Illinois Journal of Mathematics
Volume 54, Number 2, Summer 2010, Pages 469–483
S 0019-2082

QUANTUM SEMIGROUP COMPACTIFICATIONS AND UNIFORM CONTINUITY ON LOCALLY COMPACT QUANTUM GROUPS

PEKKA SALMI

Abstract. We introduce quantum semigroup compactifications and study the universal quantum semigroup compactification of a coamenable locally compact quantum group. If G is a classical locally compact group, the universal semigroup compactification corresponds to the C*-algebra of the bounded left uniformly continuous functions on G, so we study the analogous C*-algebra associated with a locally compact quantum group.

1. Introduction

In topology, a compactification of a locally compact space X is a compact space that includes a dense homeomorphic copy of X. If we replace X by a locally compact group G, then we naturally expect that the group structure on G is somehow represented in the compactification. As the classical Bohr compactification shows, if we want that a compactification is also a topological group, we cannot require that the compactification includes a topologically isomorphic copy of G, that is, we have to loosen the topological requirements of compactification. On the other hand, the multiplication of G can be extended to a compact space in such a way that G keeps both its topological and algebraic structure. However, the compact space itself is not a group anymore—just a semigroup. Such an object is called a semigroup compactification; we define them properly in Section 3, but, for a thorough treatment of the theory, the reader is referred to [5].

The aim of this paper is to define the notion of semigroup compactification for locally compact quantum groups and give a description of the universal quantum semigroup compactification. The latter object corresponds, in the

Received March 24, 2008; received in final form February 27, 2011.
Research supported by Academy of Finland.
2010 *Mathematics Subject Classification.* Primary 46L89. Secondary 43A99, 46L65.

case of a classical locally compact group G, to the C*-algebra $\mathrm{LUC}(G)$ of the bounded left uniformly continuous functions. So our attempt is to define the C*-algebra $\mathrm{LUC}(\mathbb{G})$ of the left uniformly continuous functions, so to speak, for a locally compact quantum group $\mathbb{G}$. The definition is based on the work of Ng [23]. On the dual side, Granirer [11] has defined the so-called bounded uniformly continuous functionals, which form a C*-subalgebra $\mathrm{UCB}(\widehat{G})$ of the group von Neumann algebra $\mathrm{VN}(G)$. We shall show that if $\mathbb{G}$ is the dual $\widehat{G}$ of an amenable locally compact group G, then $\mathrm{LUC}(\mathbb{G})$ agrees with $\mathrm{UCB}(\widehat{G})$. More generally, we shall compare $\mathrm{LUC}(\mathbb{G})$ with the space $L^\infty(\mathbb{G})L^1(\mathbb{G})$, which is defined through the standard action of $L^1(\mathbb{G})$ on its dual $L^\infty(\mathbb{G})$. Along the way, we develop some basic results about the completely contractive Banach algebra $\mathrm{LUC}(\mathbb{G})^*$. For example, it includes an isomorphic copy of $\mathrm{C}_0(\mathbb{G})^*$, the dual of the reduced C*-algebra associated with the locally compact quantum group $\mathbb{G}$. The definition of the quantum semigroup compactification encompasses the quantum Bohr compactification defined recently by Sołtan [28].

Using a different approach, also Runde [27] has studied uniform continuity on locally compact quantum groups (which I found out after this work was completed). Theorem 5.3 of the present paper should be compared with Theorem 5.2 of [27]: in both results the spaces $L^\infty(\mathbb{G})L^1(\mathbb{G})$ and $\mathrm{LUC}(\mathbb{G})$ are compared, but the conditions put on $\mathbb{G}$ are very different. (It should be noted that the space which we denote by $L^\infty(\mathbb{G})L^1(\mathbb{G})$ is denoted by $\mathrm{LUC}(\mathbb{G})$ in [27].)

2. Preliminaries

Throughout this paper, $\mathbb{G}$ denotes a locally compact quantum group in the sense of Kustermans and Vaes [16]. That is to say that $\mathbb{G}$ consists of a von Neumann algebra $L^\infty(\mathbb{G})$ with the following additional structure. There is a unital normal $*$-homomorphism $\Gamma : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$ that is *coassociative* in the sense that

$$(\Gamma \otimes \mathrm{id})\Gamma = (\mathrm{id} \otimes \Gamma)\Gamma$$

(here, as elsewhere, $\overline{\otimes}$ denotes the von Neumann algebra tensor product and id denotes the identity map). The map Γ is called the *comultiplication* of $\mathbb{G}$. We require also that there exist a left and a right Haar weight, ϕ and ψ, on $L^\infty(\mathbb{G})$. The reader is referred to [31, 16] for details.

Every locally compact group G induces, of course, a locally compact quantum group. It consists of the usual $L^\infty(G)$, the comultiplication

$$\Gamma(f)(g_1, g_2) = f(g_1 g_2) \quad \big(f \in L^\infty(G), g_1, g_2 \in G\big),$$

and the left and the right Haar measures on G. We say that such a locally compact quantum group is a *classical group*. The *dual of a classical group* is formed by the von Neumann algebra $\mathrm{VN}(G)$ generated by the the left regular representation λ of G, the comultiplication

$$\Gamma(\lambda(g)) = \lambda(g) \otimes \lambda(g) \quad (g \in G),$$

and the Plancherel weight [30, Section VII.3], which acts as both the left and the right Haar weight.

Let $L^2(\mathbb{G})$ be the Hilbert space that is obtained by applying the GNS-construction to the pair $(L^\infty(\mathbb{G}), \phi)$. This Hilbert space is isomorphic with the one coming from $(L^\infty(\mathbb{G}), \psi)$, and we make no distinction between the two. We identify $L^\infty(\mathbb{G})$ with its isomorphic image in $\mathrm{B}(L^2(\mathbb{G}))$, the bounded operators on $L^2(\mathbb{G})$. There is a unitary operator V on the Hilbert space tensor product $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ such that V satisfies the *pentagonal relation*

$$V_{12}V_{13}V_{23} = V_{23}V_{12} \tag{2.1}$$

(where we use the standard leg numbering notation: for example V_{13} is V acting on the first and the third component of $L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$) and determines the comultiplication via

$$\Gamma(x) = V(x \otimes 1)V^* \quad \big(x \in L^\infty(\mathbb{G})\big).$$

The norm closure of

$$\{(\omega \otimes \mathrm{id})V; \omega \in \mathrm{B}(L^2(\mathbb{G}))_*\}$$

is a C*-algebra, which we denote by $\mathrm{C}_0(\mathbb{G})$. The C*-algebra $\mathrm{C}_0(\mathbb{G})$ is the reduced C*-algebraic version of the locally compact quantum group $\mathbb{G}$ introduced in [15]. The weak* closure of $\mathrm{C}_0(\mathbb{G})$ is $L^\infty(\mathbb{G})$ and the comultiplication Γ maps $\mathrm{C}_0(\mathbb{G})$ to $M(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G}))$—the multiplier algebra of the spatial C*-algebra tensor product $\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G})$. In general, we denote the multiplier algebra of a C*-algebra A by $M(A)$, but in the special case of $A = \mathrm{C}_0(\mathbb{G})$ we shall write $\mathrm{C}_b(\mathbb{G})$ for $M(\mathrm{C}_0(\mathbb{G}))$. If $\mathbb{G} = G$ is a classical group, then, of course, $\mathrm{C}_0(\mathbb{G})$ is the C*-algebra $\mathrm{C}_0(G)$ of the continuous functions on G vanishing at infinity, and $\mathrm{C}_b(\mathbb{G})$ is the C*-algebra $\mathrm{C}_b(G)$ of the bounded continuous functions on G.

