Illinois Journal of Mathematics
Volume 53, Number 4, Winter 2009, Pages 1033–1049
S 0019-2082

HARDY SPACES IN REINHARDT DOMAINS, AND HAUSDORFF OPERATORS

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ABSTRACT. We give criteria for a function to be in the Hardy space on a bounded complete Reinhardt domain. Using these and known one-dimensional results, we obtain boundedness conditions for Hausdorff operators on Hardy spaces in Reinhardt domains. The only known earlier result for the polydisk is a paticular case of the obtained results.

1. Introduction

The aim of this paper is to obtain criteria for a function to be in the Hardy space on a bounded complete Reinhardt domain. In the obtained multivariate results, one-dimensional conditions are essentially used. As an application, we give far-going multidimensional extensions of conditions for Hausdorff operators to be bounded in Hardy spaces.

Hausdorff means, the Cesàro means among them, have been known for a long time in connection with summability of number series. For Hausdorff summability of power series in one variable, strong results were obtained in [5], [6] (some of them can be found in [15]; for Cesàro means, see [17], [18]). These results are generalized to several dimensions in [3], the only known multidimensional generalization, where the sole case of the polydisk is considered, while the spaces are H^p, $1 \le p < \infty$.

Received February 10, 2009; received in final form December 2, 2009.

The authors are indebted to the ORT Braude College, the Gelbart and Emmy Noether Institutes of Bar-Ilan University, ESF Network Programme HCAA, the Edmund Landau Minerva Center of the Hebrew University, US NSF, and University of Miami, for their support of the conference COMPLEX ANALYSIS AND DYNAMICAL SYSTEMS—IV, Naharia, Israel, May 2009, where the first presentation of the paper took place. The authors also thank L. Frerick and J. Wengenroth for helpful discussions, and the referee for thorough reading and important remarks and suggestions.

2000 *Mathematics Subject Classification.* Primary 47B38, 42B10. Secondary 46E30.

We extend these results to Hardy spaces on a rather wide class of domains—the Reinhardt domains. Sufficient conditions for the boundedness of Hausdorff type operators turn out to be necessary for a smaller subclass of domains, still quite wide. Of course, the polydisk is among them.

The approach is different from that in dimension one in [5], [6] and in several dimensions in [3], where estimates of special composition operators ensured the desired results. The point is that estimating (and sometimes even finding) such composition operators might be an extremely difficult task in the multivariate setting. Our approach is based on an inductive argument (see Main Lemma below) where one-dimensional results can be directly applied.

The outline of the paper is as follows. In the first section, we give necessary basics on Hardy spaces in dimension one and present known and certain new notions on Reinhardt domains and Hardy spaces in several dimensions. In the second section, we prove two versions of the main lemma in which a criterion is given for a function to belong to the Hardy space H^p, $0<p<\infty$, and separately to H^∞. In the third section, we present recent results on Hausdorff operators in the one-dimensional case in the form given in [6] and then define multivariate operators of Hausdorff type. In the first subsection of the following section, we prove sufficient conditions for the boundedness of Hausdorff type operators in Hardy spaces on a wide class of Reinhardt domains. In the second subsection of that section, we discuss the case of the polydisk. Next, we give necessary conditions for the boundedness of Hausdorff type operators in Hardy spaces on a smaller class of Reinhardt domains. However, the polydisk belongs to this subclass. In the last subsection of that section, we present necessary and sufficient conditions for H^∞. In the last section, certain concluding remarks are given.

By C, we will denote constants that may depend only on a considered domain and may be different even in the same occurrence.

2. Hardy spaces

2.1. One-dimensional results. Let $U_\rho=\{\zeta : |\zeta|<\rho\}$ be the disk of radius ρ in the complex plane $\mathbb{C}$. For $1\le p<\infty$, the Hardy space H^p is the space of analytic functions $f : U_1\to\mathbb{C}$ such that

$$\|f\|_{H^p}=\sup_{r<1}\left(\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p}<\infty;$$

for $p=\infty$, the definition is the same with usual modification of the norm. Such a defined H^p is a Banach space (and Hilbert for $p=2$). If $1\le p\le q\le\infty$, then $H^1\supset H^p\supset H^q$. Functions $f\in H^p$ possess boundary values (nontangential limits) $f(e^{i\theta})$ which are p-integrable on ∂U_1.

It is well known that for functions $\varphi(\zeta)$ holomorphic in U_ρ, the integral

$$\int_{|\zeta|=r<\rho} |\varphi(\zeta)|^p |d\zeta| \tag{1}$$

is a nondecreasing function of r (see, e.g., [8, Theorem 2.12]).

