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CR SINGULARITIES OF REAL FOURFOLDS IN $\mathbb{C}^3$

ADAM COFFMAN

ABSTRACT. CR singularities of real 4-submanifolds in complex 3-space are classified by using local holomorphic coordinate changes to transform the quadratic coefficients of the real analytic defining equation into a normal form. The quadratic coefficients determine an intersection index, which appears in global enumerative formulas for CR singularities of compact submanifolds.

1. Introduction

If a real 4-manifold M is embedded in $\mathbb{C}^3$, then for each point x on M there are two possibilities: the tangent 4-plane at x is either a complex hyperplane in $\mathbb{C}^3$, so M is said to be "CR singular" at x, or it is not, so M is said to be "CR generic" at x. This article considers the local extrinsic geometry of a real analytically embedded M near a CR singular point, by finding invariants under biholomorphic coordinate changes. The main result is a classification of CR singularities, via a list of normal forms for the quadratic part of the local defining function. The matrix algebra leading to the classification is worked out in Section 2, and then summarized in Section 7 after the geometric interpretation is developed in Sections 3–5.

The analysis of normal forms near CR singular points is part of the program of studying the local equivalence problem for real m-submanifolds of $\mathbb{C}^n$. In this paper, we consider the $m = 2n - 2$ case ("codimension 2"), focusing on real 4-manifolds in $\mathbb{C}^3$, since the $m = n = 2$ case is well-known and larger dimensions seem to lead to difficult computations.

In Section 3, we recall some of Lai's formulas relating the global topology of a real submanifold to the number of its CR singular points, counted with sign according to an intersection number. In Section 4, we derive a simple expression that calculates the intersection index in terms of the coefficients in

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the local defining equation, generalizing the well-known $m = n = 2$ case, where CR singular points of compact surfaces in $\mathbb{C}^2$ can be counted according to their elliptic or hyperbolic nature as determined by the local Bishop invariant. Section 6 gives some concrete examples of compact real 4-manifolds immersed in $\mathbb{C}^3$ or $\mathbb{C}P^3$ to illustrate the enumerative formulas and local invariants.

2. Normal forms for CR singularities

Let $n \geq 2$ and $m = 2n-2$, so a real m-submanifold M of a complex n-manifold has real codimension 2. Considering n in general shows how the well-known $(m,n) = (2,2)$ case is related to the higher-dimensional cases, including $(4,3)$. In this section, we are only interested in a small coordinate neighborhood, so we let the ambient complex space be $\mathbb{C}^n$, with coordinates $z_1, \ldots, z_n$. We also use the abbreviations $z = (z_1, \ldots, z_{n-1})^T$ and $\vec{z} = (z_1, \ldots, z_n)^T$ (both column vectors). The real and imaginary parts of the coordinate functions are labeled $z_k = x_k + iy_k$.

2.1. Local defining equations and transformations. We begin by assuming M is a real analytic $(2n-2)$-submanifold in $\mathbb{C}^n$ with a CR singularity at some point: the tangent space at that point is a complex hyperplane. We are interested in the invariants of M under local biholomorphic transformations. By a translation that moves the CR singular point to the origin $\vec{0}$, and then a complex linear transformation of $\mathbb{C}^n$, the tangent $(2n-2)$-plane $T_{\vec{0}}M$ can be assumed to be the $(z_1, \ldots, z_{n-1})$-subspace. Then there is some neighborhood Δ of the origin in $\mathbb{C}^n$ so that the defining equation of M in Δ is in the form of a graph over a neighborhood $\mathcal{D}$ of the origin in the complex subspace $T_{\vec{0}}M$:

$$z_n = h(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}) = h(z, \bar{z}),$$

where $h(z, \bar{z})$ is a complex valued real analytic function defined for $z \in \mathcal{D} \subseteq T_{\vec{0}}M$, and vanishing to second order at $z = (0, \ldots, 0)^T$. Once M is in this "standard position," the complex defining function $h(z, \bar{z})$ is of the following form:

$$h(z, \bar{z}) = z^T Q z + \bar{z}^T R z + \bar{z}^T S \bar{z} + e(z, \bar{z}), \tag{2.1}$$

where Q, R, S are complex $(n-1) \times (n-1)$ coefficient matrices, z^T and $\bar{z}^T$ are row vectors, and $e(z, \bar{z})$ is a real analytic function on $\mathcal{D}$ vanishing to third order at $z = (0, \ldots, 0)^T$. The matrices Q and S can be assumed to be complex symmetric. It can also be assumed that $\mathcal{D}$ is small enough so that the function $h(z, \bar{z})$ can be expressed as the restriction to $\{(z, w) \in \mathcal{D} \times \mathcal{D} : w_k = \bar{z}_k\}$ of the multi-indexed series:

$$h(z, w) = z^T Q z + w^T R z + w^T S w + \sum_{|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \geq 3} e^{\boldsymbol{\alpha\beta}} z^{\boldsymbol{\alpha}} w^{\boldsymbol{\beta}}, \tag{2.2}$$

which converges on some set

$$\mathcal{D}_c = \{(z,w) : |z_k| < \epsilon, |w_k| < \epsilon\} \subseteq \mathbb{C}^{2n-2} \tag{2.3}$$

to a complex analytic function.

DEFINITION 2.1. A (formal, with multi-indexed complex coefficient $e^{\boldsymbol{\alpha\beta}}$) monomial of the form $e^{\boldsymbol{\alpha\beta}} z^{\boldsymbol{\alpha}} w^{\boldsymbol{\beta}} = e^{\boldsymbol{\alpha\beta}} z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} w_1^{\beta_1} \cdots w_{n-1}^{\beta_{n-1}}$ has "degree" $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| = \alpha_1 + \cdots + \alpha_{n-1} + \beta_1 + \cdots + \beta_{n-1}$. A power series (convergent or formal) $e(z,w) = \sum e^{\boldsymbol{\alpha\beta}} z^{\boldsymbol{\alpha}} w^{\boldsymbol{\beta}}$ with $e^{\boldsymbol{\alpha\beta}} = 0$ for all terms of degree $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| < \mathbf{d}$ is abbreviated $e(z,w) = O(\mathbf{d})$.

The above notation applies to expressions of the form $e(z,\bar{z})$, for example, the last term in equation (2.1) is $e(z,\bar{z}) = O(3)$.

We consider the effect of a coordinate change of the following form:

$$\begin{aligned} \tilde{z}_1 &= z_1 + p_1(z_1, \ldots, z_n), \\ &\vdots \\ \tilde{z}_n &= z_n + p_n(z_1, \ldots, z_n), \end{aligned} \tag{2.4}$$

abbreviated $\vec{\tilde{z}} = \vec{z} + \vec{p}(\vec{z})$, where $p_1(\vec{z}), \ldots, p_n(\vec{z})$ are holomorphic functions defined by series centered at $\vec{0}$ with no linear or constant terms. Since this transformation of $\mathbb{C}^n$ has its linear part equal to the identity map, it is invertible on some neighborhood of the origin and preserves the form of (2.1). In the following calculations, we neglect considering the size of that neighborhood, and consider only points close enough to the origin.

As the first special case of a transformation of the form (2.4) to be used, let $p_1(\vec{z}), \ldots, p_{n-1}(\vec{z})$ be identically zero, so $\tilde{z} = z$, and let $p_n(\vec{z})$ be a homogeneous quadratic polynomial in $z_1, \ldots, z_{n-1}$, so $p_n(\vec{z}) = z^T Q' z$ for some complex symmetric $(n-1) \times (n-1)$ matrix Q'. Given a point on M near $\vec{0}$, its coordinates $\vec{z} = (z_1, \ldots, z_n)^T$ satisfy $z_n - h(z,\bar{z}) = 0$. The new coordinates at that point satisfy:

$$\begin{aligned} \tilde{z}_n &= z_n + p_n(\vec{z}) \\ &= z^T Q z + \bar{z}^T R z + \bar{z}^T S \bar{z} + e(z,\bar{z}) + p_n(\vec{z}) \\ &= \tilde{z}^T (Q + Q') \tilde{z} + \bar{\tilde{z}}^T R \tilde{z} + \bar{\tilde{z}}^T S \bar{\tilde{z}} + e(\tilde{z}, \bar{\tilde{z}}). \end{aligned} \tag{2.5}$$

So, such a quadratic transformation changes the coefficient matrix Q, but all the other coefficients of the new equation, $\tilde{z}_n - \tilde{h}(\tilde{z}, \bar{\tilde{z}}) = 0$, are the same. Choosing $Q' = \bar{S} - Q$ (and dropping the tilde notation), the defining equation in the new coordinates is:

$$z_n = z^T \bar{S} z + \bar{z}^T R z + \bar{z}^T S \bar{z} + e(z,\bar{z}), \tag{2.6}$$

so the first and third terms have a real valued sum and $e(z,\bar{z})$ is still $O(3)$.