Next we consider the extension of functions to multiplier algebras. Let A and B be C*-algebras. A $*$-homomorphism $\phi : A \to M(B)$ is said to be *nondegenerate* if the linear span of $\phi(A)B$ is dense in B (note that if A has a unit, then ϕ is nondegenerate if and only if it is unital). If ϕ is nondegenerate, it can be extended uniquely to a function $\phi : M(A) \to M(B)$ that is strictly continuous on bounded sets. Recall that the strict topology on $M(A)$ is induced by the seminorms $x \mapsto \|ax\| + \|xa\|$ where a runs through the elements of A. Also bounded linear functionals and their slices (such as $\mu \otimes \mathrm{id}$ where $\mu \in A^*$) admit unique extensions that are strictly continuous on bounded sets. We shall often use these extensions without explicit mention. See [14, Section 7], [22, Appendix A], or [17, Chapter 2] for further details on these matters.

The dual $C_0(\mathbb{G})^*$ is a completely contractive Banach algebra with respect to the multiplication

$$\mu * \nu = (\mu \otimes \nu)\Gamma \quad \big(\mu, \nu \in C_0(\mathbb{G})^*\big),$$

where $\mu \otimes \nu \in M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))^*$. We denote the predual of $L^\infty(\mathbb{G})$ by $L^1(\mathbb{G})$, which is a closed ideal in $C_0(\mathbb{G})^*$.

The action of $C_0(\mathbb{G})$ on its dual, defined by

$$\langle \mu \cdot a, b\rangle = \langle \mu, ab\rangle \quad \big(\mu \in C_0(\mathbb{G})^*, a, b \in C_0(\mathbb{G})\big),$$

can be restricted to $L^1(\mathbb{G})$. We see that every f in $L^1(\mathbb{G})$ admits a decomposition $f = g \cdot a$ with g in $L^1(\mathbb{G})$ and a in $C_0(\mathbb{G})$ (by Cohen's factorization theorem [6, Theorem I.11.10]). By taking weak* limits, it follows that

$$\langle f, x\rangle = \langle g, ax\rangle \quad \big(x \in L^\infty(\mathbb{G})\big),$$

which implies that f is strictly continuous when restricted to $C_b(\mathbb{G})$. Similarly, the weak*-continuous slice map $f \otimes \mathrm{id} : L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is strictly continuous on $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G})) \subseteq L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$.

We shall use some language from operator space theory; the reader is referred to [8, 24] for details on operator spaces.

The spatial tensor product of C*-algebras, or operator spaces, is denoted just by $\otimes$. Also the Hilbert space tensor product is denoted with the same symbol, but that should not lead to any confusion. The von Neumann algebra tensor product is denoted by $\overline{\otimes}$.

3. Quantum semigroup compactifications

Let G be a locally compact group. A *semigroup compactification* of G is a pair (S, ϕ) such that:

- S is a semigroup and is equipped with a compact topology,
- $\phi : G \to S$ is a continuous homomorphism,
- $\phi(G)$ is dense in S,
- the maps $s \mapsto st$ and $s \mapsto \phi(g)s$ are continuous on S for every fixed t in S and g in G.

There is a one-to-one correspondence between the semigroup compactifications of G and the so-called m-admissible C*-subalgebras of $C_b(G)$ [5, Theorem 3.1.7]. The C*-algebra corresponding to (S, ϕ) is just $\phi^*(C_b(S))$, where ϕ^* is the dual map $f \mapsto f \circ \phi$ from $C_b(S)$ to $C_b(G)$. A C*-subalgebra of $C_b(G)$ is m-admissible if it is unital, left translation invariant, and left m-introverted. Our approach is to define suitable notions of invariance and introversion for C*-subalgebras of $C_b(\mathbb{G})$. A starting point is to define a coaction of a locally compact quantum group on a C*-algebra.

It follows from the Ellis–Lawson joint continuity theorem [5, Theorem 1.4.2] that if (S, ϕ) is a semigroup compactification of G, then the function

$$(g, s) \mapsto \phi(g)s : G \times S \to S$$

is jointly continuous (this result depends on two things: that G is a group and that G is locally compact). In other words, G acts continuously on its compactification S.

Following [23], we define a *coaction* of a locally compact quantum group $\mathbb{G}$ on an operator space X as a completely bounded map $\alpha : X \to M(\mathrm{C}_0(\mathbb{G}) \otimes X)$ such that $(\Gamma \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\alpha$. There are only two special cases in which we need this concept. In the first case, X is a unital C*-algebra, $M(\mathrm{C}_0(\mathbb{G}) \otimes X)$ is the usual multiplier algebra, and α is a unital $*$-homomorphism. In the second case, X is a closed subspace of $\mathrm{C}_b(\mathbb{G})$,

$$M\big(\mathrm{C}_0(\mathbb{G}) \otimes X\big) = \big\{u \in M\big(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G})\big); \\ (a \otimes 1)u, u(a \otimes 1) \in \mathrm{C}_0(\mathbb{G}) \otimes X \ \forall a \in \mathrm{C}_0(\mathbb{G})\big\}, \tag{3.1}$$

and α is the restriction of the comultiplication Γ to X. So $\mathrm{C}_0(\mathbb{G}) \otimes X$ is viewed as a $\mathrm{C}_0(\mathbb{G})$-bimodule and $M(\mathrm{C}_0(\mathbb{G}) \otimes X)$ is defined with respect to the module actions. The slice maps of bounded functionals have unique extensions to $M(\mathrm{C}_0(\mathbb{G}) \otimes X)$ that are strictly continuous with respect to the module actions of $\mathrm{C}_0(\mathbb{G})$; this can be verified similarly as with the usual multiplier algebras. The reader is referred to [23, Section 1] for a proper treatment of coactions in a general setting.

