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REGULARITY FOR COMPLETE AND MINIMAL GABOR SYSTEMS ON A LATTICE

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ABSTRACT. Nonsymmetrically weighted extensions of the Balian–Low theorem are proved for Gabor systems $\mathcal{G}(g,1,1)$ that are complete and minimal in $L^2(\mathbb{R})$. For $g \in L^2(\mathbb{R})$, it is proved that if $3 < p \leq 4 \leq q < \infty$ satisfy $3/p + 1/q = 1$ and $\int |x|^p |g(x)|^2\,dx < \infty$ and $\int |\xi|^q |\widehat{g}(\xi)|^2\,d\xi < \infty$ then $\mathcal{G}(g,1,1) = \{e^{2\pi inx} g(x-k)\}_{k,n\in\mathbb{Z}}$ cannot be complete and minimal in $L^2(\mathbb{R})$. For the endpoint case $(p,q) = (3,\infty)$, it is proved that if $g \in L^2(\mathbb{R})$ is compactly supported and $\int |\xi|^3 |\widehat{g}(\xi)|^2\,d\xi < \infty$ then $\mathcal{G}(g,1,1)$ is not complete and minimal in $L^2(\mathbb{R})$. These theorems extend the work of Daubechies and Janssen from the case $(p,q) = (4,4)$. Further refinements and optimal examples are also provided.

1. Introduction

Given $g \in L^2(\mathbb{R})$ and constants a, $b > 0$, the associated *Gabor system* is $\mathcal{G}(g,a,b) = \{g_{m,n}\}_{m,n\in\mathbb{Z}} \subset L^2(\mathbb{R})$, where

$$g_{m,n}(x) = e^{2\pi imbx} g(x-na), \quad m,n \in \mathbb{Z}.$$

Thus, $\mathcal{G}(g,a,b)$ consists of time-frequency shifts of the window function g along the lattice $a\mathbb{Z} \times b\mathbb{Z}$ in the time-frequency plane. Gabor systems are of interest in theory and practice because they can be used to perform signal analysis and to provide decompositions of function spaces, for example, see [11], [12], [15]. In particular, it is important to determine for which $g \in L^2(\mathbb{R})$ and a, $b > 0$ the Gabor system $\mathcal{G}(g,a,b)$ has desirable basis or spanning properties. Frame theory is a standard setting in which to study such issues, for example, see [8].

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A collection $\{f_n\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is a *frame* for $L^2(\mathbb{R})$ with *frame bounds* $0 < A \le B < \infty$ if

$$\forall f \in L^2(\mathbb{R}), \quad A\|f\|_2^2 \le \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \le B\|f\|_2^2.$$

A standard result in frame theory states that if $\{f_n\}_{n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ then there exists a (possibly nonunique) *dual frame* $\{\widetilde{f}_n\}_{n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ such that

$$\forall f \in L^2(\mathbb{R}), \quad f = \sum_{n\in\mathbb{Z}} \langle f, \widetilde{f}_n\rangle f_n = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle \widetilde{f}_n, \tag{1.1}$$

where the series converge unconditionally in $L^2(\mathbb{R})$.

A collection $\{g_n\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is *minimal* in $L^2(\mathbb{R})$ if for each $N \in \mathbb{Z}$ we have $g_N \notin \overline{\mathrm{span}}\{g_n : n \in \mathbb{Z}, n \ne N\}$. It is *complete* if $\overline{\mathrm{span}}\{g_n\}_{n\in\mathbb{Z}} = L^2(\mathbb{R})$. A sequence that is both minimal and complete is said to be *exact*. Every exact system $\{f_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ has a unique dual system $\{g_n\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R})$ that satisfies the biorthogonality conditions $\langle f_j, g_k\rangle = \delta_{j,k}$. In general, however, exactness is a rather weak condition that is not even enough to guarantee signal expansions such as in Equation (1.1).

All frames are complete, but a frame is minimal if and only if it is the image of an orthonormal basis under a continuous, invertible map of $L^2(\mathbb{R})$ onto itself, and in this case it is called a *Riesz basis*. Thus, the class of exact frames coincides with the class of Riesz bases. In particular, every orthonormal basis is a Riesz basis, and every Riesz basis is an exact system. However, not all exact systems are Riesz bases.

The Balian–Low theorem states that if $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$ then the window function g must be poorly localized in either time or frequency. There are several variations on the Balian–Low theorem; the classical version as proved in [2], [3], [9], [21] is as follows. We use a Fourier transform normalized as $\widehat{g}(\xi) = \int g(x)e^{-2\pi i\xi x}\,dx$.

THEOREM 1.1 (Balian–Low theorem). *If $g \in L^2(\mathbb{R})$ satisfies*

$$\int |x|^2 |g(x)|^2\,dx < \infty \quad \textit{and} \quad \int |\xi|^2 |\widehat{g}(\xi)|^2\,d\xi < \infty,$$

then $\mathcal{G}(g, 1, 1)$ is not a Riesz basis for $L^2(\mathbb{R})$.

The following theorem addresses optimality in the Balian–Low theorem, see [4]. For $(p, p') = (2, 2)$ it shows that the Balian–Low theorem is essentially sharp, and for other values of (p, p') it provides examples of Gabor orthonormal bases $\mathcal{G}(g, 1, 1)$ with nonsymmetric time-frequency localization. Other related sharpness theorems can be found in [6], [7], [18].

Theorem 1.2. *Fix $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and let $2 < s$. There exists $g \in L^2(\mathbb{R})$ such that $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and such that*

$$\int \frac{|x|^p}{\log^s(2+|x|)} |g(x)|^2 \, dx < \infty \quad \textit{and} \quad \int \frac{|\xi|^{p'}}{\log^s(2+|\xi|)} |\widehat{g}(\xi)|^2 \, d\xi < \infty.$$

For $(p,p') \neq (2,2)$, Theorem 1.2 raised the natural question of whether there exist versions of the Balian–Low theorem for nonsymmetric weight pairs $|x|^p$, $|\xi|^{p'}$. Some initial progress appeared in [5], and the question was answered in full generality by Gautam in [14], with the following theorem.

Theorem 1.3. *Let $g \in L^2(\mathbb{R})$ be given.*

(1) *Fix $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. If*

$$\int |x|^p |g(x)|^2 \, dx < \infty \quad \textit{and} \quad \int |\xi|^{p'} |\widehat{g}(\xi)|^2 \, d\xi < \infty,$$

then $\mathcal{G}(g,1,1)$ is not a Riesz basis for $L^2(\mathbb{R})$.

