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EXTENSION AND RESTRICTION OF HOLOMORPHIC FUNCTIONS ON CONVEX FINITE TYPE DOMAINS

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Abstract. We consider holomorphic functions on a non-singular subvariety of a smoothly bounded convex domain of finite type. A sufficient and necessary condition is proved for such a function to have an extension to a p-integrable holomorphic function on the whole domain. This is shown under transversallity assumption and certain non-degeneracy condition of the subvariety.

1. Introduction

In this paper, we study the extension problem for smoothly bounded convex finite type domains in $\mathbb{C}^n, n > 1$. Such a domain will be denoted by D and assumed to be of type M. Throughout the paper, A stands for a complex submanifold of some open neighbourhood of $\bar{D}$. Since D is convex it is pseudoconvex and, consequently, D is a domain of holomorphy. Therefore, it is a consequence of the Cartan's theorems A and B and the Oka's Coherence theorem that for each function f holomorphic in $D \cap A$, there exists a holomorphic function F in D such that $F|_{D\cap A} = f$. Thus, the question whether each function holomorphic in $D \cap A$ is the restriction of a function which is holomorphic in D, has a positive answer in the case which we are considering.

The answer to the same question is more involved, if we impose more subtle properties on functions which are taken into account. A good example is the extension problem for bounded holomorphic functions in the same class of domains, that is, smoothly bounded convex finite type domains. Namely, it was proved recently in [1] that in this case the answer can be formulated in terms of rather delicate properties of the nonisotropic pseudodistance. It is

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worthy of mentioning at this moment that these conditions are satisfied if A is a complex linear affine hypersurface [11].

In [19], the author considered the case when A is a complex linear affine hypersurface and the functions belong either to the Hardy space or to some weighted Bergman space. These results were used to obtain essentially sharp pointwise estimates of the Szegö kernel from below on the diagonal in $\mathbb{C}^2$. Also, it was proved that if the restriction operator maps the Hardy space onto the Bergman space in $D \cap A$, then D is strictly pseudoconvex. Here, we intend to investigate a similar problem for a general complex submanifold A, which is assumed to cut transversally the boundary of D.

There are two results which should be mentioned as a motivation for our study. The celebrated theorem of Ohsawa [23] (cf. also [24] and [10] for an alternative proof) states that each L^2-integrable holomorphic function admits an extension to an L^2-integrable holomorphic function provided D is a bounded pseudoconvex domain. On the other hand, the finite type assumption on D suggests that one may expect some gain of regularity. The gain should be equal to $\frac{2}{M}$, where M stands for the type of the domain, when measured in an isotropic way on weighted L^p-spaces. The word 'gain' should be understood in a correct way. Namely, under the finite type assumption one expects that there is a strictly larger class of functions than $H^2(D \cap A)$, which admit an extension to an L^2-integrable holomorphic function. However, the problem which we consider is not of isotropic nature, since the geometry of a domain D is not isotropic. This is the reason for considering measures different from $\operatorname{dist}(\cdot, bD)^{2/M} dV_A$. The second source of motivation is the manuscript [12], where the Authors investigated restrictions in the 'gain of regularity' principle in the problem of extensions of holomorphic functions.

The results of the paper hold under the following assumptions on the variety A:

Assumption 1 (Non-singularity and transversallity condition). *A is a nonsingular subvariety of $\tilde{V} \supset \bar{D}$ of the form*

$$A = \{z \in \tilde{V} : f(z) = 0\},$$

where f is a holomorphic function in an open neighbourhood $\tilde{V}$ of $\bar{D}$ satisfying:

(i) $\partial f \neq 0$ *in* $\tilde{V}$,
(ii) $\partial f \wedge \partial r \neq 0$ *in an open set* $\tilde{U}$ *containing* bD.

The symbol r stands for a defining function of D, which is assumed to be smooth, nondegenerate on bD. Also, without loss of generality, we may and will assume that $r = p_D - 1$, where p_D stands for Minkowski functional of D. We often use the symbol ϱ to denote $|r(\cdot)|$, which is uniformly comparable with $\operatorname{dist}(\cdot, bD)$.

Observe that instead of assuming that A is the zero set of a function f we might as well assume that A is a codimension one subvariety of an open neighbourhood of $\bar{D}$. Indeed, since D is convex it is a domain of holomorphy, which would imply that A is a complete intersection.

ASSUMPTION 2 (Nondegeneracy condition I). *$A \subset \mathbb{C}^n$ satisfies Assumption 1 and either*:

(i) $n = 2$, *or*

(ii) $n > 2$ *and there exists an open set U and a constant $\epsilon > 0$ such that $A \cap bD \subset U$ and*

$$|Z_2^{\varepsilon} f(p)| \geq c > 0,$$

for each $p \in U$ and $0 < \varepsilon < \epsilon$.

The symbol $Z_l^{\varepsilon} f$ stands for the differential

$$Z_l^{\varepsilon} f(p) := \frac{\partial}{\partial \lambda} f(p + \lambda v_l)\bigg|_{\lambda=0},$$

with v_l equal to the lth vector in the ε-extremal basis at p.

ASSUMPTION 3 (Nondegeneracy condition II). *$A \subset \mathbb{C}^n$ satisfies Assumption 1 and either*:

(i) $n = 2$, *or*

(ii) $n > 2$ *and there exists an open neighbourhood U of the set $\mathcal{N}(\mathcal{L}_r)$, where*

$$\mathcal{N}(\mathcal{L}_r) := \left\{ p \in bD : \exists_{\xi \neq 0} \sum_{j,k=1}^{n} \frac{\partial^2 r(p)}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k = 0 \right\},$$

and a constant $\epsilon > 0$, such that for each $p \in U \cap D \cap A$ and $2 \leq l \leq n$

$$|Z_l^{\varepsilon} f(p)| \geq c > 0 \tag{1}$$

with a uniform constant c if $0 < \varepsilon < \epsilon$.

To make the meaning of these assumptions clear, we recall the definition of the ε-extremal basis at a given point $p \in D$. The first vector of the ε-basis is essentially equal to the normalized complex gradient, the second vector is chosen in the following way: find a point q_2 with $r(q_2) = r(p) + \varepsilon$ such that $q_2 - p$ is perpendicular to the complex gradient and such that the maximal distance from p to $\{r(\zeta) = r(p) + \varepsilon\}$ is achieved at q_2. The vector v_2 is defined as a unit vector in the direction of the vector $q_2 - p$. The procedure is continued until the basis is complete (we invite the Reader to consult [21], [22] for definitions and basic properties of the basis and to [17], [18] for a complete discussion concerning these objects, cf. also [5]).

Before we proceed observe that Assumptions 3 and 2 do not depend on a particular representation of the variety A. Let

$$A = \{z \in \tilde{V} : f_1(z) = 0\} = \{z \in \tilde{V} : f_2(z) = 0\} \tag{2}$$

with functions f_1 and f_2 which satisfy Assumption 1. Assume additionally that either Assumption 3 or 2 is satisfied by the function f_1. We claim that then the same condition is satisfied by f_2. Indeed, since $\partial f_1 \neq 0 \neq \partial f_2$ in $\tilde{V}$, it follows that f_1/f_2 is a zero free holomorphic function in $\tilde{V}$. Since D is convex, we may assume that $\tilde{V}$ is simply connected. Hence, there exists a holomorphic function h such that $f_2 = e^h f_1$. Consequently, it holds

$$\begin{aligned} |Z_l^\epsilon f_2(p)| &= |(Z_l^\epsilon h)e^h f_1(p) + e^h Z_l^\epsilon f_1(p)| \\ &= |e^h (Z_l^\epsilon f_1)(p)| \geq c > 0 \end{aligned}$$

for $p \in U \cap D \cap A$, which proves the claim in both cases.

We can now formulate our main results. The symbol R_A stands for the operator of restriction to the hypersurface A and $H^p_A(D)$ is the subspace of $H^p(D)$ consisting of all functions which vanish on A.

THEOREM 1. *Let $D \subset \mathbb{C}^n, n > 1$ be a smoothly bounded convex finite type domain and let f be a holomorphic function in some neighbourhood of $\bar{D}$ and*

$$A = \{z \in \tilde{V} \supset \bar{D} : f(z) = 0\}.$$

Assume that Assumption 2 holds.

Then, there exists a positive, finite Borel measure μ supported on $A \cap D$ such that the following sequence

$$0 \longrightarrow H^p_A(D) \hookrightarrow H^p(D) \xrightarrow{R_A} H^p(D \cap A, \mu) \longrightarrow 0 \tag{3}$$

is exact and splits for $1 \leq p < \star$, where, as usual, $\star$ refers to the BMO space. The latter means that there exists a linear extension operator

$$E^N : H^p(D \cap A, \mu) \to H^p(D)$$

such that $R_A E^N = id_{A \cap D}$. As a result, in the category of Banach spaces, it holds

$$H^p(D) \cong H^p(D \cap A, \mu) \oplus H^p_A(D), \quad 1 \leq p < \infty. \tag{4}$$

THEOREM 2. *Let $D \subset \mathbb{C}^n, n > 1$ be a smoothly bounded convex finite type domain and let f be a holomorphic function in some neighbourhood of $\bar{D}$ and*

$$A = \{z \in \tilde{V} \supset \bar{D} : f(z) = 0\}.$$

Assume that Assumption 3 holds.

Then, there exists a positive, finite Borel measure μ supported on $A \cap D$ such that the following sequence

$$0 \longrightarrow H^1_A(D) \hookrightarrow H^1(D) \xrightarrow{R_A} H^1(D \cap A, \mu) \longrightarrow 0 \tag{5}$$

is exact and splits, i.e. there exists a linear extension operator

$$E^N : H^1(D \cap A, \mu) \to H^1(D)$$

such that $R_A E^N = id_{A\cap D}$. *As a result, in the category of Banach spaces, it holds*

$$H^1(D) \cong H^1(D\cap A,\mu) \oplus H^1_A(D). \tag{6}$$

We will explain the notation now. Throughout the paper, the volume measure in $\mathbb{C}^n, n>1$ is denoted by V and V_A is the measure corresponding to the canonical Riemannian volume form on A. The latter means that this volume form is the volume form for the real part of the Hermitian metric on A, which is the restriction to A of the euclidean metric in $\mathbb{C}^n$. The symbol $H^{\kappa,p}(D)$ is the weighted Bergman space, that is,

$$H^{\kappa,p}(D) := \left\{ f\in H(D) :\ \int_D |f|^p \varrho^\kappa\, dV < \infty \right\},$$

and $H^p(D\cap A,\mu)$ is by definition the space of all functions holomorphic in $D\cap A$, which are p-integrable with respect to the measure μ. Also, $H^p_A(D)$ stands for the subspace of $H^p(D)$ consisting of all functions, which vanish on A. We shall restrict our attention mostly to the case $\kappa=0$. However, the Reader easily notices that many results which are proved in the paper hold or have a counterpart when $\kappa>-1$.

The first step in the proof of Theorem 1 is a result, which provides characterization of positive measures for which the inclusion

$$\begin{aligned} R_A[H^p(D)] &\subset H^p(D\cap A,\mu) \\ &:= \left\{ g\in H(D\cap A) :\ \int_{D\cap A} |g|^p\, d\mu < \infty \right\} \end{aligned}$$

holds. Importantly, the proof of neither this result nor the next one requires either Assumption 3 or Assumption 2—these results are proved solely under the transversallity condition 1.

Theorem 3. *Assume that* D *is a smoothly bounded convex domain of finite type in* $\mathbb{C}^n, n>1$ *and* A *is a complex submanifold in some neighbourhood of* $\bar D$ *of the form*

$$A = \{z\in \tilde V :\ f(z)=0\},$$

satisfying the nonsingularity and transversallity Assumption 1.

Denote by R_A *the restriction operator to* A *and let* μ *be a positive Borel measure supported on* $D\cap A$. *Then*

$$R_A[H^2(D)] \subset H^2(D\cap A,\mu),$$

if and only if for sufficiently small $c>0$

$$\sup\left\{ \frac{\mu(P)}{V(P)} :\ P = P_{c\varrho}(\zeta), \zeta\in D \right\} < \infty. \tag{7}$$

Furthermore, if condition (7) *holds, then*

$$R_A[H^{\kappa,p}(D)] \subset H^p(D\cap A,\varrho^\kappa\mu),$$

for any $1 \le p < \infty$ *and* $\kappa > -1$.

We will show that the measure $|\partial f|_\kappa^2\, dV_A$ satisfies condition (7). The symbol $|\cdot|_\kappa$ stands for a suitably defined non-isotropic norm of a $(1,0)$-form. This implies that

$$\begin{aligned} R_A[H^p(D)] &\subset H^p(D\cap A, |\partial f|_\kappa^2\, dV_A) \\ &= \left\{ g \in H(D\cap A) : \int_{D\cap A} |g|^p |\partial f|_\kappa^2\, dV_A < \infty \right\}. \end{aligned}$$

We warn the Reader at this moment that although we use the same symbol as in [4] for the non-isotropic norm, our definition differs from the one given in that paper.