2.2. Multivariate results. Hardy classes of holomorphic functions of several complex variables are usually defined on bounded domains $D \in \mathbb{C}^n$ as follows. If the boundary ∂D is smooth, then the class $H^p(D)$ consists of the functions f holomorphic in D such that

$$\varlimsup_{\varepsilon\to 0} \int_{\partial D} |f(z-\varepsilon\nu_z)|^p \, d\sigma(z) < \infty, \tag{2}$$

where ν_z is the external unit normal vector to ∂D at the point z, and $d\sigma(z)$ is an element of the $(2n-1)$-dimensional surface ∂D (see, e.g., [9], [19]). However, the definition for the polydisk $U^n = \{z : |z_j| < 1, j = 1, \dots, n\}$ usually differs from the general definition and is defined by the following condition instead of (2)

$$\varlimsup_{r\to 1} \int_{\mathbb{T}^n} |f(rz)|^p \left| \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \right| < \infty, \tag{3}$$

where $\mathbb{T}^n = \{z : |z_j| = 1, j = 1, \dots, n\}$ and $0 < r < 1$ (see, e.g., [3], [14]).

Let us consider bounded complete Reinhardt domains $D \subset \mathbb{C}^n$. They appear naturally as domains of convergence of multidimensional power series

$$\sum_{|\alpha|\geq 0} c_\alpha z^\alpha, \tag{4}$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, with all α_j nonnegative integers. Here $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We now define the Hardy class $H^p(D)$ to be that of functions $f(z)$ holomorphic in D and satisfying

$$\lim_{r\to 1-} \int_{\partial D_r} |f(z)|^p \, d\sigma(z) < \infty, \tag{5}$$

where $0 < r < 1$, $D_r = rD$ is the rth homothety of D, and $d\sigma(z)$ is an element of the $(2n-1)$-dimensional surface ∂D_r. Since the integral (5) can be representable by integrating first over the circles $\ell \cap \partial D_r$, where each ℓ is a complex line passing through the origin, and then by integrating over the set of such lines with respect to the corresponding positive measure, it is also a nondecreasing function of r. This explains why the usual limit is used in (5) instead of the upper limit; by the way, the usual limit can analogously be written in (3) in place of the upper limit.

Let us consider the family of parallel complex lines

$$m_k = \{z = (z_1, \dots, z_{k-1}, t, z_{k+1}, \dots, z_n), t \in \mathbb{C}\} \tag{6}$$

crossing the domain D. The intersection of each of these lines with D is a disk. For $0<r<1$, let us consider the set

$$\bigcup_{\{m_k\}} (m_k \cap \partial D_r). \tag{7}$$

We will say that the domain D is *k-tame* if the limit as $r\to 1-$ of the set (7) is exactly the whole ∂D. For example, the ball $\{z : |z|<1\}$, where $|z| = (|z_1|^2+\cdots+|z_n|^2)^{1/2}$, is a k-tame domain for each k, $1\le k\le n$, while the polydisk U^n is not k-tame for any k. However, it is true for U^n that

$$\lim_{r\to 1-} \bigcup_{1\le k\le n} \bigcup_{\{m_k\}} (m_k \cap \partial D_r) = \partial D. \tag{8}$$

There is a need to define additional types of Reinhardt domains. First, we will call the domain D *quasi-tame* if (8) holds true.

Lemma 1. *Every bounded complete Reinhardt domain is quasi-tame.*

Proof. Formula (8) is valid for a finite union of polydisks centered in the origin. Every bounded complete Reinhardt domain D is the limit of expanding sequence of domains, each of them being the union of such polydisks. The boundary of D is the limit of the boundaries of these domains, which completes the proof. □

Further, a complete bounded Reinhardt domain D is called *k-cylindric*, $1\le k\le n$, if $D\subset\{z : |z_k|<\rho\}$ for some $\rho>0$, and ∂D contains a piece of the hyper-surface $\Gamma_k(\rho)=\{z : |z_k|=\rho\}$, that is, $\Gamma_k(\rho)\cap\{z : (z_1,\dots,z_{k-1},z_{k+1},\dots,z_n)\in R\}\subset\partial D$, where R is a domain in $\mathbb{C}^{n-1}$.

3. Main Lemma and related results

In this section, we present our main results. Their analog for the bundle of complex lines passing through the origin was obtained for complete bounded Reinhardt domains in [1], [2] and reads as follows.

Let A be a bundle of complex lines passing through 0. For each line α from A the intersection $D\cap\alpha$ is a disk.

Lemma 2. *For f to belong to $H^p(D)$ it is necessary and sufficient that*:
(1) *for almost all $\alpha\in A$, the function $f|_{D\cap\alpha}$ belongs to $H^p(D\cap\alpha)$ and*
(2) *f belongs to L^p on ∂D.*

This lemma allows one to easily extend certain classical results to the multidimensional case, for example, the well-known Smirnov theorem (1928): *If $f\in H^p(U_1)$ and $f\in L^q(\partial U_1)$, $q>p$, then $f\in H^q(U_1)$.*

Let us prove a multidimensional version. Assume that $f\in H^p(D)$ and $f\in L^q(\partial D)$. Then (1) yields that $f\in H^p(D\cap\alpha)$ for almost all $\alpha\in A$. It follows from $f\in L^q(\partial D)$ and Fubini's theorem that $f|_{D\cap\alpha}$ in L^q on the boundary of this intersection for almost all α. By the one-dimensional Smirnov theorem $f \in H^q(D \cap \alpha)$ for almost all α. Again applying Lemma 2, we obtain $f \in H^q(D)$, as desired.

