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SOME REFLECTIONS ON PROVING GROUPS RESIDUALLY TORSION-FREE NILPOTENT. I

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ABSTRACT. The objective of this note is to give some alternative proofs that the intersection of the normal subgroups with torsion-free nilpotent factor groups of free groups is the identity. Our approach gives rise to a new family of one-relator groups with this property. This family includes, among others, the fundamental groups of orientable surfaces. Our hope is that the idea involved will lead to new insights into these well-known groups, and, even more important for us, a proof that free $\mathbb{Q}$-groups have the same property.

1. Introduction

1.1. Magnus' proof revisited. Recall that a group is said to be residually torsion-free nilpotent if the intersection of its normal subgroups with torsion-free nilpotent quotients is trivial. Proving that a given group is residually torsion-free nilpotent is usually hard. In 1935, Wilhelm Magnus [11] introduced associative algebras together with topology into the mix making it possible to prove that free groups are residually torsion-free nilpotent. In order to explain his proof, we need to recall a related definition. To this end, let R be an augmented associative algebra over $\mathbb{Q}$, the field of rational numbers. So by definition, R is unitary and contains an ideal A, called the augmentation ideal of R, such that $R/A \cong \mathbb{Q}$. We say that R is residually nilpotent if the powers of A intersect in 0. Now the group algebra $\mathbb{Q}G$ of a group G is an augmented algebra where the augmentation ideal consists of those elements of $\mathbb{Q}G$ with coefficient sum 0. Then the group G turns out to be residually torsion-free nilpotent if and only if $\mathbb{Q}G$ is residually nilpotent. The powers of the augmentation ideal of an augmented associated unitary algebra can be used as the basis of a topology which allows us to form the

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completion $\widehat{R}$ of R. The group of units of $\widehat{R}$ is usually quite rich and often provides one with a good supply of residually torsion-free nilpotent groups. In the event that $R = \mathbb{Q}G$, where G is the free group on $X = \{x_i \mid i \in I\}$, then it is not hard to see that

$$\widehat{R} = \langle\langle \xi \mid i \in I \rangle\rangle,$$

is the algebra of power series in the free (and hence, noncommuting) variables $\xi_i = 1 - x_i$ $(i \in I)$. This allows us to reexpress x_i^{-1} as a power series which takes the form

$$x_i^{-1} = (1 - \xi_i)^{-1} = 1 + \xi_i + \xi_i^2 + \cdots .$$

It follows that we have proved that free groups are subgroups of the group of units of such completions $\widehat{R}$. This makes it easy to prove that free groups are residually torsion-free nilpotent since commutators of larger and larger weight get closer and closer to the identity. This brilliant insight is due to Magnus [11]. The idea was generalized by A. I. Malcev in a beautifully written paper [13] in which he proved, in particular, that free products of residually torsion-free nilpotent groups are residually torsion-free nilpotent.

1.2. $\mathbb{Q}$-groups. We observe next that the set H consisting of those elements $h \in \widehat{R}$ of the form $h = 1 - \alpha$, where the constant term of α is zero, is a $\mathbb{Q}$-group, that is, a group in which extraction of nth-roots is uniquely possible for every positive integer n. $\mathbb{Q}$-groups form a variety of algebras and so free objects exist. These are the free $\mathbb{Q}$-groups. If F is the free $\mathbb{Q}$-group on X, then the mapping which sends x_i to $1 - \xi_i$ can be continued to a homomorphism of F into $\widehat{R}$. It turns out that free $\mathbb{Q}$-groups are residually torsion-free nilpotent if and only if this homomorphism is a monomorphism. Now free $\mathbb{Q}$-groups are the end result of repeatedly freely adjoining nth roots to a free group and the resultant groups. In 1968, I could only show that one adjunction did give rise to a residually torsion-free nilpotent group [2]. The argument was quite technical and seems hard to generalize.