Next, we consider some linear transformations of $\mathbb{C}^n$, but only those which fix, as a set, the complex tangent hyperplane $T_{\vec{0}}M = \{z_n = 0\}$, so they are of the form

$$\vec{\tilde{z}}_{n\times 1} = C_{n\times n}\vec{z}_{n\times 1} = \begin{pmatrix} c_{1,1} & \dots & c_{1,n-1} & c_{1,n} \\ \vdots & & \vdots & \vdots \\ c_{n-1,1} & \dots & c_{n-1,n-1} & c_{n-1,n} \\ 0 & \dots & 0 & c_{n,n} \end{pmatrix} \vec{z}_{n\times 1}, \tag{2.7}$$

with complex entries and nonzero determinant, so $c_{n,n} \neq 0$. The inverse matrix has the block form

$$C^{-1} = \begin{pmatrix} & & & * \\ & A_{(n-1)\times(n-1)} & & \vdots \\ & & & * \\ 0 & \dots & 0 & c_{n,n}^{-1} \end{pmatrix},$$

where the block A in C^{-1} does not depend on the entries $c_{1,n}, \dots, c_{n,n}$. In the special case where $c_{1,n} = \dots = c_{n-1,n} = 0$, C has a block diagonal pattern and so does its inverse, so $z = A\tilde{z}$. In the coordinate system defined by such a linear transformation, the new defining equation is

$$\begin{aligned} \tilde{z}_n &= c_{n,n} z_n \\ &= c_{n,n} \cdot \left(z^T \bar{S} z + \bar{z}^T R z + \bar{z}^T S \bar{z} + e(z,\bar{z}) \right) \\ &= c_{n,n} \cdot \left(\tilde{z}^T A^T \bar{S} A \tilde{z} + \bar{\tilde{z}}^T \bar{A}^T R A \tilde{z} + \bar{\tilde{z}}^T \bar{A}^T S \bar{A} \bar{\tilde{z}} + \tilde{e}(\tilde{z}, \bar{\tilde{z}}) \right), \end{aligned} \tag{2.8}$$

where the new higher order part is still real analytic but may have a different domain of convergence.

If the coefficients $c_{1,n}, \dots, c_{n-1,n}$ were nonzero, they would contribute only terms of degree at least 3, not affecting the quadratic terms in (2.8). Similarly, allowing a coordinate change with nonlinear terms, as in (2.4), would only introduce terms of degree at least 3, or, as in (2.5), arbitrarily alter the first quadratic term. So, under a general transformation,

$$\vec{\tilde{z}} = C\vec{z} + \vec{p}(\vec{z}) \tag{2.9}$$

[which combines (2.4) and (2.7), and preserves the standard position, (2.1)], the only interesting effect on the quadratic part of $h(z,\bar{z})$ is that the coefficient matrices are transformed as:

$$(R,S) \mapsto (c_{n,n}\bar{A}^T R A, c_{n,n}\bar{A}^T S \bar{A}). \tag{2.10}$$

The first invariant to notice is the pair $(\mathrm{rank}(R), \mathrm{rank}(S))$. The rank of the concatenated matrix $(R|S)_{(n-1)\times(2n-2)}$ is also an invariant under this action.

The group of invertible matrices A has $(n-1)^2$ complex dimensions, and the group of scalars $c_{n,n}$ is one-dimensional; however, if A is a real multiple λ of the identity matrix $\mathbb{1}$, then its action can be canceled by choosing $c_{n,n} = \lambda^{-2}$. So, there are at most $2((n-1)^2+1) - 1 = 2n^2 - 4n + 3$ real parameters in the group action. The coefficient matrices R and S have $(n-1)^2$ and $(n-1)n/2$ complex dimensions, for a total of $3n^2-5n+2$ real dimensions. The number of coefficients always exceeds the number of parameters in the group action, so we expect infinitely many equivalence classes of matrix pairs, distinguished by continuous invariants.

2.2. Degrees of flatness. We continue with the assumption that M is a real analytic submanifold of $\mathbb{C}^n$ with real codimension 2.

DEFINITION 2.2. A manifold M in standard position (2.1) has a defining function $h(z,\bar{z})$ in a "quadratically flat normal form" if the quadratic part, $z^TQz+\bar{z}^TRz+\bar{z}^TS\bar{z}$, of its defining function is a real valued polynomial. A manifold $M\subseteq\mathbb{C}^n$ with a CR singular point $\vec{x}\in M$ is "quadratically flat" at $\vec{x}$ if, after M is put into standard position (2.1) by a complex affine transformation $\vec{\tilde{z}}=L_{n\times n}\cdot(\vec{z}-\vec{x})$, there is a local holomorphic coordinate change (2.9) such that in the new coordinates, M has a defining function in a quadratically flat normal form.

The definition of "quadratically flat normal form" is equivalent to $Q=\bar{S}$ and $R=\bar{R}^T$ (so R is Hermitian symmetric). Considering (2.6) and the transformation rule (2.10), for M in standard position, the "quadratically flat" property is equivalent to R being a complex scalar multiple of a Hermitian symmetric matrix.

The notion of quadratic flatness is the $\mathbf{d}=2$ special case of the following generalization to higher degree.

DEFINITION 2.3. For $\mathbf{d}\geq 2$, a real analytic manifold M in standard position (2.1) has a defining function in a "$\mathbf{d}$-flat normal form" if the defining equation in a neighborhood of $\vec{0}$ is

$$z_n=h(z,\bar{z})=\mathbf{r}(z,\bar{z})+O(\mathbf{d}+1)$$

for some real valued polynomial $\mathbf{r}(z,\bar{z})$. A manifold $M\subseteq\mathbb{C}^n$ with a CR singular point $\vec{x}\in M$ is "$\mathbf{d}$-flat" at $\vec{x}$ if, after M is put into standard position (2.1) by a complex affine transformation $\vec{\tilde{z}}=L_{n\times n}\cdot(\vec{z}-\vec{x})$, there is a local holomorphic coordinate change (2.9) such that in the new coordinates, M has a defining function in a $\mathbf{d}$-flat normal form.

DEFINITION 2.4. A real analytic $(2n-2)$-manifold $M\subseteq\mathbb{C}^n$ is "formally flattenable" at a CR singular point $\vec{x}\in M$ if it is $\mathbf{d}$-flat at $\vec{x}$ for every $\mathbf{d}\geq 2$.

DEFINITION 2.5. A manifold $M\subseteq\mathbb{C}^n$ with a CR singular point $\vec{x}\in M$ is "holomorphically flat" at $\vec{x}$ if, after M is put into standard position (2.1) by a complex affine transformation $\vec{\tilde{z}}=L_{n\times n}\cdot(\vec{z}-\vec{x})$, there is a local holomorphic coordinate change (2.9) such that in the new coordinates, the defining function (2.1) is real valued.

By the definition, if M is holomorphically flat near a CR singular point, then there is a local coordinate system around the point so that a neighborhood of $\vec{0}$ in M is contained in the real hyperplane $\{\mathrm{Im}(z_n)=0\}$. By the well-known normal form result of É. Cartan that a real analytic nonsingular Levi flat hypersurface is locally biholomorphically equivalent to a real hyperplane, the local notion of M being holomorphically flat at $\vec{x}$ is equivalent to the (more coordinate-free) property that there exists a real analytic nonsingular Levi flat hypersurface containing a neighborhood of $\vec{x}$ in M.

2.3. The $m=n=2$ case. When $m=n=2$, M is a real surface in $\mathbb{C}^2$ with a CR singular point. For M in standard position (2.1), the coefficient matrices are size 1×1, and can be written as complex constant coefficients. The action of (2.10) becomes $(R,S)\mapsto (c_{2,2}|\alpha|^2R, c_{2,2}\bar{\alpha}^2S)$ for nonzero complex constants $c_{2,2}$ and α, where $A_{1\times 1}=(\alpha)$. If $R\neq 0$, then (R,S) can then be transformed into $(1,\gamma_1)$, $\gamma_1\geq 0$. If $R=0$, then there are two normal forms: $(0,1)$ and $(0,0)$. The quadratic normal forms for the defining function of M are then:

$$\begin{aligned} z_2 &= z_1\bar{z}_1+\gamma_1\cdot(z_1^2+\bar{z}_1^2)+e(z_1,\bar{z}_1), \quad \gamma_1\geq 0, \quad \text{or} \\ z_2 &= z_1^2+\bar{z}_1^2+e(z_1,\bar{z}_1), \quad \text{or} \\ z_2 &= e(z_1,\bar{z}_1). \end{aligned} \tag{2.11}$$

So, γ_1 is the well-known Bishop invariant [Bishop] and the second case is $\gamma_1=+\infty$. This calculation of the quadratic normal forms shows that any surface M with a CR singular point is quadratically flat (Definition 2.2) at that point.

The normalization of the cubic terms depends on γ_1; all the cases $0\leq\gamma_1\leq\infty$ are surveyed in [C_3] Section 5, and we recall a few examples here.

For $\gamma_1\in(0,\frac{1}{2})\cup(\frac{1}{2},1)\cup(1,\infty)$, it was shown by [MW] that the cubic terms of $e(z_1,\bar{z}_1)$ can be eliminated by a holomorphic coordinate change near the origin, and so M is 3-flat. [There may be $O(4)$ terms that cannot be eliminated or made real valued.]

Any M with $\gamma_1=\frac{1}{2}$ is 3-flat; although there may be some cubic terms that cannot be eliminated by a holomorphic coordinate change, such terms can always be made real valued. For $\gamma_1=1$, there are some M which are not 3-flat.

It was also shown by [MW], and [HK] respectively, that for $0<\gamma_1<\frac{1}{2}$, and $\gamma_1=0$, M is holomorphically flat. Results of [MW] and [Gong] show that there exist some surfaces M with $\gamma_1>\frac{1}{2}$ which are formally flattenable but not holomorphically flat.