We say that a closed subspace $X \subseteq \mathrm{C}_b(\mathbb{G})$ is *left invariant* if the comultiplication Γ on $\mathrm{C}_b(\mathbb{G})$ defines a coaction on X, that is, if $\Gamma(x) \in M(\mathrm{C}_0(\mathbb{G}) \otimes X)$ for every x in X. If $\mathbb{G} = G$ is a classical locally compact group, then $M(\mathrm{C}_0(G) \otimes X) = \mathrm{C}_b(G, X)$, and X is left invariant if and only if X is left translation invariant and consists of left uniformly continuous functions. That X consists of left uniformly continuous functions in the classical case is not a real restriction for us because the left uniformly continuous functions form the maximal left-introverted subspace of $\mathrm{C}_b(G)$.

A left-invariant subspace $X \subseteq \mathrm{C}_b(\mathbb{G})$ is said to be *left introverted* if

$$\nu x := (\mathrm{id} \otimes \nu)\Gamma(x)$$

is in X for every x in X and ν in X^*. Note that νx is well defined because X is left invariant and $\mathrm{id} \otimes \nu : M(\mathrm{C}_0(\mathbb{G}) \otimes X) \to \mathrm{C}_b(\mathbb{G})$. In the classical case $((\mathrm{id} \otimes \nu)\Gamma(x))(g) = \langle \nu, \ell_g x \rangle$, where ℓ_g denotes the left translation by g in G, so this definition really does agree with the classical definition of left introversion, which goes back to Day [7].

If X is a left-introverted subspace of $\mathrm{C}_b(\mathbb{G})$, then X^* is a Banach algebra with respect to the multiplication defined by

$$\langle \mu\nu, x \rangle = \langle \mu, \nu x \rangle = \langle \mu, (\mathrm{id} \otimes \nu)\Gamma(x) \rangle \quad (\mu, \nu \in X^*, x \in X).$$

It is not difficult to show that the multiplication on X^* is actually completely contractive when X^* is equipped with the dual operator space structure. It is important to note that $\mu\nu$ defined above is *not* $(\mu \otimes \nu)\Gamma$ but rather $\mu(\mathrm{id} \otimes \nu)\Gamma$.

Every right translation $\mu \mapsto \mu\nu$ is weak*-continuous on X^*, but the same is not necessarily true for the left translations.

Finally, we define a *quantum semigroup compactification* of $\mathbb{G}$ to be a left-invariant, left-introverted, unital C*-subalgebra of $C_b(\mathbb{G})$. This definition is slightly stronger than in the classical case: "left m-introverted" has changed to "left introverted". A left translation invariant subspace X of $C_b(G)$ is said to be left m-introverted if $\mu x \in X$ for every x in X and for every multiplicative mean μ on X. This is enough to give a semigroup structure to the spectrum of X, which is exactly the set of all multiplicative means on X, but is too restrictive in the noncommutative case. It should be mentioned that, for example, distal functions on the integers form a left-m-introverted C*-algebra that is not left introverted [5, Exercise 4.6.15].

The quantum Bohr compactification defined by Sołtan [28] is an example of a quantum semigroup compactification. The quantum Bohr compactification $AP(\mathbb{G})$ is the norm closure in $C_b(\mathbb{G})$ of the matrix elements of admissible finite-dimensional representations of $\mathbb{G}$. The comultiplication of $\mathbb{G}$ maps the unital C*-algebra $AP(\mathbb{G})$ to $AP(\mathbb{G}) \otimes AP(\mathbb{G})$, and $AP(\mathbb{G})$ is a compact quantum group. It is immediate that $AP(\mathbb{G})$ is left invariant and left introverted, and so a quantum semigroup compactification of $\mathbb{G}$ in our sense. As another example, we shall study the universal quantum semigroup compactification in the next section.

It is also possible to give a seemingly more general definition of a quantum semigroup compactification. Start with a coaction α of $\mathbb{G}$ on a unital C*-algebra X such that α is a unital $*$-homomorphism. Define a *compactification map* θ from $\mathbb{G}$ to X to be a unital $*$-isomorphism $\theta : X \to C_b(\mathbb{G})$ such that

$$\Gamma\theta = (\mathrm{id} \otimes \theta)\alpha \tag{3.2}$$

and

$$(\mathrm{id} \otimes \nu)\alpha(x) \in \theta(X) \quad (x \in X, \nu \in X^*) \tag{3.3}$$

(since θ is unital, $\mathrm{id} \otimes \theta$ is nondegenerate and can be extended to a map $M(C_0(\mathbb{G}) \otimes X) \to M(C_0(\mathbb{G}) \otimes C_b(\mathbb{G})))$. Then the triple (X, α, θ) is an *abstract quantum semigroup compactification* of $\mathbb{G}$. Compared with the definition of a semigroup compactification (S, ϕ) of a classical group G, the C*-algebra X corresponds to $C_b(S)$ and the map θ to ϕ^*. Condition (3.3) gives X^* a semigroup structure, and in the classical case it does so to the spectrum S of $C_b(S)$. Condition (3.2) corresponds to ϕ being a homomorphism. It might seem unnecessary to require that θ is a $*$-isomorphism—and not a mere $*$-homomorphism—but this property corresponds precisely with the requirement that $\phi(G)$ is dense in S. All in all, this abstract definition does not properly generalize the situation because the compactification map θ carries all the structure between X and $\theta(X) \subseteq C_b(\mathbb{G})$.

4. Left uniformly continuous elements

Let G be a locally compact group. A function x in $\mathrm{C}_b(G)$ is *left uniformly continuous* if the map $g \mapsto \ell_g x : G \to \mathrm{C}_b(G)$, where ℓ_g denotes the left translation by g, is norm-continuous. Noting that $M(\mathrm{C}_0(G) \otimes \mathrm{C}_b(G)) = \mathrm{C}_b(G, \mathrm{C}_b(G))$ and $\Gamma(x)(g) = \ell_g x$ for every g in G, it is natural to define that, for a locally compact quantum group $\mathbb{G}$,

$$\mathrm{LUC}(\mathbb{G}) = \big\{x \in \mathrm{C}_b(\mathbb{G}); \Gamma(x) \in M\big(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_b(\mathbb{G})\big)\big\}.$$

A similar definition appears in the context of coactions in [23, Lemma A1]. In the next section we see that $\mathrm{LUC}(\mathbb{G})$ agrees with Granirer's $\mathrm{UCB}(\widehat{G})$ when $\mathbb{G}$ is the dual of an amenable locally compact group G.

In general, $\mathrm{LUC}(\mathbb{G})$ is a unital C*-subalgebra of $\mathrm{C}_b(\mathbb{G})$. Moreover, $\mathrm{LUC}(\mathbb{G})$ includes $\mathrm{C}_0(\mathbb{G})$ because $(a \otimes 1)\Gamma(b)$ and $\Gamma(b)(a \otimes 1)$ are in $\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G})$ for every a and b in $\mathrm{C}_0(\mathbb{G})$ by [16, Proposition 1.6].