In a completely similar way one can prove the next theorem. To formulate it, let us remark the following. If $f \in H^p(D)$, then in almost every section $D \cap \alpha$ this function will have angular boundary values on the boundary of this section, that is, on the circle. Therefore, boundary values of f almost everywhere on ∂D can be understood in the sense of $(2n-1)$-dimensional measure. We will denote these boundary values by f as well.

THEOREM 3. *If* $0 < p < \infty$, *then*

$$\|f\|_p^p = \int_{\partial D} |f(\zeta)|^p \, d\sigma(z)$$

and

$$\lim_{r \to 1-} \int_{\partial D} |f(r\zeta) - f(\zeta)|^p \, d\sigma(z) = 0.$$

This theorem immediately yields the following corollary.

COROLLARY 4. *If* $f \in H^p(D)$ *and* $0 < p < \infty$, *then for* $\zeta \in \partial D$

$$\lim_{r \to 1-} \|f(r\zeta) - f(\zeta)\|_p = 0,$$

and polynomials are dense in $H^p(D)$.

We note that Lemma 2, Smirnov's theorem for Reinhardt domains, Theorem 3 and other multidimensional analogs of the one-dimensional classical theorems were mentioned in [1], [2]. The reason we give the proof of the multidimensional Smirnov theorem is to illustrate the method, while Theorem 3 is given since Corollary 4 will be used below.

It is worth mentioning that later a multidimensional generalization of Smirnov's theorem was proved in [11] for bounded domains D in $\mathbb{C}^n$ with Lyapunov's boundary, that is, $\partial D \in C^{1+\varepsilon}$ with some $\varepsilon > 0$.

We also note a result in [20] in the spirit of the last assertion of Corollary 4: it is proved there that if $f \in H^p(D)$, $1 \le p < \infty$, D is a strictly pseudoconvex domain with C^3 boundary, then there exists a sequence $\{f_n\}$, $n = 1, 2, \dots$, of functions, holomorphic in $\overline{D}$, that converges in the H^p-sense to f.

3.1. The case $p < \infty$. We first consider the case when $0 < p < \infty$.

MAIN LEMMA FOR $0 < p < \infty$. *Let D be a bounded complete Reinhardt domain, k-tame with k being a fixed integer, $1 \le k \le n$. For a function f holomorphic in D to belong to the class $H^p(D)$, it is necessary and sufficient that*:

(1) *for almost all complex lines m_k the restriction of the function $f(z)$ to the disk $m_k \cap D = Q_k$ belongs to the Hardy class $H^p(Q_k)$ and*

(2) *the function f is L^p summable on ∂D, that is,*

$$\int_{\partial D} |f(z)|^p \, d\sigma(z) < \infty. \tag{9}$$

Here, we understand the values $f(z)$, $z \in \partial D$, as angular boundary values on the circles $m_k \cap \partial D$ which by (1) exist almost everywhere for almost all m_k, that is, almost everywhere on ∂D.

Proof of Main Lemma for $0 < p < \infty$. First, let $f \in H^p(D)$. Denoting [cf. (6)]

$$\psi(m_k, r) = \int_{m_k \cap \partial D_r} |f(z)|^p |dt|$$

and taking into account that this function is nondecreasing in r, we get, by the Lebesgue theorem (see, e.g., [16, Theorem 12.6]),

$$\lim_{r \to 1-} \int_{\{m_k\}} \psi(m_k, r) \, d\mu = \int_{\{m_k\}} \lim_{r \to 1-} \psi(m_k, r) \, d\mu, \tag{10}$$

where $d\mu$ is the measure corresponding to the equality

$$\int_{\{m_k\}} \int_{m_k \cap \partial D} |f(z)|^p |dt| \, d\mu = \int_{\partial D} |f(z)|^p \, d\sigma. \tag{11}$$

Since the left-hand side in (10) is finite because of (5), the function under the integral sign on the right-hand side of (10) is finite for almost all m_k. This implies condition (1). On the other hand, the well-known property of functions from the Hardy spaces H^p in the disk (see, e.g., [10])

$$\lim_{r \to \rho} \int_{|\zeta| = r < \rho} |\varphi(\zeta)|^p |d\zeta| = \int_{|\zeta| = \rho} |\varphi(\zeta)|^p |d\zeta| \tag{12}$$

implies that the integral on the right-hand side of (10) is just (11).

Conversely, let (1) and (2) hold true. The same Lebesgue theorem yields (10). By (12),

$$\lim_{r \to 1-} \psi(m_k, r) = \int_{m_k \cap \partial D} |f(z)|^p |dt|$$

for almost all m_k. Therefore, the right-hand side in (10) is just the left-hand side in (11). This yields, by (10) and (2), the finiteness of the left-hand side in (10). Hence, (5) holds true, that is, $f \in H^p(D)$. The proof is complete. □

LEMMA 5. *Let D be a bounded complete Reinhardt domain. For a function f holomorphic in D to belong to the class $H^p(D)$, it is necessary and sufficient that* (2) *holds true and* (1) *from Main Lemma holds true for all k.*

Proof. The proof repeats that of the previous lemma, and making use of the fact that

$$\lim_{r \to 1-} \bigcup_{\{m_k\}} (m_k \cap \partial D_r) \subset \partial D$$

and then of (8). This completes the proof. □

3.2. The case $p = \infty$. Contrary to many other situations, here the case $p = \infty$ is easier and less restrictive.