(2) *If g has compact support and satisfies*

$$\int |\xi| |\widehat{g}(\xi)|^2 \, d\xi < \infty,$$

then $\mathcal{G}(g,1,1)$ is not a Riesz basis for $L^2(\mathbb{R})$.

The following theorem generalizes the Balian–Low theorem to Gabor systems $\mathcal{G}(g,1,1)$ that are exact but need not be frames, see [9]. The full extent of the Balian–Low theorems for complete and minimal systems proved in [9] is quite elegant and appears to have been somewhat overlooked in the subsequent literature.

Theorem 1.4. *If $g \in L^2(\mathbb{R})$ satisfies*

$$\int |x|^4 |g(x)|^2 \, dx < \infty \quad \textit{and} \quad \int |\xi|^4 |\widehat{g}(\xi)|^2 \, d\xi < \infty,$$

then $\mathcal{G}(g,1,1)$ is not exact in $L^2(\mathbb{R})$.

Other versions of the Balian–Low theorem for exact Gabor systems appear in [9], [17], and quantify time-frequency concentration in terms of membership in modulation spaces or Wiener amalgam spaces instead of Heisenberg-like products.

2. Main results

Our first main result extends Theorem 1.4 to nonsymmetric (p,q) weight pairs. Interestingly, in contrast to Theorem 1.3 for Riesz bases, which relies on the standard dual index relation $\frac{1}{p} + \frac{1}{p'} = 1$, here the relation between p and q is $\frac{3}{p} + \frac{1}{q} = 1$. The proof of Theorem 2.1 will be given in Section 4.

THEOREM 2.1. *If $g \in L^2(\mathbb{R})$ satisfies*

$$\int |x|^p |g(x)|^2 \, dx < \infty \quad \text{and} \quad \int |\xi|^q |\widehat{g}(\xi)|^2 \, d\xi < \infty, \tag{2.1}$$

where

$$3 < p \leq 4 \quad \text{and} \quad \frac{3}{p} + \frac{1}{q} = 1,$$

then $\mathcal{G}(g,1,1)$ is not exact.

The same result holds if the roles of p and q are interchanged in (2.1).

Our next main result addresses the endpoint case $(3,\infty)$, which is not covered by Theorem 2.1. The proof of Theorem 2.2 will be given in Section 5.

THEOREM 2.2. *If $g \in L^2(\mathbb{R})$ is compactly supported and satisfies*

$$\int |\xi|^3 |\widehat{g}(\xi)|^2 \, d\xi < \infty, \tag{2.2}$$

then $\mathcal{G}(g,1,1)$ is not exact in $L^2(\mathbb{R})$.

The same result holds if the compact support condition is assumed to hold for $\widehat{g}$ and the decay condition (2.2) is assumed to hold for g.

The final part of this paper, Section 6, is devoted to constructively demonstrating that the $(3,\infty)$ endpoint Balian–Low theorem for exact Gabor systems obtained in Theorem 2.2 is sharp. Specifically, we will show in that section that for any sufficiently small $\varepsilon > 0$ there exists a function $g \in L^2(\mathbb{R})$ such that g is supported in $[0,2]$, $\int |\xi|^{3-\varepsilon} |\widehat{g}(\xi)|^2 \, d\xi < \infty$, and $\mathcal{G}(g,1,1)$ is exact.

3. Background: The Zak transform

In this section, we review some facts that we will need about the Zak transform, which is an important tool in the analysis of Gabor systems.

The Zak transform of $g \in L^2(\mathbb{R})$ is the function Zg of two variables defined by

$$Zg(x,\xi) = \sum_{n \in \mathbb{Z}} g(x-n) e^{2\pi i n \xi}, \quad x, \xi \in \mathbb{R}.$$

The Zak transform has many properties which make it useful for studying Gabor systems. A measurable function $G : \mathbb{R}^2 \to \mathbb{C}$ is said to be *quasiperiodic* if

$$\forall x, \xi \in \mathbb{R}, \quad G(x, \xi + 1) = G(x,\xi) \quad \text{and} \quad G(x+1,\xi) = G(x,\xi) e^{2\pi i \xi}.$$

It is straightforward to verify that the Zak transform Zg of $g \in L^2(\mathbb{R})$ is a quasiperiodic function. Without loss of information, one may therefore concentrate on the restriction of Zg to the unit square $[0,1]^2$ since its values on $\mathbb{R}^2$ are then uniquely determined by quasiperiodic extension. With this identification, Z defines a unitary mapping $Z : L^2(\mathbb{R}) \to L^2([0,1]^2)$, see [15]. This unitarity yields the following result that allows us to equate properties of $\mathcal{G}(g,a,b)$ with properties of Zg.

THEOREM 3.1. *Given $g \in L^2(\mathbb{R})$, the following statements hold.*

(1) $\mathcal{G}(g,1,1)$ *is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|Zg| = 1$ a.e.*
(2) $\mathcal{G}(g,1,1)$ *is a Riesz basis for $L^2(\mathbb{R})$ if and only if there exist $0 < A \leq B < \infty$ such that $A \leq |Zg| \leq B$ a.e.*
(3) $\mathcal{G}(g,1,1)$ *is exact in $L^2(\mathbb{R})$ if and only if $1/Zg \in L^2([0,1]^2)$.*
(4) $\mathcal{G}(g,1,1)$ *is complete in $L^2(\mathbb{R})$ if and only if $Zg \neq 0$ a.e.*

It is useful to note that the conditions of Theorem 3.1 are invariant under the Fourier transform because of the Zak transform identity $Z\widehat{g}(x,\xi) = e^{2\pi i x\xi} Zg(-\xi, x)$. For example, since our main focus will be on exact systems, the following equivalent result holds.

(3′) $\mathcal{G}(g,1,1)$ is exact in $L^2(\mathbb{R})$ if and only if $1/Z\widehat{g} \in L^2([0,1]^2)$.

The utility of the Zak transform for studying Gabor systems extends well beyond the list of properties in Theorem 3.3. For example, the Zak transform can also be used to characterize Gabor systems that are Bessel sequences, frames, or Schauder bases, see [15], [17].

Given its close connection with Gabor systems, it is not surprising that the Zak transform may be interpreted as a time-frequency representation. For example, the following result follows from Theorem 4.4 in [16] and Theorem 1 in [10].