The primary aim of this note is to describe some new ways of proving groups residually torsion-free nilpotent in the hope that they will shed some light on how to prove free $\mathbb{Q}$-groups are residually torsion-free nilpotent. Although our efforts have only been minimally successful, they have led to a new family of one-relator groups that are residually torsion-free nilpotent which include, among others, the fundamental groups of orientable surfaces. They provide potentially new insights into these well-known groups. One of our ways of proving free groups residually torsion-free nilpotent is to make use of wreath products and work of A. I. Lichtman [9], suggesting that an exploration of an analogue of wreath products for $\mathbb{Q}$-groups will be worth-while.

2. Summary of results

We give first two proofs that free groups are residually torsion-free nilpotent.

2.1. A combinatorial proof of residual nilpotence. The first of these in, Section 4, makes no use of Magnus' original proof [11] described above. It, as well as the proof of our main theorem, Theorem 1, follows very closely the argument in [4]. It is hard to describe the proof without giving most of the details, which we have chosen to do here. Despite its simplicity it reduces the proof of residual torsion-free nilpotence to that of an allied group providing a very different perspective as to why free groups turn out to be residually torsion-free nilpotent.

2.2. Using wreath products. The second proof, in Section 6, has been organized so that it requires only a few lines. It makes use of work of A. I. Lichtman [9], which itself depends on a second theorem of Magnus [12], which tacitly makes use of wreath products. However, this proof can be adjusted so that it too can be made completely elementary. It is worth pointing out that the same sort of argument can be used to prove various other residual properties of free groups. Perhaps even more important here is that it suggests that an exploration of analogues of wreath products in the category of $\mathbb{Q}$-groups could turn out to be quite interesting. It promises to give rise to a number of new results about $\mathbb{Q}$-groups. Among these would be a proof that free $\mathbb{Q}$-groups are residually torsion-free nilpotent. In addition, it would provide a means for understanding the monoid of varieties of $\mathbb{Q}$-groups. In the case of every-day groups, the corresponding monoid is free (Shmelkin [17] and independently, B. H., Hanna and Peter Neumann [15]).

2.3. Our main theorem. We will prove in Section 5 the following theorem.

THEOREM 1. *Let $Y = \{b, \dots, c\}$ and let w be a Y-word. Then*

$$G = \langle t, a, b, \dots, c; t^{-1}at = aw \rangle$$

is residually torsion-free nilpotent.

The proof of Theorem 1 follows readily along the lines of our direct proof that free groups are residually torsion-free nilpotent. It can be considerably generalized but the additional complexity does not seem worth-while at this point.

Before commenting on some possible impacts of the method of proof of Theorem 1, it is worth recording two of its immediate consequences, both of which are well known.

2.4. Some consequences of Theorem 1 and its proof. First, we have the following corollary.

Corollary 2. *The fundamental groups G_k of two dimensional orientable surfaces are residually torsion-free nilpotent.*

We need only note here that the G_k can be presented in the form

$$G_k = \langle t, a, x_1, y_1, \dots, x_k, y_k \mid t^{-1}at = a[x_1, y_1] \cdots [x_k, y_k] \ (k \geq 0) \rangle.$$

This was proved first in [1] and at about the same time, by Karen N. Frederick [6]. The residual nilpotence of the G_k is a corollary of a more general theorem, namely that the family of finite presentations of the form

$$\langle a, b, \dots, c, t; [w, t] \rangle$$

define residually free groups provided only that the words w represent elements in the free group on $a, b, \dots, c$ which are not proper powers. In particular then it follows, as noted for the first time in [1], that the G_k are residually free.

Second, we have the following corollary.

Corollary 3. *The groups*

$$\langle a, b, c; [a, b] = c^n \rangle$$

are residually torsion-free nilpotent.

