathbf{A} is strongly mixing if and only if*

$$(5.1) \quad \lim_{n \rightarrow \infty} \Delta_n(a \otimes b) = \mu(a)\tilde{\mu}(b)$$

for all $a \in A$ and $b \in \tilde{A}$.

Proof. The system \mathbf{A} is strongly mixing if and only if $\lim_{n \rightarrow \infty} \langle U_n x, y \rangle = \langle x, \Omega \rangle \langle \Omega, y \rangle$ for all $x, y \in H$, but in turn this is equivalent to (5.1), since $\tilde{A}\Omega$ is dense in H . □

We can also view (5.1) as saying that the sequence (Δ_n) of joinings converges pointwise to the joining $\mu \odot \tilde{\mu}$.

Next, we are going to use this result to prove a version of Ornstein’s criterion for strong mixing (in the measure theoretic setting) [18, Theorem 2.1] for W^* -dynamical systems. Its worth mentioning that although Ornstein’s paper [18] doesn’t explicitly deal with joinings, it did lead to Rudolph’s seminal work [21] on joinings and both papers have been very influential in further developments in classical ergodic theory.

But first we need the following lemma.

LEMMA 5.2. *Consider a system \mathbf{A} which is not weakly mixing, but with (A, μ, α_n) ergodic for every $n \in \mathbb{N}$ (the action of \mathbb{Z} in this case is given by $j \mapsto (\alpha_n)^j$). Then for every $k > 0$ there exists a projection $P \in A$, left fixed by*

the modular automorphism group associated with μ , such that $0 < \mu(P) < 1/k$, $P\alpha_n(P) = \alpha_n(P)P$ for all n , and

$$(5.2) \quad \limsup_{n \rightarrow \infty} \mu(\alpha_n(P)P) > k\mu(P)^2,$$

or equivalently,

$$(5.3) \quad \limsup_{n \rightarrow \infty} \Delta_n(P \otimes (J\pi(P)J)) > k(\mu \odot \tilde{\mu})(P \otimes (J\pi(P)J)),$$

where J is the modular conjugation associated with $(\pi(A), \Omega)$.

Proof. The proof is divided into two parts. Part (i) proves the existence of a projection $P \in A$ such that $0 < \mu(P) < 1/k$, $P\alpha_n(P) = \alpha_n(P)P$ for all n , and (5.2) is satisfied. Part (ii), for which I am indebted to the referee, proves that the construction in (i) yields a projection P left fixed by the modular automorphism group associated with μ , and that this invariance ensures the equivalence of (5.2) and (5.3).

(i) Using the notation in Definition 2.5, but denoting U_1 simply as U for simplicity, and correspondingly α_1 as α , it follows from the fact that \mathbf{A} is ergodic but not weakly mixing that U has an eigenvalue $\chi \in \mathbb{C} \setminus \{1\}$ with corresponding eigenvector of the form $u\Omega$ for some $u \in A$ which means (see [23, Theorem 2.5]) that $\alpha(u) = \chi u$, where for simplicity of notation we have identified A with $\pi(A)$ and hence making π in Definition 2.5 the identity mapping $A \rightarrow A$ (we can do this since μ is faithful).

Without loss, we can assume that u is unitary. Namely, $\alpha(u^*u) = \bar{\chi}\chi u^*u = u^*u$, so $u^*u \in \mathbb{C}1$, since \mathbf{A} is ergodic. It follows that $u^*u = \|u^*u\|1 = \|u\|^2 1$, since $u^*u \geq 0$. Since $u \neq 0$, we can assume that $u^*u = 1$ by renaming $u/\|u\|$ as u . In the same way ergodicity and this normalization procedure gives $uu^* = 1$.

Note that $u \notin \mathbb{C}1$, since $\chi \neq 1$. Since $\alpha^n(u) = \chi^n u$ while (A, μ, α^n) is ergodic, it follows that $\chi^n \neq 1$ for all $n \in \mathbb{Z}$.