THEOREM 3.2. *Let $\varepsilon > 0$ and $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ be given. If $g \in L^2(\mathbb{R})$ satisfies*

$$\int |x|^{p+\varepsilon} |f(x)|^2 \, dx < \infty \quad \text{and} \quad \int |\xi|^{p'+\varepsilon} |\widehat{f}(\xi)|^2 \, d\xi < \infty$$

then the Zak transform Zg is continuous on $\mathbb{R}^2$.

The importance of the continuity of Zg comes from the following property of quasiperiodic functions [15].

THEOREM 3.3. *If $G : \mathbb{R}^2 \to \mathbb{C}$ is quasi-periodic and continuous then G has a zero.*

Consequently, if $g \in L^2(\mathbb{R})$ and Zg is continuous, then by Theorem 3.1, $\mathcal{G}(g,1,1)$ cannot be a Riesz or orthonormal basis for $L^2(\mathbb{R})$. The hypotheses in the Balian–Low theorem do not imply that Zg is continuous, and the proof of Theorem 1.1 requires a more subtle analysis of Zg.

The hypotheses corresponding to the endpoint case $\varepsilon = 0$ in Theorem 3.2 also do not imply that Zg is continuous, but this can be addressed in terms of Sobolev spaces. Given $r, s > 0$ and $G \in L^2(\mathbb{R}^2)$, we say that G is in the mixed Sobolev space $S_{r,s}(\mathbb{R}^2)$ if

$$\|G\|_{S_{r,s}} = \left(\int\!\!\int (|u|^r + |v|^s + 1) |\widehat{G}(u,v)|^2 \, du \, dv \right)^{1/2} < \infty.$$

The following theorem was obtained in [14].

THEOREM 3.4. *If $r, s > 0$ and $g \in L^2(\mathbb{R})$ satisfy*

$$\int |x|^r |g(x)|^2 \, dt < \infty \quad \text{and} \quad \int |\xi|^s |\widehat{g}(\xi)|^2 \, d\xi < \infty,$$

then for any smooth compactly supported function $\varphi \in C_c^\infty(\mathbb{R}^2)$ we have $\varphi Zg \in S_{r,s}(\mathbb{R}^2)$.

This again leads to the conclusion that $\mathcal{G}(g,1,1)$ cannot be a Riesz basis for $L^2(\mathbb{R})$.

4. Nonsymmetric Balian–Low theorem for exact Gabor systems

We will prove Theorem 2.1 in this section.

In addition to the usual notations of analysis, the notation $A \lesssim B$ will mean that there exists an absolute constant C such that $A \le CB$. We allow the absolute constant C to depend on other fixed locally defined quantities when appropriate, and will not explicitly point out these dependencies since they are usually clear from context.

Proof of Theorem 2.1. I. For the given range of parameters p, q we have $q > p'$, so it follows from Theorem 3.2 that Zg is continuous. Therefore, by Theorem 3.3, the quasiperiodic function Zg has a zero, and hence $\mathcal{G}(g,1,1)$ is not a Riesz basis. We must show that $\mathcal{G}(g,1,1)$ is not exact. Without loss of generality, we assume that the zero of Zg is located at the origin, that is, $Zg(0,0) = 0$.

II. We shall estimate the behavior of $Zg(x,y)$ near $(0,0)$. It follows from Theorem 3.4 that Zg is locally in the mixed Sobolev space $S_{p,q}(\mathbb{R}^2)$. That is, if $\varphi \in C_c^\infty(\mathbb{R}^2)$ is any smooth cutoff function and $G = \varphi Zg$, then

$$\iint |\widehat{G}(\gamma,\xi)|^2 (1 + |\gamma|^p + |\xi|^q) \, d\gamma \, d\xi < \infty.$$

We take a smooth cutoff function $\varphi \in C_c^\infty(\mathbb{R}^2)$ that is identically one on an open neighborhood of $(0,0)$. It then suffices to estimate $G = \varphi Zg$ near $(0,0)$. We calculate that

$$\begin{aligned} |G(x,y)| &= |G(x,y) - G(0,0)| \\ &\le \iint |\widehat{G}(\gamma,\xi)| \left|1 - e^{2\pi i (x\gamma + y\xi)}\right| d\gamma \, d\xi \\ &\lesssim \left(\iint |\widehat{G}(\gamma,\xi)|^2 (1 + |\gamma|^p + |\xi|^q) \, d\gamma \, d\xi \right)^{1/2} \\ &\qquad \times \left(\iint \frac{\sin^2 \pi (x\gamma + y\xi)}{1 + |\gamma|^p + |\xi|^q} \, d\gamma \, d\xi \right)^{1/2} \\ &= \|G\|_{S_{p,q}} \left(\iint \frac{\sin^2 \pi (x\gamma + y\xi)}{1 + |\gamma|^p + |\xi|^q} \, d\gamma \, d\xi \right)^{1/2}. \end{aligned} \tag{4.1}$$

So, we need to bound the integral

$$\begin{aligned}
I &= \int\!\!\int \frac{\sin^2 \pi(x\gamma + y\xi)}{1+|\gamma|^p+|\xi|^q}\, d\gamma\, d\xi \\
&\lesssim \int\!\!\int \frac{\sin^2 \pi(x\gamma)}{1+|\gamma|^p+|\xi|^q}\, d\gamma\, d\xi + \int\!\!\int \frac{\sin^2 \pi(y\xi)}{1+|\gamma|^p+|\xi|^q}\, d\gamma\, d\xi = I_1 + I_2.
\end{aligned}$$

III. The following change of variables formula holds for $s > 1$, $r > 0$ and will be useful in our subsequent computations:

$$\begin{aligned}
\int_0^\infty \frac{1}{1+|\gamma|^r+|\xi|^s}\, d\xi &= \frac{1}{1+|\gamma|^r} \int_0^\infty \frac{1}{1+\left|\frac{\xi}{(1+|\gamma|^r)^{1/s}}\right|^s}\, d\xi \\
&= (1+|\gamma|^r)^{\frac{1}{s}-1}\left(\int_0^\infty \frac{1}{1+|u|^s}\, du\right).
\end{aligned} \tag{4.2}$$