2.4. The $m=4$, $n=3$ case.

2.4.1. *A quadratic normal form.* In this, the main case of this paper, M is a 4-manifold in $\mathbb{C}^3$, which we assume is given in the form (2.6). The quadratic coefficient matrices R, S are size 2×2, so there are 7 independent complex coefficients, and, as previously calculated, 9 real parameters in the group action (2.10). One expects that in general, attempting to put the pair (R, S) into a normal form leaves 5 continuous real invariants.

We choose to begin the normalization by considering the action of the transformation $R_{2\times 2} \mapsto c\bar{A}^T RA$, where $c = c_{3,3}$ is a nonzero scalar and A is an invertible 2×2 complex matrix. Conveniently, the problem of finding representative matrices for the orbits of this action has already been solved in [C$_2$].

PROPOSITION 2.6 ([C$_2$], Theorem 4.3). *Given a complex 2×2 matrix R, there is exactly one of the following normal forms N such that $N = c\bar{A}^T RA$ for some nonzero complex c and some invertible complex $A_{2\times 2}$:*

(1) $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, $0 \le \theta \le \pi$;
(2) $\begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$, $0 \le \tau < 1$;
(3) $\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$;
(4) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$;
(5) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

For most values of the invariants θ, τ, these normal forms are not Hermitian symmetric—unless R is already a complex multiple of a Hermitian matrix, one would not expect $c\bar{A}^T RA$ to be Hermitian. So, unlike the $m = n = 2$ case from Section 2.3, the quadratic part of the defining function $h(z, \bar{z})$ generally cannot be made real valued by a holomorphic coordinate change. For a manifold in standard position (2.1), the following are equivalent: (I) M is quadratically flat at $\vec{0}$; (II) R is a multiple of a Hermitian matrix; (III) $N = c\bar{A}^T RA$, where N is one of the following normal forms from the proposition: Case (1) with $\theta = 0$ or π, Case (4), or Case (5).

Using N from the above proposition and introducing the notation $P = 2\bar{S}_{2\times 2}$, (2.6) becomes:

$$z_3 = (\bar{z}_1, \bar{z}_2) N \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathrm{Re}\left((z_1, z_2) P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) + e(z_1, \bar{z}_1, z_2, \bar{z}_2). \tag{2.12}$$

The linear action of C (2.7), (2.8), followed by another holomorphic transformation (2.5), preserves the form of the defining equation (2.12), and acts on the coefficient matrices by the transformation:

$$(N, P) \mapsto (c\bar{A}^T NA, \bar{c}A^T PA). \tag{2.13}$$

Since N is already normalized, to find a normal form for (2.12), we consider only pairs (c, A) that preserve N: $N = c\bar{A}^T NA$. Since this depends on the nature of the various matrices N appearing in the proposition, we proceed in cases.

In each case, let $P = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, with complex entries a, b, d, and let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, with complex entries and nonzero determinant.

Case (1a). For $N=\left(\begin{smallmatrix}1&0\\0&e^{i\theta}\end{smallmatrix}\right)$, with $0<\theta<\pi$, if $N=c\bar{A}^TNA$, then c is real (this follows from calculating determinants, for example), and

$$N=c\bar{A}^TNA=c\begin{pmatrix}\bar\alpha&\bar\gamma\\\bar\beta&\bar\delta\end{pmatrix}\begin{pmatrix}1&0\\0&e^{i\theta}\end{pmatrix}\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$$

implies

$$\begin{aligned}1&=c\cdot(\alpha\bar\alpha+\gamma\bar\gamma e^{i\theta}),\\0&=\alpha\bar\beta+\gamma\bar\delta e^{i\theta},\\e^{i\theta}&=c\cdot(\beta\bar\beta+\delta\bar\delta e^{i\theta}).\end{aligned}\tag{2.14}$$

It follows that $\gamma=\beta=0$, $c=|\alpha|^{-2}$, and $|\delta|=|\alpha|$. So the action of (2.13) is that P can be transformed to:

$$\bar{c}A^TPA=\frac{1}{|\alpha|^2}\begin{pmatrix}\alpha&0\\0&\delta\end{pmatrix}\begin{pmatrix}a&b\\b&d\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\delta\end{pmatrix}=\begin{pmatrix}a\frac{\alpha^2}{|\alpha|^2}&b\frac{\alpha\delta}{|\alpha\delta|}\\b\frac{\alpha\delta}{|\alpha\delta|}&d\frac{\delta^2}{|\delta|^2}\end{pmatrix}.$$

When a and d are both nonzero, α and δ can be chosen to rotate them onto the positive real axis. The value of b cannot be normalized any further except that P with positive a, d, and complex b is equivalent to the matrix with the same a, d, but opposite value for b.

If $a=0$ or $d=0$, then α and δ can be chosen to transform P into a matrix with all nonnegative entries.

Case (1b). For $N=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, the $\theta=0$ case of Proposition 2.6, c is real as in the previous case, and by the calculation analogous to (2.14) with $\theta=0$, in fact c is positive, so $c=\frac{+1}{|\det(A)|}$. The equation $N=c\bar{A}^TNA=\overline{(\sqrt{c}A)}^TN(\sqrt{c}A)$ shows A is a real multiple of a unitary matrix, and conversely if $A=rU$ for some real r and unitary U, then $(c,A)=(r^{-2},rU)$ stabilizes N. So the action of (2.13) is that P can be transformed to:

$$\bar{c}A^TPA=\frac{1}{|\det(A)|}A^TPA=r^{-2}(rU)^TP(rU)=U^TPU.$$

The normal form problem for P is thus reduced to finding a normal form for a complex symmetric 2×2 matrix under the relation of congruence by a unitary matrix. This problem has a well-known solution by Takagi ([HJ], Section 4.4, see also Theorem 5 of [Hua]), which says that a complex symmetric matrix has a diagonal normal form under unitary congruence, with nonnegative real entries. These entries can be reordered by a unitary transformation, so a normal form is $\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right)$ with $0\le a\le d$.

Case (1c). For $N=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, the $\theta=\pi$ case of Proposition 2.6, c is real. Briefly neglecting c, we consider the group of invertible matrices A preserving N. The condition $\bar{A}^TNA=N$ is equivalent to $AN\bar{A}^T=N$. For any symmetric coefficient matrix P, define an auxiliary matrix $B=\bar{P}^TNP$, which is

Hermitian symmetric. Then the congruence action $P' = A^T P A$ transforms the product:

$$B' = \overline{P'}^T N P' = \overline{(A^T P A)}^T N(A^T P A) = \bar{A}^T \bar{P}^T \bar{A} N A^T P A = \bar{A}^T \bar{P}^T N P A,$$

so B' is related to B by Hermitian congruence. So, the action (2.13) of A on (N, P) (with $c = 1$) has been temporarily replaced by the action of simultaneous Hermitian congruence on the pair (N, B) of Hermitian symmetric matrices. This normal form problem is considered by [HJ], and more recently by [HS] and [LR]. Recalling that N is Hermitian congruent to the matrix $N' = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ [the $\tau \to 1^-$ limit of Case (2) from the proposition], the result from [HJ], [HS], [LR] is that for any Hermitian B, the pair (N, B) is equivalent under simultaneous Hermitian congruence to exactly one pair from the following list:

- $\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} k_1 & 0 \\ 0 & k_2 \end{smallmatrix}\right)\right)$, $k_1, k_2 \in \mathbb{R}$;
- $\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & k \\ k & 1 \end{smallmatrix}\right)\right)$, $k \in \mathbb{R}$;
- $\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & x+iy \\ x-iy & 0 \end{smallmatrix}\right)\right)$, $x \in \mathbb{R}$, $y > 0$.

We continue with Case (1c) by splitting into subcases corresponding to the above three intermediate normal forms.

Case (1ci). There is some nonsingular matrix A so that the Hermitian pair $(N, \bar{P}^T N P)$ is simultaneously diagonalized and $P' = A^T P A$ is a complex symmetric matrix $\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$ [using the same place-holding letters as in Case (1a) even though P has been transformed once already] satisfying

$$B' = \overline{P'}^T N P' = \overline{\begin{pmatrix} a & b \\ b & d \end{pmatrix}}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}. \tag{2.15}$$

By another transformation, of the form $e^{i\xi}\mathbb{1}$ [which does not affect the pair (N, B')], we may assume that the entry b of P' satisfies $b \geq 0$. It then follows from expanding (2.15) that the entries of P' must satisfy either $b = 0$ or $d = \bar{a}$. In the $b = 0$ case, a transformation of the form $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$, with $|\alpha| = |\delta| = 1$, preserves N and puts P' into a diagonal normal form with nonnegative real entries. By a transformation of the form $(c, A) = \left(-1, \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)\right)$, these entries can be interchanged, so a unique normal form is $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with $0 \leq a \leq d$.