MAIN LEMMA FOR $p = \infty$. *Let D be a bounded complete Reinhardt domain and k be a fixed integer, $1 \leq k \leq n$. For a function f holomorphic in D to belong to the class $H^\infty(D)$, it is sufficient that:*

(1) *for almost all complex lines m_k, the restriction of the function $f(z)$ to the disk $m_k \cap D = Q_k$ belongs to the Hardy class $H^\infty(Q_k)$ and*

(2) *the function f is $L^\infty(\partial D)$, that is,*

$$\operatorname*{ess\,sup}_{\partial D} |f(z)| = B < \infty; \tag{13}$$

and it is necessary that:

(1′) *for all complex lines m_k, the restriction of the function $f(z)$ to Q_k belongs to the Hardy class $H^\infty(Q_k)$ and*

(2′) *there holds*

$$\sup_{\partial D} |f(z)| = B < \infty. \tag{14}$$

For a function f holomorphic in D to belong to the class $H^\infty(D)$, it is necessary and sufficient that (2) holds true and (1′) holds true for all m_k.

Proof. Let $f \in H^\infty(D)$, then (14) holds true, and moreover

$$\sup_{D} |f(z)| = B < \infty. \tag{15}$$

By this, we have $f|_{Q_k} \in H^\infty(Q_k)$.

Conversely, let (1) and (2) hold true. Then it immediately follows from (13) that $\sup_{\partial Q_k} |f(z)| \leq B$ for almost all m_k; on the other hand, $f|_{Q_k} \in H^\infty(Q_k)$ for almost all m_k. Therefore, for almost all m_k

$$\sup_{Q_k} |f(z)| \leq B. \tag{16}$$

Since a complete Reinhardt domain is the union of polydisks, (15) follows from (16) by continuity.

The last assertion of the lemma is proved by repeating the above argument. The proof is complete. □

3.3. A remark on operators. The following remark will be helpful in our argument.

REMARK 1. Let $0 < p < \infty$. It is worth mentioning for the sequel that if A is a linear bounded operator in the space $H^p(U_1)$, in the disk of radius 1, it will be such in the space $H^p(U_\rho)$, in the disk of radius ρ (since we define the operator by its action on polynomials and polynomials are dense in $H^p(U_\rho)$, it also acts in $H^p(U_\rho)$). Indeed, it follows from both the definition and the basic property of H^p that

$$\|\varphi\|_{H^p(U_\rho)} = \left(\int_{|\zeta|=\rho} |\varphi(\zeta)|^p |d\zeta|\right)^{1/p}.$$

Conformal mapping of the unit disk U_1 onto the disk U_ρ is the homothety $\zeta \to \rho\zeta$. By this the function $\varphi(\zeta)$ holomorphic in U_ρ turns out to be the function $\varphi(\rho\zeta)$ holomorphic in U_1. Further,

$$\int_{|\zeta|=\rho} |\varphi(\zeta)|^p |d\zeta| = \rho \int_{|\zeta|=1} |\varphi(\rho\zeta)|^p |d\zeta|,$$

therefore

$$\|\varphi\|_{H^p(U_\rho)} = \|\varphi(\rho\zeta)\|_{H^p(U_1)}\rho^{1/p}. \tag{17}$$

If the operator A is linear and bounded in $H^p(U_1)$ with

$$\|A\varphi(\rho\,\cdot)\|_{H^p(U_1)} \le K\|\varphi(\rho\,\cdot)\|_{H^p(U_1)}, \tag{18}$$

then it follows from (17) that for the same operator in $H^p(U_\rho)$

$$\|A\varphi\|_{H^p(U_\rho)} \le K\|\varphi\|_{H^p(U_\rho)}. \tag{19}$$

Moreover, comparing (17) and (18), we see that the norm of A is the same in both $H^p(U_1)$ and $H^p(U_\rho)$.

4. Hausdorff operators

4.1. One-dimensional notions and results. After the appearance of the paper [12], general Hausdorff summability of power series was studied in [5] and then in [6] as follows.

Let Δ be the forward difference operator defined on scalar sequences $\mu = (\mu_n)_{n=0}^\infty$ by $\Delta\mu_n = \mu_n - \mu_{n+1}$ and $\Delta^k\mu_n = \Delta(\Delta^{k-1}\mu_n)$ for $k = 1, 2, \dots$ with $\Delta^0\mu_n = \mu_n$.