It follows, as noted in [5] as well as in [2], that a commutator in a free group is a proper power only if it is trivial. This was first proved by Magnus, Karrass and Solitar in [8]. These and other results about equations in free groups can be deduced on knowing that free groups are residually finite p-groups, a less demanding requirement than being residually torsion-free nilpotent.

An inspection of the proof of Theorem 1 suggests that it may well allow one to find some finer residual properties of a number of groups. We will focus here only on surface groups. It is well known that surface groups have solvable conjugacy problem. It probably follows from the proof of Theorem 1 that a little more is true, namely that two elements are conjugate if and only if they are conjugate in every nilpotent quotient. Moreover, it is easy to find an algorithm which determines whether or not an element in a surface group is a proper power. The proof of Theorem 1 suggests again that more is true, namely that an element in a surface group is a proper power if and only if it is a proper power modulo every term of its lower central series. More interestingly, it may allow us to determine the quotient of the group automorphisms of a surface group which are trivial modulo the derived group by the subgroup of inner automorphisms.

We need to recall some notation and definitions.

3. Notation and definitions

Let G be a group and let $x_1, x_2, \ldots$ be elements of G. We denote the commutator $x_1^{-1}x_2^{-1}x_1x_2$ by $[x_1, x_2]$ and define, for $n > 1$,

$$[x_1, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}].$$

If H and K are subgroups of G, we define

$$[H, K] = gp([h, k] \mid h \in H, k \in K).$$

The *lower central series*

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots$$

of G is defined inductively by setting

$$\gamma_{n+1}(G) = [\gamma_n(G), G].$$

As usual, G is *nilpotent* if $\gamma_{c+1}(G) = 1$ for some c, with the least such c the *class* of G. Now the elements of finite order in a nilpotent group H form a normal subgroup $\mathrm{tor}(H)$, the *torsion subgroup* of H. So it makes sense to define

$$\overline{\gamma}_n(G)/\gamma_n(G) = \mathrm{tor}\big(G/\gamma_n(G)\big).$$

We define

$$\overline{\gamma}_\omega(G) = \bigcap_{n=1}^{\infty} \overline{\gamma}_n(G).$$

Then it follows that a group G is residually torsion-free nilpotent if $\overline{\gamma}_\omega(G) = 1$. So the residually torsion-free nilpotent groups consist of those groups G such that for each $g \in G,\ g \neq 1$, there exists a normal subgroup N of G, which may depend on g, such that $g \notin N$ and G/N is a torsion-free nilpotent group. For convenience, we denote the class of torsion-free nilpotent groups by $\mathcal{T}$ and the class of residually torsion-free nilpotent groups by $r\mathcal{T}$. We will need also the class $\mathcal{T}_c$ of torsion-free nilpotent groups of class at most c. We term a group in $r\mathcal{T}$ free in the class $r\mathcal{T}$ if it can be generated by a set X such that every map from X into an $r\mathcal{T}$ group can be continued to a homomorphism. Of course we already know that the free groups in $r\mathcal{T}$ are the absolutely free groups. Our objective here is to give a proof of this theorem which differs from the one given by Magnus. We note that at this point all we can assert is that the free groups in $r\mathcal{T}$ are the quotients $F/\overline{\gamma}_\omega(F)$ where F is a free group. Similarly, the free groups in $\mathcal{T}_c$ are the quotients $(F/\gamma_{c+1}(F))/\mathrm{tor}(F/\gamma_{c+1}(F))$.

4. Our first proof of Magnus' theorem that free groups are residually torsion-free nilpotent

Here, we describe our first, direct proof that free groups are residually torsion-free nilpotent.

The proof uses the trick that Magnus introduced in his proof of the Freiheitssatz [10] and a small variation of the idea used in the paper [4].

4.1. The start of the proof. We shall have need of a variation of a family of torsion-free nilpotent groups $H(Y,n,c)$ introduced in [2] which have to be adjusted for the purposes that we have in mind here.