Notice that if A satisfies (2), that is, A is the zero set of functions f_1 and f_2, each of which satisfies Assumption 1, then the Borel measures

$$|\partial f_1|_\kappa^2\, dV_A \quad \text{and} \quad |\partial f_2|_\kappa^2\, dV_A$$

are equivalent. Indeed, there exists a function $h \in H(\tilde{V})$ such that $f_2 = e^h f_1$ and, as a result, on $D \cap A$

$$|\partial f_2|_\kappa^2\, dV_A = |e^h|^2 \cdot |\partial f_1|_\kappa^2\, dV_A \lesssim |\partial f_1|_\kappa^2\, dV_A.$$

Naturally, the same argument shows that on $D\cap A$ it holds $|\partial f_1|_\kappa^2\, dV_A \lesssim |\partial f_2|_\kappa^2\, dV_A$ and proves the claim.

Another obvious candidate for a measure μ satisfying condition (7) is $w(\zeta)\, dV_A$, where the function w is equal to

$$w(\zeta) := \sup\left\{ \frac{V(P)}{V_A(P\cap A)} : P = P_\varepsilon(\zeta), \varepsilon \le c\varrho \right\} \sim \frac{V(P_\varrho(\zeta))}{V_A(P_\varrho(\zeta)\cap A)}.$$

One can easily show that this choice of the measure is actually equivalent to $|\partial f|_\kappa^2\, dV_A$. Also, it is worth of noticing that condition (7) implies that the positive Borel measure μ is finite—this fact is proved as Corollary 1 below.

Next, we prove that there always exists an extension operator of gain $\varrho^{2/M}$ with $L^1(D)$ as the target space.

THEOREM 4. *Assume that D is a smoothly bounded convex domain of finite type in $\mathbb{C}^n, n > 1$ and A is a complex submanifold in some neighbourhood of $\bar{D}$ of the form*

$$A = \{z : f(z) = 0\},$$

with holomorphic f such that $\partial f \ne 0$ in $\bar{D}$ and $\partial f \wedge \partial r \ne 0$ in some neighbourhood of bD. For each sufficiently large N, there exists a measure ν_N and an extension operator

$$E^N : L^1(D\cap A, \nu_N) \to L^1(D).$$

Furthermore, the measure ν_N is equal to $|\partial f|_\kappa^2\, dV_A + \mathfrak{h}_N\, dV_A$ with $\mathfrak{h}_N \lesssim \varrho^{2/M}$. The latter estimate is uniform with respect to N.

In the next step, we shall show that Assumption 3 implies that the measures ν_N from Theorem 4 satisfy condition (7) provided N is sufficiently large.

THEOREM 5. *Assume that D is a smoothly bounded convex domain of finite type in $\mathbb{C}^n, n > 1$ and A is a complex submanifold in some neighbourhood of $\bar{D}$ of the form*

$$A = \{z : f(z) = 0\}.$$

(i) *If A satisfies Assumption 2, then for each N the measure ν_N satisfies condition (7).*
(ii) *If A satisfies Assumption 3, then there exists $N_0 \in \mathbb{N}$ such that the measures ν_N defined in Theorem 4 satisfy condition (7) provided $N \geq N_0$.*

Importantly, in both cases in Theorem 5 we prove that

$$\nu_N \lesssim |\partial f|_\kappa^2 \, dV_A.$$

The next step in the proof of Theorem 1 is the following theorem.

THEOREM 6. *Assume that D is a smoothly bounded convex domain of finite type in $\mathbb{C}^n, n > 1$ A is a complex submanifold in some neighbourhood of $\bar{D}$ of the form*

$$A = \{z : f(z) = 0\},$$

with f satisfying Assumption 2. For any N the operator E^N maps continuously $L^\infty(D \cap A)$ into $\mathrm{BMO}(D)$.

Theorem 6 needs explanation. Namely the symbol $\mathrm{BMO}(D)$ stands for the space of all locally integrable functions g for which

$$\sup \frac{1}{V(P)} \int_P |g - g_P| \, dV < \infty,$$

where the supremum is taken over all polydisks $P = P_\varepsilon(\zeta)$ with $\varepsilon \leq c\varrho(\zeta)$ and as usual g_P stands for the mean value of g over P. The constant c is chosen according to conditions from Assumption 4 below. The proof of Theorem 1 is completed by the interpolation argument which we recall as Theorem 9.

The fact that the real interpolating spaces between $\mathrm{BMO}(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ are isomorphic to $L^p(\mathbb{R}^n)$ is classic [16] (cf. also [6]). The case of spaces of homogeneous type was studied in [20]. Unfortunately, we cannot simply invoke this result, since we work in a domain. Consequently, in the definition of BMO we took into account only polydisks $P_\varepsilon(\zeta)$ with $\varepsilon \leq \varrho(\zeta)$. However, hardly surprisingly, this is not an obstacle for the interpolation result to hold true.

A Whitney type cover consisting of non-isotropic polydisks—the cover of $D \cap A$ rather than D—is the main tool in Theorem 3. Although the corresponding argument is rather standard, there are two aspects of the proof, which should be emphasized. Namely, we cannot use any compactness argument, since the choice of an extremal basis is not continuous (cf. [17] for an example). Secondly, the cover should be locally finite and, as is well-known, this depends on the doubling property or more precisely on asymptotic behaviour of volumes of P_ε as ε tends to 0. It was proved in [18] that $V(P_\varepsilon(\zeta)) \sim \varepsilon^{\sum_{j=1}^n (1/m_j)} = \varepsilon^\nu$, where $(m_1, \dots, m_n)$ is the multitype of bD_ϱ at ζ (cf. [7]). Hence, the exponent ν may change abruptly from point to point. As we shall see, both obstacles turn out to be of technical nature only.

The extension operator from Theorems 5 and 6 is the one constructed by B. Berndtsson [3] (cf. also [2]) with the support function of K. Diederich and J. E. Fornaess [8].

2. Preliminaries

The definition of the extremal basis was sketched in the Introduction. We will not recall definitions of non-isotropic polydisks $P_\varepsilon(\zeta)$ or directional distances τ, since they can be found in many places—most notably in [21], [22], where they were defined or in [4], where their importance was established. We refer the Reader to [17] and [18] for a complete discussion of properties of these objects. We emphasize only that once they are defined D turns into a space of homogeneous type (this statement should be taken with some care, since there exist different notions of this object in the literature). We will keep to the notation introduced in these papers. For instance, $\tau_k(\zeta, \varepsilon)$ stands as usual for $\tau(\zeta, v_k, \varepsilon)$, where v_k is the kth vector of the ε-basis at ζ. Recall that with this notation

$$\varepsilon \sim \tau_1(\zeta, \varepsilon) \le \tau_n(\zeta, \varepsilon) \le \dots \le \tau_2(\zeta, \varepsilon) \lesssim \varepsilon^{1/M}, \tag{8}$$

with uniform constants. The estimate explains, at least to some degree, the meaning of Assumption 3.

Let $\delta : \bar{D} \times \bar{D} \to \mathbb{R}_+$ be defined in the following way

$$\delta(z, \zeta) := \inf\{\varepsilon > 0 : \zeta \in P_\varepsilon(z)\}.$$

The function δ is a pseudodistance, that is,

$$\begin{aligned} &\delta(z, \zeta) = 0, \quad \text{if and only if} \quad z = \zeta, \\ &\delta(z, \zeta) \sim \delta(\zeta, z), \\ &\delta(z, \zeta) \lesssim \delta(z, \xi) + \delta(\xi, \zeta) \end{aligned} \tag{9}$$

with uniform constants. These properties play an important role in the construction of the Whitney cover. Importantly, properties stated in (9) hold up to some uniform constant.

We will make use of the properties of the pseudoballs $P_\varepsilon(\zeta)$, often without explicitly referring to them. More specifically, for each c there exists $b = b(c)$ such that

$$P_{c\varepsilon}(z) \subset bP_\varepsilon(z), \qquad cP_\varepsilon(z) \subset P_{b\varepsilon}(z)$$

for $z \in D$ and $\varepsilon > 0$.

If $P_\varepsilon(w) \cap P_\varepsilon(z) \neq \emptyset$, then

$$P_\varepsilon(w) \subset C P_\varepsilon(z), \tag{10}$$

with a uniform constant C for each $\varepsilon > 0$. There exists $c_6 > 0$ such that

$$P_{c_6\,\varrho(z)}(z) \subset\subset D, \tag{11}$$

for each $z \in D$. In order to simplify the notation, we will assume that this constant is equal to 1. Also,

$$\tau(w, v, \varepsilon) \sim \tau(z, v, \varepsilon)$$

uniformly for $w \in P_\varepsilon(z)$. Since $\tau_1(\zeta, \varepsilon) \sim \varepsilon$, it follows from (11) that if $z \in P_{c_6\,\varrho(\zeta)}(\zeta)$, then $\varrho(z) \sim \varrho(\zeta)$.

Another important fact is that

$$V(P_\varepsilon(w)) \sim V(P_\varepsilon(z)), \tag{12}$$

if $w \in P_\varepsilon(z)$. Naturally, $V(P_\varepsilon(z))$ stands for the euclidean volume of $P_\varepsilon(z)$.

Remark 1. In general, it is not true that if $\delta < \varepsilon$, then $P_\delta(z) \subset P_\varepsilon(z)$. This is the reason for a specific construction of a cover $\{P^i\}_{i\in\mathbb{N}}$ below. This fact is also important in the proof of Theorem 6 below.

Recall that $\mathrm{BMO}(D)$ is the space of all locally integrable functions in D such that

$$\sup_P \frac{1}{V(P)} \int_P |g - g_P| < \infty,$$

where the supremum above is taken over the family of all polydisks $P = P_\varepsilon(\zeta)$ centred at $\zeta \in D$ with $\varepsilon \leq c\varrho(\zeta)$.

Assumption 4. *Given $P = P_\varepsilon(\zeta)$ we denote by P^* the polydisk $P_{b(2)\varepsilon}(\zeta)$, where $b(2)$ is the constant which satisfies the following property*

$$2P_\varepsilon(\zeta) \subset P_{b(2)\varepsilon}(\zeta).$$

We will assume that c in the definition of the space BMO *is chosen in such a way that*:

(i) *for any $P = P_\varepsilon(\zeta)$, it holds $P^* \subset\subset D$,*
(ii) *for any $P = P_\varepsilon(\zeta)$ and $z \in P$, it holds $CP_\varepsilon(z) \subset\subset D$, where C here stands for the constant in the engulfing property* (10).

Next, we recall the concept of the multitype from [7]. Let Γ_n be the set of all n-tuples $\Lambda = (\lambda_1, \dots, \lambda_n)$ of elements of closed real line such that:

(1) $-\infty < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \infty$,

(2) for each k either λ_k is infinite, or there is a set of nonnegative integers $\{a_1, \dots, a_n\}$ with $a_k > 0$, such that

$$\sum_{j=1}^{k} \frac{a_j}{\lambda_j} = 1.$$

The set Γ_n is called the set of weights. It can be ordered lexicographically. A weight $\Lambda \in \Gamma_n$ is called distinguished if there exist holomorphic coordinates $(z_1, \dots, z_n)$ about p such that p is mapped to the origin and such that

$$\sum_{j=1}^{n} \frac{\alpha_j + \beta_j}{\lambda_j} < 1 \quad \Longrightarrow \quad \frac{\partial^{|\alpha|+|\beta|} r(p)}{\partial z_1^{\alpha_1} \, \partial \bar{z}_1^{\beta_1} \cdots \partial z_n^{\alpha_n} \, \partial \bar{z}_n^{\beta_n}} = 0.$$

Definition 1. The multitype $\mathcal{M}(bD, p)$ of a point p is defined to be the lexicographically smallest weight

$$\mathcal{M} = (m_1(p), \dots, m_n(p)) \in \Gamma_n$$

such that $\mathcal{M} \geq \Lambda$ for every distinguished weight Λ.

The following fact was proved in [18].

Theorem 7 (Corollary 2.21 in [18]). *Assume that D is a smoothly bounded convex domain of finite type in $\mathbb{C}^n, n > 1$ and $(m_1, \dots, m_n)$ is the multitype of bD at ζ, then*

$$V(P_\varepsilon(\zeta)) \sim \varepsilon^{\sum_{j=1}^{n} (2/m_j)},$$

with uniform constants.

Crucial to our analysis is the construction of support functions from [8]. Although we shall rather make use of estimates on S and the corresponding Leray decomposition, we recall the result below.

Theorem 8 (Diederich, Fornaess). *Let $D \subset\subset \mathbb{C}^n$ be a smooth convex domain of finite type M and r a convex defining function of D in a neighbourhood U of bD. Then the function $S(z, \zeta) \in C^\infty(\bar{D} \times U)$, holomorphic in z, constructed in [9], has the following property:*

Let, for $\zeta \in U, n_\zeta$ denote the outer unit normal to the level set $\{r = r(\zeta)\}$ and let v be any unit vector complex tangential to this level set at ζ.