Denote the spectrum of u by $\sigma(u)$ and let E be the spectral measure relative to $(\sigma(u), H)$ with

$$u = \int \iota dE$$

where $\iota : \sigma(u) \rightarrow \sigma(u)$ denotes the identity map (consult [16, Section 2.5] for a clear exposition of the spectral theory that we are using here). Note that from the definition of the spectrum of an element it follows that $\sigma(\alpha^n(u)) = \sigma(u)$, hence for $v \in \sigma(u)$ we have $\chi^n v \in \sigma(u)$. But $\chi^m v \neq \chi^n v$ and hence

$$E(\{\chi^m v\})E(\{\chi^n v\}) = E(\{\chi^m v\} \cap \{\chi^n v\}) = 0$$

for any integers $m \neq n$. Setting

$$\tilde{\chi} : \sigma(u) \rightarrow \sigma(u) : v \mapsto \chi v$$

and defining spectral measures $F := \alpha \circ E$ and $D := E \circ \tilde{\chi}^{-1}$ relative to $(\sigma(u), H)$, one can verify that $\int \iota dF = \chi u = \int \iota dD$ and hence by uniqueness of the spectral measure we have $\alpha \circ E = E \circ \tilde{\chi}^{-1}$ and more generally

$$(5.4) \quad \alpha^n \circ E = E \circ \tilde{\chi}^{-n}$$

for all $n \in \mathbb{Z}$. Putting all this together, we find that

$$\alpha^m(E(\{v\}))\alpha^n(E(\{v\})) = 0$$

for all integers $m \neq n$, hence $P_n := \alpha^1(E(\{v\})) + \dots + \alpha^n(E(\{v\}))$ is a projection and so $0 \leq n\mu(E(\{v\})) \leq 1$ for every $n \in \mathbb{N}$, which means

$$(5.5) \quad E(\{v\}) = 0$$

for all $v \in \sigma(u)$.

In the remainder of the proof, for any set V in the unit circle we will simply write $E(V)$ instead of $E(V \cap \sigma(u))$, and we will also use the notation $P_{(\theta_1, \theta_2]} := E(e^{i(\theta_1, \theta_2]})$ for any interval $(\theta_1, \theta_2]$. Consider $-\pi < \theta_1 < \theta_2 \leq \pi$. By (5.4)

$$\alpha^n(P_{(\theta_1, \theta_2]}) = P_{(\theta_1 + \text{Arg } \chi^{-n}, \theta_2 + \text{Arg } \chi^{-n})}$$

But for any $\varepsilon > 0$ there are arbitrarily large values of n such that $|\text{Arg } \chi^{-n}| < \varepsilon$ and hence such that $\alpha^n(P_{(\theta_1, \theta_2]})P_{(\theta_1, \theta_2]} \geq P_{(\theta_1 + \varepsilon, \theta_2 - \varepsilon]}$. Furthermore, since μ is normal while $\langle \Omega, E(\cdot)\Omega \rangle$ is a usual positive measure, one can show that $\lim_{n \rightarrow \infty} \mu(P_{(\theta_1, \theta_1 + 1/n]}) = 0$, and by also employing (5.5) one similarly finds $\lim_{n \rightarrow \infty} \mu(P_{(\theta_2 - 1/n, \theta_2]}) = 0$. Combining this with the fact that $P_{(\theta_1, \theta_2]} - P_{(\theta_1 + \varepsilon, \theta_2 - \varepsilon]} = P_{(\theta_1, \theta_1 + \varepsilon]} + P_{(\theta_2 - \varepsilon, \theta_2]}$ it follows that for any ε' we can choose ε small enough that $\mu(P_{(\theta_1, \theta_2]} - P_{(\theta_1 + \varepsilon, \theta_2 - \varepsilon]}) < \varepsilon'$ and therefore there are arbitrarily large values of n such that

$$\mu(\alpha^n(P_{(\theta_1, \theta_2]})P_{(\theta_1, \theta_2]}) > \mu(P_{(\theta_1, \theta_2]}) - \varepsilon'$$