4.2. The groups $H(Y,n,c)$. The groups $H(Y,n,c)$ are infinite cyclic extensions of a family of groups $N = N(Y,n,c)$ which depend on three parameters, a set Y and two positive integers n and c. The groups $N = N(Y,n,c)$ are free in the class $\mathcal{T}_c$, freely generated by a set

$$\{y_1, \ldots, y_n \mid y \in Y\}$$

indexed by the integers $\{1, \ldots, n\}$ and the elements $y \in Y$. We will need the following lemma.

LEMMA 4. *Let N be as above and for each $y \in Y$, let*

$$z(y,1) = y_1 y_2, \qquad z(y,2) = y_2 y_3, \qquad \ldots,$$
$$z(y,n-1) = y_{n-1} y_n, \qquad z(y,n) = y_n.$$

Then

$$z(y,1), \qquad \ldots, \qquad z(y,n)$$

freely generate the group N, viewed as a free group in the class $\mathcal{T}_c$.

In order to prove the lemma, we note that any set of elements of a nilpotent group which generates it modulo its derived group, generates the whole group. Since the elements $z(y,h)$ generate N modulo its derived group, they generate N. So the map which sends y_h to $z(y,h)$ for $y \in Y$ and $h = 1, \ldots, n$, defines an epimorphism μ of N. Notice that μ then induces an epimorphism of $gp(y_1, \ldots, y_n)$ and since finitely generated nilpotent groups are hopfian, μ is monic on $gp(y_1, \ldots, y_n (y \in Y))$. Hence, μ is an automorphism of N which suffices to prove the lemma.

We come now to the definition of the groups $H(Y,n,c)$. Each of them is an extension of the corresponding group $N = N(Y,n,c)$ by the infinite cyclic group generated by t which acts on N by μ:

$$t^{-1} y_1 t = y_1 y_2, \qquad \ldots, \qquad t^{-1} y_{n-1} t = y_{n-1} y_n, \qquad t^{-1} y_n t = y_n$$

for $y \in Y$. So

$$[t, y_1] = y_2, \qquad [t, y_2] = y_3, \qquad \ldots, \qquad [t, y_{n-1}] = y_n, \qquad [t, y_n] = 1.$$

It follows readily from these relations, that modulo $\gamma_2(N)$, $H(Y,n,c)$ is nilpotent of class n. Hence, by a theorem of P. Hall [7], $H(Y,n,c)$ is also nilpotent. It is clearly torsion-free. So we have proved the following.

LEMMA 5. *The groups $H(Y,n,c)$ are torsion-free nilpotent groups for every choice of Y, n and c.*

We shall have need of the following observation about the conjugates of the elements y_i of $H(Y,n,c)$:

LEMMA 6. *The conjugates*

$$t^{-1}y_1t, \qquad t^{-2}y_1t^2, \qquad \ldots, \qquad t^{-n}y_1t^n$$

are linearly independent modulo the derived group of $N(Y,n,c)$ *for each* $y \in Y$.

Each of the conjugates $t^{-i}y_1t^i$ can be expressed, modulo the derived group of N, as a product w_iy_i, where w_i is a word in $y_1,\ldots,y_{i-1}$ for $i=0,\ldots,n$. This suffices to prove the lemma.

4.3. The end of the direct proof of Magnus' theorem. Let F be the free group, freely generated by the set X and let $f \in F$ be a nontrivial element of F. We need to prove that there exists a normal subgroup K of F such that $f \notin K$ with F/K torsion-free nilpotent. The proof is by induction on the length ℓ of f. We can assume that $f \in F'$, the derived group of F for if $f \notin F'$, then we can take $K = F'$.