Define

$$a_{\alpha\beta}(\zeta, v) := \frac{\partial^{\alpha+\beta}}{\partial \lambda^\alpha \, \partial \bar{\lambda}^\beta} r(\zeta + \lambda v) \bigg|_{\lambda = 0}.$$

Then there are constants $K, c, d > 0$, such that one has for all points z written as $z = \zeta + \mu n_\zeta + \lambda v$ with $\mu, \lambda \in \mathbb{C}$ the estimate

$$\Re S(z, \zeta) \leq -\left| \frac{\Re \mu}{2} \right| - \frac{K}{2} (\Im \mu)^2 - c \sum_{j=2}^{M} \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta, v)| |\lambda|^j$$
$$+ d \sup\{0, r(z) - r(\zeta)\}.$$

Next, one defines C^∞ functions $Q_j : \bar{D} \times \bar{D} \to \mathbb{C}$ such that

$$S(z,\zeta) = \sum_{j=1}^{n} Q_j(z,\zeta)(z_j - \zeta_j)$$

and Q_j are holomorphic in the first variable. We will denote the corresponding Leray form by Q, that is,

$$Q = \sum_{j=1}^{n} Q_j \, d\zeta_j. \tag{13}$$

Making use of the non-isotropic polydisks $P_\varepsilon(\zeta)$ one constructs suitable covers. Namely, there exist uniform constants $C, c > 0$ such that

$$P^i = P^i_\varepsilon(\zeta) := CP_{2^i\varepsilon}(\zeta) \setminus cP_{2^{i-1}\varepsilon}(\zeta),$$

$i \in \mathbb{Z}, i \lesssim \lceil \log_2 \frac{\varepsilon_0}{\varepsilon} \rceil$ is a cover of $P_{\varepsilon_0}(\zeta)$. Also, the following estimate was proved first in [9] but we repeat the formulation after [13] in a form more appropriate for our study.

Lemma 1 (Lemma 3.2 in [13]). *For all $z, \zeta \in U$, where U stands for some open neighbourhood of bD, it holds*

$$|S(z,\zeta)| \gtrsim \varepsilon$$

for $\zeta \in P^0_\varepsilon(z)$ or $z \in P^0_\varepsilon(\zeta)$.

We will make heavy use of estimate on $\overline{\partial} Q$ in the ε-extremal coordinates. Such estimates were obtained in [9] for the first time (Theorem 5.1 *ibid.*), also in [13] (Lemma 3.3, Lemma 3.4 *ibid.*). However, since in our case both variables may change, we shall refer to Lemma 3.1 in [14] most often.

Lemma 2 (Lemma 3.1 in [14]). *Let Q be the form defined in (13) and $\zeta_0 \in D$ sufficiently close to bD. Assume that the ε-extremal coordinates have been chosen at ζ_0 and that Φ^* is the unitary transformation such that $w^* = \Phi^*(z - \zeta_0)$. Also, let $\eta^* = \Phi^*(\zeta - \zeta_0)$ and define*

$$Q^*(w^*, \eta^*) := \bar{\Phi}^* Q\big(\zeta_0 + (\bar{\Phi}^*)^T w^*, \zeta_0 + (\bar{\Phi}^*)^T \eta^*\big). \tag{14}$$

There exists a uniform constant C such that

$$|Q^*_k(w^*, \eta^*)| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)},$$

$$\left| \frac{\partial}{w^*_i} Q^*_k(w^*, \eta^*) \right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)\tau_i(\zeta_0, \varepsilon)},$$

$$\left| \frac{\partial}{\partial \bar{\eta}_j} Q_k(w^*, \eta^*) \right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)\tau_j(\zeta_0, \varepsilon)},$$

$$\left| \frac{\partial^2}{\partial w^*_i \, \partial \eta^*_j} Q^*_k(w^*, \eta^*) \right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)\tau_i(\zeta_0, \varepsilon)\tau_j(\zeta_0, \varepsilon)},$$

provided $|\eta_j| \leq C\tau_j(\zeta_0, \varepsilon)$ *for all* j, $|w_1^*| \leq C$ *and* $|w_j| \leq C\tau_j(\zeta_0, \varepsilon)$ *for* $j = 2, \ldots, n$.

This lemma was proved in [14] under the assumption that $\zeta_0 \in bD$. However, the proof works also for ζ_0 sufficiently close to bD. We shall refer to estimates for Q given in Lemma 2 as estimates of Q at a point ζ_0.

3. Proofs and auxiliary results

As was stated in the Introduction, we need to construct a Whitney type cover of $A \cap D$ consisting of the non-isotropic polydisks.

PROPOSITION 1. *Assume that* D *is a smoothly bounded convex domain of finite type in* $\mathbb{C}^n, n > 1$ *and* A *is a subvariety satisfying Assumption* 1.

For each $C_3 > 0$ *and for each* c_2, C_2, c_5, C_5 *there exists a family of non-isotropic polydisks* $\mathfrak{P}$ *and positive constants* $C_1, C_4 > 0$ *such that*:

(i) *The polydisks* $P \in \mathfrak{P}$ *are disjoint.*

(ii) *The polydisks* $C_1 P, P \in \mathfrak{P}$ *form a cover of* $A \cap D$ *and* $100C_1 < C_3$.

(iii) *For any* $P = P_\varepsilon(z) \in \mathfrak{P}$

$$c_2 \varepsilon \leq \operatorname{dist}\big(z, b(D \cap A)\big) \sim \operatorname{dist}\big(P, b(D \cap A)\big) \leq C_2 \varepsilon.$$

(iv)

$$\sup_{\eta \in D} \#\{P \in C_3 \mathfrak{P} : P \ni \eta\} < C_4.$$

(v) *For any* $P = P_\varepsilon(z) \in \mathfrak{P}$

$$c_5 \varepsilon \leq \operatorname{dist}\big(z, b(D \cap A)\big) \sim \operatorname{dist}\big(C_3 P, b(D \cap A)\big) \leq C_5 \varepsilon.$$

Proof. Fix now $C_3 > 0$ and define a family of all polydisks

$$P_{c\varrho(z)}(z) \tag{15}$$

with $z \in D \cap A$ and c small enough to guarantee that $C_3 P_{c\varrho(z)}(z) \subset\subset D$. Let $\bar{\mathfrak{P}}$ be a family of all those polydisks. Notice that we can choose c small enough in order to guarantee that for any c_5, C_5

$$c_5 \varepsilon \leq \operatorname{dist}\big(C_3 P, b(D \cap A)\big) \leq C_5 \varepsilon$$

for any $P = P_\varepsilon \in \bar{\mathfrak{P}}$. This follows from comments which appear in Preliminaries after (11). As a result, the constant c in the definition of $P_{c\varrho(z)}(z)$ can be chosen in such a way that for any c_2, C_2

$$c_2 \varepsilon \leq \operatorname{dist}\big(P_\varepsilon(z), b(D \cap A)\big) \leq C_2 \varepsilon.$$

Next, we pick a subcollection $\mathfrak{P} \subset \bar{\mathfrak{P}}$ in the following way. Define

$$s_0 = \sup\{\varepsilon : P = P_\varepsilon \in \bar{\mathfrak{P}}\}$$

and choose $P^0 = P_{\varepsilon_0}$ in such a way that $\varepsilon_0 \geq s_0/2$. Assuming that $P^0, \dots, P^{k-1}$ have been chosen define

$$s_k = \sup\left\{\varepsilon : P = P_\varepsilon \in \bar{\mathfrak{P}}, P \cap \bigcup_{i\leq k-1} P^i = \emptyset\right\}$$

and choose $P^k = P_\varepsilon$ with $\varepsilon \geq s_k/2$. This completes the construction of the family

$$\mathfrak{P} = \{P_{\varepsilon_1}(z_1), P_{\varepsilon_2}(z_2), \dots\}.$$

Notice that properties (i), (iii) and (v) are satisfied automatically.

We will show that there exists C_1 such that $C_1 P$ is a cover of $D \cap A$. In order to prove this pick $z \in D \cap A$ and consider the polydisk $P := P_{c\varrho(z)}(z)$ with c exactly the same as in (15) and denote $\rho = c\varrho(z)$. Pick the largest natural number k such that $\varepsilon_k \geq \rho/2$. Such a number k does exist since ε_k tends to 0, since

$$V(D) \geq \sum_{i=1}^{\infty} V(P_{\varepsilon_i}(z_i)) \gtrsim \sum_{i=1}^{\infty} \varepsilon_i^{n+1}.$$

Here, we made use of the fact that $V(P_\varepsilon(z)) \sim \prod_{i=1}^n \tau_i^2(z,\varepsilon)$ and the estimates $\tau_1(z,\varepsilon) \sim \varepsilon, \tau_i(z,\varepsilon) \lesssim \varepsilon^{1/2}, i \geq 2$.

With this choice of k, it must hold

$$P \cap \bigcup_{i=1}^{k} P^i \neq \emptyset. \tag{16}$$

Indeed, if (16) does not hold, then

$$\varepsilon_{k+1} \geq \frac{s_{k+1}}{2} \geq \frac{\rho}{2},$$

the latter inequality being a consequence of definition of s_{k+1} and (16). This is impossible in view of definition of the number k.

This implies that there exists ζ and $P_{\varepsilon_i}(z_i)$ such that

$$\delta(z_i, \zeta) \leq \varepsilon_i,$$
$$\delta(z, \zeta) \leq \rho.$$

As a result,

$$\delta(z, z_i) \leq C(\rho + \varepsilon_i) \leq 3C\varepsilon_i, \tag{17}$$

which means that there exists C_1 such that $z \in C_1 P^i$. This completes the proof of (ii). Notice that we may assume that for instance $100C_1 < C_3$ changing if necessary c in (15), since the constant C in (17) is uniform.

To complete the proof, it suffices that to show (iv). In order to accomplish this task, observe that

$$\delta(z) \sim \operatorname{dist}\big(z, b(D \cap A)\big) \sim \operatorname{dist}(z, bD), \tag{18}$$

where

$$\delta(z) := \inf\{\delta(z,\zeta), \zeta \in bD\}.$$

One should notice that it is here where we make use of Assumption 1.

Indeed, there exist constants c_6 and C_6 such that for any $z \in D$

$$P_{c_6\varrho(z)}(z) \cap bD = \emptyset \quad \text{and} \quad P_{C_6\varrho(z)}(z) \cap bD \neq \emptyset.$$

Both statements are consequences of (11) and comments which follow after (11).

The first property implies that for any $\zeta \in bD$, it holds $\delta(z,\zeta) \geq c_6\varrho(z)$. Thus, obviously $\delta(z) \geq c_6\varrho(z)$. As for the second, if $\zeta \in P_{C_6\varrho(z)}(z) \cap bD$, then

$$\begin{aligned}\delta(z) &\leq \delta(z,\zeta) = \inf\{\varepsilon > 0 : P_\varepsilon(z) \ni \zeta\}\\ &\leq C_6\varrho(z).\end{aligned}$$

This justifies (18).

We intend now to complete the proof of (iv). Fix $\eta \in D \cap A$ and denote $t = \operatorname{dist}(\eta, b(D \cap A))$. Let $P = P_\rho(z) \in \mathfrak{P}$ satisfy $\eta \in C_3P$. We have

$$c_2\rho \leq d\big(P_\rho(z), b(D \cap A)\big) \sim d\big(z, b(D \cap A)\big) \leq C\delta(z) \leq C\delta(z,\xi)$$

for any $\xi \in bD$. Thus,

$$c_2\rho \leq C\delta(z,\xi) \leq C\big(\delta(z,\eta) + \delta(\eta,\xi)\big)$$

and, as a result,

$$\begin{aligned}c_2\rho &\leq C\big(\delta(z,\eta) + \delta(\eta)\big) \leq C\delta(z,\eta) + C\operatorname{dist}\big(\eta, b(D \cap A)\big)\\ &\leq CC_3\rho + Ct.\end{aligned}$$

Also,

$$t = \operatorname{dist}\big(\eta, b(D \cap A)\big) \leq C\delta(\eta) \leq C\delta(\eta,z) + C\delta(z,\xi)$$

with $\xi \in bD$, and

$$t \leq CC_3\rho + C\operatorname{dist}\big(z, b(D \cap A)\big) \leq CC_3\rho + CC_2\rho.$$

Thus, if $\eta \in C_3P_\rho(z)$, then

$$\rho \geq (CC_3 + CC_2)^{-1}t. \tag{19}$$

Observe that for any $\zeta \in P = P_\rho(z)$ with $\eta \in C_3P$

$$\delta(\zeta,\eta) \leq C\delta(\zeta,z) + C\delta(z,\eta) \leq C\rho + CC_3\rho$$

and $C^{-1}(c_2 - CC_3)\rho \leq t$. Recall at this moment that we can make c_5, C_5 and also c_2 arbitrary large without affecting C_3 whatsoever.

Denote

$$I = \{i \in \mathbb{N} : \eta \in P^i = P_{\varepsilon_i}(z_i) \in \mathfrak{P}\}$$

and notice that we have proved so far that for any $i, j \in I$

$$\delta(z_i, z_j) \leq C\delta(z_i,\eta) + C\delta(\eta,z_j) \leq C(1+C_3)\varepsilon_i + C(1+C_3)\varepsilon_j \leq Ct.$$

This means that for a universal constant C and any $i, j \in I$

$$P_{\varepsilon_i}(z_i) \subset P_{Ct}(z_j).$$

As a result, for any $j \in I$

$$\sum_{i\in I} V(P_{\varepsilon_i}(z_i)) \le V(P_{Ct}(z_j)),$$

because $P_{\varepsilon_i}(z_i), \varepsilon_i = c\varrho(z_i)$ were chosen in Proposition 1 to be disjoint. It is worth repeating at this moment that while C depends on C_3, it does not depend on η.