In the $d = \bar{a}$ case, the same type of diagonal transformation with $\delta = 1/\alpha$ puts P' into the form $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ with $b > 0$ and $a \geq 0$. However, this simplifies even further, and is not always different from the previous ($b = 0$) case. When $0 < b < a$, a transformation of the form

$$(c, A) = \left(\frac{-1}{2\sqrt{a^2 - b^2}(-a + \sqrt{a^2 - b^2})}, \begin{pmatrix} b & -a + \sqrt{a^2 - b^2} \\ -a + \sqrt{a^2 - b^2} & b \end{pmatrix}\right)$$

preserves N and diagonalizes P' to $\sqrt{a^2-b^2}\cdot\mathbb{1}$. When $0\le a<b$, N and P' cannot be simultaneously diagonalized, but a transformation of the form

$$(c,A)=\left(\frac{1}{4b\sqrt{b^2-a^2}},\begin{pmatrix}-a+i(b+\sqrt{b^2-a^2}) & b-\sqrt{b^2-a^2}-ia\\ b-\sqrt{b^2-a^2}-ia & -a+i(b+\sqrt{b^2-x^2})\end{pmatrix}\right)$$

preserves N and takes P' to $\frac{-i}{b}\sqrt{b^2-a^2}(a-i\sqrt{b^2-a^2})N'$, which can be rotated by $A=e^{i\theta}\mathbb{1}$ to $\sqrt{b^2-a^2}N'$. In the $0<a=b$ case, P' has rank 1 but (N,P') is not simultaneously diagonalizable, so it is inequivalent to the $b=0$ case with rank 1, where $0=a<d$. A transformation of the form

$$(c,A)=\left(\frac{1}{4a},\begin{pmatrix}1+a & 1-a\\ 1-a & 1+a\end{pmatrix}\right)$$

preserves N and normalizes a to 1.

Case (1cii). If there is no transformation simultaneously diagonalizing (N,B), then there is some nonsingular matrix A so that $\bar{A}^TNA=N'$, $A^TPA=P'$, and $B'=\bar{A}^TBA$ equals either the second or third normal form from the above list—in this subcase we consider the second. The property $\bar{A}^TNA=N'$ is equivalent to $\bar{A}N'A^T=N$, so

$$B'=\bar{A}^TBA=\bar{A}^T\bar{P}^TNPA=\bar{A}^T\bar{P}^T\bar{A}N'A^TPA=\overline{(A^TPA)}^T N'(A^TPA).$$

With notation as in the previous case, $P'=A^TPA$ is a complex symmetric matrix $\left(\begin{smallmatrix}a & b\\ b & d\end{smallmatrix}\right)$ satisfying

$$B'=\overline{P'}^TN'P'=\overline{\begin{pmatrix}a & b\\ b & d\end{pmatrix}}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ b & d\end{pmatrix}=\begin{pmatrix}0 & k\\ k & 1\end{pmatrix}. \tag{2.16}$$

By another transformation, of the form $e^{i\xi}\mathbb{1}$ [which does not affect the pair (N',B')], we may assume that the entry a of P' satisfies $a\ge 0$. However, it follows from expanding the product in (2.16) that there are no solutions of (2.16) with $a>0$, so $a=0$. Then (2.16) becomes

$$\overline{\begin{pmatrix}0 & b\\ b & d\end{pmatrix}}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & b\\ b & d\end{pmatrix}=\begin{pmatrix}0 & b\bar{b}\\ b\bar{b} & b\bar{d}+\bar{b}d\end{pmatrix}=\begin{pmatrix}0 & k\\ k & 1\end{pmatrix},$$

so neither b nor d is 0. By yet another scalar transformation, we may assume that $b>0$, without changing the RHS of the above equation, so b is an invariant determined by k, and the equality of entries $b\cdot(d+\bar{d})=1$ implies $\mathrm{Re}(d)\neq 0$. A transformation $(c,A)=\left(\frac{1}{r},\left(\begin{smallmatrix}1 & is\\ 0 & r\end{smallmatrix}\right)\right)$, with r and s real, preserves N' and transforms $P'=\left(\begin{smallmatrix}0 & b\\ b & d\end{smallmatrix}\right)$ into $\left(\begin{smallmatrix}0 & b\\ b & 2bis+rd\end{smallmatrix}\right)$. Since $\mathrm{Re}(d)\neq 0$ and $b>0$, r and s can be chosen to normalize d to 1.

Case (1ciii). The third case starts with the same steps as the previous case, with $\bar{A}^TNA=N'$ and $A^TPA=P'=\left(\begin{smallmatrix}a & b\\ b & d\end{smallmatrix}\right)$ satisfying

$$B'=\overline{P'}^TN'P'=\overline{\begin{pmatrix}a & b\\ b & d\end{pmatrix}}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ b & d\end{pmatrix}=\begin{pmatrix}0 & x+iy\\ x-iy & 0\end{pmatrix}, \tag{2.17}$$

with $y>0$. By another transformation, of the form $e^{i\xi}\mathbb{1}$ [which does not affect the pair (N', B')], we may assume that the entry a of P' satisfies $a \geq 0$. Then it follows from expanding the product in (2.17) that $a>0$, $b=0$, and $ad = x+iy$, an invariant quantity. A transformation $(c, A) = \left(a, \left(\begin{smallmatrix} 1/a & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$ preserves N' and transforms $P' = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ into $\left(\begin{smallmatrix} 1 & 0 \\ 0 & ad \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x+iy \end{smallmatrix}\right)$.

Case (2a). For $N = \left(\begin{smallmatrix} 0 & 1 \\ \tau & 0 \end{smallmatrix}\right)$, with $0<\tau<1$, if $N = c\bar{A}^T NA$, then c is real, and

$$N = c\bar{A}^T NA = c\begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

implies

$$\begin{aligned} 0 &= \alpha\bar{\gamma}\tau + \gamma\bar{\alpha}, \\ 0 &= \beta\bar{\delta}\tau + \bar{\beta}\delta, \\ 1 &= c\cdot(\beta\bar{\gamma}\tau + \bar{\alpha}\delta), \\ \tau &= c\cdot(\gamma\bar{\beta} + \alpha\bar{\delta}\tau). \end{aligned} \tag{2.18}$$

It follows that $\gamma = \beta = 0$, $\bar{\alpha}\delta$ is real, and $c = (\bar{\alpha}\delta)^{-1}$. So the action of (2.13) is that P can be transformed to:

$$\bar{c}A^T PA = \frac{1}{\bar{\alpha}\delta}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} a\alpha/\bar{\delta} & b\alpha/\bar{\alpha} \\ b\alpha/\bar{\alpha} & d\delta/\bar{\alpha} \end{pmatrix}.$$

When $b \neq 0$, α can be chosen to rotate it onto the positive real axis. Then, using real α and δ, a can be either scaled onto the unit circle or the origin; if $a=0$, then d can be scaled onto the unit circle or the origin. The resulting normal form is unique except that (a, d) is equivalent to the pair $(-a, -d)$.

If $b=0$ and $a \neq 0$, then α and δ can be chosen to transform a into 1, leaving $d \in \mathbb{C}$ as an invariant. If $b=a=0$, α and δ can be chosen to transform d into 1 or 0.

Case (2b). For $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, the $\tau = 0$ case of Proposition 2.6, if $N = c\bar{A}^T NA$, then c does not have to be real, but the calculation is similar to the previous (2a) case. The $\tau=0$ analogue of (2.18) implies $\gamma=\beta=0$ and $c = (\bar{\alpha}\delta)^{-1}$. So the action of (2.13) is that P can be transformed to:

$$\bar{c}A^T PA = (\alpha\bar{\delta})^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \bar{\delta}^{-1}\begin{pmatrix} a\alpha & b\delta \\ b\delta & d\frac{\delta^2}{\alpha} \end{pmatrix}.$$

When $b \neq 0$, δ can be chosen to rotate it onto the positive real axis. Then, using real δ and a complex number α, d can be transformed to 1, leaving $a \in \mathbb{C}$ as an invariant, or to 0 and then a can be transformed to 1 or 0.

If $b=0$ and $d \neq 0$, then α and δ can be chosen to transform d into 1 and a to a nonnegative invariant. If $b=d=0$, α and δ can be chosen to transform a into 1 or 0.

Case (3). For $N = \left(\begin{smallmatrix} 0 & 1 \\ 1 & i \end{smallmatrix}\right)$, the group of $\left(c, \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\right)$ such that $N = c\bar{A}^T NA$ is exactly the set with $\gamma = 0$, $\alpha = \delta \neq 0$, $\alpha\bar{\beta} + \bar{\alpha}\beta = 0$, and $c|\alpha|^2 = 1$. Then

$$\bar{c}A^T PA = \frac{1}{|\alpha|^2} \begin{pmatrix} a\alpha^2 & a\alpha\beta + b\alpha^2 \\ a\alpha\beta + b\alpha^2 & a\beta^2 + 2b\alpha\beta + d\alpha^2 \end{pmatrix}.$$

If $a \neq 0$, then β can eliminate b, α can rotate a to the positive real axis, and the complex number in the d position is an invariant. If $a = 0$ and $b \neq 0$, then β can eliminate d and α can rotate b to the positive real axis. If $a = b = 0$, then α can rotate d to the nonnegative real axis.

Case (4). For $N = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, the group of $\left(c, \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\right)$ such that $N = c\bar{A}^T NA$ is exactly the set with $\beta = 0$, $\delta \neq 0$, and $c|\alpha|^2 = 1$. The entry d of P can be normalized to 1 or 0. In the $d = 1$ case, γ can eliminate any b entry, and α can rotate a onto the nonnegative real axis. In the $d = 0$ case, if $b \neq 0$, then (a, b) can be normalized to $(0, 1)$; if $b = 0$, then α can rotate a onto the nonnegative real axis.