IV. To find an upper bound on I, we begin by estimating the integral I_1:

$$I_1 \lesssim \int_0^\infty \int_0^\infty \frac{\sin^2 \pi(x\gamma)}{1+|\gamma|^p+|\xi|^q}\, d\xi\, d\gamma = \int_{\frac{1}{|x|}}^\infty \int_0^\infty + \int_0^{\frac{1}{|x|}} \int_0^\infty = J_1 + J_2.$$

First, note that $\frac{3}{p} + \frac{1}{q} = 1$ and $3 < p \le q < \infty$, so applying equation (4.2) we obtain

$$\begin{aligned}
J_1 &\le \int_{1/|x|}^\infty \int_0^\infty \frac{1}{1+|\gamma|^p+|\xi|^q}\, d\xi\, d\gamma \\
&= \int_{1/|x|}^\infty (1+|\gamma|^p)^{\frac{1}{q}-1}\left(\int_0^\infty \frac{1}{1+|u|^q}\, du\right) d\gamma \\
&\lesssim \int_{1/|x|}^\infty |\gamma|^{\frac{p}{q}-p}\, d\gamma \\
&= \int_{1/|x|}^\infty |\gamma|^{-3}\, d\gamma \lesssim |x|^2.
\end{aligned}$$

Next, if we assume $|x| < \frac{1}{2}$ then by again applying Equation (4.2) we have

$$\begin{aligned}
J_2 &\le \pi^2|x|^2 \int_0^{1/|x|} \int_0^\infty \frac{|\gamma|^2}{1+|\gamma|^p+|\xi|^q}\, d\xi\, d\gamma \\
&= \pi^2|x|^2 \int_0^{1/|x|} |\gamma|^2(1+|\gamma|^p)^{\frac{1}{q}-1}\left(\int_0^\infty \frac{1}{1+|u|^q}\, du\right) d\gamma \\
&\lesssim |x|^2 \int_0^{1/|x|} (1+|\gamma|^p)^{\frac{2}{p}+\frac{1}{q}-1}\, d\gamma \\
&= |x|^2 \int_0^{1/|x|} \frac{1}{(1+|\gamma|^p)^{1/p}}\, d\gamma \\
&\lesssim |x|^2(1+\log|x|^{-1}) \lesssim |x|^2|\log|x||.
\end{aligned}$$

It follows that for $|x|$ near 0 we have:

$$I_1 \lesssim |x|^2 + |x|^2 |\log|x|| \lesssim |x|^2 |\log|x||.$$

V. Next, we estimate the integral I_2 by writing

$$\begin{aligned} I_2 &\lesssim \int_0^\infty \int_0^\infty \frac{\sin^2 \pi(y\xi)}{1+|\gamma|^p + |\xi|^q}\, d\gamma\, d\xi \\ &= \int_{\frac{1}{|y|}}^\infty \int_0^\infty + \int_0^{\frac{1}{|y|}} \int_0^\infty = K_1 + K_2. \end{aligned}$$

The estimates for K_1 and K_2 are similar to the ones for J_1 and J_2, but there are differences due to the asymmetry between p and q. In particular, we have $\frac{3}{p} + \frac{1}{q} = 1$ (but not $\frac{3}{q} + \frac{1}{p} = 1$) and $3 < p \le 4 \le q < \infty$.

Since $\frac{1}{q} + \frac{3}{p} = 1$, we have $q = \frac{p}{p-3}$ and therefore $\frac{q}{p} - q = \frac{1-p}{p-3} = F(p)$. For p in the interval $(3, 4]$, we have $F(p) \le -3$. Using this and proceeding as for J_1, we obtain for $|y| < 1$ that

$$K_1 \lesssim \int_{1/|y|}^\infty |\xi|^{\frac{q}{p} - q}\, d\xi \lesssim \int_{1/|y|}^\infty |\xi|^{-3}\, d\xi \lesssim |y|^2.$$

Similarly, for $|y| < \frac{1}{2}$,

$$\begin{aligned} K_2 &\lesssim |y|^2 \int_0^{1/|y|} (1+|\xi|^q)^{\frac{2}{q} + \frac{1}{p} - 1}\, d\xi \\ &= |y|^2 \int_0^{1/|y|} (1+|\xi|^q)^{\frac{1}{q}(2 + \frac{q}{p} - q)}\, d\xi \\ &\le |y|^2 \int_0^{1/|y|} (1+|\xi|^q)^{-\frac{1}{q}}\, d\xi \\ &\lesssim |y|^2 (1 + \log|y|^{-1}) \lesssim |y|^2 |\log|y||, \end{aligned}$$

where we have again used the fact that $\frac{q}{p} - q \le -3$. Combining these estimates for K_1 and K_2, we have

$$I_2 \lesssim |y|^2 + |y|^2 |\log|y|| \lesssim |y|^2 |\log|y||.$$

VI. Combining the estimates for I_1 and I_2 and using Equation (4.1), we see that in a neighborhood of $(0,0)$ we have

$$|G(x,y)|^2 \lesssim I \lesssim I_1 + I_2 \lesssim |x|^2 |\log|x|| + |y|^2 |\log|y||. \tag{4.3}$$

In particular, these estimates show that one almost has a Sobolev embedding into the space of Lipschitz continuous functions. More importantly for us,

$$\int_0^1 \int_0^1 \frac{1}{|G(x,y)|^2}\, dx\, dy \gtrsim \int_0^{1/2} \int_0^{1/2} \frac{1}{|x|^2 |\log|x|| + |y|^2 |\log|y||}\, dx\, dy = \infty.$$

It follows that $1/Zg \notin L^2[0,1]^2$, and hence that $\mathcal{G}(g,1,1)$ is not exact.