It follows that we can assume that one of the elements of X, say s, occurs with exponent sum 0 in f. Let P be the normal closure in F of $X' = X - \{s\}$. Then P is freely generated by the elements

$$x_i = s^{-i}xs^i \qquad (x \in X', i \in \mathbb{Z}).$$

Notice that $f \in P$. Therefore, on replacing f by a conjugate by s^i if necessary, f can expressed as a word f' in the generators

$$x_i = s^{-i}xs^i \qquad (x \in X', i \in \{1,\ldots,n\}).$$

We observe that the length of f' is at most $\ell - 2$, since s occurs in f with exponent sum 0. So, inductively, we can find $c > 0$ such that

$$f \notin \overline{\gamma}_{c+1}(P).$$

We now choose the set Y introduced previously to have the same cardinality as X' equipped with a matching $\phi\colon X' \longrightarrow Y$. We now define a homomorphism μ of F onto $H(Y,n,c)$ as follows:

$$\mu : s \mapsto t, \qquad x \mapsto x\phi \qquad (x \in X').$$

μ induces a surjection

$$\mu_* : F/\overline{\gamma}_{c+1}(P) \longrightarrow H(Y,n,c)$$

which maps $x_i\overline{\gamma}_{c+1}(P)$ onto $(x\phi)_i$ for $i = 1,\ldots,n$ and $x \in X'$. The restriction of μ_* to

$$gp(x_i\overline{\gamma}_{c+1}(P) \mid i = 1,\ldots,n, x \in X'),$$

is an isomorphism between

$$gp(x_i\overline{\gamma}_{c+1}(P) \mid i = 1,\ldots,n, x \in X'),$$

and $N(Y,n,c)$ by Lemma 6. Hence,

$$f\mu = f'\mu_* \neq 1.$$

So if K is the kernel of μ, F/K is torsion-free and nilpotent and $f \notin K$. This completes the proof of the theorem.

5. A family of residually torsion-free nilpotent one-relator groups

We arrive finally at the proof of Theorem 1, which is modeled on the argument described in Section 4, and should be compared with the work of P. C. Wong in [18].

5.1. The groups $J = J(Y^+, n, c, w)$. We will need a slight variation of the groups $H(Y, n, c)$ introduced in Section 4. We label these groups $J = J(Y^+, n, c, w)$, where here w is an arbitrary word in the generators Y. The groups J are infinite cyclic extensions of a corresponding family of groups $O = O(Y^+, n, c, w)$. The groups O are free nilpotent of class c freely generated by the elements

$$\{a\} \cup \{y_1, \dots, y_n \mid y \in Y\}$$

indexed as before by the integers $\{1, \dots, n\}$, the elements $y \in Y$ and a single extra element a.

Each of the groups J is an extension of the corresponding group O by the infinite cyclic group generated by an element t acting on O as follows:

$$t^{-1}at = aw, \qquad t^{-1}y_1t = y_1y_2, \qquad \dots,$$
$$t^{-1}y_{n-1}t = y_{n-1}y_n, \qquad t^{-1}y_nt = y_n$$

for $y \in Y$. It follows from these relations that modulo $\gamma_2(O)$, J is nilpotent of class at most $c+1$—we have only to compute its upper centrals series. Hence, again using the theorem of P. Hall [7], J is also nilpotent. It is clearly torsion-free. So we have proved the following lemma.

LEMMA 7. *The groups $J(Y^+, n, c, w)$ are torsion-free nilpotent groups for every choice of Y, n, c and w.*

Notice again as before, that the following lemma holds.

LEMMA 8. *The elements*

$$t^{-1}y_1t, \qquad t^{-2}y_1t^2, \qquad \dots, \qquad t^{-n}y_1t^n \qquad (y \in Y)$$

together with a are linearly independent modulo the derived group of O.

5.2. The proof of Theorem 1. We are now in position to prove Theorem 1. To this end consider the normal closure L in G of the elements in the set $\{a\} \cup Y$. Now $t^{-1}at = aw$ in G, where $w \in gp(Y)$. It follows from Magnus' proof of his Freiheitsatz that L is a free group, freely generated by a together with the conjugates $y_i = t^{-i}yt^i$ of the elements $y \in Y$.