Now suppose that there is a $\delta > 0$ such that $\mu(P_{(\theta_1, \theta_2]}) = 0$ or $\mu(P_{(\theta_1, \theta_2]}) \geq \delta$ for all $-\pi < \theta_1 < \theta_2 \leq \pi$. With $V(m, r) := (-\pi + 2\pi(r - 1)/m, -\pi + 2\pi r/m]$, we have $\sum_{r=1}^m \mu(P_{V(m, r)}) = 1$, hence each

$$\mathcal{I}_m := \{V(m, r) : \mu(P_{V(m, r)}) \geq \delta, r \in \{1, \dots, m\}\}$$

contains at least one element, and we have a sequence of intervals $I_m \in \mathcal{I}_{2^m}$ with $I_{m+1} \subset I_m$. But then $\langle \Omega, E(\bigcap_{m=1}^\infty I_m)\Omega \rangle \geq \delta$ contradicting (5.5). We conclude that for any $k' > k > 0$ there are $-\pi < \theta_1 < \theta_2 \leq \pi$ such that $0 < \mu(P_{(\theta_1, \theta_2]}) < 1/k'$. With $P := P_{(\theta_1, \theta_2]}$, we have $P\alpha^n(P) = \alpha^n(P)P$ from (5.4), completing part (i) of the proof.

(ii) We continue with the notation in (i).

By [25, Corollary VIII.1.4] $\alpha \circ \sigma_t = \sigma_t \circ \alpha$, where $t \mapsto \sigma_t$ is the modular automorphism group associated with μ . With u and χ as before, it follows that $\alpha(\sigma_t(u)) = \chi\sigma_t(u)$. Together with $\alpha(u) = \chi u$, this implies that

$$\sigma_t(u) = \lambda_t u$$

for some $\lambda_t \in \mathbb{C}$, according to [23, Lemma 2.1(3)], for every $t \in \mathbb{R}$. Note that $|\lambda_t| = 1$. From the group property of σ_t it is easily verified that $\lambda_{s+t} = \lambda_s \lambda_t$. Since $t \mapsto \langle x, \sigma_t(u)y \rangle = \langle \Delta^{-it}x, u\Delta^{-it}y \rangle$ is continuous for all $x, y \in H$, where Δ is the modular operator associated with (A, Ω) , it follows that $t \mapsto \lambda_t$ is continuous. Therefore,

$$\lambda_t = e^{i\theta t}$$

for all $t \in \mathbb{R}$ for some $\theta \in \mathbb{R}$; see for example [20, p. 12]. It follows that $\Delta^{it}u\Omega = \sigma_t(u)\Omega = e^{i\theta t}u\Omega$, hence by the definition of $J\Delta^{1/2}$ (see, for example, [3, Section 2.5.2])

$$Ju^*\Omega = J(J\Delta^{1/2})u\Omega = \Delta^{1/2}u\Omega = e^{\theta/2}u\Omega$$

and by taking the norm both sides we conclude that $e^{\theta/2} = 1$ and therefore $\theta = 0$. This proves that

$$\sigma_t(u) = u$$

for all $t \in \mathbb{R}$.

Note that the fixed point algebra of the modular automorphism group is itself a von Neumann algebra (as is the fixed point algebra of any system) and since u is in this fixed point algebra as shown above, it follows that its spectral projections are too. In particular,

$$\sigma_t(P) = P$$

for all $t \in \mathbb{R}$. This means that $\Delta^{it}P\Omega = P\Delta^{it}\Omega = P\Omega$ and therefore

$$JP\Omega = J(J\Delta^{1/2})P^*\Omega = \Delta^{1/2}P\Omega = P\Omega$$

so

$$\begin{aligned} \Delta_n(P \otimes (JPJ)) &= \mu_\Delta(\alpha^n(P) \otimes (JPJ)) = \langle \Omega, \alpha^n(P)JPJ\Omega \rangle \\ &= \langle \Omega, \alpha^n(P)P\Omega \rangle = \mu(\alpha^n(P)P). \end{aligned}$$