Finally, note that the same theorem holds with the roles of p and q interchanged in (2.1) because of condition (3′) following Theorem 3.1. □

REMARK 4.1. For perspective on the upper bound

$$I=\int_0^\infty\int_0^\infty \frac{\sin^2\pi(x\gamma+y\xi)}{1+|\gamma|^p+|\xi|^q}\,d\gamma\,d\xi \lesssim |x|^2|\log|x||+|y|^2|\log|y||$$

used in the proof of Theorem 2.1, we note the following lower bound. Assume that $0\le x,\ y\le 1$, and let

$$\Omega=\Omega_{x,y}=\left\{(\gamma,\xi):0\le x\gamma\le\frac{1}{4}\text{ and }0\le y\xi\le\frac{1}{4}\right\}.$$

Then

$$\begin{aligned}
I &\ge \iint_\Omega \frac{\sin^2\pi(x\gamma+y\xi)}{1+|\gamma|^p+|\xi|^q}\,d\gamma\,d\xi \gtrsim \iint_\Omega \frac{(x\gamma+y\xi)^2}{1+|\gamma|^p+|\xi|^q}\,d\gamma\,d\xi\\
&\gtrsim |x|^2\int_0^{1/(4x)}\int_0^{1/(4y)} \frac{|\gamma|^2}{1+|\gamma|^p+|\xi|^q}\,d\gamma\,d\xi\\
&\quad+|y|^2\int_0^{1/(4x)}\int_0^{1/(4y)} \frac{|\xi|^2}{1+|\gamma|^p+|\xi|^q}\,d\gamma\,d\xi\\
&\gtrsim |x|^2\int_0^{1/4}\int_0^{1/4} \frac{|\gamma|^2}{1+|\gamma|^p+|\xi|^q}\,d\gamma\,d\xi\\
&\quad+|y|^2\int_0^{1/4}\int_0^{1/4} \frac{|\xi|^2}{1+|\gamma|^p+|\xi|^q}\,d\gamma\,d\xi\\
&\gtrsim |x|^2+|y|^2.
\end{aligned}$$

REMARK 4.2. As noted after (4.3), the condition (2.1) with $\frac{3}{p}+\frac{1}{q}=1$ implies that the Zak tranform Zg is almost Lipschitz continuous. If one instead imposes the stronger condition $\frac{3}{p}+\frac{1}{q}<1$ in Theorem 2.1 then similar computations as in the proof of Theorem 2.1 show that the Zak transform Zg is Lipschitz continuous. This should be compared with the condition $\frac{1}{p}+\frac{1}{p'}=1$ in Theorem 1.3. The proof of Theorem 1.3 involves an embedding of the Zak transform Zg into the space of vanishing mean oscillation (which often serves as a reasonable substitute for the space of continuous functions). If one imposes the stronger condition $\frac{1}{p}+\frac{1}{p'}<1$ then by Theorem 3.2 one has that the Zak transform is continuous. In view of this, Theorems 1.1, 1.3, 1.4, and 2.1 deal with "critical" time-frequency regularities.

5. Endpoint regularity for exact Gabor systems

This section is devoted to proving Theorem 2.2, which addresses the endpoint case $(3,\infty)$ in Theorem 2.1.

We make use of the following additional notation. For $g \in L^2(\mathbb{R})$ and $s > 0$, we shall say that $g \in H^s(\mathbb{R})$ if $\|g\|_{H^s} = (\int |\xi|^{2s} |\widehat{g}(\xi)|^2 \, d\xi)^{1/2} < \infty$. Also, χ_S will denote the characteristic function of a set $S \subseteq \mathbb{R}$.

Proof of Theorem 2.2. I. It follows from Theorem 3.2 that Zg is continuous on $\mathbb{R}^2$ and hence that Zg has a zero. We assume without loss of generality that $Zg(0,0) = 0$. We will show that $1/Zg$ cannot be in $L^2[0,1]^2$, which in turn implies that $\mathcal{G}(g,1,1)$ is not exact.

II. Assume that $g \in H^{3/2}(\mathbb{R})$. Then $\widehat{g} \in L^1(\mathbb{R})$, so g is continuous and $g \in L^\infty(\mathbb{R})$. Since g is compactly supported, its Zak transform is given by the finite series

$$Zg(x,\xi) = \sum_{n=-N}^{N} g(x-n) e^{2\pi i n \xi}, \quad (x,\xi) \in [0,1]^2,$$

for some $N \geq 0$. Therefore,

$$\begin{aligned}
(5.1) \quad & |Zg(x,\xi)|^2 \\
&= |Zg(x,\xi) - Zg(0,0)|^2 \\
&\lesssim |Zg(x,\xi) - Zg(0,\xi)|^2 + |Zg(0,\xi) - Zg(0,0)|^2 \\
&\lesssim \sum_{n=-N}^{N} |g(x-n) - g(0-n)|^2 + \sum_{n=-N}^{N} |e^{2\pi i n \xi} - 1|^2 \big\| g\chi_{[n,n+1]} \big\|_\infty^2 \\
&\lesssim \sum_{n=-N}^{N} |g(x-n) - g(0-n)|^2 + |\xi|^2 \sum_{n=-N}^{N} |\pi n|^2 \big\| g\chi_{[n,n+1]} \big\|_\infty^2 \\
&\lesssim \sum_{n=-N}^{N} |g(x-n) - g(0-n)|^2 + |\xi|^2 \|g\|_\infty^2.
\end{aligned}$$

III. By Hölder's inequality, we have for all x, $y \in \mathbb{R}$ that

$$\begin{aligned}
|g(x) - g(y)|^2 &= \left| \int \widehat{g}(\xi) (e^{2\pi i x \xi} - e^{2\pi i y \xi}) \, d\xi \right|^2 \\
&\lesssim \left(\int (|\xi|^3 + 1) |\widehat{g}(\xi)|^2 \, d\xi \right) \left(\int \frac{\sin^2 \pi (x-y)\xi}{|\xi|^3 + 1} \, d\xi \right) \\
&\lesssim \|g\|_{H^{3/2}}^2 \int \frac{\sin^2 \pi (x-y)\xi}{|\xi|^3 + 1} \, d\xi.
\end{aligned}$$

We will estimate

$$\int \frac{\sin^2 \pi (x-y)\xi}{|\xi|^3 + 1} \, d\xi = \int_{|\xi| \leq 1/|x-y|} + \int_{|\xi| \geq 1/|x-y|} = I_1 + I_2.$$

If $|x-y|<\frac{1}{2}$ then

$$I_1 \lesssim \int_{|\xi|\le 1/|x-y|} \frac{|x-y|^2|\xi|^2}{|\xi|^3+1}\,d\xi$$
$$\lesssim |x-y|^2\big(1+\big|\log|x-y|\big|\big)$$
$$\lesssim |x-y|^2\big|\log|x-y|\big|.$$

Likewise, if $|x-y|<\frac{1}{2}$ then

$$I_2 \lesssim \int_{|\xi|>1/|x-y|} \frac{1}{|\xi|^3+1}\,d\xi \lesssim |x-y|^2.$$

It follows from our bounds on I_1 and I_2 that for $|x|\le \frac{1}{2}$ we have

$$|g(x-n)-g(0-n)|^2 \lesssim |x|^2|\log|x|| + |x|^2 \lesssim |x|^2|\log|x||. \tag{5.2}$$

IV. Combining Equations (5.1) and (5.2), we have for (x,ξ) near $(0,0)$ that

$$|Zg(x,\xi)|^2 \lesssim |x|^2|\log|x|| + |\xi|^2.$$

Consequently,

$$\int_0^1\int_0^1 \frac{1}{|Zg(x,\xi)|^2}\,dx\,d\xi \gtrsim \int_0^{1/2}\int_0^{1/2} \frac{1}{|x|^2|\log|x||+|\xi|^2}\,dx\,d\xi = \infty.$$

Hence, $1/Zg \notin L^2[0,1]^2$, so $\mathcal{G}(g,1,1)$ is not exact.