Setting $c_{n,k} = \binom{n}{k}\Delta^{n-k}\mu_k$, $k \le n$, we define the Hausdorff matrix $H = H_\mu$ with generating sequence μ to be the lower triangular matrix with the entries

$$H_\mu(i,j) = \begin{cases} 0, & i < j, \\ c_{i,j}, & i \ge j. \end{cases}$$

It induces two operators on spaces of power series which are formally given by

$$H_\mu f(z) = H_\mu\left(\sum_{n=0}^\infty a_n z^n\right) = \sum_{n=0}^\infty \left(\sum_{k=0}^n c_{n,k}a_k\right) z^n,$$

which is obtained by letting the matrix H_μ multiply the Taylor coefficients of f, and

$$A_\mu f(z) = A_\mu\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} c_{n,k}a_n\right)z^k,$$

which is obtained by letting the transposed matrix $A_\mu = H_\mu^*$ to act on the Taylor coefficients of f. Such a matrix A_μ is called a quasi-Hausdorff matrix. The convergence of the power series $A_\mu f$ is more delicate than that of H_μ. However, it is clear that if f is a polynomial then $A_\mu f$ is also a polynomial. If the space considered contains the polynomials, we may ask whether A_μ extends to a bounded operator on the corresponding space. This is the case for H^p, $0<p<\infty$, where polynomials are dense.

An important special case of such matrices occurs when μ_n is the moment sequence of a finite (positive) Borel measure μ on $(0,1]$:

$$\mu_n = \int_0^1 t^n\, d\mu(t), \quad n = 0,1,\dots.$$

In this case for $k \le n$

$$c_{n,k} = \binom{n}{k}\int_0^1 t^k(1-t)^{n-k}\, d\mu(t).$$

Various choices of the measure μ give rise to well-known classical matrices. For example, when μ is the Lebesgue measure one has the Cesàro matrix.

The following two theorems give criteria for boundedness of the Hausdorff and quasi-Hausdorff matrices on H^p (see in [6], Theorems 2.4 and 2.3, respectively).

THEOREM A. *Let μ be a finite positive Borel measure on $(0,1]$. Then $H_\mu : H^1 \to H^1$ is a bounded operator if and only if*

$$\int_0^1 \big(1+\ln(1/t)\big)\, d\mu(t) < +\infty. \tag{20}$$

If $1<p\le\infty$, then $H_\mu : H^p \to H^p$ is a bounded operator if and only if

$$\|H_\mu\|_{H^p\to H^p} = \int_0^1 t^{1/p-1}\, d\mu(t) < \infty. \tag{21}$$

THEOREM B. *Let μ be a finite positive Borel measure on $(0,1]$ and $1\le p<\infty$. Then $A_\mu : H^p \to H^p$ defines a bounded operator if and only if*

$$\|A_\mu\|_{H^p\to H^p} = \int_0^1 t^{-1/p}\, d\mu(t) < +\infty. \tag{22}$$

4.2. Multivariate Hausdorff operators. Let us consider a natural multidimensional analog of the Hausdorff type operators by defining it on the power series (4), representing functions holomorphic in D, as

$$(\mathcal{H}_\mu f)(z)=\sum_{|\alpha|\ge 0}\left(\sum_{\beta\le\alpha}\prod_{j=1}^{n}h_{\alpha_j,\beta_j}(\mu_j)c_\beta\right)z^\alpha \tag{23}$$

for the Hausdorff operator, while for the quasi-Hausdorff operator

$$(\mathcal{A}_\mu f)(z)=\sum_{|\alpha|\ge 0}\left(\sum_{\beta\ge\alpha}\prod_{j=1}^{n}h_{\alpha_j,\beta_j}(\mu_j)c_\beta\right)z^\alpha, \tag{24}$$

where $\beta\le\alpha$ and $\beta\ge\alpha$ means that $\beta_j\le\alpha_j$ and $\beta_j\ge\alpha_j$, respectively, for all $j=1,\dots,n$. Here, as above in dimension one, $h_{\alpha_j,k_j}(\mu_j)=\binom{\alpha_j}{k_j}\Delta^{\alpha_j-k_j}\mu_j(k_j)$, $k_j\le\alpha_j$, with $\mu_j(k_j)$ being the moment sequence of a finite (positive) Borel measure μ_j on $(0,1]$:

$$\mu_j(k_j)=\int_0^1 t^{k_j}\,d\mu_j(t),\quad k_j=0,1,\dots.$$

In this case for $k_j\le\alpha_j$

$$h_{\alpha_j,k_j}(\mu_j)=\binom{\alpha_j}{k_j}\int_0^1 t^{k_j}(1-t)^{\alpha_j-k_j}\,d\mu_j(t).$$

This extension of the one-dimensional Hausdorff operators to several dimensions was first suggested in [3].

While (23) is well defined for $p\le\infty$, for (24) the definition is correct when $p<\infty$.

Various choices of the measures μ_j give rise to the well-known classical matrices. For example, when all μ_j are the Lebesgue measures one has the multidimensional Cesàro matrix, of the classical form in the case when D is the polydisk.