Furthermore,

$$\mu \odot \tilde{\mu}(P \otimes (JPJ)) = \mu(P)\langle \Omega, JPJ\Omega \rangle = \mu(P)^2.$$

The equivalence of (5.2) and (5.3) now follows. □

Now we are in a position to state and prove our version of Ornstein’s criterion. In its proof, we encounter a C^* -dynamical system with invariant state, that is, a (C, ρ, τ) with (C, τ) as in Definition 4.3 and where ρ is any state on C with $\rho \circ \tau_g = \rho$ for all $g \in G$ (with $G = \mathbb{Z}$ the relevant case). We will refer to such a (C, ρ, τ) as a C^* -dynamical system as well. For such a C^* -dynamical system, *weak mixing* is defined in the same way as for W^* -dynamical systems in Definition 2.5, but we call it *ergodic* if the fixed point space of the unitary representation of τ on the Hilbert space H of the GNS construction of (C, ρ) is one dimensional, that is, $\dim\{x \in H : U_gx = x \text{ for all } g \in G\} = 1$.

THEOREM 5.3. *Let \mathbf{A} be a system such that (A, μ, α_n) is ergodic for every $n \in \mathbb{N}$. (Alternatively, we could assume that \mathbf{A} is weakly mixing.) Furthermore, assume that there is a real number $k > 0$ such that*

$$\limsup_{n \rightarrow \infty} \Delta_n(c^*c) \leq k\mu \odot \tilde{\mu}(c^*c)$$

for all $c \in A \odot \tilde{A}$. Then \mathbf{A} is strongly mixing.

Proof. Note that \mathbf{A} is weakly mixing, for if it was not, our assumptions would contradict Lemma 5.2 for $c = P \otimes (J\pi(P)J)$. In the rest of the proof, we only need weak mixing of \mathbf{A} , rather than the ergodicity of (A, μ, α_n) for all $n \in \mathbb{N}$.

We now follow the basic argument presented in [5, Theorem 4.3] for the measure theoretic case, and we work in the setting of the maximal C^* -algebraic tensor product as explained at the beginning of Section 4. Since $J(\mathbf{A}, \tilde{\mathbf{A}})$ is weakly* compact by Proposition 4.2, the sequence (Δ_n) has a cluster point ω in $J(\mathbf{A}, \tilde{\mathbf{A}})$ in the weak* topology. From our assumptions, it follows that $\omega \leq k\mu \otimes_m \tilde{\mu}$.

Note that $\tilde{\mathbf{A}}$ is weakly mixing, since \mathbf{A} is. Also recall that a C^* -dynamical system (C, ρ, τ) for an action of \mathbb{Z} is weakly mixing if and only if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\rho(a\tau_n(b)) - \rho(a)\rho(b)| = 0$$

for all $a, b \in C$; see, for example, [17, Proposition 5.4] or [7, Definition 2.3 and Proposition 3.4], and keep in mind that $(\{1, \dots, N\})_N$ is a Følner sequence in \mathbb{Z} . We now use this characterization of weak mixing to show that the C^* -dynamical system $\mathbf{A} \otimes_m \tilde{\mathbf{A}} := (A \otimes_m \tilde{A}, \mu \otimes_m \tilde{\mu}, \alpha \otimes_m \tilde{\alpha})$ is weakly mixing. It will be convenient to write $\rho := \mu \otimes_m \tilde{\mu}$ and $\tau := \alpha \otimes_m \tilde{\alpha}$. For any