Finally, note that by condition (3′) following Theorem 3.1, the same result holds if the compact support condition is assumed to hold for $\widehat{g}$ and the decay condition (2.2) is assumed to hold for g. □

It is clear that if $\mathcal{G}(g,1,1)$ is complete then the support of g must have at least unit measure, that is, $|\operatorname{supp}(g)|\ge 1$. The following result is a simple refinement of Theorem 2.2 for exact Gabor systems with minimal support.

THEOREM 5.1. *If $g\in L^2(\mathbb{R})$ is supported within $[0,1]$ and satisfies*

$$\int |\xi|^2|\widehat{g}(\xi)|^2\,d\xi < \infty,$$

then $\mathcal{G}(g,1,1)$ is not exact in $L^2(\mathbb{R})$.

Proof. Since $g\in H^1(\mathbb{R})$ it follows from the Sobolev embedding theorem, for example, [20, Theorem 8.5, p. 205], that g is continuous and

$$\forall x,y\in\mathbb{R},\quad |g(x)-g(y)| \le \|g'\|_2|x-y|^{1/2}.$$

Since g is continuous and supported on $[0,1]$, we have $Zg(x,\xi)=g(x)$ for all $0\le x,\ \xi\le 1$. Since $g(0)=0$, we also have $|Zg(x,\xi)|=|g(x)|=|g(x)-g(0)|\lesssim |x|^{1/2}$, and hence

$$\int_0^1\int_0^1 \frac{1}{|Zg(x,\xi)|^2}\,dx\,d\xi \gtrsim \int_0^1\int_0^1 \frac{1}{|x|}\,dx\,d\xi = \infty.$$

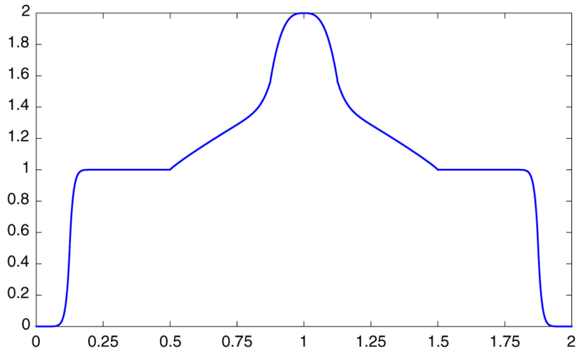
Therefore, $\mathcal{G}(g,1,1)$ is not exact. □

6. Examples

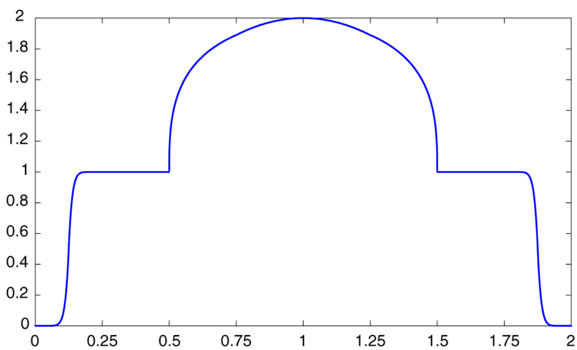
The following example will show that the $(3,\infty)$ endpoint Balian–Low theorem (Theorem 2.2) is sharp in the sense that for any sufficiently small $\varepsilon > 0$ there exists a function $g \in L^2(\mathbb{R})$ such that g is supported in $[0,2]$, $\int |\xi|^{3-\varepsilon} |\widehat{g}(\xi)|^2 \, d\xi < \infty$, and $\mathcal{G}(g,1,1)$ is exact.

EXAMPLE 6.1. Fix $0 < \alpha < 1$, and let $g = g_\alpha \in L^2(\mathbb{R})$ be any function that satisfies the following properties (1)–(8):

(1) $\operatorname{supp}(g) = [0,2]$,
(2) g is real-valued and continuous on $\mathbb{R}$,



(a)



(b)

FIGURE 1. Typical functions g_α from Example 6.1. Part (a) gives a rough sketch of an example when α is very close to 1. Part (b) gives a rough sketch of an example for a moderate value of $0 < \alpha < 1$ (the “Taj Mahal” function).

(3) g is C^∞ at all $x \in \mathbb{R}$ except at $x = \frac{1}{2}$ and $x = \frac{3}{2}$,
(4) $g(x) = 1$ for $x \in [\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{2}, \frac{7}{4}]$,
(5) there exists $0 < \Delta < \frac{1}{8}$ such that:

$$g(x) = \begin{cases} 1 + \left(x - \frac{1}{2}\right)^\alpha, & \text{for } x \in \left[\frac{1}{2}, \frac{1}{2} + \Delta\right], \\ 1 + \left(\frac{3}{2} - x\right)^\alpha, & \text{for } x \in \left[\frac{3}{2} - \Delta, \frac{3}{2}\right], \end{cases}$$

(6) $g(x) = g(2 - x)$ for all $0 \le x \le 1$,
(7) g is nondecreasing on $[0, 1]$ and positive on $(0, 2)$,
(8) $g(1) = 2$ and $g(1/8) = g(15/16) = \frac{1}{2}$.

We shall prove in Theorem 6.3 that the function $g = g_\alpha$ satisfies:

- $\mathcal{G}(g, 1, 1)$ is exact, and
- if $\varepsilon > 0$ is fixed and $\alpha = \alpha(\varepsilon)$ is sufficiently close to 1 then

$$\int |\xi|^{3-\varepsilon} |\widehat{g_\alpha}(\xi)|^2 \, d\xi < \infty.$$

In particular, g is essentially optimal with respect to Theorem 2.2.