For any $j \in I$, it holds

$$\sum_{i\in I} V(P_{\varepsilon_i}(z_i)) \le V(P_{Ct}(z_j)) \sim (Ct)^{\sum_{k=1}^{n}(2/m_k^j)}, \tag{20}$$

where $(m_1^j, \ldots, m_n^j)$ stands for the multitype at z_j. This implies that

$$\sum_{i\in I} V(P_{\varepsilon_i}(z_i)) \le C \inf\left\{(Ct)^{\sum_{k=1}^{n}(2/m_k^j)} : j\in I\right\} \le C(Ct)^s,$$

where

$$s := \sup\left\{\sum_{k=1}^{n} \frac{2}{m_k^j} : j \in I\right\}. \tag{21}$$

As for the left-hand side of (20), observe

$$\sum_{i\in I} V(P_{\varepsilon_i}(z_i)) \gtrsim C\sum_{i\in I} \varepsilon_i^{\sum_k (2/m_k^i)} \gtrsim \sum_{i\in I} t^{\sum_k (2/m_k^i)}$$

in view on estimates on ρ proved in (19). Notice that for any $c > 0$ there exists $C > 0$ such that $x^s \le Cx^{\tilde{s}}$ for any $x \in [0, c]$ and $\tilde{s} \ge 2 + \frac{2(n-1)}{M}$ – the number s was defined in (21). Therefore, since D is bounded we may write

$$C(Ct)^s \ge \operatorname{card} I \cdot (\varepsilon)^s \gtrsim \operatorname{card} I \cdot t^s.$$

This implies that

$$\operatorname{card} I \lesssim C^s,$$

with uniform constants, where as above

$$s = \sup\left\{\sum_{k=1}^{n} \frac{2}{m_k^j} : j \in I\right\}.$$

Therefore, as might have been expected $\operatorname{card} I \lesssim C^{n+1}$. $\square$

Before we prove Theorem 3, we use Proposition 1 to show the following easy observation.

COROLLARY 1. *Assume that D is a smoothly bounded convex domain of finite type in $\mathbb{C}^n, n>1$ and A satisfies Assumption 1. Let μ be a positive Borel measure supported on $D\cap A$, which satisfies condition (7) from Theorem 3. Then μ is a finite measure.*

Proof. Choose a Whitney cover $\mathfrak{P}$ as in Proposition 1 and observe that

$$\mu(D\cap A)\le C\sum_{P\in C_1\mathfrak{P}}\mu(P)\le C\sum_{P\in C_1\mathfrak{P}}V(P)\le CV(D),$$

which proves the claim. ☐

Proof of Theorem 3. Let $1\le p<\infty$ and consider a cover $\mathfrak{P}=\{P_{c\varrho(z_k)}(z_k)\}_{k\in\mathbb{N}}$ from Proposition 1. Without loss of generality, we may assume that c in the definition of $\mathfrak{P}$ is small enough to guarantee that for $P\in C_1\mathfrak{P}$ condition (7) holds true, that is,

$$\sup\left\{\frac{\mu(P\cap A)}{V(P)}:P\in C_1\mathfrak{P}\right\}<\infty.$$

For $g=R_AG$, we have

$$\begin{aligned}\int_{D\cap A}|g|^p\varrho^\kappa\,d\mu&=\int_{D\cap A}|G|^p\varrho^\kappa\,d\mu\le C\sum_{P\in C_1\mathfrak{P}}\int_{P\cap A}|G|^p\varrho^\kappa\,d\mu\\&\le C\sum_{P\in C_1\mathfrak{P}}\sup\left\{\frac{1}{\varrho^\kappa(\tilde P)V(\tilde P)}\int_{\tilde P}|G|^p\varrho^\kappa\,dV\right\}\mu_\kappa(P\cap A),\end{aligned}$$

where the supremum for a given $P\in C_1\mathfrak{P}$ is taken over all polydisks $\tilde P$ centred at a point $\zeta\in P$ with radius equal to $\tilde c\varrho(P)$. This time the constant $\tilde c$ is chosen in such a way that $\tilde P\subset\subset D$ and $\tilde P\subset C_3P$, where C_3 is a constant from Proposition 1. The symbol $\tilde{\mathfrak{P}}(P)$ below with $P\in\mathfrak{P}$ stands for the family of all polydisks $\tilde P=P_\varepsilon(\zeta)$ with $\zeta\in P$ and $\varepsilon=\tilde c\varrho(P)$. Thus, if condition (7) holds, then

$$\begin{aligned}(22)\quad&\int_{D\cap A}|g|^p\varrho^\kappa\,d\mu\\&\le C\sup\left\{\frac{\mu(P\cap A)}{V(\tilde P)}:P\in\mathfrak{P},\tilde P\in\tilde{\mathfrak{P}}(P)\right\}\sum_{P\in C_1\mathfrak{P}}\int_{C_3P}|G|^p\varrho^\kappa\,dV\\&\le C\sup\left\{\frac{\mu(P\cap A)}{V(P)}:P\in\mathfrak{P}\right\}\int_D|G|^p\varrho^\kappa\,dV\\&\le C\int_D|G|^p\varrho^\kappa\,dV\end{aligned}$$

with uniform constants. The second inequality is a consequence of Proposition 1. Also, for any $\zeta\in P_\varepsilon(z)$, it holds $V(P_\varepsilon(z))\sim V(P_\varepsilon(\zeta))$ and, as a result, if $\zeta\in P\in\mathfrak{P}$ and $P=P_\varepsilon$, then

$$V(P_\varepsilon(\zeta))\sim V(P).$$

On the other hand, $V(P_\varepsilon(z)) \sim V(P_{c\varepsilon}(z))$ with constants which depend only on c—this follows from Theorem 7. This shows that $V(\tilde{P}) \sim V(P)$ for any $P \in \mathfrak{P}$ and explains the second inequality in (22). We proved that condition (7) implies that

$$R_A[H^{\kappa,p}(D)] \subset \left\{ g \in H(D \cap A) : \int_{D\cap A} |g|^p \varrho^\kappa \, d\mu < \infty \right\}.$$

Now we intend to show the inverse implication provided $p = 2$ and $\kappa = 0$. Assume that $R_A[H^2(D)] \subset H^2(D \cap A, \mu)$ for a positive Borel measure μ on $D \cap A$. This implies, by the closed graph theorem that R_A maps continuously $H^2(D)$ into $H^2(D \cap A, \mu)$. The proof of this part of the theorem will be completed once we know that there exist constants $c_1, c_2 > 0$ such that for each $\zeta \in D$ there is a function $G \in H^2(D)$ of norm 1 such that

$$|G(z)|^2 \geq c_1 \frac{1}{V(P_\varrho(\zeta))} \tag{23}$$

for each $z \in P_{c_2\varrho(\zeta)}(\zeta)$. Indeed, if such functions exist then for each $\zeta \in D$

$$c_1 \frac{\mu(P_{c_2\varrho}(\zeta) \cap A)}{V(P_\varrho(\zeta))} \leq \int_{P_{c_2\varrho}(\zeta)\cap A} |G|^2 \, d\mu \leq \int_{D\cap A} |G|^2 \, d\mu \leq C \int_D |G|^2 \, dV \leq C,$$

with uniform $C > 0$, which completes the proof. Thus, it suffices to show existence of functions satisfying (23). An obvious candidate is the Bergman kernel $B(\cdot, \zeta)$. Namely, define for a fixed $\zeta \in D$ the function G as

$$G(z) := \frac{B(z,\zeta)}{\sqrt{B(\zeta,\zeta)}}.$$

Obviously G is of norm 1 in $H^2(D)$. Also, it is a consequence of Theorem 3.4 in [22] that G satisfies (23) when $z = \zeta$. Theorem 5.2 *ibid.* gives a bound for derivatives of B, which suffices to prove existence of c_1, c_2 for which (23) holds on $P_{c_2\varrho}$. □

Now we intend to prove Theorems 4 and 6. First, decompose f as

$$f(z) - f(\zeta) = \sum_{i=1}^n h_i(\zeta, z)(z_i - \zeta_i) \tag{24}$$

with holomorphic functions h_i. Since D is convex in order to obtain functions h_i, we may simply write

$$\begin{aligned} f(z) - f(\zeta) &= \int_0^1 \frac{d}{dt} f\big(\zeta + t(z-\zeta)\big) \, dt \\ &= \sum_{i=1}^n \int_0^1 \partial_i f\big(\zeta + t(z-\zeta)\big) \, dt \cdot (z_i - \zeta_i). \end{aligned}$$

Obviously, functions h_i are smooth in both variables in $\bar{D}$. Denote by S the support function constructed in [8]. Also, let ϕ be a smooth cut-off function supported in some compact subset of D and identically equal to 0 outside a larger compact subset of D. Define

$$\mathcal{Q}(\zeta,z)=\frac{1}{r(\zeta)}\left(\big(1-\phi(\zeta)\big)\sum_{i=1}^{n}Q_i(z,\zeta)\,d\zeta_i+\phi(\zeta)\sum_{i=1}^{n}\frac{\partial r}{\partial\zeta_i}(\zeta)\,d\zeta_i\right).$$

With this notation, one has the following result.

PROPOSITION 2 ([3]). *For any $N>1$ the integral operator E^N defined on any function $g\in H^\infty(D\cap A)$ for any $z\in D$ by*

$$\begin{aligned}&E^N g(z)\\&\quad:=\int_{D\cap A}g(\zeta)\\&\qquad\times\left[(dV)^{\#}\rfloor\frac{r^{N+n-1}(\zeta)}{(r(\zeta)+(1-\phi(\zeta))S(z,\zeta)+\phi(\zeta)\sum_{i=1}^{n}\frac{\partial r}{\partial\zeta_i}(\zeta)(\zeta_i-z_i))^{N+n-1}}\right.\\&\qquad\left.\times(\overline{\partial}\mathcal{Q}(\zeta,z))^{n-1}\wedge\frac{\overline{\partial f(\zeta)}\wedge\sum_{i=1}^{n}h_i(\zeta,z)\,d\zeta_i}{|\partial f|^2}\right]dV_A(\zeta)\end{aligned}$$

is a linear extension operator. Furthermore, the function $E^N g$ is continuous on $\bar{D}\setminus(bD\cap A)$.

We repeated the formulation after [11]. The symbol $\rfloor$ above stands for the interior multiplication. Also, $\#$ has the same meaning here as in differential geometry, that is, $dV^{\#}$ is the unique element of the exterior power ΛTD of the tangent bundle such that $(dV^{\#},dV)=1$, where here $(\cdot,\cdot)$ stands for the duality between ΛTD and $\Lambda^{*}TD$ induced by the standard Euclidean metric.

Observe that g in Proposition 2 is assumed to be bounded, which is not the case in our situation. Obviously, this is not an obstacle if A is affine, since we may assume that $0\in D\cap A$ and consider functions $g_t(\zeta):=g(t\zeta), t<1$. This suffices to complete the proof in that case, since D is convex and $g_t\to g$ in $L^p(\nu), p<\infty$ for the measures which we will consider. This argument does not work in case of a general variety A.

Instead, we may shrink the domain only, without changing A or g, in order to be able to apply results from [3]. This is why we will divide the proof of Theorem 4 into two steps. First, we will show that E^N from Proposition 4 maps boundedly $L^1(D\cap A,\nu_N)$ into $L^1(D)$. This fact will be used in the next step to prove that $R_AE^N=\mathrm{id}$ on $H^1(D\cap A,\nu_N)$, which also implies that $R_AE^N=\mathrm{id}$ on $H^p(D\cap A,\nu_N)$ for $1<p<\infty$ since the measures ν_N are finite (cf. Corollary 1). It is worth mentioning that this argument works because the assumption $g\in H^1(D\cap A,\nu_N)$ and the fact

$$H^1(D\cap A,\nu_N)\subset H^1(D\cap A,\varrho^{2/M}dV_A)$$

provides us with a bound on the growth of g in terms of ϱ.

Proof of Theorem 4. We will show that the operator E^N maps boundedly $L^1(D\cap A,\nu_N)$ into $L^1(D)$ for a suitably defined measure ν_N. Importantly, at this moment we do not claim that E^N is an extension operator on $H^1(D\cap A,\nu_N)$ with values in $H^1(D)$.