In order to prove that $\mathrm{LUC}(\mathbb{G})$ is a quantum semigroup compactification, we need to assume that $\mathrm{C}_0(\mathbb{G})$ has the following slice map property introduced by Wassermann [32] as property S. We say that a C*-algebra A has the *slice map property* if, for every C*-algebra B and its C*-subalgebra C, we have that $x \in A \otimes C$ whenever $(\mu \otimes \mathrm{id})(x) \in C$ for every μ in A^*. Every nuclear C*-algebra has the slice map property by [33]. A locally compact quantum group $\mathbb{G}$ is *coamenable* if $L^1(\mathbb{G})$ has a bounded approximate identity. A detailed study of coamenability can be found in [4], where it is shown in Theorems 3.2 and 3.3 that coamenability of $\mathbb{G}$ implies that $\mathrm{C}_0(\mathbb{G})$ is nuclear. So, in particular, if $\mathbb{G}$ is coamenable, $\mathrm{C}_0(\mathbb{G})$ has the slice map property. Note that every classical group is coamenable and the dual of a classical group is coamenable if and only if the group is amenable (by a famous result of Leptin [21]).

In the classical case, the LUC-compactification (that is, the spectrum of the left uniformly continuous functions) is the universal semigroup compactification of a given locally compact group. The following result shows that this universal property carries over to the quantum version of the LUC-compactification.

THEOREM 4.1. *Suppose that* $\mathrm{C}_0(\mathbb{G})$ *has the slice map property. Then* $\mathrm{LUC}(\mathbb{G})$ *is the universal quantum semigroup compactification of* $\mathbb{G}$ *in the sense that every other quantum semigroup compactification of* $\mathbb{G}$ *is included in* $\mathrm{LUC}(\mathbb{G})$.

Proof. The C*-algebra $\mathrm{LUC}(\mathbb{G})$ is left invariant because the comultiplication Γ maps $\mathrm{LUC}(\mathbb{G})$ to $M(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{LUC}(\mathbb{G}))$ by [23, Lemma A1]. Let $\mu \in \mathrm{LUC}(\mathbb{G})^*$ and $x \in \mathrm{LUC}(\mathbb{G})$. For every a in $\mathrm{C}_0(\mathbb{G})$,

$$\begin{aligned}(a \otimes 1)\Gamma(\mu x) &= (a \otimes 1)(\mathrm{id} \otimes \mathrm{id} \otimes \mu)\big((\Gamma \otimes \mathrm{id})\Gamma(x)\big) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes \Gamma)\big((a \otimes 1)\Gamma(x)\big)\end{aligned}$$

by coassociativity. Since $(a\otimes 1)\Gamma(x)$ is in $\mathrm{C}_0(\mathbb{G})\otimes\mathrm{LUC}(\mathbb{G})$, the function $\mathrm{id}\otimes\Gamma$ maps it to $\mathrm{C}_0(\mathbb{G})\otimes M(\mathrm{C}_0(\mathbb{G})\otimes\mathrm{LUC}(\mathbb{G}))$. Then applying $\mathrm{id}\otimes\mathrm{id}\otimes\mu$ gives an element of $\mathrm{C}_0(\mathbb{G})\otimes\mathrm{C}_b(\mathbb{G})$. Hence, $(a\otimes 1)\Gamma(\mu x)$ is in $\mathrm{C}_0(\mathbb{G})\otimes\mathrm{C}_b(\mathbb{G})$, and similarly $\Gamma(\mu x)(a\otimes 1)$ is in $\mathrm{C}_0(\mathbb{G})\otimes\mathrm{C}_b(\mathbb{G})$. Therefore, $\mu x\in\mathrm{LUC}(\mathbb{G})$, and so $\mathrm{LUC}(\mathbb{G})$ is left introverted.

By the definition of $\mathrm{LUC}(\mathbb{G})$, any left-invariant subspace of $\mathrm{C}_b(\mathbb{G})$ is included in $\mathrm{LUC}(\mathbb{G})$, which is therefore the universal quantum semigroup compactification of $\mathbb{G}$. □

Example. A locally compact quantum group $\mathbb{G}$ is said to be *discrete* if the dual of $\mathbb{G}$ is compact (i.e., $\mathrm{C}_0(\widehat{\mathbb{G}})$ has an identity) or, equivalently, if $L^1(\mathbb{G})$ has an identity [26]. When $\mathbb{G}$ is discrete, $\mathrm{C}_0(\mathbb{G})$ is a direct sum (c_0-direct sum to be precise) of full matrix algebras [34, 25]:

$$\mathrm{C}_0(\mathbb{G})=\bigoplus_{\alpha\in I}M_{n_\alpha}.$$

(This last condition is, in fact, equivalent with discreteness.) In this case, $\mathrm{C}_b(\mathbb{G})$ is the direct product (ℓ^∞-direct sum) of the same algebras:

$$\mathrm{C}_b(\mathbb{G})=\prod_{\alpha\in I}M_{n_\alpha}.$$

Next, we show that if $\mathbb{G}$ is discrete, then $\mathrm{LUC}(\mathbb{G})=\mathrm{C}_b(\mathbb{G})$.

Note first that

$$M\big(\mathrm{C}_0(\mathbb{G})\otimes\mathrm{C}_0(\mathbb{G})\big)=\prod_{\alpha,\beta\in I}M_{n_\alpha}\otimes M_{n_\beta},$$

and let $u=(u_{\alpha,\beta})$ be an element of $M(\mathrm{C}_0(\mathbb{G})\otimes\mathrm{C}_0(\mathbb{G}))$. Fix $a=(a_\alpha)$ in $\mathrm{C}_0(\mathbb{G})$ such that $a_\alpha\neq 0$ for only finitely many α's, and let $1=(1_\beta)$ be the identity in $\mathrm{C}_b(\mathbb{G})$. Then

$$(a\otimes 1)u=\big((a_\alpha\otimes 1_\beta)u_{\alpha,\beta}\big)$$

is such that $(a_\alpha\otimes 1_\beta)u_{\alpha,\beta}\neq 0$ for only finitely many α's. It follows that $(a\otimes 1)u$ is in $\mathrm{C}_0(\mathbb{G})\otimes\mathrm{C}_b(\mathbb{G})$, and by taking limits we see that this holds for any a in $\mathrm{C}_0(\mathbb{G})$.