We shall need the following standard lemma whose proof we include for the sake of completeness.

Lemma 6.2. *Fix $\alpha > -\frac{1}{2}$ and $\delta > 0$. Let $\varphi \in C_c^\infty(\mathbb{R})$ be a smooth function supported on $[-4\delta, 4\delta]$ that is identically 1 on $[-2\delta, 2\delta]$, and let $f(x) = f_\alpha(x) = x^\alpha \chi_{[0,\infty]}(x) \varphi(x)$. Then*

$$\forall 0 < s < \frac{2\alpha + 1}{2}, \qquad \int |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi < \infty.$$

Proof. Case 1: $-\frac{1}{2} < \alpha \le \frac{1}{2}$. We shall use the fact (e.g., see [23, Prop. 4]) that if $h \in L^2(\mathbb{R})$ and $0 < s < 1$ then

$$\int |\xi|^{2s} |\widehat{h}(\xi)|^2 \, d\xi < \infty \quad \iff \quad \iint \frac{|h(x+t) - h(x)|^2}{|t|^{2s+1}} \, dx \, dt < \infty. \tag{6.1}$$

If $0 < s < \frac{2\alpha+1}{2}$ then $0 < s < 1$ and so we may apply Equation (6.1) to the function $f = f_\alpha$. Since

$$\int_{|t| \ge \delta} \int \frac{|f(x+t) - f(x)|^2}{|t|^{2s+1}} \, dx \, dt \lesssim \|f\|_2^2 \int_{|t| \ge \delta} \frac{1}{|t|^{2s+1}} \, dt < \infty,$$

we only need show that

$$\int_{-\delta}^{\delta} \int \frac{|f(x+t) - f(x)|^2}{|t|^{2s+1}} \, dx \, dt = \int_{-\delta}^{\delta} \frac{1}{|t|^{2s+1}} \int |f(x+t) - f(x)|^2 \, dx \, dt < \infty. \tag{6.2}$$

Hence, our goal is to estimate

$$I(t) = \int |f(x+t) - f(x)|^2 \, dx. \tag{6.3}$$

Note that $I(t) = I(-t)$, so it suffices to consider $0 < t < \delta$. Now, since f is compactly supported on $[0, 4\delta]$ and has a bounded derivative on (δ, ∞), we have $|f(x+t) - f(x)| \lesssim |t|$ for $x > \delta$. Therefore,

$$\int_\delta^\infty |f(x+t) - f(x)|^2\, dx \lesssim \int_\delta^{4\delta} |t|^2\, dx \lesssim |t|^2. \tag{6.4}$$

Next, consider the subcase $-1/2 < \alpha < 1/2$. Since $|(x+t)^\alpha - x^\alpha| \le |\alpha||t||x|^{\alpha-1}$, it follows that

$$\begin{aligned}
\int_{-\infty}^\delta & |f(x+t) - f(x)|^2\, dx \\
&= \int_{-t}^t |f(x+t) - f(x)|^2\, dx + \int_t^\delta |(x+t)^\alpha - x^\alpha|^2\, dx \\
&\lesssim \int_{-t}^t (|x+t|^{2\alpha} + |x|^{2\alpha})\, dx + \int_t^\delta |t|^2 |x|^{2\alpha-2}\, dx \\
&\lesssim |t|^{2\alpha+1} + |t|^2.
\end{aligned} \tag{6.5}$$

Hence,

$$\begin{aligned}
I(t) &= \int_\delta^\infty |f(x+t) - f(x)|^2\, dx + \int_{-\infty}^\delta |f(x+t) - f(x)|^2\, dx \\
&\lesssim |t|^{2\alpha+1} + |t|^2,
\end{aligned}$$

so

$$\int_{-\delta}^\delta \frac{I(t)}{|t|^{2s+1}}\, dt \lesssim \int_{-\delta}^\delta \frac{|t|^{2\alpha+1} + |t|^2}{|t|^{2s+1}}\, dt < \infty.$$

In the subcase when $\alpha = 1/2$, one proceeds similarly as above to obtain $I(t) \lesssim |t|^2 + |t|^2 |\log t|$, so the desired conclusion also holds for $\alpha = 1/2$.

Case 2: $\frac{1}{2} < \alpha$. Take an integer $n \ge 1$ such that $\alpha \in (n - \frac{1}{2}, n + \frac{1}{2}]$. Note that $f, f^{(1)}, \dots, f^{(n-1)}$ are all absolutely continuous and in $L^1(\mathbb{R})$, and we also have $f^{(n)} \in L^1(\mathbb{R})$. Therefore, $\widehat{f^{(n)}}(\xi) = (2\pi i \xi)^n \widehat{f}(\xi)$. Moreover, $f^{(n)}(x) = c_1 x^{\alpha-n} \varphi_1(x) \chi_{[0,\infty)}(x)$ for some $c_1 \in \mathbb{R}$ and $\varphi_1 \in C_c^\infty(\mathbb{R})$ that is supported on $[-4\delta, 4\delta]$ and equals 1 on $[-2\delta, 2\delta]$. Since $\alpha - n \in (-\frac{1}{2}, \frac{1}{2}]$, it follows from Case 1 that, for any $0 < s < \frac{2(\alpha-n)+1}{2}$,

$$\begin{aligned}
(2\pi)^n \int |\xi|^{2s+2n} |\widehat{f}(\xi)|^2\, d\xi &= \int |\xi|^{2s} |(2\pi i \xi)^n \widehat{f}(\xi)|^2\, d\xi \\
&= \int |\xi|^{2s} \left|\widehat{f^{(n)}}(\xi)\right|^2 d\xi < \infty.
\end{aligned}$$

Therefore,

$$\int |\xi|^{2s} |\widehat{f}(\xi)|^2\, d\xi < \infty \quad \text{for } n < s < \frac{2\alpha+1}{2}. \tag{6.6}$$

Equation (6.6) extends to the remaining range of parameters, $0 < s \le n$, by noting that if $0 < a < b$ then we have

$$\int |\xi|^a |\widehat{f}(\xi)|^2 \, d\xi \le \int_{-1}^{1} |\widehat{f}(\xi)|^2 \, d\xi + \int |\xi|^b |\widehat{f}(\xi)|^2 \, d\xi. \qquad \square$$