We have

$$\begin{aligned}\int_D\left|\int_{D\cap A}g(\zeta)E^N(\zeta,z)\,dV_A(\zeta)\right|dV(z)\\ &\leq\sum_{i=0}^{\infty}\int_{D\cap A}|g(\zeta)|\int_{P^{-i}(\zeta)}|E^N(\zeta,z)|\,dV(z)\,dV_A(\zeta)\\ &\quad+\int_{D\cap A}|g(\zeta)|\sum_{i=1}^{C\lceil\log_2(1/\varrho)\rceil}\int_{P^i(\zeta)}|E^N(\zeta,z)|\,dV(z)\,dV_A(\zeta),\end{aligned}$$

where $P^i=P^i(\zeta):=CP_{2^i\varrho}(\zeta)\setminus cP_{2^{i-1}\varrho}(\zeta), i\in\mathbb{N}$ and, as usually, $\varrho=\varrho(\zeta)$. For a fixed $\zeta\in D\cap A$ the kernel E^N can be estimated in the following way

$$\begin{aligned}(25)\quad |E^N(\zeta,z)|&\lesssim\sum_{l,k}\frac{\varrho^N}{(\varrho+2^i\varrho)^{N+n-1}}\\ &\qquad\times\frac{(2^i\varrho)^{n-1}}{\prod_{\alpha\neq l}\tau_\alpha(\zeta,2^i\varrho)\prod_{\beta\neq k}\tau_\beta(\zeta,2^i\varrho)}|\overline{Z_l^{\zeta,i}f(\zeta)}h_k(\zeta,z)|\\ &\lesssim\frac{1}{2^{iN}}\sum_l\left(\frac{|Z_l^{\zeta,i}f(\zeta)|^2}{\prod_{\alpha\neq l}\tau_\alpha^2(\zeta,2^i\varrho)}+\frac{|h_l(\zeta,z)|^2}{\prod_{\beta\neq l}\tau_\beta^2(\zeta,2^i\varrho)}\right)\end{aligned}$$

provided $z\in P^i(\zeta)$. The symbol $Z_l^{\zeta,i}$ stands for the derivative

$$Z_l^{\zeta,i}f(\zeta):=\frac{\partial}{\partial\lambda}f(\zeta+\lambda v_l)\bigg|_{\lambda=0},$$

where v_l is the lth vector of the $(2^i\varrho)$-extremal basis at ζ. Notice that h_l in (25) is again the lth coordinate in the decomposition (24) of f in the $(2^i\varrho)$-extremal basis.

Remark 2 (Technical assumption). In order to simplify the notation, we will suppress denoting dependence of derivatives either on ζ or i. Thus, we will often simply write ∂_l below instead of $Z_l^{\zeta,i}$ once ζ and i are fixed.

In order to obtain estimate (25), one needs to express the kernel in $(2^i\varrho)$-extremal coordinates at ζ choosing a suitable unitary transformation as in (14) and invoke Lemma 1 and Lemma 2. We will proceed in a similar way in other estimates from below without an explicit reference either to Lemma 1 or Lemma 2.

We obtain

$$\int_{P^i} |E^N(\zeta,z)|\,dV \\ \le \frac{1}{2^{iN}} \sum_l \biggl(\frac{|\partial_l f(\zeta)|^2 V(P^i)}{\prod_{\alpha\ne l} \tau_\alpha(\zeta,2^i\varrho)} + \frac{1}{\prod_{\beta\ne l}\tau_\beta(\zeta,2^i\varrho)} \int_{P^i} |h_l(\zeta,z)|^2\,dV \biggr).$$

As for the first term, we may write

$$\frac{1}{2^{iN}} \sum_l \frac{|\partial_l f(\zeta)|^2 V(P^i)}{\prod_{\alpha\ne l}\tau_\alpha(\zeta,2^i\varrho)} \le \frac{1}{2^{iN}} \sum_l |\partial_l f(\zeta)|^2 \tau_l^2(\zeta,2^i\varrho) \lesssim \frac{1}{2^{i(N-2/M)}} |\partial f|_\kappa^2,$$

where $|\omega|_\kappa$ stands for the non-isotropic norm of $(1,0)$-covector ω, which is defined in the following way

$$\Bigl| \sum \omega_k \bigl(Z_k^{\zeta,\varrho(\zeta)}\bigr)^* \Bigr|_\kappa (\zeta) := \Bigl(\sum |\omega_\kappa|^2 \tau_k^2(\zeta,\varrho) \Bigr)^{1/2}$$

where the symbol $(Z_k^{\zeta,\varrho(\zeta)})^*$ stands for the covector dual to $Z_k^{\zeta,\varrho(\zeta)}$.

If we denote

$$d\mathfrak{h}_+ := \Biggl[\sum_l \sum_{i=1}^{\lceil \log_2 (1/\varrho) \rceil} \frac{1}{2^{iN}} \frac{1}{\prod_{\beta\ne l} \tau_\beta^2(\zeta,2^i\varrho)} \int_{P^i} |h_k(\zeta,z)|^2\,dV \Biggr] dV_A,$$

then we obtain the following estimate

$$\sum_{i=1}^{\lceil \log_2 (1/\varrho) \rceil} \int_{D\cap A} |g(\zeta)| \int_{P^i} |E^N(\zeta,z)|\,dV\,dV_A \\ \le \int_{D\cap A} |g(\zeta)|\cdot|\partial f|_\kappa^2\,dV_A + \int_{D\cap A} |g(\zeta)|\,d\mathfrak{h}_+.$$

In $P^{-i} := CP_{2^{-i}\varrho}(\zeta) \setminus cP_{2^{-i-1}\varrho}(\zeta)$, we can estimate the kernel E^N in a similar way. Namely,

$$|E^N(\zeta,z)| \le \sum_l \frac{\varrho^N}{\varrho^{N+n-1}} \frac{(2^{-i}\varrho)^{n-1}}{\prod_{\alpha\ne l}\tau_\alpha^2(\zeta,2^{-i}\varrho)} \bigl(|\partial_l f(\zeta)|^2 + |h_l(\zeta,z)|^2 \bigr) \\ \lesssim \frac{1}{2^{i(n-1)}} \sum_l \frac{|\partial_l f(\zeta)|^2 + |h_l(\zeta,z)|^2}{\prod_{\alpha\ne l}\tau_\alpha^2(\zeta,2^{-i}\varrho)},$$

and, as a result,

$$\int_{P^{-i}(\zeta)} |E^N(\zeta,z)|\,dV \\ \le \frac{1}{2^{i(n-1)}} \sum_l \biggl(|\partial f(\zeta)|_\kappa^2 + \frac{1}{\prod_{\alpha\ne l}\tau_\alpha^2(\zeta,2^{-i}\varrho)} \int_{P^{-i}} |h_l(\zeta,z)|^2\,dV \biggr).$$

To sum this up, we have proved

$$\int_D |E^N g(z)|\, dV \leq C \int_{D\cap A} |g(\zeta)||\partial f|_\kappa^2(\zeta)\, dV + \int_{D\cap A} |g(\zeta)|\mathfrak{h}\, dV_A,$$

where,

$$\mathfrak{h} := \mathfrak{h}_+ + \left[\sum_{i=0}^{\infty} \frac{1}{2^{i(n-1)}} \sum_l \frac{1}{\prod_{\alpha\neq l} \tau_\alpha^2(\zeta, 2^{-i}\varrho)} \int_{P^{-i}(\zeta)} |h_l(\zeta, z)|^2\, dV(z)\right] dV_A.$$

In other words, $E^N : L^1(D\cap A, \nu) \to L^1(D)$, where $d\nu := |\partial f|_\kappa^2\, dV_A + \mathfrak{h}\, dV_A$.

We shall prove the claims concerning the measure ν. For future reference, we single out the following lemma.

LEMMA 3. *Assume that D is a smoothly bounded convex domain of finite type in $\mathbb{C}^n, n > 1$ and A satisfies Assumption 1. Then*

$$\sup\left\{\frac{\mu_f(P\cap A)}{V(P)} : P = P_{\varrho(\zeta)}(\zeta), \zeta \in D\cap A\right\} < \infty,$$

where $d\mu_f = |\partial f|_\kappa^2\, dV_A$.

Proof. Fix P and notice that

$$|\partial f|_\kappa^2(\zeta) \sim \max\{|\partial_l f(\zeta)|^2 \tau_l^2(\zeta, \varrho) : 1 \leq l \leq n\}. \tag{26}$$

Recall that (26) is written according to the convention from Remark 2 and ∂_l in (26) is actually equal to $Z_l^{\zeta,\varrho(\zeta)}$.

Recall that at a given point z with $f(z) = 0$ the volume form dV_A can be estimated by

$$dV_A \lesssim \frac{dx_1\, dy_1 \cdots \widehat{dx_i\, dy_i} \cdots dx_n\, dy_n}{|\bigwedge_{j\neq i}(dx_j \wedge dy_j) \wedge d\Re(f\circ\Phi^*) \wedge d\Im(f\circ\Phi^*)|}, \tag{27}$$

where $v_j^* = dx_j + \sqrt{-1}\, dy_j$ are the ε-extremal coordinates and Φ^* is the corresponding unitary change of coordinates. The same estimate was used in [11] in the proof of extension of H^∞ functions.

We will denote by P_j a subset of P consisting of all $z \in P$ such that

$$0 < \max\{|\partial_l f(z)|^2 \tau_l^2(z, \varrho(z)) : 1 \leq l \leq n\} = |\partial_j f(z)|^2 \tau_j^2(z, \varrho(z)).$$

The first inequality is a consequence of Assumption 1.

Observe that with this notation (27) implies the following estimate

$$\int_P |\partial f|_\kappa^2\, dV_A$$
$$= \sum_{i=1}^{n} \int_{P_i} |\partial f|_\kappa^2\, dV_A$$

$$\lesssim \sum_{i=1}^{n} \int_{\substack{|v_j^*| \le \tau_j(\zeta,\varrho) \\ j \ne i, v^* = v^*(z), z \in P_j}} \frac{|\partial_i f(z)|^2 \tau_i^2(z,\varrho(z))\, dx_1\, dy_1 \cdots \widehat{dx_i\, dy_i} \cdots dx_n\, dy_n}{|\bigwedge_{j\ne i}(dx_j \wedge dy_j) \wedge d\Re(f\circ\Phi^*) \wedge d\Im(f\circ\Phi^*)|}$$

$$\le nC \prod_{i=1}^{n} \tau_i^2(\zeta,\varrho) \sim V(P).$$

This proves the lemma. □

At this moment, it suffices to show that $\mathfrak{h} \lesssim \varrho^{2/M}$.

$$\begin{aligned}
\mathfrak{h}(\zeta) &= \sum_l \sum_{i=1}^{\lceil \log_2(1/\varrho)\rceil} \frac{1}{2^{iN}} \frac{1}{\prod_{\beta\ne l} \tau_\beta^2(\zeta, 2^i\varrho)} \int_{P^i} |h_l(\zeta,z)|^2\, dV(z) \\
&\quad + \sum_l \sum_{i=0}^{\infty} \frac{1}{2^{i(n-1)}} \frac{1}{\prod_{\beta\ne l}\tau_\beta^2(\zeta,2^{-i}\varrho)} \int_{P^{-i}} |h_l(\zeta,z)|^2\, dV(z) \\
&\le C \sum_l \sum_{i=1}^{\lceil \log_2(1/\varrho)\rceil} \frac{\tau_l^2(\zeta,2^i\varrho)}{2^{iN}} + C\sum_l\sum_{i=0}^{\infty} \frac{\tau_l^2(\zeta,2^{-i}\varrho)}{2^{i(n-1)}} \le C\varrho^{2/M}.
\end{aligned}$$

Thus, we have shown that E^N maps $L^1(D\cap A,\nu_N)$ into $L^1(D)$ and the measure ν_N constructed in the proof satisfies the required estimates. We also need to show that E^N is an extension operator on $H^1(D\cap A,\nu_N)$, that is, that $R_A E^N = \mathrm{id}$ on $H^1(D\cap A,\nu_N)$. We intend to prove this fact now. The Reader may want to skip this part of the proof. However, the author feels it is reasonable at least to sketch the argument.

The main motivation behind the integral representations constructed in [3] and [2] was to have kernels, which vanish on the boundary. Not surprisingly, the equality

$$R_A E^N|_{H^1(D\cap A,\nu_N)} = \mathrm{id}|_{H^1(D\cap A,\nu_N)}, \tag{28}$$

follows from the fact that for sufficiently large N a function $\varrho^N g$ is bounded in $D\cap A$ for any $g\in H^1(D\cap A,\nu_N)$.