In particular, if $a\in\mathrm{C}_0(\mathbb{G})$ and $x\in\mathrm{C}_b(\mathbb{G})$, then $(a\otimes 1)\Gamma(x)\in\mathrm{C}_0(\mathbb{G})\otimes\mathrm{C}_b(\mathbb{G})$ and similarly $\Gamma(x)(a\otimes 1)\in\mathrm{C}_0(\mathbb{G})\otimes\mathrm{C}_b(\mathbb{G})$. This shows that $x\in\mathrm{LUC}(\mathbb{G})$, and so $\mathrm{LUC}(\mathbb{G})=\mathrm{C}_b(\mathbb{G})$.

Another compactification of a discrete quantum group is presented in [29].

We record here a simple lemma for an easy reference. A variant of this lemma is well known and can be found, for example, in [3]. Recall that $M(\mathrm{C}_0(\mathbb{G})\otimes X)$ is defined by (3.1).

LEMMA 4.2. *Let X be a closed subspace of $\mathrm{C}_b(\mathbb{G})$. If $\mu\in\mathrm{C}_0(\mathbb{G})^*$ and $\nu\in X^*$, then*

$$\mu(\mathrm{id}\otimes\nu)=\nu(\mu\otimes\mathrm{id})=\mu\otimes\nu$$

on $M(\mathrm{C}_0(\mathbb{G})\otimes X)$.

The following decomposition theorem is a generalization of results concerning classical groups [10, 9] and duals of classical groups [20]. In particular, it shows that $\mathrm{C}_0(\mathbb{G})^*$ can be considered as a subalgebra of $\mathrm{LUC}(\mathbb{G})^*$, a fact we shall frequently use later on.

THEOREM 4.3. *Let X be a closed left-introverted subspace of $\mathrm{C}_b(\mathbb{G})$ such that $\mathrm{C}_0(\mathbb{G}) \subseteq X$. Then there is a completely isometric algebra isomorphism $\tau : \mathrm{C}_0(\mathbb{G})^* \to X^*$ such that*

$$X^* = \tau(\mathrm{C}_0(\mathbb{G})^*) \oplus \mathrm{C}_0(\mathbb{G})^\perp.$$

The annihilator $\mathrm{C}_0(\mathbb{G})^\perp$ is a weak-closed ideal in X^*.*

Proof. For any μ in $\mathrm{C}_0(\mathbb{G})^*$, let $\tau(\mu)$ be the unique extension of μ to X that is strictly continuous on bounded sets. Since μ can be written as $\mu'.a$ where $\mu' \in \mathrm{C}_0(\mathbb{G})^*$ and $a \in \mathrm{C}_0(\mathbb{G})$, we have that

$$\langle \tau(\mu), x\rangle = \langle \mu', ax\rangle \quad (x \in X).$$

We begin by showing that τ is a homomorphism. First, we consider $\tau(\nu)x$ for ν in $\mathrm{C}_0(\mathbb{G})^*$ and x in X. Let $f \in L^1(\mathbb{G})$ and let (c_α) be a bounded net in $\mathrm{C}_0(\mathbb{G})$ that converges strictly to x. By Lemma 4.2,

$$\langle \tau(\nu)x, f\rangle = \langle \tau(\nu), (f \otimes \mathrm{id})\Gamma(x)\rangle = \lim \langle \nu, (f \otimes \mathrm{id})\Gamma(c_\alpha)\rangle$$

because both Γ and $f \otimes \mathrm{id}$ are strictly continuous on bounded sets. Continuing the calculation, we have that

$$\langle \tau(\nu)x, f\rangle = \lim \langle f, (\mathrm{id} \otimes \nu)\Gamma(c_\alpha)\rangle = \langle (\mathrm{id} \otimes \nu)\Gamma(x), f\rangle$$

where $\mathrm{id} \otimes \nu : M(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{G})) \to \mathrm{C}_b(\mathbb{G})$.

Now let $\mu \in \mathrm{C}_0(\mathbb{G})^*$. By the previous calculation,

$$\langle \tau(\mu)\tau(\nu), x\rangle = \langle \tau(\mu), (\mathrm{id} \otimes \nu)\Gamma(x)\rangle,$$

which shows that $\tau(\mu)\tau(\nu)$ is strictly continuous on bounded sets. Therefore, it suffices to prove that $\tau(\mu)\tau(\nu) = \mu * \nu$ on $\mathrm{C}_0(\mathbb{G})$, but, for every a in $\mathrm{C}_0(\mathbb{G})$,

$$\langle \tau(\mu)\tau(\nu), a\rangle = \langle \mu, (\mathrm{id} \otimes \nu)\Gamma(a)\rangle = \langle \mu * \nu, a\rangle.$$

To see that τ is a complete isometry, let $[\mu_{ij}] \in M_n(\mathrm{C}_0(\mathbb{G})^*)$, that is, $[\mu_{ij}]$ is an $n \times n$ matrix with entries in $\mathrm{C}_0(\mathbb{G})^*$. Obviously $\|\tau^{(n)}[\mu_{ij}]\|_n \geq \|[\mu_{ij}]\|_n$, where $\tau^{(n)}$ denotes the nth amplification of τ, so we only need to show the converse. Fix $\varepsilon > 0$. By Smith's lemma [8, Proposition 2.2.2],

$$\left\|\tau^{(n)}[\mu_{ij}]\right\|_n = \|[\tau(\mu_{ij})]\|_{\mathrm{CB}(X,M_n)} = \sup_{[x_{kl}]} \|[\langle \tau(\mu_{ij}), x_{kl}\rangle]\|_{M_{n^2}},$$

where the supremum runs through all $[x_{kl}]$ in $M_n(X)$ with norm less than or equal to 1. By Cohen's factorization theorem, we may choose a from

$C_0(\mathbb{G})$ and $[\nu_{ij}]$ from $M_n(C_0(\mathbb{G})^*)$ such that $\|a\| = 1$, $\|\mu_{ij} - \nu_{ij}\| < \varepsilon/n^4$, and $\mu_{ij} = \nu_{ij}.a$. Then, continuing the preceding calculation,

$$\begin{aligned}\left\|\tau^{(n)}[\mu_{ij}]\right\|_n &= \sup_{[x_{kl}]} \|[\langle \nu_{ij}, ax_{kl}\rangle]\|_{M_{n^2}} \le \sup_{[x_{kl}]} \|[\langle \nu_{ij}, x_{kl}\rangle]\|_{M_{n^2}} \\ &\le \|[\mu_{ij}]\|_n + \sup_{[x_{kl}]} \sum_{i,j,k,l} |\langle \mu_{ij} - \nu_{ij}, x_{kl}\rangle| < \|[\mu_{ij}]\|_n + \sum_{i,j,k,l} \varepsilon/n^4 \\ &= \|[\mu_{ij}]\|_n + \varepsilon,\end{aligned}$$

as required.