$$c = \sum_{j=1}^m a_j \otimes c_j \in A \odot \tilde{A}$$

and

$$d = \sum_{j=1}^m b_j \otimes d_j \in A \odot \tilde{A}$$

we have

$$\begin{aligned} & |\rho(c\tau_n(d)) - \rho(c)\rho(d)| \\ & \leq \sum_{j=1}^m \sum_{k=1}^m |\mu(a_j\alpha_n(b_k))\tilde{\mu}(c_j\tilde{\alpha}_n(d_k)) - \mu(a_j\alpha_n(b_k))\tilde{\mu}(c_j)\tilde{\mu}(d_k)| \\ & \quad + \sum_{j=1}^m \sum_{k=1}^m |\mu(a_j\alpha_n(b_k))\tilde{\mu}(c_j)\tilde{\mu}(d_k) - \mu(a_j)\mu(b_k)\tilde{\mu}(c_j)\tilde{\mu}(d_k)| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{j=1}^m \sum_{k=1}^m \|a_j\| \|b_k\| |\tilde{\mu}(c_j \tilde{\alpha}_n(d_k)) - \tilde{\mu}(c_j) \tilde{\mu}(d_k)| \\ &\quad + \sum_{j=1}^m \sum_{k=1}^m \|c_j\| \|d_k\| |\mu(a_j \alpha_n(b_k)) - \mu(a_j) \mu(b_k)| \end{aligned}$$

therefore $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\rho(c\tau_n(d)) - \rho(c)\rho(d)| = 0$, since \mathbf{A} and $\tilde{\mathbf{A}}$ are both weakly mixing. Now consider arbitrary $a, b \in A \otimes_m \tilde{A}$ and any $\varepsilon > 0$. Then there are $c, d \in A \odot \tilde{A}$ such that in the maximal C^* -norm $\|a - c\|_m < \varepsilon$ and $\|b - d\|_m < \varepsilon$, so

$$\begin{aligned} &|\rho(a\tau_n(b)) - \rho(a)\rho(b)| \\ &\leq |\rho(c\tau_n(d)) - \rho(c)\rho(d)| \\ &\quad + |\rho((a - c)\tau_n(b))| + |\rho(c\tau_n(b - d))| + |\rho(c - a)\rho(b)| + |\rho(c)\rho(d - b)| \\ &\leq |\rho(c\tau_n(d)) - \rho(c)\rho(d)| + 2\varepsilon\|b\|_m + 2\varepsilon(\|a\|_m + \varepsilon). \end{aligned}$$

From all this, it follows that $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\rho(a\tau_n(b)) - \rho(a)\rho(b)| = 0$, i.e. $\mathbf{A} \otimes_m \tilde{\mathbf{A}}$ is weakly mixing.

From the definitions of weak mixing and ergodicity of a C^* -dynamical system, it follows that $\mathbf{A} \otimes_m \tilde{\mathbf{A}}$ is ergodic and therefore $\mu \otimes_m \tilde{\mu} \in E_{\alpha \otimes_m \tilde{\alpha}}$ by [3, Theorem 4.3.17] and Definition 4.3. However, if $\omega_1 \leq k\omega_0$ where $\omega_0 \in E_{\alpha \otimes_m \tilde{\alpha}}$ while ω_1 is an invariant state on $A \otimes_m \tilde{A}$ under $\alpha \otimes_m \tilde{\alpha}$, then $\omega_1 = \omega_0$, since if this was not the case, then $k > 1$ so $\omega_2 := (k\omega_0 - \omega_1)/(k - 1)$ is an invariant state which gives $\omega_0 = \omega_1/k + (k - 1)\omega_2/k$ contradicting $\omega_0 \in E_{\alpha \otimes_m \tilde{\alpha}}$. So $\omega = \mu \otimes_m \tilde{\mu}$ which means that $\mu \otimes_m \tilde{\mu}$ is the unique cluster point of (Δ_n) in the weak* topology. Hence,

$$w * -\lim_{n \rightarrow \infty} \Delta_n = \mu \otimes_m \tilde{\mu}$$

in $J(\mathbf{A}, \tilde{\mathbf{A}})$. Therefore, \mathbf{A} is strongly mixing by Proposition 5.1. □

Note that the following partial converse is of course also true, namely if \mathbf{A} is strongly mixing, then it is weakly mixing and $w * -\lim_{n \rightarrow \infty} \Delta_n = \mu \otimes_m \tilde{\mu}$.