Case (5). Under an arbitrary congruence transformation, the rank is the only invariant of a complex symmetric matrix P under a congruence transformation $A^T PA$. The three normal forms are:

- $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$;
- $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$;
- $P = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

All these results on normal forms for the matrix pairs (N, P) are summarized by Table 1 in Section 7.

2.4.2. *An alternative quadratic normal form.* Instead of choosing to normalize the R matrix first in (2.6), we could have chosen to normalize the symmetric matrix S. Then the calculations start off in a simpler way, since first applying the congruence transformation to the complex symmetric matrix S, the three normal forms for $\bar{A}^T S\bar{A}$ are exactly as in the above Case (5).

In the first case, $S = \mathbb{1}$, the normalization problem for (R, S) reduces to the problem of finding a normal form for R under the action $R \mapsto c\bar{A}^T RA$, where (c, A) satisfies $c\bar{A}^T \bar{A} = \mathbb{1}$. This is the generic form of S, so one still expects five continuous real parameters in any collection of representative matrix pairs, but we do not attempt to find such normal forms.

In the second case, the pair (c, A) stabilizing S and acting on R satisfies

$$c\bar{A}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

leading to another normal form problem for R which we again do not pursue.

In the last case, where S is the zero matrix, the R matrix can then be put into one of the normal forms N from Proposition 2.6.

For the special case where R is Hermitian, [I] gives a list of 2×2 normal forms for (R, S), following this approach of normalizing S first.

2.4.3. *One example of a cubic normal form.* With M in standard position and the quadratic part of the defining function in normal form (2.12), we can consider its cubic terms. In the expansion

$$z_3 = (\bar{z}_1, \bar{z}_2)N\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathrm{Re}\left((z_1, z_2)P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) + e_3(z, \bar{z}) + e(z, \bar{z}),$$

$e_3(z, \bar{z})$ is the cubic part, and $e(z, \bar{z}) = O(4)$. In particular,

$$\begin{aligned} e_3 = {} & e^{3000} z_1^3 + e^{2100} z_1^2 \bar{z}_1 + e^{1200} z_1 \bar{z}_1^2 + e^{0300} \bar{z}_1^3 \\ & + e^{2010} z_1^2 z_2 + e^{1110} z_1 \bar{z}_1 z_2 + e^{0210} \bar{z}_1^2 z_2 + e^{2001} z_1^2 \bar{z}_2 \\ & + e^{1101} z_1 \bar{z}_1 \bar{z}_2 + e^{0201} \bar{z}_1^2 \bar{z}_2 + e^{1020} z_1 z_2^2 + e^{0120} \bar{z}_1 z_2^2 \\ & + e^{1011} z_1 z_2 \bar{z}_2 + e^{0111} \bar{z}_1 z_2 \bar{z}_2 + e^{1002} z_1 \bar{z}_2^2 + e^{0102} \bar{z}_1 \bar{z}_2^2 \\ & + e^{0030} z_2^3 + e^{0021} z_2^2 \bar{z}_2 + e^{0012} z_2 \bar{z}_2^2 + e^{0003} \bar{z}_2^3. \end{aligned} \tag{2.19}$$

The holomorphic coordinate changes that fix the origin and preserve the standard position of M are of the form

$$\begin{aligned} \tilde{z}_1 &= c_{11} z_1 + c_{12} z_2 + c_{13} z_3 + p_1^{20} z_1^2 + p_1^{11} z_1 z_2 + p_1^{02} z_2^2 \\ \tilde{z}_2 &= c_{21} z_1 + c_{22} z_2 + c_{23} z_3 + p_2^{20} z_1^2 + p_2^{11} z_1 z_2 + p_2^{02} z_2^2 \\ \tilde{z}_3 &= c_{33} z_3 + p_3^{200} z_1^2 + p_3^{110} z_1 z_2 + p_3^{020} z_2^2 + p_3^{101} z_1 z_3 + p_3^{011} z_2 z_3 \\ &\quad + p_3^{300} z_1^3 + p_3^{210} z_1^2 z_2 + p_3^{120} z_1 z_2^2 + p_3^{030} z_2^3. \end{aligned} \tag{2.20}$$

The linear coefficients are as in (2.7); the p coefficients are from (2.4). Assigning to each monomial $z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$ a "weight" $\alpha_1 + \alpha_2 + 2\alpha_3$, including terms in (2.20) of higher weight, such as z_3^2, would not contribute any changes to the quadratic or cubic coefficients of the defining function in the $\tilde{z}$ coordinates. The effect of a coordinate change (2.20) on the cubic part e_3 depends on the quadratic coefficients from N and P. If the quadratic part has already been put into a normal form, then in attempting to find a normal form for the cubic terms, one would want to use only holomorphic transformations that preserve the quadratic normal form, and this subgroup of transformations also depends on the coefficient matrices N and P. The comprehensive problem of finding cubic normal forms for every equivalence class of CR singularities seems to be difficult, so we consider just one special but interesting case.

Let M be given by this defining equation in standard position and with quadratic normal form as in Case (1b) from Section 2.4.1:

$$z_3 = z_1\bar{z}_1 + \gamma_1 \cdot (z_1^2 + \bar{z}_1^2) + z_2\bar{z}_2 + \gamma_2 \cdot (z_2^2 + \bar{z}_2^2) + e_3(z, \bar{z}) + e(z, \bar{z}). \tag{2.21}$$

The above expression is in a quadratically flat normal form: the quadratic part is real valued with $0 \le \gamma_1 \le \gamma_2$, and e_3 is as above. Now, assume further that $0 < \gamma_1 < \gamma_2$, with neither γ_1 nor γ_2 equal to $\frac{1}{2}$ or 1.

It follows from the normal form result of [MW] (as in Section 2.3) that there is a transformation of the form

$$\begin{aligned} \tilde{z}_1 &= z_1 + c_{13}z_3 + p_1^{20}z_1^2, \\ \tilde{z}_2 &= z_2, \\ \tilde{z}_3 &= z_3 + p_3^{101}z_1z_3 + p_3^{300}z_1^3 \end{aligned} \tag{2.22}$$

that eliminates the terms $e^{3000}z_1^3 + e^{2100}z_1^2\bar{z}_1 + e^{1200}z_1\bar{z}_1^2 + e^{0300}\bar{z}_1^3$ from e_3, without changing the quadratic part. Similarly, a transformation of z_2, z_3 can eliminate the cubic terms depending only on z_2, $\bar{z}_2$, without changing the quadratic part or reintroducing the cubic terms in z_1, $\bar{z}_1$. This leaves twelve monomials in (2.19) with complex coefficients.

Among the transformations (2.20), the subgroup preserving this partial normal form is given by:

$$\begin{aligned} \tilde{z}_1 &= c_{11}z_1 + p_1^{11}z_1z_2 + p_1^{02}z_2^2, \\ \tilde{z}_2 &= c_{22}z_2 + p_2^{20}z_1^2 + p_2^{11}z_1z_2, \\ \tilde{z}_3 &= c_{33}z_3 + p_3^{210}z_1^2z_2 + p_3^{120}z_1z_2^2, \end{aligned} \tag{2.23}$$

again, omitting terms of higher weight. The linear coefficients c_{11}, c_{22}, c_{33} must be real, and satisfy $c_{33} = c_{11}^2 = c_{22}^2$. The six complex coefficients of $\vec{p}$ can be arbitrary, and then after a coordinate change of the form (2.23), the defining equation becomes:

$$\tilde{z}_3 = \tilde{z}_1\bar{\tilde{z}}_1 + \gamma_1 \cdot (\tilde{z}_1^2 + \bar{\tilde{z}}_1^2) + \tilde{z}_2\bar{\tilde{z}}_2 + \gamma_2 \cdot (\tilde{z}_2^2 + \bar{\tilde{z}}_2^2) + \tilde{e}_3(\tilde{z}, \bar{\tilde{z}}) + e(\tilde{z}, \bar{\tilde{z}}),$$