For every μ in X^*, define $\mu_0 = \tau(\mu|_{C_0(\mathbb{G})})$ and $\mu_1 = \mu - \mu_0$. Then μ and μ_0 agree on $C_0(\mathbb{G})$ so $\mu_1 \in C_0(\mathbb{G})^\perp$. It is clear that this procedure gives an algebraic direct sum decomposition

$$X^* = \tau(C_0(\mathbb{G})^*) \oplus C_0(\mathbb{G})^\perp.$$

Obviously, $C_0(\mathbb{G})^\perp$ is weak*-closed. Finally, we show that it is an ideal. Let $\nu \in C_0(\mathbb{G})^\perp$ and let $\mu = \mu_0 + \mu_1$ be the decomposition of μ in X^*. For every a and b in $C_0(\mathbb{G})$,

$$b(\nu a) = (\mathrm{id} \otimes \nu)\big((b \otimes 1)\Gamma(a)\big) = 0$$

because $(b \otimes 1)\Gamma(a) \in C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$. Since b is arbitrary, it follows that $\nu a = 0$ and so $\mu\nu \in C_0(\mathbb{G})^\perp$. On the other hand,

$$\langle \nu\mu_0, a\rangle = \langle \nu, (\mathrm{id} \otimes \mu_0)\Gamma(a)\rangle = 0$$

because $\mu_0 \in \tau(C_0(\mathbb{G})^*)$. Since both $\nu\mu_0$ and $\nu\mu_1$ are in $C_0(\mathbb{G})^\perp$, also $\nu\mu$ is in $C_0(\mathbb{G})^\perp$. □

From now on, we consider $C_0(\mathbb{G})^*$ as a subalgebra of $\mathrm{LUC}(\mathbb{G})^*$ and suppress the isomorphism τ used in the preceding theorem.

The following lemma is a direct consequence of Lemma 4.2. Determining the topological center is a much harder task: see, for example, [18, 19].

LEMMA 4.4. *Suppose that* $C_0(\mathbb{G})$ *has the slice map property. Then* $C_0(\mathbb{G})^*$ *is included in the topological center of* $\mathrm{LUC}(\mathbb{G})^*$, *that is, for every* μ *in* $C_0(\mathbb{G})^*$, *the map* $\nu \mapsto \mu\nu : \mathrm{LUC}(\mathbb{G})^* \to \mathrm{LUC}(\mathbb{G})^*$ *is weak*–weak*-continuous.*

Proof. If $\mu \in C_0(\mathbb{G})^*$, $\nu \in \mathrm{LUC}(\mathbb{G})^*$, and $x \in \mathrm{LUC}(\mathbb{G})$, then

$$\langle \mu\nu, x\rangle = \langle \mu, (\mathrm{id} \otimes \nu)\Gamma(x)\rangle = \langle \nu, (\mu \otimes \mathrm{id})\Gamma(x)\rangle$$

by Lemma 4.2. The statement follows immediately. □

5. Duals of classical groups and $\mathrm{LUC}(\mathbb{G})$

In the classical setting when G is a locally compact group, $\mathrm{LUC}(G) = L^\infty(G)L^1(G)$ where the action of $L^1(G)$ on $L^\infty(G)$ comes from the first Arens product (and shall be described soon). This result inspired Granirer [11] to define the space of bounded uniformly continuous functionals by setting $\mathrm{UCB}(\widehat{G}) = \mathrm{A}(G)\cdot \mathrm{VN}(G)$, where $\mathrm{A}(G)$ is the Fourier algebra of G, which is the predual of the group von Neumann algebra $\mathrm{VN}(G)$. In this section, we study the relation between $\mathrm{LUC}(\mathbb{G})$, as we defined in the previous section, and $L^\infty(\mathbb{G})L^1(\mathbb{G})$.

Define an action of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$ by setting

$$\langle xf, g\rangle = \langle x, f*g\rangle = \langle (f\otimes \mathrm{id})\Gamma(x), g\rangle$$

whenever $x \in L^\infty(\mathbb{G})$ and $f, g \in L^1(\mathbb{G})$. Then define $L^\infty(\mathbb{G})L^1(\mathbb{G})$ to be the norm-closed linear span of elements of the form xf with x in $L^\infty(\mathbb{G})$ and f in $L^1(\mathbb{G})$. If $\mathbb{G}$ is coamenable, every member of $L^\infty(\mathbb{G})L^1(\mathbb{G})$ is of the form xf. In [13] the space $\mathrm{RUC}(\mathbb{G})$, which is the right-hand side analogue of $\mathrm{LUC}(\mathbb{G})$, is *defined* to be $L^1(\mathbb{G})\cdot L^\infty(\mathbb{G})$ (using the opposite side action). Our notation digresses in this regard and should not be confused with the notation of [13].

It should be noted that (topologically) left-invariant and (topologically) left-introverted subspaces X of $L^\infty(\mathbb{G})$ can be defined using the above action and the action defined by

$$\langle \nu x, f\rangle = \langle \nu, xf\rangle \quad \big(\nu \in X^*, x\in X, f\in L^1(\mathbb{G})\big).$$

These notions of invariance and introversion agree with the definitions in Section 3 if we consider only subspaces of $\mathrm{LUC}(\mathbb{G})$. Theorem 4.3 holds true also for these topologically left-introverted subspaces of $\mathrm{C}_b(\mathbb{G})$.

THEOREM 5.1. *Suppose that $\mathbb{G}$ is a coamenable locally compact quantum group. Then* $\mathrm{LUC}(\mathbb{G}) \subseteq L^\infty(\mathbb{G})L^1(\mathbb{G})$.

Proof. We first show that $L^1(\mathbb{G})$ is weak*-dense in $\mathrm{LUC}(\mathbb{G})^*$. Indeed, $L^1(\mathbb{G})$ is weak*-dense in its second dual $L^\infty(\mathbb{G})^*$ and restricting the functionals in $L^1(\mathbb{G})$ to $\mathrm{LUC}(\mathbb{G})$ gives a weak*-dense subspace of $\mathrm{LUC}(\mathbb{G})^*$. Since each f in $L^1(\mathbb{G})$ is strictly continuous when restricted to $\mathrm{LUC}(\mathbb{G})$, it follows that this weak*-dense copy of $L^1(\mathbb{G})$ in $\mathrm{LUC}(\mathbb{G})^*$ is the same one that is obtained in Theorem 4.3.

Since $\mathbb{G}$ is coamenable, there is an identity ε in $\mathrm{C}_0(\mathbb{G})^*$. Let $\mu \in \mathrm{LUC}(\mathbb{G})^*$ and let (f_α) be a net in $L^1(\mathbb{G})$ converging to μ in the weak* topology. The left translation by ε is weak*-continuous on $\mathrm{LUC}(\mathbb{G})^*$ by Lemma 4.4, so $\varepsilon\mu = \lim \varepsilon * f_\alpha = \lim f_\alpha = \mu$. Similarly, $\mu\varepsilon = \mu$ because the right translations are always weak*-continuous on $\mathrm{LUC}(\mathbb{G})^*$. Therefore, ε is an identity also in $\mathrm{LUC}(\mathbb{G})^*$.