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REFERENCES

- [1] D. Avitzour, *Noncommutative topological dynamics. II*, Trans. Amer. Math. Soc. **282** (1984), 121–135. MR 0728705
- [2] C. Beyers, R. Duvenhage and A. Ströh, *The Szemerédi property in ergodic W^* -dynamical systems*, J. Operator Theory **64** (2010), 35–67. MR 2669426

- [3] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics 1*, 2nd ed., Springer-Verlag, New York, 1987. MR 0887100
- [4] F. Combes, *Sur les faces d'une C^* -algèbre*, Bull. Sci. Math. (2) **93** (1969), 37–62. MR 0265947
- [5] T. de la Rue, *An introduction to joinings in ergodic theory*, Discrete Contin. Dyn. Syst. **15** (2006), 121–142. MR 2191388
- [6] R. Duvenhage, *Joinings of W^* -dynamical systems*, J. Math. Anal. Appl. **343** (2008), 175–181. MR 2409467
- [7] R. Duvenhage, *Bergelson's theorem for weakly mixing C^* -dynamical systems*, Studia Math. **192** (2009), 235–257. MR 2504840
- [8] F. Fidaleo, *An ergodic theorem for quantum diagonal measures*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (2009), 307–320. MR 2541399
- [9] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49. MR 0213508
- [10] E. Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, Providence, RI, 2003. MR 1958753
- [11] P. R. Halmos, *On automorphisms of compact groups*, Bull. Amer. Math. Soc. **49** (1943), 619–624. MR 0008647
- [12] U. Krengel, *Ergodic theorems*, Walter de Gruyter & Co., Berlin, 1985. MR 0797411
- [13] J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. (4) **33** (2000), 837–934. MR 1832993
- [14] J. Kustermans and S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand. **92** (2003), 68–92. MR 1951446
- [15] M. Lemańczyk, *Introduction to ergodic theory from the point of view of spectral theory*, Lecture Notes on the Tenth Kask Mathematics Workshop (G. H. Choe, ed.), Korea Advanced Institute of Science and Technology, Math. Res. Center, Taejon, Korea, 1995.
- [16] G. J. Murphy, *C^* -algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990. MR 1074574
- [17] C. P. Niculescu, A. Ströh and L. Zsidó, *Noncommutative extensions of classical and multiple recurrence theorems*, J. Operator Theory **50** (2003), 3–52. MR 2015017
- [18] D. S. Ornstein, *On the root problem in ergodic theory*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, University of California Press, Berkeley, 1972, pp. 347–356. MR 0399415
- [19] K. Petersen, *Ergodic theory*, Cambridge University Press, Cambridge, 1983. MR 0833286
- [20] W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Mathematics, No. 12, Interscience Publishers (a division of John Wiley and Sons), New York–London, 1962. MR 0152834
- [21] D. J. Rudolph, *An example of a measure preserving map with minimal self-joinings, and applications*, J. Analyse Math. **35** (1979), 97–122. MR 0555301
- [22] J.-L. Sauvageot and J.-P. Thouvenot, *Une nouvelle définition de l'entropie dynamique des systèmes non commutatifs*, Comm. Math. Phys. **145** (1992), 411–423. MR 1162806
- [23] E. Størmer, *Spectra of ergodic transformations*, J. Funct. Anal. **15** (1974), 202–215. MR 0377544
- [24] Ș. Strătilă, *Modular theory in operator algebras*, Editura Academiei Republicii Socialiste România, Bucharest, Abacus Press, Tunbridge Wells, 1981. MR 0696172
- [25] M. Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, vol. 125, Operator Algebras and Non-commutative Geometry, vol. 6, Springer-Verlag, Berlin, 2003. MR 1943006

- [26] S. Vaes, *Locally compact quantum groups*, Ph.D. thesis, Katholieke Universiteit Leuven, 2001.
- [27] A. van Daele, *Locally compact quantum groups. A von Neumann algebra approach*: available at [arXiv:math/0602212v1](https://arxiv.org/abs/math/0602212v1).

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