We have to prove that if u is a non-trivial element of G then there is a homomorphism of G onto a torsion-free nilpotent group for which the image of u is nontrivial. If $u \notin L$, we can take G/L to be that torsion-free nilpotent group. So we can restrict attention to the case where $u \in L$. Replacing u by a suitable conjugate of u by a power of t allows us to assume that $u \in gp(\{y_0, y_1, \ldots, y_n\ (y \in Y)\})$. Choose c so that $u \notin \gamma_{c+1}(L)$. Now let ρ be the homomorphism of G onto $J(Y^+, n, c, w)$ which maps t, a and the elements of Y to their correspondingly named images in J. Since the elements $a, y_1, \ldots, y_n$ are linearly independent modulo the derived group of O, u maps onto a nontrivial element of O, which completes the proof of Theorem 1.

6. Proving free groups residually torsion-free nilpotent using wreath products

6.1. Using Lichtmans's theorem. If one strips away some of the technicalities, there is a very simple proof that free groups are residually torsion-free. To this end, we will need the following theorem of A. I. Lichtman [9]: *if F is a free group, if $R \trianglelefteq F$ and if F/R is residually torsion-free nilpotent, then $F/[R,R]$ is also residually torsion-free nilpotent.* We choose $R = \overline{\gamma}_\omega(F)$. Notice that R is the smallest normal subgroup of F with residually torsion-free nilpotent quotient. We claim that $R = 1$. Suppose the contrary. Then by Lichtman's theorem, $F/[R,R]$ is residually torsion-free nilpotent. But $[R,R] < \overline{\gamma}_\omega(F)$, which is not possible. This completes the proof.

6.2. Where do wreath products come into play and can they be used in the case of $\mathbb{Q}$-groups? We recall first the definition of a wreath product. Suppose that A and T are subgroups of the group W. If A and T generate W and if the conjugates A^t of A by the elements $t \in T$ are distinct and generate their direct product, then W is termed the wreath product of A by T and denoted $A \wr T$. Magnus [12] proved that if F is a free group, if $R \trianglelefteq F$ and if $A = R/[R,R]$, $T = F/R$, then $F/[R,R]$ can be embedded in $W = A \wr T$. The residual torsion-free nilpotence of W follows from the residual nilpotence of the rational group algebra of F/R. This outline provides a blueprint for proving free $\mathbb{Q}$-groups residually torsion-free nilpotent. One of the key steps needed then is a theorem of S. V. Polin [16] who has proved that $\mathbb{Q}$-subgroups of free $\mathbb{Q}$-groups are again free. So what remains is finding an analogue of Magnus' theorem for $\mathbb{Q}$-groups. Now given a group G together with a normal subgroup H of G, the right-regular representation of G on the cosets of H gives rise to an embedding of G into the so-called unrestricted wreath product of H by G/N. An inspection of this observation can then be used to prove Magnus' theorem. Perhaps the same approach can be taken in the case of $\mathbb{Q}$-groups.

7. Final reflections

It seems to be difficult to find conditions that ensure that a given one-relator group is residually torsion-free nilpotent. The only result of a similar kind is that a one-relator group defined by a positive word is residually solvable [3]. Whether there are algorithms to decide if a one-relator group is residually torsion-free nilpotent or residually nilpotent or residually solvable are intriguing open problems The recent book by Mikhailov and Passi [14] contains a wealth of information and a number of interesting results which seem to be relevant but do not exactly fit into this line of investigation.

The proof in [2] that adjoining an nth root to an element in a free group which is not a proper power gives rise to a residually torsion-free nilpotent group involves understanding the kernel of a homomorphism onto an infinite cyclic group. The relevance of our approach here is that it suggests that this kernel can be approximated by a very special torsion-free nilpotent group and so can be generalized to handle more than a single adjunction.

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