We also mention that (23) can be considered as a repeated one-dimensional Hausdorff operator in each of the n variables. This feature will be pivotal in the proofs of the following results. Let us, for brevity, illustrate this for $\mathcal{H}_\mu$ in the two-dimensional case. We have

$$\begin{aligned}(\mathcal{H}_\mu f)(z,\zeta)&=\sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\left(\sum_{m=0}^{M}\sum_{n=0}^{N}h_{M,m}(\mu_1)h_{N,n}(\mu_2)c_{mn}\right)z^m\zeta^n\\&=\sum_{M=0}^{\infty}\sum_{m=0}^{M}h_{M,m}(\mu_1)\left(\sum_{N=0}^{\infty}\sum_{n=0}^{N}h_{N,n}(\mu_2)c_{mn}\zeta^n\right)z^m.\end{aligned}$$

The relation in the parenthesis on the right-hand side is the one-dimensional Hausdorff operator in the second variable generated by the measure μ_2. It depends on m and serves as the mth coefficients on which the other one-dimensional Hausdorff method generated by the measure μ_1 acts.

5. Conditions for the boundedness of Hausdorff operators

We will consider sufficient and necessary conditions for the boundedness of Hausdorff type operators on Hardy spaces separately, since they coincide only on a special subclass of Reinhardt domains.

5.1. Sufficient conditions for $1 \le p < \infty$. As often happens, sufficient conditions hold true for a wider class of objects.

We will start with results of maximal generality. By $\mathcal{H}_{\mu_k} := (\mathcal{H}_{\mu_k})(z_k)$ we will denote the operator $\mathcal{H}_{\mu_k}$ with respect to the kth variable z_k with all other variables fixed; the same for $\mathcal{A}_{\mu_k}$.

THEOREM 6. *Let a complete bounded Reinhardt domain D be k-tame. The Hausdorff operator $\mathcal{H}_{\mu_k}$ is bounded on $H^p(D)$ for $1 < p < \infty$ provided*

$$\int_0^1 s^{1/p-1}\, d\mu_k(s) < \infty \tag{25}$$

and for $p = 1$ provided

$$\int_0^1 \big(1 + \ln(1/s)\big)\, d\mu_k(s) < \infty. \tag{26}$$

THEOREM 7. *Let a complete bounded Reinhardt domain D be k-tame. The quasi-Hausdorff operator $\mathcal{A}_{\mu_k}$ is bounded on $H^p(D)$ for $1 \le p < \infty$ provided*

$$\int_0^1 s^{-1/p}\, d\mu_k(s) < \infty. \tag{27}$$

The proofs of both theorems are completely identical, therefore we shall give the proof only for one of them, H_{μ_k}.

Proof. Applying the operator $\mathcal{H}_{\mu_k}$ with respect to the kth variable z_k with all other variables fixed and taking into account the main lemma, we conclude that for almost all m_k the norm of this operator is bounded in the sense that

$$\|\mathcal{H}_{\mu_k} f\|^p_{H^p(m_k \cap D)} \le C\|f\|^p_{H^p(m_k \cap D)} = C\int_{m_k \cap \partial D} |f(z)|^p |dt|.$$

It remains to observe that by (11) the integral over the whole ∂D is equal to

$$\int_{\{m_k\}} \int_{m_k \cap \partial D} |f(z)|^p |dt|\, d\mu.$$

We derive from the Main Lemma and (11) that $\mathcal{H}_\mu f$ is continuous in $H^p(D)$. The proof is complete. $\square$

COROLLARY 8. *Let D be a bounded complete Reinhardt domain D, k-tame for all k, $1 \le k \le n$. Then the Hausdorff operator $\mathcal{H}_\mu f$ is bounded in $H^p(D)$ for $1 < p < \infty$ provided*

$$\prod_{k=1}^{n} \int_0^1 s^{1/p-1}\, d\mu_k(s) < \infty, \tag{28}$$

and for $p = 1$ provided

$$\prod_{k=1}^{n} \int_0^1 \big(1 + \ln(1/s)\big)\, d\mu_k(s) < \infty. \tag{29}$$

The Hausdorff operator $\mathcal{A}_\mu f$ is bounded in $H^p(D)$ for $1 \le p < \infty$ provided

$$\prod_{k=1}^{n} \int_0^1 s^{-1/p}\, d\mu_k(s) < \infty. \tag{30}$$

COROLLARY 9. *Let D be a bounded complete Reinhardt domain. Then the Hausdorff operator $\mathcal{H}_\mu f$ is bounded in $H^p(D)$ for $1 < p < \infty$ provided (28) holds and for $p = 1$ provided (29) holds, while the Hausdorff operator $\mathcal{A}_\mu f$ is bounded in $H^p(D)$ for $1 \le p < \infty$ provided (30) holds.*

Applying (8), we prove this corollary exactly in the same way as the theorem.