It order to prove (28), we need to recall the setup from [3] and [11]. First, let $D_t = \{z : r(tz) < 0\}$ for $t > 1$ and define the following forms

$$\begin{aligned}
s(\zeta,z) &= -r(z)\sum_{i=1}^{n}(\bar\zeta_i - \bar z_i)\, d\zeta_i + \big(1-\phi(z)\big)\overline{S(\zeta,z)}\sum_{i=1}^{n} Q_i(\zeta,z)\, d\zeta_i, \\
\mathcal{Q}^1(\zeta,z) &= \frac{1}{r(\zeta)}\left(\big(1-\phi(\zeta)\big)\sum_{i=1}^{n} Q_i(z,\zeta)\, d\zeta_i + \phi(\zeta)\sum_{i=1}^{n}\frac{\partial r}{\partial\zeta_i}(\zeta)\, d\zeta_i\right), \\
\mathcal{Q}^2_\varepsilon(\zeta,z) &= \frac{\overline{f(\zeta)}\sum_{i=1}^{n} h_i(\zeta,z)\, d\zeta_i}{|f|^2+\varepsilon}.
\end{aligned}$$

With these definitions for any $g\in C^1(\bar{D}_t)$ and $z\in D_t$, it holds

$$(29)\quad g(z)=c_n\left(\int_{bD_t} g(\zeta)K_0^\varepsilon(\zeta,z)-\int_{D_t}\overline{\partial}g(\zeta)\wedge K_0^\varepsilon(\zeta,z)-\int_{D_t} g(\zeta)P_0^\varepsilon(\zeta,z)\right),$$

where

$$\begin{aligned}P_0^\varepsilon(\zeta,z) &= C_n\frac{1}{(\langle\mathcal{Q}^1,z-\zeta\rangle+1\rangle)^{N+n}}\frac{f(z)\overline{f(\zeta)}+\varepsilon}{|f(\zeta)|^2+\varepsilon}(\overline{\partial}\mathcal{Q}^1)^n\\ &\quad+D_n\frac{1}{(\langle\mathcal{Q}^1,z-\zeta\rangle+1\rangle)^{N+n-1}}(\overline{\partial}\mathcal{Q}^1)^{n-1}\wedge(\overline{\partial}\mathcal{Q}^2_\varepsilon)\\ &:=P_0^{0,\varepsilon}(\zeta,z)+P_0^{1,\varepsilon}(\zeta,z),\end{aligned}$$

$$\begin{aligned}&K_0^\varepsilon(\zeta,z)\\ &:=\sum_{k=0}^{n-1}A_{n,k}\frac{1}{(\langle\mathcal{Q}^1,z-\zeta\rangle+1)^{N+k}}\frac{f(z)\overline{f(\zeta)}+\varepsilon}{|f(\zeta)|^2+\varepsilon}(\overline{\partial}\mathcal{Q}^1)^k\wedge\frac{s\wedge(\overline{\partial s})^{n-k-1}}{\langle s,\zeta-z\rangle^{n-k}}\\ &\quad+\sum_{k=0}^{n-2}B_{n,k}\frac{1}{(\langle\mathcal{Q}^1,z-\zeta\rangle+1)^{N+k}}(\overline{\partial}\mathcal{Q}^1)^k\wedge\frac{s\wedge(\overline{\partial s})^{n-k-2}}{\langle s,\zeta-z\rangle^{n-k-1}}\wedge(\overline{\partial}\mathcal{Q}^2_\varepsilon)\\ &:=K_0^{0,\varepsilon}(\zeta,z)+K_0^{1,\varepsilon}(\zeta,z).\end{aligned}$$

This corresponds to the choice $G_1(z)=z^{-N}$ and $G_2(z)=z$ in [3].

Consider now a function $g\in H^1(D\cap A,\nu)$, where

$$H^1(D\cap A,\nu)=\left\{g\in H(D\cap A):\int_{D\cap A}|g|\,d\nu<\infty\right\}.$$

We construct now its extension g_γ to a function which is smooth on D. This task is accomplished as follows. It is a consequence of Assumption 1 that we may treat f at each point $p\in bD$ as one of the variables. Taking an appropriate partition of unity, we may also assume, without loss of generality, that f is the last variable z_n. From the fact that $\partial f\wedge\partial r\neq 0$ and convexity of D, it follows that there exits a uniform $a>0$ such that the lense

$$L_a:=\{z:|z_n|\le a\varrho(z),(z_1,\dots,z_{n-1})\in D\cap A\}$$

is contained in D. The function g, defined on $D\cap A$ is now extended to a function on L_a simply by putting $g(z):=g(z_1,\dots,z_{n-1})$. Next, we choose a family of cut-off functions π_γ such that $\pi_\gamma=1$ on $\{x\le\gamma/2\}$ and $\pi_\gamma=0$ if $x\ge\gamma$ and define for sufficiently small γ, $g_\gamma(z)=\pi(\frac{|z_n|^2}{r^2(z)})g(z)$.

Thus, the representation formula holds for g_γ and $t>1,\varepsilon>0$. Therefore,

$$\begin{aligned}&\left|g_\gamma(z)-\int_{D_t}\overline{\partial}g_\gamma(\zeta)\wedge K_0^\varepsilon(\zeta,z)-\int_{D_t}g_\gamma(\zeta)P_0^\varepsilon(\zeta,z)\right|\\ &\quad\le C\int_{bD_t}|g_\gamma(\zeta)K_0^\varepsilon(\zeta,z)|\le\frac{C_\gamma}{\varepsilon}\|g\|_{H^1(D\cap A,\nu_N)}\mathcal{O}(t-1).\end{aligned}$$

The last estimate follows from the fact that the assumption

$$g \in H^1(D \cap A, \nu_N) \subset H^1(D \cap A, \varrho^{2/M}\, dV_A)$$

gives a bound on growth of g_γ and $\overline{\partial} g_\gamma$ in terms of ϱ, which is smaller than the order of vanishing of $K_0^\varepsilon(\zeta, z)$ for $z \in D \cap A$ and ζ approaching bD. The constant C_γ tends to ∞ when $\gamma \to 0$.

As a result, for these ε

$$\begin{aligned} &\left| |g(z) - E^N g_\gamma(z)| - \left| E^N g_\gamma(z) - \int_{D_t} \overline{\partial} g_\gamma(\zeta) \wedge K_0^\varepsilon(\zeta, z) - \int_{D_t} g_\gamma(\zeta) P_0^\varepsilon(\zeta, z) \right| \right| \\ &\quad \le \frac{C_\gamma}{\varepsilon} \|g\|_{H^1(D\cap A, \nu_N)} \mathcal{O}(t-1). \end{aligned}$$

Now, let $\varepsilon_m = \frac{1}{m}$ and $t_m = 1 + \frac{1}{m^2}, m \in \mathbb{N}$ and notice that with this choice we obtain

$$\begin{aligned} &|g(z) - E^N g_\gamma(z)| \\ &\quad = \lim_{m\to\infty} \left| E^N g_\gamma(z) - \int_{D_{t_m}} \overline{\partial} g_\gamma(\zeta) \wedge K_0^{\varepsilon_m}(\zeta, z) - \int_{D_{t_m}} g_\gamma(\zeta) P_0^{\varepsilon_m}(\zeta, z) \right|. \end{aligned}$$

On the other hand, for $z \in D$, it holds

$$\lim_{m\to\infty} \int_{D_{t_m}} g_\gamma(z) P_0^{\varepsilon_m}(\zeta, z) = \int_D g_\gamma(\zeta) P_0(\zeta, z),$$

where

$$\begin{aligned} P_0(\zeta, z) \;&=\; C_n \frac{1}{(\langle \mathcal{Q}^1, z-\zeta\rangle + 1)^{N+n}} \frac{f(z)}{f(\zeta)} (\overline{\partial}\mathcal{Q}^1)^n \\ &\quad + D_n dV^{\#} \rfloor \Bigg[\frac{1}{(\langle \mathcal{Q}^1, z-\zeta\rangle + 1)^{N+n-1}} (\overline{\partial}\mathcal{Q}^1)^{n-1} \\ &\quad \wedge \frac{\overline{\partial f(\zeta)} \wedge \sum_{i=1}^n h_i(\zeta, z)}{|\partial f|^2} \Bigg] dV_A \\ &:= P_0^0(\zeta, z) + P_0^1(\zeta, z). \end{aligned}$$

Indeed, the fact that

$$\begin{aligned} &\int_{D_{t_m}} g_\gamma(z) C_n \frac{1}{(\langle \mathcal{Q}^1, z-\zeta\rangle + 1\rangle)^{N+n}} \frac{f(z)\overline{f(\zeta)} + \varepsilon}{|f(\zeta)|^2 + \varepsilon} (\overline{\partial}\mathcal{Q}^1)^n \\ &\quad \longrightarrow \int_D g_\gamma(\zeta) P_0^0(\zeta, z), \end{aligned}$$

as $\varepsilon \to 0$, follows from the Lebesgue theorem, since $\partial f \neq 0$ in D. Also, the growth restriction on g, which is a consequence of the fact that $g \in H^1(D \cap A, \varrho^{2/M}\, dV_A)$, guarantees that $r^N(\zeta) g_\gamma(\zeta)$ vanishes as $\zeta \to bD$ provided N is large enough. As for the second term, the proof in [3] relies on the following fact. Namely, it holds

$$\frac{\varepsilon}{(|f|^2+\varepsilon)^2} \to \frac{dV_A}{|\partial f|^2}, \tag{30}$$

weakly as distributions, as $\varepsilon \to 0$. This implies that

$$\overline{\partial}\mathcal{Q}_\varepsilon^2 = \varepsilon C \frac{\overline{\partial f} \wedge \sum_{i=1}^n h_i\, d\zeta_i}{(|f|^2+\varepsilon)^2} \to c \frac{\sum_{i=1}^n h_i\, d\zeta_i \wedge \overline{\partial f}}{|\partial f|^2}\, dV_A, \tag{31}$$

as $\varepsilon \to 0$, this time weakly as currents. This means that for a fixed $z \in D$ we also have that

$$dV^{\#} \rfloor (\overline{\partial}\mathcal{Q}^1)^{n-1} \wedge (\overline{\partial}\mathcal{Q}_\varepsilon^2) \to dV^{\#} \rfloor \left[(\overline{\partial}\mathcal{Q}^1)^{n-1} \wedge \frac{\sum_{i=1}^n h_i d\zeta_i \wedge \overline{\partial f}}{|\partial f|^2} \right] dV_A$$

weakly as distributions. Therefore, for any compactly supported cut-off function ψ

$$\begin{aligned} &\left(dV^{\#} \rfloor (\overline{\partial}\mathcal{Q}^1)^{n-1} \wedge (\overline{\partial}\mathcal{Q}_\varepsilon^2), \frac{\psi g_\gamma}{(\langle \mathcal{Q}^1, z-\zeta\rangle + 1\rangle)^{N+n-1}} \right) \\ &\quad \overrightarrow{\varepsilon \to 0} \left(dV^{\#} \rfloor \left[(\overline{\partial}\mathcal{Q}^1)^{n-1} \wedge \frac{\sum_{i=1}^n h_i d\zeta_i \wedge \overline{\partial f}}{|\partial f|^2} \right] dV_A, \right. \\ &\quad \left. \times \frac{\psi g_\gamma}{(\langle \mathcal{Q}^1, z-\zeta\rangle + 1\rangle)^{N+n-1}} \right), \end{aligned}$$

where this time the bracket stands for duality between distributions and test functions. Now we can find a cut-off function ψ such that

$$\left(dV^{\#} \rfloor (\overline{\partial}\mathcal{Q}^1)^{n-1} \wedge (\overline{\partial}\mathcal{Q}_\varepsilon^2), \frac{\psi g_\gamma}{(\langle \mathcal{Q}^1, z-\zeta\rangle + 1\rangle)^{N+n-1}} \right) \tag{32}$$

is as close as we please to

$$\int_D g_\gamma(\zeta) P_0^{1,\varepsilon}(\zeta, z),$$

uniformly with respect to $\varepsilon > 0$, while the right-hand side of (32) is close to $E^N g_\gamma(z)$. Both statements follow from the first part of the proof and are again consequences of the fact that the growth of g_γ can be controlled in terms of ϱ. In the proof of the former one also makes use of the fact that the growth of $\overline{\partial} g_\gamma$ can also be controlled in terms of ϱ. Also, for $z \in D \cap A$

$$\int_D \overline{\partial} g_\gamma(\zeta) \wedge K_0^{0,\varepsilon}(\zeta, z) \to 0,$$

by Lebesgue theorem, since $A = \{f(z) = 0\}$ and

$$\int_D \overline{\partial} g_\gamma(\zeta) \wedge K_0^{1,\varepsilon}(\zeta, z) \to 0,$$

because on $D\cap A$, it holds $\overline{\partial} g_\gamma = 0$. Therefore, we have

$$|g(z) - E^N g_\gamma(z)| = \lim_{m\to 0}\left| E^N g_\gamma(z) - \int_{D_{t_m}} g_\gamma(\zeta) P_0^{\varepsilon_m}(\zeta,z)\right| = 0.$$

Therefore, letting $\gamma\to 0$ we obtain that $g(z) = E^N g(z)$ provided $z\in D\cap A$ and $g\in H^1(D\cap A, \varrho^{2/M}\,dV_A)$. This completes the proof of Theorem 4. □

Proposition 3. *Assume that D is a smoothly bounded convex and strictly pseudoconvex domain in $\mathbb{C}^n, n>1$. Then*

$$\sup\left\{\frac{\mathfrak{h}(P)}{V(P)} : P = P_\varrho(\zeta), \zeta\in D\cap A\right\} < \infty,$$

where $\mathfrak{h}(P)$ stands for $\int_P \mathfrak{h}\,dV_A$.

Proof. Under the assumption that D is strictly pseudoconvex one has $\tau_\alpha(\zeta,\varrho)\sim \varrho^{1/2}$ if $\alpha>1$. Therefore, we have

$$\mathfrak{h}(P)\lesssim \int_{P\cap A} (\varrho^2+\varrho)\,dV_A.$$

Also, since $f=0$ cuts the boundary transversally $\partial f\wedge\partial r\neq 0$ in some neighbourhood U of bD. This implies that for each $\zeta\in U$ there exists $2\le j\le n$ such that $|\partial_j f(\zeta)|\ge c$ with a uniform positive constant c. Hence, the method of Lemma 3 proves this proposition, as well. □

Proof of Theorem 5. We will prove (ii) first. Assume first that $n=2$ and transversallity condition 1 holds. Then for each $p\in U$, it holds $Tf(p)\neq 0$, where L is defined in the following way

$$T := \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1}.$$

Indeed, notice that $\partial f\wedge \partial r = (Tf)T^*\wedge\partial r$ with T^* denoting the $(1,0)$-form dual to T. Hence, Assumption 1 implies that $Tf\neq 0$ in some open neighbourhood of bD. Denote this neighbourhood by U—the same symbol will stand for the neighbourhood from Assumption 3 in case $n>2$.