Let $x \in \mathrm{LUC}(\mathbb{G})$ and let (e_α) be a net in $L^1(\mathbb{G})$ that converges to ε in the weak* topology of $\mathrm{LUC}(\mathbb{G})^*$. For every μ in $\mathrm{LUC}(\mathbb{G})^*$,

$$\langle \mu, xe_\alpha \rangle = \langle e_\alpha, \mu x \rangle \to \langle \varepsilon, \mu x \rangle = \langle \mu, x \rangle$$

by Lemma 4.2. So xe_α converges weakly to x in $\mathrm{LUC}(\mathbb{G})$. Let K be the convex hull of $\{e_\alpha\}$. It follows from the Hahn–Banach separation theorem that x is in the norm closure of $xK \subseteq L^\infty(\mathbb{G})L^1(\mathbb{G})$. But $L^\infty(\mathbb{G})L^1(\mathbb{G})$ is closed so we are done. □

As a by-product of the preceding proof, we get that $\mathrm{LUC}(\mathbb{G})^*$ is unital when $\mathbb{G}$ is coamenable.

The next lemma is proved for duals of classical groups in [12]. We denote the C*-algebra of compact operators on a Hilbert space H by $\mathrm{B}_0(H)$.

LEMMA 5.2. $L^\infty(\mathbb{G})L^1(\mathbb{G}) \subseteq \mathrm{C}_b(\mathbb{G})$.

Proof. Given x in $L^\infty(\mathbb{G})$ and f in $L^1(\mathbb{G})$, we should show that $a(xf)$ and $(xf)a$ are in $\mathrm{C}_0(\mathbb{G})$ for every a in $\mathrm{C}_0(\mathbb{G})$. Write $f = g(K\cdot)$ where $g \in \mathrm{B}(L^2(\mathbb{G}))_*$ and $K \in \mathrm{B}_0(L^2(\mathbb{G}))$. Then

$$\begin{aligned} a(xf) &= (f \otimes \mathrm{id})\big((1 \otimes a)\Gamma(x)\big) \\ &= (g \otimes \mathrm{id})\big((K \otimes a)V(x \otimes 1)V^*\big). \end{aligned}$$

The operator V is in $M(\mathrm{B}_0(L^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{G}))$ (perhaps the simplest reference for this fact is [35], knowing that V is manageable [15, Proposition 6.10]) and so $(K \otimes a)V$ is in $\mathrm{B}_0(L^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{G})$. It follows that $(K \otimes a)V(x \otimes 1)V^*$ is also in $\mathrm{B}_0(L^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{G})$, and so $a(xf) \in \mathrm{C}_0(\mathbb{G})$, as required. That also $(xf)a$ is in $\mathrm{C}_0(\mathbb{G})$ can be proved similarly. □

Baaj and Skandalis defined and studied regular multiplicative unitaries in their fundamental paper [2]. Let H be a Hilbert space. A unitary operator W on $H \otimes H$ is said to be *multiplicative* if it satisfies the pentagonal relation (2.1). Then a multiplicative unitary W is *regular* if the norm-closed linear span of

$$\{(K \otimes 1)W(1 \otimes F); K, F \in \mathrm{B}_0(H)\}$$

is $\mathrm{B}_0(H \otimes H)$. As noted in [2], Kac algebras, and so locally compact groups and their duals, are determined by regular multiplicative unitaries. However, for example, the multiplicative unitary associated with the quantum group $E_\mu(2)$ is not regular [1], so regularity is truly a restrictive assumption for the following result. Unfortunately, even the notion of manageability introduced by Woronowicz [35] does not seem to help eliminate the regularity assumption.

THEOREM 5.3. *Suppose that* $\mathbb{G}$ *is a locally compact quantum group such that the multiplicative unitary* V *is regular. Then* $L^\infty(\mathbb{G})L^1(\mathbb{G}) \subseteq \mathrm{LUC}(\mathbb{G})$.

Proof. By the preceding lemma $L^\infty(\mathbb{G})L^1(\mathbb{G}) \subseteq \mathrm{C}_b(\mathbb{G})$, so it suffices to show that $\Gamma(xf) \in M(\mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_b(\mathbb{G}))$ for every x in $L^\infty(\mathbb{G})$ and f in $L^1(\mathbb{G})$. Note first that

$$\Gamma(xf) = \Gamma\big((f \otimes \mathrm{id})\Gamma(x)\big) = (f \otimes \mathrm{id} \otimes \mathrm{id})\big((\Gamma \otimes \mathrm{id})\Gamma(x)\big)$$

by coassociativity.

Write $f = g(K \cdot F)$ where $g \in \mathrm{B}(L^2(\mathbb{G}))_*$ and $K, F \in \mathrm{B}_0(L^2(\mathbb{G}))$. For every a in $\mathrm{C}_0(\mathbb{G})$,

$$\begin{aligned}(a \otimes 1)\Gamma(xf) &= (f \otimes \mathrm{id} \otimes \mathrm{id})\big((1 \otimes a \otimes 1)(\Gamma \otimes \mathrm{id})\Gamma(x)\big)\\ &= (g \otimes \mathrm{id} \otimes \mathrm{id})\big((K \otimes a \otimes 1)V_{12}V_{13}(x \otimes 1 \otimes 1)V_{13}^*V_{12}^*(F \otimes 1 \otimes 1)\big).\end{aligned}$$

Since $V \in M(\mathrm{B}_0(L^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{G}))$, we can replace $(K \otimes a \otimes 1)V_{12}$ in the above calculation by $K \otimes a \otimes 1$ with K still in $\mathrm{B}_0(\mathbb{G})$ and a in $\mathrm{C}_0(\mathbb{G})$. Then we can transfer $1 \otimes a \otimes 1$ to right so that we obtain the term

$$(1 \otimes a \otimes 1)V_{12}^*(F \otimes 1 \otimes 1),$$

which is in $\mathrm{B}_0(L^2(\mathbb{G})) \otimes \mathrm{C}_0(\mathbb{G}) \otimes 1$ by [2, Proposition 3.6] since V is regular. Again, replace this term by $F \otimes a \otimes 1$ with F in $\mathrm{B}_0(\mathbb{G})$ and a in $\mathrm{C}_0(\mathbb{G})$. Then, continuing the calculation, we have

$$\begin{aligned}&(g \otimes \mathrm{id} \otimes \mathrm{id})\big((K \otimes 1 \otimes 1)V_{13}(x \otimes 1 \otimes 1)V_{13}^*(F \otimes a \otimes 1)\big)\\ &\quad = a \otimes (h \otimes \mathrm{id})\Gamma(x) = a \otimes xh,\end{aligned}$$

where $h = g(K \cdot F)$ is in $\mathrm{B}(L^2(\mathbb{G}))_*$ (the slight abuse of notation is not a problem here: $(h \otimes \mathrm{id})\Gamma(x)$ is well defined and agrees with xh' when h' is the restriction of h to $L^\infty(\mathbb{G})$). But $xh \in \mathrm{C}_b(\mathbb{G})$ by the preceding lemma, so we get that $(a \otimes 1)\Gamma(xf) \in \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_b(\mathbb{G})$. Similarly, $\Gamma(xf)(a \otimes 1) \in \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_b(\mathbb{G})$, and so $xf \in \mathrm{LUC}(\mathbb{G})$. □