5.2. The case of polydisk. Let us consider an interesting particular case of the polydisk U^n. Corollary 9 concerns the Hardy class $H_1^p(U^n)$, when the integral over the whole boundary ∂D is involved. However, in the study of boundary properties of holomorphic functions in the polydisk, the Hardy class $H_2^p(U^n)$ used is defined by means of the integral (3). We now wish to show that $H_1^p(U^n)$ is wider than $H_2^p(U^n)$. Without loss of generality and for the sake of simplicity, let us restrict ourselves to the case $n = 2$.

Let $f \in H_2^p(U^n)$, that is,

$$\lim_{r \to 1} \int_{\mathbb{T}^2} |f(rz)|^p \left| \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \right| < \infty.$$

Using properties of functions from the Hardy class (first of all the monotonicity of (1)), it is easy to show that for $r \le 1$

$$\int_{\mathbb{T}^2} |f(rz_1, z_2)|^p \left| \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \right| \le \int_{\mathbb{T}^2} |f(z)|^p \left| \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \right|. \tag{31}$$

Denoting the right-hand side of (31) by J and repeating in essence the arguments from the proof of the main lemma, we get that for almost all r, $0 \le r \le 1$, the function $f(z_1, z_2)$ belongs to each Hardy class $H_2^p(U^2_{(r,1)})$, where

$U^2_{(r,1)} = \{z : |z_1| < r, |z_2| < 1\}$. For the affiliation $f \in H^p_1(U^2)$, it suffices to show that

$$\lim_{r\to 1-} \int_{\partial U_r^2} |f|^p \, d\sigma < \infty.$$

The boundary ∂U_r^2 consists of the two hyper-surfaces $\Gamma_r^1 = \{z : |z_1| \le r, |z_2| = r\}$ and $\Gamma_r^2 = \{z : |z_1| = r, |z_2| \le r\}$. We have

$$\int_{\Gamma_r^j} |f|^p \, d\sigma \le CJ \int_0^1 dr = CJ$$

for both $j = 1, 2$. By this $f \in H^p_1(U^2)$. Hence,

$$H^p_1(U^2) \subset H^p_2(U^2).$$

Let us give an example to demonstrate that this inclusion is proper. Recall that for the function $f(z_1) = \sum_{k=0}^{\infty} a_k z_1^k$ to belong to the Hardy class $H^2(U_\rho)$, it is necessary and sufficient that

$$\sum_{k=0}^{\infty} |a_k|^2 \rho^{2k} < \infty.$$

This is a classical fact—it is just Parseval's identity for $H^2(U_\rho)$, see, for example, [4]. Analogously, it is easy to prove that for the function

$$f(z_1, z_2) = \sum_{k,l=0}^{\infty} a_{kl} z_1^k z_2^l$$

holomorphic in the polydisk $U^2_{(r,\rho)}$ to belong to the Hardy class H^2_2, it is necessary and sufficient that

$$\sum_{k,l=0}^{\infty} |a_{kl}|^2 r^{2k} \rho^{2l} < \infty. \tag{32}$$

For multiple Fourier series, a similar assertion is given in [21]. Similarly, for the function $f(z_1, z_2) \in H^2_1(U^2)$, it is necessary and sufficient that

$$\sum_{k,l=0}^{\infty} |a_{kl}|^2 \left[\int_0^1 r^{2k} \, dr + \int_0^1 \rho^{2l} \, d\rho \right] < \infty. \tag{33}$$

Consider the function

$$f(z_1, z_2) = \sum_{k=1} \frac{(z_1 z_2)^k}{\sqrt{k}}. \tag{34}$$

For this function, (32) is not valid when $r = \rho = 1$, while (33) holds true, that is, the function (34) does not belong to $H^2_2(U^2)$ but belongs to $H^2_1(U^2)$. The same function squared does not belong to $H^1_2(U^2)$ but belongs to $H^1_1(U^2)$.

Using a two-dimensional analog of the Hausdorff–Young inequality (for the one-dimensional case, see, for example, [4, §6.1] or [13, Chapter 2, §11.4]) for power series, it is easy to build an example of a function that, in a similar way, shows that the class $H_2^p(U^2)$ is strictly smaller than $H_1^p(U^2)$ for $1 \le p < \infty$.

However, for $p = \infty$ these classes coincide, since maximum of the absolute value of a function holomorphic in the polydisk is attained on its skeleton $\mathbb{T}^n$.

5.3. Necessary conditions. It turns out that sufficient conditions for the boundedness of Hausdorff type operators are also necessary for a smaller class of Reinhardt domains, the above defined k-cylindric domains.

THEOREM 10. *Let a complete bounded Reinhardt domain D be k-cylindric. If an operator $\mathcal{H}_{\mu_k}$ is bounded in $H^p(D)$, $1 < p < \infty$ ($p = 1$, respectively), then* (25) *holds true* ((26) *for $p = 1$, respectively).*

THEOREM 11. *Let a complete bounded Reinhardt domain D be k-cylindric. If an operator $\mathcal{A}_{\mu_k}$ is bounded in $H^p(D)$, $1 < p < \infty$, then* (27) *holds true.*

As for the case of sufficient conditions, both theorems are proved in a completely similar way. We present the proof of the first one explicitly.