Therefore, in both cases, that is, when $n=2$ and $n>2$ under Assumption 3, there exists $t_0<1$ such that

$$\Bigg\{\zeta\in D\cap A : \varrho(\zeta) < t_0,\\ \mathcal{L}_r(\zeta;\xi) = \sum_{j,k=1}^n \frac{\partial^2 r(\zeta)}{\partial z_j\,\partial\bar z_k}\xi_j\bar\xi_k = 0 \text{ for some } \xi\in\mathbb{C}^n, \xi\neq 0\Bigg\}\subset U.$$

This follows from the assumption that r is of the form $p_D - 1$.

Thus, if $\zeta \in D \cap A$ and the null space of $\mathcal{L}_r(\zeta, \cdot)$ is of positive dimension, then either $\zeta \in U$ or $\varrho(\zeta) \geq t_0 > 0$. Observe that if $\varrho(\zeta) \geq t_0$, then $\mathfrak{h}(\zeta) \leq C$ for some positive constant C. As a result, we obtain in this case

$$\sup\left\{\frac{\nu_N(P)}{V(P)} : P = P_\varrho, \varrho \geq t_0\right\} \geq c > 0.$$

Hence, we may assume that $\varrho(\zeta) < t_0$. There are two possibilities: either $\zeta \in U$ or $\zeta \notin U$. In the latter case, it holds $\tau_l(\zeta, \varrho) \sim \varrho^{1/2}$ with a uniform constant. The fact that constants are uniform is a consequence of the following formula

$$\tau(z, v, \varepsilon) \sim \min_{1 \leq k \leq M}\left\{\left(\frac{\varepsilon}{\sum_{i+j=k} |a_{ij}(z, v)|}\right)^{1/k}\right\}, \tag{33}$$

where

$$a_{ij}(z, v) = \frac{\partial^{i+j}}{\partial \lambda^i \partial \bar{\lambda}^j} r(z + \lambda v)\bigg|_{\lambda=0}.$$

Therefore, if $\varrho(\zeta) < t_0$ and $\zeta \notin U$ then we can proceed as in Proposition 3.

Thus, we may assume that $\zeta \in U$. We will show that for sufficiently large N, it holds $\mathfrak{h}_N(\zeta) \lesssim |\partial f|^2_\kappa(\zeta)$. In view of Lemma 3, this suffices to complete the proof. Consider first the term $\mathfrak{h}_+$

$$\begin{aligned}
\mathfrak{h}_+(\zeta) &= \sum_l \sum_{i=1}^{\lceil \log_2 (1/\varrho) \rceil} \frac{1}{2^{iN}} \frac{1}{\prod_{\beta \neq l} \tau_\beta^2(\zeta, 2^i \varrho)} \int_{P^i} |h_l(\zeta, z)|^2 \, dV(z) \\
&\leq C \sum_l \sum_{i=1}^{\lceil \log_2 (1/\varrho) \rceil} \frac{1}{2^{iN}} \frac{V(P^i)}{\prod_{\beta \neq l} \tau_\beta^2(\zeta, 2^i \varrho)} \leq C \sum_l \sum_{i=1}^{\lceil \log_2 (1/\varrho) \rceil} \frac{\tau_l^2(\varrho, 2^i \varrho)}{2^{iN}} \\
&\leq C \frac{1}{2^N} \sum_{l=1}^n \tau_l^2(\zeta, \varrho) \leq nC \frac{\tau_2^2(\zeta, \varrho)}{2^N}.
\end{aligned}$$

In the estimates, we made use of the fact that

$$|h_k(\zeta, z)|^2 \leq \sup_{z \in D} |\partial_k f(z)|^2 \leq C,$$

since f is holomorphic in some neighbourhood of D. Thus, we need to choose N_0 in such a way that $nC2^{-N} \leq \sqrt{c}$, where c denotes the constant in (1) from Assumption 3. With such a choice, we obtain for $\zeta \in U$

$$\mathfrak{h}_+(\zeta) \leq |Z_2 f(\zeta)|^2 \tau_2^2(\zeta, \varrho) \leq |\partial f|^2_\kappa(\zeta).$$

As for the term $\mathfrak{h}_-$, notice that

$$|Z_l^{\zeta,0} f(z)| \leq |Z_l^{\zeta,0} f(\zeta)| + \sum_{k=1}^n |Z_k^{\zeta,0} Z_l^{\zeta,0} f(\xi_l)| \tau_l(\zeta, \varepsilon) \leq |Z_l^{\zeta,0} f(\zeta)| + C\varepsilon^{1/M}$$

for $z \in P_\varepsilon(\zeta)$ and some $\xi_1, \ldots, \xi_n \in P_\varepsilon(\zeta)$. Choose $\varepsilon > 0$ here in such a way that it satisfies the condition $C\varepsilon^{1/m} \leq c$ with c as in (1). Also, there exists j

such that $2^j\varepsilon \geq t_0$. Consequently, for $z \in P(\zeta)$ with $\zeta \in U$ and $\varrho(\zeta) < t_0$ we obtain

$$|Z_l f(z)| \leq C|Z_l f(\zeta)|$$

for $z \in P(\zeta)$. We are now ready to complete the proof of (ii).

$$\begin{aligned}\mathfrak{h}_-(\zeta) &= \sum_{i=0}^{\infty} \frac{1}{2^{i(n-1)}} \frac{1}{\prod_{\beta\neq l} \tau_\beta^2(\zeta, 2^{-i}\varrho)} \int_{P^{-i}} |h_l(\zeta,z)|^2\, dV(z) \\ &\lesssim \sum_{i=0}^{\infty} \sum_l \frac{\tau_l^2(\zeta, 2^{-i}\varrho)}{2^{i(n-1)}} |Z_l^{\zeta,0} f(\zeta)|^2 \leq |\partial f|_\kappa^2(\zeta).\end{aligned}$$

Now, we intend to prove (i). Actually, the situation under Assumption 2 is easier. We will show that $\mathfrak{h}\, dV_A$ satisfies condition (7). For future reference, we single out the method as a lemma.

LEMMA 4. *Assume that D is a smoothly bounded convex domain of finite type and A satisfies Assumptions 1 and 2. Then for any $\zeta \in A\cap D$ and $\varepsilon > 0$, it holds*

$$\sum_{l=1}^{n} V_A(A\cap P\cap D)\tau_l^2(\zeta,\varepsilon) \leq CV(P),$$

where $P = P_\varepsilon(\zeta)$.

Proof. We can assume that ε here is small enough to guarantee that $P_\varepsilon(\zeta) \subset U$, where U is the neighbourhood from Assumption 2. Indeed, the statement is trivial for ε bounded from below. Also, for each l, it holds $\tau_l(\zeta,\varepsilon) \lesssim \tau_2(\zeta,\varepsilon)$ and

$$\begin{aligned}&V_A(A\cap P\cap D)\\ &\lesssim \int_{\substack{|v_j^*(z)|\leq \tau_j(\zeta,\varepsilon)\\ j\neq 2}} \frac{dx_1\, dy_1\, \widehat{dx_2\, dy_2} \cdots dx_n\, dy_2}{|\bigwedge_{j\neq 2}(dx_j\wedge dy_j)\wedge d\Re(f\circ\Phi^*)\wedge d\Im(f\circ\Phi^*)|}\\ &\lesssim \prod_{j\neq 2} \tau_j^2(\zeta,\varepsilon).\end{aligned}$$

This proves the lemma, since $\prod_{j=1}^n \tau_j^2(\zeta,\varepsilon) \sim V(P)$. □

Notice that

$$\mathfrak{h}(\zeta) \leq C \sum_{i=1}^{C\lceil \log_2 (1/\varrho)\rceil} \frac{1}{2^{iN}} \sum_l \tau_l^2(\zeta, 2^i\varrho) + C\sum_{i=0}^{\infty} \frac{1}{2^{i(n-1)}} \sum_l \tau_l(\zeta, 2^{-i}\varrho)$$

and, as a result, for $P = P_\varepsilon(z)$ with $\varepsilon \leq c\varrho(z)$

$$\int_{A\cap P} \mathfrak{h}(\zeta)\, dV_A \leq C\int_{A\cap P} \sum_l \tau_l^2(\zeta,\varepsilon)\, dV_A \leq C\sum_l \tau_l^2(z,\varepsilon) V_A(A\cap P)$$

and application of Lemma 4 completes the proof of (i) and the proof of the whole Theorem 5. □

Proof of Theorem 6. Fix $P=P_\varepsilon(\xi)$ and notice that we have

$$\begin{aligned}
&\frac{1}{V(P)}\int_P |E^N g(z)-(E^N g)_P|\,dV(z)\\
&\leq \frac{1}{V(P)^2}\int_{P\times P}|E^N g(z)-E^N g(\zeta)|\,dV(z)\,dV(\zeta)\\
&\leq \frac{2\|g\|_{L^\infty}}{V(P)^2}\\
&\quad\times\sum_{i=0}^{C\lceil\log_2(1/\varepsilon)\rceil}\int_{P\times P}\int_{D\cap A\cap P^i}|E^N(\eta,\zeta)-E^N(\eta,z)|\,dV_A(\eta)\,dV(z)\,dV(\zeta),
\end{aligned}$$

where $P^i := CP_{2^{i+1}\varepsilon}(\xi)\setminus cP_{2^i\varepsilon}(\xi), i=1,\ldots$ and $P^0=P^*$. Naturally, for any $\eta,\zeta,z\in D$

$$|E^N(\eta,\zeta)-E^N(\eta,z)|\leq |E^N(\eta,\zeta)-E^N(\eta,\xi)|+|E^N(\eta,\xi)-E^N(\eta,z)|.$$

Hence,

$$\begin{aligned}
&\int_{P\times P}\int_{D\cap A\cap P^i}|E^N(\eta,\zeta)-E^N(\eta,z)|\,dV_A(\eta)\,dV(z)\,dV(\zeta)\\
&\quad\leq 2V(P)\int_P\int_{D\cap A\cap P^i}|E^N(\eta,\zeta)-E^N(\eta,\xi)|\,dV_A(\eta)\,dV(\zeta).
\end{aligned}$$

As a result, for any sequence $\xi_i\in D$ with $i\in\mathbb{N}_0$, it holds

$$\begin{aligned}
&\frac{1}{V(P)}\int_P |E^N g(z)-(E^N g)_P|\,dV\\
&\leq \frac{4\|g\|_{L^\infty}}{V(P)}\\
&\quad\times\sum_{i=0}^{C\lceil\log_2(1/\varepsilon)\rceil}\int_{D\cap A\cap P^i}\int_P|E^N(\eta,\zeta)-E^N(\eta,\xi_i)|\,dV_A(\eta)\,dV(\zeta).
\end{aligned}\tag{34}$$

Now we need to choose points ξ_i. As for the case of integral over $P^*\cap A$, we proceed as follows: if $P^*\cap A=\emptyset$, then there is nothing to prove, otherwise let $\xi_0\in P^*\cap A$. Recall that the engulfing property (10) ensures that in this case there is a uniform C such that $P^*\subset CP_\varepsilon(\xi_0)$ and $V(P)\sim V(CP_\varepsilon(\xi_0))$ by (20). Therefore,

$$\begin{aligned}
\mathcal{I} &:= \frac{1}{V(P)}\int_P\int_{D\cap A\cap P^*}|E^N(\zeta,z)-E^N(\zeta,\xi_0)|\,dV(\zeta)\,dV(z)\\
&\leq \frac{1}{V(P)}\int_{A\cap D\cap CP^*(\xi_0)}\int_{CP(\xi_0)}|E^N(\zeta,z)-E^N(\zeta,\xi_0)|\,dV(z)\,dV_A(\zeta)
\end{aligned}$$

$$\leq \frac{1}{V(P)} \int_{A\cap D\cap CP^*(\xi_0)} \int_{CP(\xi_0)} |E^N(\zeta,z)|\,dV(z)\,dV_A(\zeta)$$
$$+ \frac{1}{V(P)} \int_{A\cap D\cap CP^*(\xi_0)} \int_{CP(\xi_0)} |E^N(\zeta,\xi_0)|\,dV(z)\,dV_A(\zeta).$$

It follows from Lemma 2 and Lemma 1 that we have the following estimate for the kernel in $CP_\varepsilon(\xi_0)$

$$E^N(\zeta,z) \lesssim \sum_l \frac{1}{\varrho^{n-1}(\zeta)} \frac{\varepsilon^{n-1}}{\prod_{\alpha\neq l}\tau_\alpha^2(\xi_0,\varepsilon)} \big(|\partial_l f(\zeta)|^2 + |h_l(\zeta,z)|\big)$$
$$\leq \sum_l \frac{1}{\varrho^{n-1}(\zeta)} \frac{\varepsilon^{n-1}}{\prod_{\alpha\neq l}\tau_\alpha^2(\xi_0,\varepsilon)} \Big(|\partial_l f(\zeta)|^2 + \sup_{CP_\varepsilon(\xi_0)} |\partial_l f|^2\Big).$$

There are two cases, which should be taken into account. Namely,

$$\frac{1}{V(P)} \int_{A\cap D\cap CP^*(\xi_0)} \int_{CP(\xi_0)} \sum_l \frac{|\partial_l f(\zeta)|^2}{\varrho^{n-1}(\zeta)} \frac{\varepsilon^{n-1}}{\prod_{\alpha\neq l}\tau_\alpha^2(\xi_0,\varepsilon)}\,dV_A(\zeta)\,dV(z)$$
$$\leq \frac{1}{V(P)} \int_{A\cap D\cap CP^*(\xi_0)} \sum_l |\partial_l f(\zeta)|^2 \tau_l^2(\xi_0,\varepsilon)\,dV_A$$
$$\lesssim \frac{1}{V(P)} \int_{A\cap D\cap CP^*(\xi_0)} |\partial f(\zeta)|_\kappa^2\,dV_A,$$

since $\tau(\xi_0,\varepsilon)\sim\tau(\zeta,\varepsilon)$ in $CP^*(\xi_0)$. Recall that according to Lemma 3 the measure $|\partial f|_\kappa^2\,dV_A$ satisfies condition (7). This completes the argument in this case, since

$$\int_{A\cap D\cap CP^*(\xi_0)} |\partial f(\zeta)|_\kappa^2\,dV_A \lesssim V(CP^*(\xi_0)) \sim V(P).$$

The estimate of terms involving the functions h_l follows from Lemma 4.