In particular, if $\mathbb{G} = \widehat{G}$ is a dual of a classical locally compact group G, then $\mathrm{UCB}(\widehat{G}) = L^\infty(\mathbb{G})L^1(\mathbb{G}) \subseteq \mathrm{LUC}(\mathbb{G})$ and the equality holds if G is amenable.

Acknowledgments. I thank the referee for helpful comments and suggestions and in particular for pointing out the references [25] and [29].

References

[1] S. Baaj, *Représentation régulière du groupe quantique des déplacements de Woronowicz*, Recent advances in operator algebras (Orléans, 1992), Astérisque, no. 232, 1995, pp. 11–48. MR 1372523

[2] S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres*, Ann. Sci. École Norm. Sup. (4) **26** (1993), 425–488. MR 1235438

[3] E. Bédos, G. J. Murphy and L. Tuset, *Amenability and coamenability of algebraic quantum groups*, Int. J. Math. Math. Sci. **31** (2002), 577–601. MR 1931751

[4] E. Bédos and L. Tuset, *Amenability and co-amenability for locally compact quantum groups*, Internat. J. Math. **14** (2003), 865–884. MR 2013149

[5] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on semigroups*, John Wiley & Sons, New York, 1989. MR 0999922

[6] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin, 1973. MR 0423029
[7] M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544. MR 0092128
[8] E. G. Effros and Z.-J. Ruan, *Operator spaces*, Clarendon Press, Oxford, New York, 2000. MR 1793753
[9] M. Filali, *On the ideal structure of some algebras with an Arens product*, Recent progress in functional analysis (Valencia, 2000), North-Holland Math. Stud., vol. 189, North-Holland, Amsterdam, 2001, pp. 289–297. MR 1861765
[10] F. Ghahramani, A. T.-M. Lau, and V. Losert, *Isometric isomorphisms between Banach algebras related to locally compact groups*, Trans. Amer. Math. Soc. **321** (1990), 273–283. MR 1005079
[11] E. E. Granirer, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **189** (1974), 371–382. MR 0336241
[12] E. E. Granirer, *Density theorems for some linear subspaces and some C^*-subalgebras of* $\mathrm{VN}(G)$, Convegno sull'Analisi Armonica e Spazi di Funzioni su Gruppi Localmente Compatti (Rome, 1976), Symp. Math., vol. 22, Academic Press, London, 1977, pp. 61–70. MR 0487287
[13] Z. Hu, M. Neufang and Z.-J. Ruan, *Multipliers on a new class of Banach algebras, locally compact quantum groups, and topological centres*, Proc. Lond. Math. Soc. (3) **100** (2010), 429–458. MR 2595745
[14] J. Kustermans, *One-parameter representations on C*-algebras*, Preprint, 1997; available at `arXiv:funct-an/9707009`.
[15] J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. (4) **33** (2000), 837–934. MR 1832993
[16] J. Kustermans and S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand. **92** (2003), 68–92. MR 1951446
[17] E. C. Lance, *Hilbert C*-modules. A toolkit for operator algebraists*, Cambridge University Press, Cambridge, 1995. MR 1325694
[18] A. T.-M. Lau, *Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups*, Math. Proc. Cambridge Phil. Soc. **99** (1986), 273–283. MR 0817669
[19] A. T.-M. Lau and V. Losert, *The C^*-algebra generated by operators with compact support on a locally compact group*, J. Funct. Anal. **112** (1993), 1–30. MR 1207935
[20] A. T.-M. Lau, R. J. Loy and G. A. Willis, *Amenability of Banach and C^*-algebras on locally compact groups*, Studia Math. **119** (1996), 161–178. MR 1391474
[21] H. Leptin, *Sur l'algèbre de Fourier d'un groupe localement compact*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968), A1180–A1182. MR 0239002
[22] T. Masuda, Y. Nakagami and S. L. Woronowicz, *A C*-algebraic framework for quantum groups*, Internat. J. Math. **14** (2003), 903–1001. MR 2020804
[23] C.-K. Ng, *Cohomology of Hopf C*-algebras and Hopf von Neumann algebras*, Proc. London Math. Soc. (3) **83** (2001), 708–742. MR 1851088
[24] G. Pisier, *Introduction to operator space theory*, Cambridge University Press, Cambrigde, 2003. MR 2006539
[25] P. Podleś and S. L. Woronowicz, *Quantum deformation of Lorentz group*, Comm. Math. Phys. **130** (1990), 381–431. MR 1059324
[26] V. Runde, *Characterizations of compact and discrete quantum groups through second duals*, J. Operator Theory **60** (2008), 415–428. MR 2464219
[27] V. Runde, *Uniform continuity over locally compact quantum groups*, J. Lond. Math. Soc. (2) **80** (2009), 55–71. MR 2520377
[28] P. M. Sołtan, *Quantum Bohr compactification*, Illinois J. Math. **49** (2005), 1245–1270. MR 2210362

[29] P. M. Sołtan, *Compactifications of discrete quantum groups*, Algebr. Represent. Theory **9** (2006), 581–591. MR 2272699

[30] M. Takesaki, *Theory of operator algebras II*, Springer-Verlag, Berlin, 2003. MR 1943006

[31] A. Van Daele, *Locally compact quantum groups. A von Neumann algebra approach*, Preprint, 2006; available at arXiv:math/0602212.

[32] S. Wassermann, *The slice map problem for C^*-algebras*, Proc. London Math. Soc. (3) **32** (1976), 537–559. MR 0410402

[33] S. Wassermann, *A pathology in the ideal space of $L(H) \otimes L(H)$*, Indiana Univ. Math. J. **27** (1978), 1011–1020. MR 0511255

[34] S. L. Woronowicz, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 845–884. MR 1616348

[35] S. L. Woronowicz, *From multiplicative unitaries to quantum groups*, Internat. J. Math. **7** (1996), 127–149. MR 1369908

Pekka Salmi, University of Oulu, Department of Mathematical Sciences, PL 3000, FI-90014 Oulun yliopisto, Finland

E-mail address: pekka.salmi@iki.fi