THEOREM 6.3. *Fix $0 < \alpha < 1$. If $g = g_\alpha \in L^2(\mathbb{R})$ is a compactly supported function as described in Example 6.1, then $\mathcal{G}(g,1,1)$ is exact in $L^2(\mathbb{R})$ and*

$$\int |\xi|^{2s} |\widehat{g}(\xi)|^2 \, d\xi < \infty, \quad 0 < s < \frac{2\alpha+1}{2}. \tag{6.7}$$

Proof. I. Note that Zg is continuous on $\mathbb{R}^2$ since g is continuous and compactly supported. Since g is supported on $[0,2]$ we have

$$Zg(x,\xi) = g(x) + g(x+1)e^{-2\pi i \xi}, \quad (x,\xi) \in [0,1]^2.$$

With g defined as in Example 6.1 , it is not difficult to see that if $0 \le x,\, \xi \le 1$ then

$$Zg(x,\xi) = 0 \quad \Longleftrightarrow \quad (x,\xi) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Also, note that $|g(x)|/|g(x+1)| \ne 1$ for all $x \in [0,\frac{1}{2}) \cup (\frac{1}{2},1]$, and $g(x) \ne 0$ for all $0 < x < 2$. Since g is continuous, there exists an $\eta > 0$ such that

$$0 < \eta \le \big||g(x)|^2 - |g(x+1)|^2\big|, \quad x \in \left[0, \frac{1}{2} - \Delta\right] \cup \left[\frac{1}{2} + \Delta, 1\right]. \tag{6.8}$$

II. We shall make use of the following formula, for example, see [19, Exercise 17, p. 207], to show that $1/Zg \in L^2[0,1]^2$: for all $A \in \mathbb{R}\backslash\{-1,1\}$,

$$\int_0^1 \frac{d\xi}{|A + e^{-2\pi i \xi}|^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{1 + A^2 - 2A\cos\theta} = \frac{1}{|1 - A^2|}.$$

Since $|g(x)|/|g(x+1)| \ne 1$ for a.e. $x \in [0,1]$, we have

$$\begin{aligned}
\|1/Zg\|^2_{L^2[0,1]^2} &= \int_0^1 \int_0^1 \frac{d\xi \, dx}{|g(x) + g(x+1)e^{-2\pi i \xi}|^2} \\
&= \int_0^1 \frac{1}{|g(x+1)|^2} \left(\int_0^1 \frac{d\xi}{\left|\frac{g(x)}{g(x+1)} + e^{-2\pi i \xi}\right|^2} \right) dx \\
&= \int_0^1 \frac{1}{|g(x+1)|^2} \frac{dx}{\left|1 - |\frac{g(x)}{g(x+1)}|^2\right|} \\
&= \int_0^1 \frac{dx}{\big||g(x+1)|^2 - |g(x)|^2\big|} \\
&= \int_{\frac{1}{2}-\Delta}^{\frac{1}{2}+\Delta} + \int_{[0,\frac{1}{2}-\Delta] \cup [\frac{1}{2}+\Delta,1]} = I_1 + I_2.
\end{aligned}$$

It follows immediately from Equation (6.8) that $I_2 < \infty$.

III. For $x \in [\frac{1}{2} - \Delta, \frac{1}{2} + \Delta] \subset [\frac{1}{4}, \frac{3}{4}]$, we have $|g(x) + g(x+1)| \geq 2$. Since $0 < \alpha < 1$, it follows that

$$\begin{aligned} I_1 &= \int_{\frac{1}{2}-\Delta}^{\frac{1}{2}+\Delta} \frac{dx}{|g(x+1) - g(x)||g(x+1) + g(x)|} \\ &\lesssim \int_{\frac{1}{2}-\Delta}^{\frac{1}{2}+\Delta} \frac{dx}{|g(x+1) - g(x)|} \\ &\lesssim \int_{\frac{1}{2}}^{\frac{1}{2}+\Delta} \frac{dx}{(x - \frac{1}{2})^{\alpha}} < \infty. \end{aligned}$$

Combining this with our bound for I_2, it follows that $1/Zg \in L^2[0,1]^2$. So $\mathcal{G}(g,1,1)$ is exact.

IV. It remains to show that Equation (6.7) holds. Since $g(x)$ is compactly supported on $[0,2]$ and is C^∞ except at $x = \frac{1}{2}$ and $x = 3/2$ it suffices to work locally near these two points. In view of the definition of $g(x)$ near $x = \frac{1}{2}$ and $x = 3/2$, Equation (6.7) follows from Lemma 6.2. □

The following example addresses optimality in Theorem 5.1.

EXAMPLE 6.4. Fix $\varepsilon > 0$. Example 5.11 and Theorem 6.1 in [17] constructively show that there exists a function $g \in L^2(\mathbb{R})$ such that

- $\operatorname{supp}(g) = [0,1]$,
- $\mathcal{G}(g,1,1)$ is exact in $L^2(\mathbb{R})$ (in fact, it is a Schauder basis for $L^2(\mathbb{R})$), and
- $\int |\xi|^{2-\varepsilon} |\widehat{g}(\xi)|^2 \, d\xi < \infty$.

Thus, Theorem 5.1 is essentially sharp.

We note that the lattice structure of $\mathcal{G}(g,1,1)$ has played an essential role in our analysis of time-frequency localization of exact Gabor systems. In contrast to lattice Gabor systems, it is known that there exist irregular Gabor systems $\mathcal{G}(g,\Lambda)$, where Λ is not a lattice, that are exact and where the window function $g(x) = e^{-\pi x^2}$ is the Gaussian, see [1]. In particular, window functions for irregular exact Gabor systems can be very nicely localized in both time and frequency.

A concrete example of an exact Gabor system with the Gaussian can be obtained as follows. It is known that if $g(x) = e^{-\pi x^2}$ then $\mathcal{G}(g,1,1)$ is complete in $L^2(\mathbb{R})$, and remains complete if any single element is removed from this system, see [13, Theorem 3.41]. Moreover, $\mathcal{G}(g,1,1)$ does not remain complete if any two elements are removed. In particular, $\mathcal{G}(g,1,1)\backslash\{g\}$ is exact but does not have the full structure of a lattice Gabor system.

As a final remark, recall that Theorems 1.3 and 2.1 provide Balian–Low theorems for Riesz bases and exact systems in terms of two different scales of parameters. An interesting question is to determine what happens "between" Riesz bases and exact systems, and to appropriately interpret the intermediate parameter scales. While this article was under review, the authors learned of the recent preprint [22] which sheds substantial insight into this question.

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