Proof of Theorem 10. Let us consider in D the subclass of $H^p(U_\rho)$ which consists of holomorphic functions of one variable z_k from the Hardy class in the disk U_ρ. These functions do belong to the whole $H^p(D)$ as well, since the integral over ∂D_r is the sum of two integrals, one over $\Gamma_k(r\rho) \cap \partial D_r$ and the other over $\partial D_r \setminus \Gamma_k(r\rho)$. In the second integral, we have $|z_k| < \rho$, and our functions are continuous at these points. The first integral is a repeated integral, for which the inner integral tends to the norm in $H^p(D \cap m_k)$ as $r \to 1-$, while the external integral is that over m_k with respect to an appropriate positive measure. By the k-cylindricity, the norm in $H^p(D \cap m_k)$ is the same in this case, since our functions depend only upon z_k. Therefore, the operator $\mathcal{H}_{\mu_k}$ is also continuous when acting on functions of one variable from $H^p(U_\rho)$. Applying now the known and given above one-dimensional necessary (and sufficient) results, we conclude on the necessity of (25) (and (26) for $p = 1$, respectively). The proof is complete. □

As above, extending an assumed restriction to all k yields a general result, this time necessary.

COROLLARY 12. *If a complete bounded Reinhardt domain D is k-cylindric for all k, $1 \le k \le n$. Then the condition* (28) [*or, relatively,* (29)] *is necessary for the boundedness of $\mathcal{H}_\mu$ in $H^p(D)$ when $1 < p < \infty$ (or, correspondingly, when $p = 1$).*

The necessary condition for the boundedness of $\mathcal{A}_\mu$ when $1 \le p < \infty$ is (30).

And, finally, let us give as a corollary a very special partial result earlier obtained in [3] for a smaller Hardy class.

COROLLARY 13. *If a domain D is the polydisk, then the condition* (28) [*or, relatively,* (29)] *is necessary and sufficient for the boundedness of $\mathcal{H}_\mu$ in $H^p(D)$ when $1<p<\infty$* (*or, correspondingly, when $p=1$*).

The necessary and sufficient condition for the boundedness of $\mathcal{A}_\mu$ when $1\le p<\infty$ is (30).

5.4. Necessary and sufficient conditions for $p=\infty$. Since the version of the main lemma for $p=\infty$ is less restrictive, so is its application to Hausdorff operators as well.

THEOREM 14. *Let D be a complete bounded Reinhardt domain. The Hausdorff operator $\mathcal{H}_{\mu_k}$ is bounded on $H^\infty(D)$ if and only if*

$$\int_0^1 s^{-1}\,d\mu_k(s)<\infty. \tag{35}$$

Proof. The proof of sufficiency goes along the same lines as that of the corresponding theorems for $p<\infty$, Theorems 6 and 7.

Let us go on to the necessity. Let ρ be the maximal radius of Q_k. Considering the function of one variable z_k in the disk U_ρ from the class $H^\infty(U_\rho)$, we conclude that it is also from $H^\infty(D)$. If the operator $\mathcal{H}_{\mu_k}$ (or $\mathcal{A}_{\mu_k}$) is bounded in $H^\infty(D)$, it is necessary that it be bounded on the class of all above-mentioned functions from $H^\infty(U_\rho)$, since

$$\sup|f||_{Q_k}=\sup|f||_D.$$

The necessity follows from this and the corresponding result for functions of one variable. □

We are now in a position to give the multidimensional result in full generality.

COROLLARY 15. *Let D be a bounded complete Reinhardt domain. Then the Hausdorff operator $\mathcal{H}_\mu f$ is bounded in $H^\infty(D)$ if and only if*

$$\prod_{k=1}^n\int_0^1 s^{-1}\,d\mu_k(s)<\infty. \tag{36}$$

6. Concluding remarks

To summarize the obtained results: we have proved the main lemma and corresponding related results as criteria for a function to belong to Hardy spaces on a wide class of Reinhardt domains.

It easily follows from the main lemma that most classical theorems on H^p classes in the disk are true for $H^p(D)$ for the corresponding Reinhardt domains D as well (Smirnov's theorem, for example, and others; see [4], [7], [10], [13]).

We have applied the mentioned criteria to study Hausdorff operators. These operators, in various settings, became more popular in the last decade. Our sufficient results for the boundedness of Hausdorff operators on Hardy spaces turn out to be necessary for a smaller class of Reinhardt domains. For the polydisk we have necessary and sufficient conditions, which was obtained earlier for a smaller Hardy class. On the other hand, for the unit ball in $\mathbb{C}^n$ we have only sufficient conditions.

Sharpness of the sufficient assumptions and, moreover, of the necessary assumptions is an interesting open problem.

Finally, let us mention that we have generalized one-dimensional results for Hausdorff operators to all n variables. Of course, we are able to formulate and prove corresponding results for any group of variables but omit this. First, it is a routine task, and, secondly, at present we see no application of such results.

Boundedness of quasi-Hausdorff operators in H^∞ is proved in a much more sophisticated way in dimension one (see [6]) and is open in several dimensions.

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