$$\begin{aligned} \tilde{e}_3 = {} & \left(\frac{e^{1110}}{c_{22}} - \frac{p_1^{11}}{c_{11}c_{22}}\right)\tilde{z}_1\bar{\tilde{z}}_1\tilde{z}_2 + \left(\frac{e^{1101}}{c_{22}} - \frac{\overline{p_1^{11}}}{c_{11}c_{22}}\right)\tilde{z}_1\bar{\tilde{z}}_1\bar{\tilde{z}}_2 \\ & + \left(\frac{e^{2001}}{c_{22}} - \frac{p_2^{20}}{c_{11}^2}\right)\tilde{z}_1^2\bar{\tilde{z}}_2 + \left(\frac{e^{0210}}{c_{22}} - \frac{\overline{p_2^{20}}}{c_{11}^2}\right)\bar{\tilde{z}}_1^2\tilde{z}_2 \\ & + \left(\frac{e^{1011}}{c_{11}} - \frac{p_2^{11}}{c_{11}c_{22}}\right)\tilde{z}_1\tilde{z}_2\bar{\tilde{z}}_2 + \left(\frac{e^{0111}}{c_{11}} - \frac{\overline{p_2^{11}}}{c_{11}c_{22}}\right)\bar{\tilde{z}}_1\tilde{z}_2\bar{\tilde{z}}_2 \\ & + \left(\frac{e^{1002}}{c_{11}} - \frac{\overline{p_1^{02}}}{c_{22}^2}\right)\tilde{z}_1\bar{\tilde{z}}_2^2 + \left(\frac{e^{0120}}{c_{11}} - \frac{p_1^{02}}{c_{22}^2}\right)\bar{\tilde{z}}_1\tilde{z}_2^2 \\ & + \left(\frac{e^{2010}}{c_{22}} - \frac{2\gamma_1 p_1^{11}}{c_{11}c_{22}} - \frac{2\gamma_2 p_2^{20}}{c_{11}^2} + \frac{p_3^{210}}{c_{11}^2c_{22}}\right)\tilde{z}_1^2\tilde{z}_2 \\ & + \left(\frac{e^{0201}}{c_{22}} - \frac{2\gamma_1 \overline{p_1^{11}}}{c_{11}c_{22}} - \frac{2\gamma_2 p_2^{20}}{c_{11}^2}\right)\bar{\tilde{z}}_1^2\bar{\tilde{z}}_2 \\ & + \left(\frac{e^{1020}}{c_{11}} - \frac{2\gamma_1 p_1^{02}}{c_{22}^2} - \frac{2\gamma_2 p_2^{11}}{c_{11}c_{22}} + \frac{p_3^{120}}{c_{11}c_{22}^2}\right)\tilde{z}_1\tilde{z}_2^2 \\ & + \left(\frac{e^{0102}}{c_{11}} - \frac{2\gamma_1 \overline{p_1^{02}}}{c_{22}^2} - \frac{2\gamma_2 \overline{p_2^{11}}}{c_{11}c_{22}}\right)\bar{\tilde{z}}_1\bar{\tilde{z}}_2^2. \end{aligned} \tag{2.24}$$

In general, there are not enough parameters in (2.23) to put these twelve coefficients into a sparse normal form. We turn to yet a further special case.

Suppose that after M has been partially normalized as in (2.22), so that

$$e^{3000} = e^{2100} = e^{1200} = e^{0300} = e^{0030} = e^{0021} = e^{0012} = e^{0003} = 0,$$

eight of the remaining twelve coefficients satisfy the following conditions:

$$\begin{aligned} e^{1110} &= \overline{e^{1101}}, \\ e^{2001} &= \overline{e^{0210}}, \\ e^{1011} &= \overline{e^{0111}}, \\ e^{1002} &= \overline{e^{0120}}. \end{aligned} \tag{2.25}$$

This condition holds if (but not only if) the partially normalized e_3 is real valued, so the defining function of M is in a 3-flat normal form. Then, by inspection of (2.24), there is a transformation with complex coefficients p_1^{02}, p_1^{11}, p_2^{11}, p_2^{20} that can normalize the all the e^{1110}, ..., coefficients in (2.25) to 0. A transformation using p_3^{120} and p_3^{210} can then change the e^{2010} and e^{1020} coefficients to any value, in particular, to the complex conjugates of e^{0201} and e^{0102}, so that e_3 can be brought to the following real valued normal form:

$$\tilde{e}_3 = \overline{e^{0201}}\tilde{z}_1^2\tilde{z}_2 + e^{0201}\bar{\tilde{z}}_1^2\bar{\tilde{z}}_2 + \overline{e^{0102}}\tilde{z}_1\tilde{z}_2^2 + e^{0102}\bar{\tilde{z}}_1\bar{\tilde{z}}_2^2. \tag{2.26}$$

The coefficients e^{0201}, e^{0102} are not invariants since a real linear re-scaling by c_{11} and $c_{22} = \pm c_{11}$ is still possible.

There is a different, more useful, statement about the conditions under which the above normal form can be achieved.

THEOREM 2.7. *Suppose the real* 4*-manifold* M *in* $\mathbb{C}^3$ *is in standard position with defining equation in the quadratically flat normal form* (2.21) *with* $0 < \gamma_1 < \gamma_2$, *and neither* γ_1 *nor* γ_2 *equal to* $\frac{1}{2}$, *and with cubic terms* e_3 *as in* (2.19) *that satisfy the conditions*

$$e^{1200} = \overline{e^{2100}}, \tag{2.27}$$

$$\begin{aligned} e^{0021} &= \overline{e^{0012}}, \\ e^{1110} &= \overline{e^{1101}}, \\ e^{2001} &= \overline{e^{0210}}, \\ e^{1011} &= \overline{e^{0111}}, \\ e^{1002} &= \overline{e^{0120}}. \end{aligned} \tag{2.28}$$

Then there is a holomorphic coordinate change (2.20) *that puts the cubic part into the* 3*-flat normal form* $\tilde{e}_3$ (2.26).

Proof. The last four out of the above six conditions are copied from (2.25). The proof of the theorem is to proceed as above, first eliminating the cubic terms in $z_1, \bar{z}_1$ only. The condition (2.27) allows this to be done by a transformation of the form (2.22), but with $p_3^{101} = 0$. (This is possible even in the $\gamma_1 = 1$ case, cf. Example 5.6 of [C3].) Similarly, the cubic terms in $z_2, \bar{z}_2$ only can be eliminated by a transformation of the form

$$\begin{aligned} \tilde{z}_1 &= z_1, \\ \tilde{z}_2 &= z_2 + c_{23} z_3 + p_2^{02} z_2^2, \\ \tilde{z}_3 &= z_3 + p_3^{030} z_2^3, \end{aligned} \tag{2.29}$$

without reintroducing any of the previously eliminated terms. Again, the condition (2.28) (and $\gamma_2 \neq \frac{1}{2}$) means that a term $p_3^{011} z_2 z_3$ is not needed in (2.29). These first two steps may alter the numerical values of the remaining coefficients, but the claim is that if the cubic coefficients $e^{1110}, \ldots, e^{0120}$ satisfy the reality conditions (2.25) at the start of the process, then the new corresponding coefficients continue to satisfy those conditions. The calculation to verify this is straightforward but omitted; however, the assumption $p_3^{101} = p_3^{011} = 0$ is crucial: if either were nonzero, cubic terms not satisfying (2.25) could appear in the new defining equation. The rest of the normalization proceeds exactly as above. $\square$

The conclusion of the above theorem holds in particular when M has the specified quadratic normal form and is also 3-flat.

2.5. Flatness in higher dimensions. Proposition 2.6 showed that for the $m = 4$, $n = 3$ case considered in Section 2.4, R cannot, in general, be put into a Hermitian normal form. Similarly for higher dimensions $n > 2$, $m = 2n - 2$, quadratic flatness is a nongeneric property for codimension 2 manifolds with CR singularities. Higher degree flatness is even more nongeneric.

Consider the quadratically flat case, where M is in standard position in $\mathbb{C}^n$ and the coefficient matrix R happens to be Hermitian symmetric. This property of R is preserved by the action of $(c_{n,n}, A)$ from (2.7) if and only if $c_{n,n}$ is real—unless R is the zero matrix, in which case there is no such condition on $c_{n,n}$.

For a nonzero Hermitian matrix R, the transformations (2.10) [or (2.13)] of the defining function that preserve the property of being in a quadratically flat normal form have the following action: $R \mapsto c_{n,n} \bar{A}^T R A$, a congruence transformation of R, followed by real scalar multiplication. The Hermitian property of R is preserved, along with its rank. Congruence transformations also preserve the signature (p, q): the number of positive, and negative, eigenvalues; R is "definite" if $p = n$ or $q = n$. Multiplying by a negative scalar $c_{n,n}$ interchanges the p and q quantities. The rank $\rho(R)$, the number $\sigma(R) = |p - q|$ (from which p and q can be recovered, modulo switching), and the property of definiteness (or indefiniteness) are invariants of R under the action of this transformation group.

If R is the zero matrix, then $c_{n,n}R$ is still Hermitian even if $c_{n,n}$ is not a real number. M is quadratically flat: the quadratic part of $h(z,\bar{z})$ can be made real valued by a transformation of the form (2.5).

Including both cases $R\neq 0$, $R=0$, we can conclude that if M has a defining function $h(z,\bar{z})$ in a quadratically flat normal form, then any holomorphic coordinate change that preserves the property that $h(z,\bar{z})$ is in a quadratically flat normal form must leave invariant the rank $\rho(R)$ and the number $\sigma(R)=|p-q|$. In particular, these quantities are also invariants of a defining function in a **d**-flat normal form, or which is real valued (the holomorphically flat case), under holomorphic coordinate changes that preserve the flatness property of the defining function. The determinant of R is real and transforms as: $\det(c_{n,n}\bar{A}^TRA)=c_{n,n}^{n-1}|\det(A)|^2\det(R)$, so the sign of $\det(R)$ is also an invariant if n is odd.

For arbitrary dimensions $m=2n-2$, $n\geq 2$, if the $(n-1)\times(n-1)$ coefficient matrix R in (2.6) is Hermitian and definite (so that R transforms by congruence to the identity matrix $\mathbb{1}$), the result of Takagi is that then there is a unitary coordinate change diagonalizing S, with real entries $0\leq \gamma_1\leq\cdots\leq\gamma_{n-1}$ on the diagonal. These numbers are called "generalized Bishop invariants" by [HY], since the $0\leq\gamma_1<\infty$ normal form (2.11) from Section 2.3 is the $n=2$ special case. The definite normal form from Case (1b) of Section 2.4.1 is the $n=3$ case. The $R=\mathbb{1}$, $S=0_{(n-1)\times(n-1)}$ case, where $z_n=z_1\bar{z}_1+\cdots+z_{n-1}\bar{z}_{n-1}+O(3)$, is considered by [HY].