Hence, under Assumption 3, we obtain

$$\frac{1}{V(P)} \textstyle\int_{D\cap A\cap CP^*(\xi_0)} \int_{CP(\xi_0)} |E^N(\zeta,z) - E^N(\zeta,\xi_0)|\,dV \lesssim 1.$$

We have not used any cancellation property so far. This plays a role in $P^i = CP_{2^{i+1}\varepsilon}(\xi)\setminus cP_{2^i\varepsilon}(\xi)$, where in (34) we simply take $\xi_i=\xi$. Again, if there exists ζ_i belonging to $A\cap D\cap P^i$, then there is $C>0$ such that $P^i\subset CP_{2^{i+1}\varepsilon}(\zeta_i)$ and we estimate the kernel in $CP_{2^{i+1}\varepsilon}(\zeta_i)$. The latter statement means that we choose $(2^{i+1}\varepsilon)$-basis at ζ_i, find the unitary transformation Φ^* and refer to Lemma 2 as far as estimates of $(\overline{\partial}\mathcal{Q}^1)^{n-1}$ are concerned. However, the terms $S(z,\zeta)$ and $S(\xi,\zeta)$ in the denominator will be estimated under the assumption that $\zeta\in P^i(\xi)$.

One easily observes that one of the terms, which appear in $|E^N(\zeta,z)-E^N(\zeta,\xi)|$—and we will restrict our attention to a typical term—can be estimated by the following expression

$$\mathcal{J}=\sum_{j+l=N+n-2}\frac{\varrho^{N+n-1}(\zeta)|S(z,\zeta)-S(\xi,\zeta)|}{(r(\zeta)+S(z,\zeta))^{N+n-1-l}(r(\zeta)+S(\xi,\zeta))^{N+n-1-j}} \times\left|(\overline{\partial}\mathcal{Q}^1(\zeta,z))^{n-1}\wedge\overline{\partial f(\zeta)}\wedge\sum_{k=1}^{n}h_k(\zeta,z)\,d\zeta_k\right|. \tag{35}$$

Observe that

$$|S(z,\zeta)-S(\xi,\zeta)|=\left|\sum_\alpha\partial_\alpha S(z',\zeta)(z_\alpha-\xi_\alpha)\right|$$

with $z'\in P_\varepsilon(\xi)$

$$=\left|\sum_\alpha\partial_\alpha\left[\sum_\beta Q_\beta(z',\zeta)(z'_\beta-\zeta_\beta)\right](z_\alpha-\xi_\alpha)\right|\le 2^{i+1}\varepsilon\sum_\alpha\frac{\tau_\alpha(\xi,\varepsilon)}{\tau_\alpha(\xi,2^{i+1}\varepsilon)}$$
$$\lesssim 2^{i+1}2^{-i/M}\varepsilon$$

since $z\in P_\varepsilon(\xi)$ and $\zeta\in P_{2^{i+1}\varepsilon}(\xi)$. The corresponding decomposition of S is with respect to the $(2^{i+1}\varepsilon)$-basis at ξ and the estimate follows from Lemma 2.

Thus, for the term (35), we have the following estimate in $CP_{2^i\varepsilon}(\zeta_i)$

$$\mathcal{J}\le C2^{-i/M}\sum_l\frac{\varrho^N}{\varrho^N}\frac{1}{(2^i\varepsilon)^n}\frac{(2^i\varepsilon)^n}{\prod_{\beta\ne l}\tau_\beta(\zeta_i,2^i\varepsilon)}\big(|\partial_l f(\zeta)|^2+|h_l(\zeta,z)|^2\big).$$

This follows again from Lemma 1 and Lemma 2. Therefore,

$$\frac{1}{V(P)}\sum_{i=1}^{C\lceil\log_2(1/\varepsilon)\rceil}\int_{A\cap P^i}\int_P\mathcal{J}(z,\zeta,\xi)\,dV_A(\zeta)\,dV(z)$$
$$\le\sum_{i=1}^{C\lceil\log_2(1/\varepsilon)\rceil}2^{-i/M}\int_{A\cap CP_{2^{i+1}}(\zeta_i)}\sum_l\frac{|\partial_l f(\zeta)|^2+\sup_{CP_{2^{i+1}\varepsilon}(\zeta_i)}|\partial_l f|^2}{\prod_{\beta\ne l}\tau_\beta(\zeta_i,2^i\varepsilon)}\,dV_A(\zeta)$$
$$\lesssim\sum_{i=1}^{C\lceil\log_2\frac{1}{\varepsilon}\rceil}2^{-i/M}\left(1+\sum_l\frac{1}{\prod_{\beta\ne l}\tau_\beta(\zeta_i,2^i\varepsilon)}\int_{A\cap CP_{2^{i+1}}(\zeta_i)}dV_A\right),$$

by (27) and the method of Lemma 3.

Now it is a consequence of Assumption 2 and estimate (27) that

$$\sum_l\frac{1}{\prod_{\beta\ne l}\tau_\beta(\zeta_i,2^i\varepsilon)}\int_{A\cap CP_{2^{i+1}\varepsilon}(\zeta_i)}dV_A\le C\sum_l\frac{\prod_{\beta\ne 2}\tau_\beta(\zeta_i,2^i\varepsilon)}{\prod_{\beta\ne l}\tau_\beta(\zeta_i,2^i\varepsilon)}\lesssim 1,$$

by the method of Lemma 4. This completes the proof for (35) and the whole proof, since similar estimates hold for other terms, as well. $\square$

As was stated in the Introduction the proof of Theorem 1 is based on interpolation. We formulate the result, which we need below

THEOREM 9. *Assume that D is a smoothly bounded convex domain of finite type in $\mathbb{C}^n$ and let ν be a positive measure on $D \cap A$, where A is a complex submanifold satisfying Assumption 1. Assume that T is a bounded operator on the following pairs of spaces*

$$T : L^1(D \cap A, \nu) \to L^1(D),$$
$$T : L^\infty(D) \to \mathrm{BMO}(D).$$

Then, T maps boundedly $L^p(\nu)$ into $L^p(D)$ for $1 < p < \infty$.

From the Riesz–Thorin theorem it follows that the interpolation spaces between L^1 and L^∞ constructed by means of the real method are isomorphic to L^p spaces. Thus, in order to prove the theorem is suffices to prove that $[L^1(D), \mathrm{BMO}(D)]_\theta \cong L^p$ with $\frac{1}{p} = 1 - \theta$. Since the proof follows a well-known pattern, instead of writing down details we refer the reader to [16] and [20] for proofs in a standard situation (one may also want to consult [15] for a different approach). The first method boils down to computing the K functional, while the second reduces the problem to constructing a Whitney type cover in the spirit of Proposition 1 and the Marcinkiewicz interpolation theorem.

Proof of Theorem 1. We proved that under Assumption 2 for sufficiently large N, the operator E^N is a continuous operator between the following pairs of spaces

$$E^N : L^1(D \cap A, \nu_N) \to L^1(D),$$
$$E^N : L^\infty(D) \to \mathrm{BMO}(D).$$

Theorem 9 implies that E^N maps $L^p(D \cap A, \nu_N)$ into $L^p(D)$ for $1 < p < \infty$. Consequently, for each $1 \leq p < \infty$

$$R_A[H^p(D)] = H^p(D \cap A, \nu),$$

which shows that the sequence (3) is exact. Indeed, for any $f \in H^p(D \cap A, \nu)$, it holds $E^N f \in H^p(D)$. This implies that $f = R_A E^N f \in R_A(H^p(D))$. In other words, we have $R_A[H^p(D)] = H^p(D \cap A, \nu)$. Obviously, the rest follows from this easily. □

Proof of Theorem 2. Following immediately from Theorem 3 and Theorem 5 by arguments similar to those which prove Theorem 1. □

REFERENCES

[1] W. Alexandre, *Problémes d'extension dans les domains convexes de type fini*, Math. Z. **253** (2006), 263–280. MR 2218700

[2] B. Berndtsson and M. Andersson, *Henkin–Ramirez formulas with weight factors*, Ann. Inst. Fourier (Grenoble) **32** (1982), 91–110. MR 0688022

[3] B. Berndtsson, *A formula for interpolation and division in* $\mathbb{C}^n$, Math. Ann. **263** (1983), 399–418. MR 0707239

[4] J. Bruna, P. Charpentier and Y. Dupain, *Zero varieties for the Nevanlinna class in convex domains of finite type*, Annals of Math. **147** (1998), 391–415. MR 1626753

[5] J. Bruna, A. Nagel and S. Waigner, *Convex hypersurfaces and Fourier transforms*, Annals of Math. **127** (1988), 333–365. MR 0932301

[6] C. Bennett and R. Sharpley, *Weak type inequalities for* H^p *and BMO*, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, RI, 1979, pp. 201–229. MR 0545259

[7] D. Catlin, *Boundary invariants of pseudoconvex domains*, Annals of Math. **120** (1984), 529–586. MR 0769163

[8] K. Diederich and J. E. Fornaess, *Support functions for convex domains of finite type*, Math. Z. **230** (1999), 145–164. MR 1671870

[9] K. Diederich, B. Fischer and J. E. Fornass, *Hölder estimates on convex domains of finite type*, Math. Z. **232** (1999), 43–61. MR 1714279

[10] K. Diederich and G. Herbort *An alternative proof of an extension theorem of T. Ohsawa*, Michigan Math. J. **46** (1999), 347–360. MR 1704201

[11] K. Diederich and E. Mazzilli, *Extension of bounded holomorphic functions in convex domains*, Manuscripta Math. **105** (2001), 1–12. MR 1885811

[12] K. Diederich and E. Mazzilli, *Extension and restriction of holomorphic functions*, Ann. Inst. Fourier (Grenoble) **47** (1997), 1079–1099. MR 1488245

[13] B. Fischer, L^p *estimates on convex domains of finite type*, Math. Z. **236** (2001), 401–418. MR 1815835

[14] B. Fischer, *Nonisotropic Hölder estimates on convex domains of finite type*, Michigan Math. J. **52** (2004), 219–239. MR 2043407

[15] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 2. Edizioni della Normale, Pisa, 2005. MR 2192611

[16] R. Hanks, *Interpolation by the real method between BMO,* $L^\alpha(0<\alpha<\infty)$*, and* $H^\alpha(0<\alpha<\infty)$, Indiana Univ. Math. J. **26** (1977), 679–689. MR 0448052

[17] T. Hefer, *Hölder and* L^p *estimates for* $\overline{\partial}$ *on convex domains of finite type depending on Catlin's multitype*, Math. Z. **242** (2002), 367–398. MR 1980628

[18] T. Hefer, *Extremal bases and Hölder estimates for* $\overline{\partial}$ *on convex domains of finite type*, Michigan Math. J. **52** (2004), 573–602. MR 2097399

[19] M. Jasiczak, *Extension of holomorphic functions with applications*, submitted.

[20] G. Li and W. Li, *Interpolation spaces between* L^1 *and BMO on spaces of homogeneous type*, J. Math. Res. Exposition **27** (2007), 525–532. MR 2349498

[21] J. D. McNeal, *Convex domains of finite type*, J. Funct. Anal. **108** (1992), 361–373. MR 1176680

[22] J. D. McNeal, *Estimates on the Bergman kernels of convex domains*, Adv. Math. **109** (1994), 108–139. MR 1302759

[23] T. Ohsawa and K. Takegoshi, *On the extension of* L^2 *holomorphic functions*, Math. Z. **195** (1987), 197–204. MR 0892051

[24] T. Ohsawa, L^2 *extension theorems—backgrounds and a new result*, Finite or infinite dimensional complex analysis and applications, Kyushu University Press, Fukuoka, 2005, pp. 261–274. MR 2359704

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