For P complex symmetric and R Hermitian symmetric, but not necessarily definite, a description of canonical representatives for the equivalence relation $(R,P)\sim(\bar{A}^TRA, A^TPA)$ is given by [E]. Allowing scalar multiplication, as in (2.13), with real $c_{n,n}$ as above, is different only in that (R,P) is equivalent to $(-R,-P)$ under the action of (2.13), while they may be inequivalent according to [E]. In the 2×2 case, the choices made by [I] and [E] for canonical forms are different from those in the calculations of Section 2.4.1, Cases (1b), (1c), (4), (5). However, it is straightforward to check how our list of normal forms, as summarized in Examples 7.3–7.7, corresponds (modulo scalars) to the systems of canonical forms in [I], [E].

3. Topological considerations

We recall some well-known general facts on Grassmannian manifolds, and also recall from [L] some topological properties of real m-submanifolds of complex n-manifolds: the local property of "general position" for real submanifolds, and also global properties measured by characteristic classes. The Grassmannian constructions and formulas of [L], in the special case of the topology of real surfaces immersed in complex surfaces ($m=n=2$), are also reviewed by [BF], [Bohr], [F], [ABKLR], Section 8.5.3. In this section, we work with connected, m-dimensional, smooth manifolds M, not necessarily real analytic.

3.1. The Grassmannian construction. For $0<m<2n$, let $G(m,\mathbb{R}^{2n})$ denote the Grassmannian manifold of m-dimensional real linear subspaces of $\mathbb{R}^{2n}$. The real dimension of $G(m,\mathbb{R}^{2n})$ is $m\cdot(2n-m)$.

Similarly, let $SG(m,R^{2n})$ denote the manifold of oriented m-dimensional subspaces of $\mathbb{R}^{2n}$. Its real dimension is also $m\cdot(2n-m)$. There is a two-to-one covering map $\mathcal{F}: SG(m,\mathbb{R}^{2n})\to G(m,\mathbb{R}^{2n})$ given by forgetting the orientation, and an involution $\mathcal{R}: SG(m,\mathbb{R}^{2n})\to SG(m,\mathbb{R}^{2n})$ given by reversing the orientation.

For any immersion ι of a real m-manifold M into $\mathbb{R}^{2n}$, we can define the Gauss map $\boldsymbol{\gamma}: M\to G(m,\mathbb{R}^{2n}): x\mapsto T_{\iota(x)}M$. If M has an orientation, then the corresponding map is denoted $\boldsymbol{\gamma}_s: M\to SG(m,\mathbb{R}^{2n})$.

When m is even and there is a "complex structure operator," J, a real linear map $\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ such that $J\circ J=-\mathbb{1}$, then $G(m,\mathbb{R}^{2n})$ contains $\mathcal{C}$, the set of J-invariant subspaces. $\mathcal{C}$ is a submanifold: it is the image in $G(m,\mathbb{R}^{2n})$ of the inclusion embedding of the complex Grassmannian manifold $\mathbb{C}G(m/2,\mathbb{C}^n)$ of complex subspaces in $\mathbb{C}^n$.

In the case $m=2n-2$, the manifold $\mathbb{C}G(n-1,\mathbb{C}^n)$ is homeomorphic to the complex projective space $\mathbb{C}P^{n-1}$, and the real dimension of $\mathcal{C}$ is $2(n-1)$, half the dimension $4(n-1)$ of $G(2n-2,\mathbb{R}^{2n})$. The inverse image $\mathcal{F}^{-1}(\mathcal{C})\subseteq SG(2n-2,\mathbb{R}^{2n})$ is a disjoint union $\mathcal{C}^+\cup\mathcal{C}^-$, where $\mathcal{C}^+$ is the set of oriented subspaces whose orientation agrees with that given by the complex structure, and $\mathcal{C}^-$ is the set of subspaces where these orientations are opposite (so they could be called "anticomplex" subspaces). Each component $\mathcal{C}^+$ and $\mathcal{C}^-$ is the image of an embedding of $\mathbb{C}G(n-1,\mathbb{C}^n)$, with real dimension, and codimension, equal to $2(n-1)$. The $\mathcal{C}^+$ submanifold has a natural orientation.

For $0<m<2n$ and $0\le j\le n$, given an immersion of an m-submanifold $\iota: M\to\mathbb{R}^{2n}=\mathbb{C}^n$, let N_j denote the subset

$$N_j=\{x\in M:\ \dim_{\mathbb{R}}(T_xM\cap J\cdot T_xM)\ge 2j\},$$

so at a point $x\in N_j$, the tangent space T_xM contains a J-invariant subspace of real dimension $2j$, which is a complex subspace of complex dimension j in $\mathbb{C}^n$.

For $m=2n-2$, $N_{n-2}=M$ and $N_{n-1}=\boldsymbol{\gamma}^{-1}(\mathcal{C})$ is the CR singular locus. The immersion is in "general position" at $x\in M$ if the Gauss map $\boldsymbol{\gamma}: M\to G(2n-2,\mathbb{R}^{2n})$ meets $\mathcal{C}$ transversely at $\boldsymbol{\gamma}(x)$. M is trivially in general position at all its CR generic points, and the immersion is said to be in general position if it is in general position at every point. Then, by counting dimensions, an immersion in general position has only isolated CR singular points, and if M is compact, then the CR singular locus N_{n-1} is a finite set.

If M has an orientation, then the CR singular locus is the same set as $(\mathcal{F}\circ\boldsymbol{\gamma}_s)^{-1}(\mathcal{C})=\boldsymbol{\gamma}_s^{-1}(\mathcal{F}^{-1}(\mathcal{C}))=\boldsymbol{\gamma}_s^{-1}(\mathcal{C}^+\cup\mathcal{C}^-)$, which is a disjoint union $\boldsymbol{\gamma}_s^{-1}(\mathcal{C}^+)\cup\boldsymbol{\gamma}_s^{-1}(\mathcal{C}^-)$. So, $N_{n-1}=N_{n-1}^+\cup N_{n-1}^-$, where $N_{n-1}^+=\boldsymbol{\gamma}_s^{-1}(\mathcal{C}^+)$ and $N_{n-1}^-=\boldsymbol{\gamma}_s^{-1}(\mathcal{C}^-)$. The local and global notions of "general position" as defined previously are equivalent to the analogous transverse meeting of $\boldsymbol{\gamma}_s$ with $\mathcal{C}^+\cup\mathcal{C}^-$ in $SG(2n-2,\mathbb{R}^{2n})$. At each point of N_{n-1}^+ in general position, there is an oriented intersection number, ±1, of $\boldsymbol{\gamma}_s$ with the oriented submanifold $\mathcal{C}^+$: the intersection number at $\boldsymbol{\gamma}_s(x)$ is be denoted the index, $ind(x)$.

To define the intersection index at a point x of N_{n-1}^-, [L] makes a choice of orientation for $\mathcal{C}^-$ which is opposite to that induced by $\mathcal{R}:\mathcal{C}^+\to\mathcal{C}^-$. Then $ind(x)$, the intersection number of $\boldsymbol{\gamma}_s$ with $\mathcal{C}^-$ at $\boldsymbol{\gamma}_s(x)$, is equal to the intersection number of $\mathcal{R}\circ\boldsymbol{\gamma}_s$ with $\mathcal{C}^+$ at $\mathcal{R}(\boldsymbol{\gamma}_s(x))$. Equivalently, if M^- denotes the manifold M with its orientation reversed, with Gauss map $\boldsymbol{\gamma}_s':M^-\to SG(2n-2,\mathbb{R}^{2n})$, then $ind(x)$ is equal to the intersection number of $\boldsymbol{\gamma}_s'$ with $\mathcal{C}^+$ at $\boldsymbol{\gamma}_s'(x)$.

3.2. Bundle maps. More generally, let M be a smooth, oriented manifold with real dimension $2n-2$, and let F be a smooth, oriented real vector bundle over M with $2n$-dimensional fibers. Then the space of oriented real $(2n-2)$-subspaces of fibers of F forms a "Grassmann bundle" $SG(2n-2,F)$ over M—on a local coordinate patch U of M where F can be trivialized as $U\times\mathbb{R}^{2n}$, the Grassmann bundle is of the form $U\times SG(2n-2,\mathbb{R}^{2n})$. Suppose F admits a smooth complex structure operator J. Then for each point $x\in M$, the J_x-invariant subspaces form sets $\mathcal{C}_x^+$ and $\mathcal{C}_x^-$ in the fiber over x, giving a pair of smooth bundles $\mathcal{C}^+$, $\mathcal{C}^-$ of complex Grassmannians over M; each total space has codimension $2n-2$ in the total space $SG(2n-2,F)$. If T is another smooth, oriented real vector bundle over M, with $(2n-2)$-dimensional fibers, then a nonsingular bundle map $\mu:T\to F$ induces a section $\boldsymbol{\gamma}_\mu:M\to SG(2n-2,F):x\mapsto\mu(T_x)$, generalizing the Gauss map. The transverse intersection of $\boldsymbol{\gamma}_\mu$ with $\mathcal{C}^+$ and $\mathcal{C}^-$ defines a notion of "general position" for T, and the intersection numbers define the index of generally isolated points x where $\boldsymbol{\gamma}_\mu(x)=\mu(T_x)$ is a J_x-invariant subspace of F_x.

A special case of this bundle construction is where T is the tangent bundle TM of M, and ι is an immersion of M in an almost complex manifold $\mathcal{A}$, with real dimension $2n$ and a smooth complex structure operator J on $T\mathcal{A}$. Then the differential of ι defines a smooth bundle map $\mu:TM\to F=\iota^*T\mathcal{A}$, inducing $\boldsymbol{\gamma}_\mu:M\to SG(2n-2,F)$. A point $x\in M$ is a CR singular point of ι if $\boldsymbol{\gamma}_\mu(x)=\mu(T_xM)$ is a J-invariant subspace of $T_{\iota(x)}\mathcal{A}$.

If, additionally, $T\mathcal{A}$ admits a positive definite Riemannian metric g, then there is an oriented real 2-plane normal bundle νM orthogonal to TM in $\iota^*T\mathcal{A}$, and further, if g has the property that J is an isometry with respect to g, then T_xM is a complex hyperplane in $\iota^*T_x\mathcal{A}$ if and only if ν_xM is a complex line. Such a metric g can be chosen for any $(T\mathcal{A}, J)$, although we do not use it except to define the normal bundle.

3.3. Characteristic class formulas. Following the notation of [F], for an immersion $\iota : M \to \mathcal{A}$ (or bundle map $\mu : T \to F$) as in the previous subsection, we define index sums:

$$I_+ = \sum_{x \in N^+_{n-1}} ind(x),$$
$$I_- = \sum_{x \in N^-_{n-1}} ind(x).$$

When M is compact and the immersion (or bundle map) is in general position, I_+ and I_- are finite sums of ± 1 terms. Then I_+, I_-, $I_+ + I_-$, and $I_+ - I_-$ are all invariants of the homotopy class of ι (or μ). Reversing the orientation of M interchanges the values of I_+ and I_-, so $I_+ + I_-$ is the same and $I_+ - I_-$ has the opposite sign.

If, instead of following [L] as in Section 3.1, we make the other choice of orientation for $\mathcal{C}^-$, then I_- has the opposite sign and the quantities $I_+ + I_-$ and $I_+ - I_-$ are switched. Some of the following examples show that this is the choice of orientation that corresponds to index sums appearing in enumerative formulas of [Webster], [HL_1], [HL_2], [Domrin], and [C_1].

Our notation for characteristic classes in cohomology is copied from [L]: denote the Euler class of the tangent bundle TM by Ω, denote the Euler class of the normal bundle $\nu M \to M$ of the immersion by $\widetilde{\Omega}$, and denote the total Chern class of the pullback bundle $\iota^*(T\mathcal{A}, J) \to M$ by $1 + c_1 + c_2 + \cdots + c_n$.

Proposition 3.1 ([L]). *For an immersion ι of a compact, oriented $(2n-2)$-manifold M in general position in an almost complex $2n$-manifold $\mathcal{A}$,*

$$I_+ = \int_M \frac{1}{2}\left(\Omega + \sum_{r=0}^{n-1} \widetilde{\Omega}^r c_{n-1-r}\right).$$

Corollary 3.2. *With the orientation convention for $\mathcal{C}^-$ as in [L],*

$$I_- = \int_M \frac{1}{2}\left(\Omega + \sum_{r=0}^{n-1} (-1)^{r+1} \widetilde{\Omega}^r c_{n-1-r}\right),$$
$$I_+ + I_- = \int_M \left(\Omega + \sum_{k=0}^{\lfloor n/2 \rfloor - 1} \widetilde{\Omega}^{2k+1} c_{n-2k-2}\right),$$
$$I_+ - I_- = \int_M \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \widetilde{\Omega}^{2k} c_{n-2k-1}.$$

Proof. I_- is calculated by reversing the orientation of the tangent and normal bundles TM and νM, which switches the sign of Ω and $\widetilde{\Omega}$. Then, applying the formula from the proposition, integrating over M^- gives the opposite of the integral over M. □

EXAMPLE 3.3. When $\mathcal{A}=\mathbb{C}^n$, the Chern classes are trivial, so for M immersed in general position,

$$I_+ = \int_M \frac{1}{2}(\Omega + \widetilde{\Omega}^{n-1}),$$
$$I_- = \int_M \frac{1}{2}\big(\Omega + (-1)^n \widetilde{\Omega}^{n-1}\big).$$

If, in addition, ι is an embedding, then $\widetilde{\Omega}$ is the zero class, so $I_+ = I_- = \frac{1}{2}\chi(M)$, where $\chi(M)$ is the Euler characteristic of M ([L], Theorem 4.10).

EXAMPLE 3.4. For $n=2$, ι is an immersion of a compact, oriented real surface M in a 4-manifold $\mathcal{A}$ with an almost complex structure J. Complex or anticomplex points with Lai's index $+1$ are "elliptic," or "hyperbolic" with index -1. Again following the notation of [F] and [ABKLR], if e_+ (respectively, e_-) is the number of elliptic points with positively (negatively) oriented complex tangent spaces and h_+ (h_-) is the number of positive (negative) hyperbolic points, and $e=e_+ + e_-$, $h=h_+ + h_-$, then for M in general position,

$$I_+ = e_+ - h_+ = \int_M \frac{1}{2}(\Omega + c_1 + \widetilde{\Omega}),$$
$$I_- = e_- - h_- = \int_M \frac{1}{2}(\Omega - c_1 + \widetilde{\Omega}),$$
$$I_+ + I_- = e - h = \int_M (\Omega + \widetilde{\Omega}),$$
$$I_+ - I_- = (e_+ - e_-) - (h_+ - h_-) = \int_M c_1.$$

The last formula was also proved by [Webster] and is a special case of a degeneracy locus formula of [HL_2] and [Domrin]. When ι is an embedding into $\mathcal{A}=\mathbb{C}^2$, $\widetilde{\Omega}=0$, so the formulas are $e_+ - h_+ = e_- - h_- = \frac{1}{2}\chi(M)$, which were known to [Bishop] and [Wells] in the special cases where M is an embedded sphere or torus. Immersions with double points are considered by [BF] and [Bohr].

EXAMPLE 3.5. For $n=3$, ι is an immersion of a compact, oriented real 4-manifold M in a 6-manifold $\mathcal{A}$ with an almost complex structure J. For M in general position, we adapt from the previous example the notation e_+, h_+ for counting elements of N_2^+, and e_-, h_- for elements of N_2^-. (However, the terms "elliptic" and "hyperbolic" are not adapted here from $n=2$ to $n\geq 3$, they appear in Section 7 in a more specific application.)

$$\begin{aligned} I_+ = e_+ - h_+ &= \int_M \frac{1}{2}(\Omega + c_2 + \widetilde{\Omega} c_1 + \widetilde{\Omega}^2), \\ I_- = e_- - h_- &= \int_M \frac{1}{2}(\Omega - c_2 + \widetilde{\Omega} c_1 - \widetilde{\Omega}^2), \\ I_+ + I_- = e - h &= \int_M (\Omega + \widetilde{\Omega} c_1), \\ I_+ - I_- = (e_+ - e_-) - (h_+ - h_-) &= \int_M (c_2 + \widetilde{\Omega}^2). \end{aligned}$$

There are also formulas

$$I_+ - I_- = \int_M (c_2 + p_1 \nu M) \tag{3.1}$$

$$= \int_M (c_1^2 - c_2 - p_1 TM), \tag{3.2}$$

where p_1 is the first Pontrjagin class of the normal bundle νM or tangent bundle TM. The equality (3.1) follows from the well-known identity of characteristic classes $\widetilde{\Omega}^2 = p_1 \nu M$. Formula (3.2) is a special case of the degeneracy locus formulas of [HL$_2$] and [Domrin]—the calculation establishing the equivalence of (3.1) and (3.2) appears in [C$_1$].

Example 3.6. Not every compact, oriented 4-manifold can be immersed in $\mathbb{R}^6 \cong \mathbb{C}^3$—for example, $\mathbb{C}P^2$ cannot [Hirsch]. If such a 4-manifold M is immersed in general position in $\mathbb{C}^3$, then $I_+ + I_- = \chi(M)$ and

$$I_+ - I_- = \int_M p_1 \nu M = \int_M -p_1 TM = -p_1 M,$$

the opposite of the first Pontrjagin number of M [HL$_1$]. The number $\int_M p_1 \nu M$ is also three times the algebraic number of triple points of an immersion in general position [Herbert]. A compact, oriented 4-manifold admitting a CR generic immersion in $\mathbb{C}^3$ must have $\chi(M) = p_1 M = 0$; the converse problem, finding a sufficient condition for the existence of a CR generic immersion, is considered by [JL].

Example 3.7. Consider the complex projective space $\mathcal{A} = \mathbb{C}P^3$, and a nonsingular, degree $\boldsymbol{d}$ complex hypersurface $Y \subseteq \mathbb{C}P^3$. In terms of the hyperplane class H in the cohomology ring of $\mathbb{C}P^3$, there are well-known formulas ([GH], Sections 1.1, 3.4, 4.6) for Chern classes: $c_2(T\mathbb{C}P^3|_Y) = 6H^2$, $c_1(T\mathbb{C}P^3|_Y) = 4H$, and $c_1 \nu Y = \boldsymbol{d} H$. Since $\int_Y H^2 = \boldsymbol{d}$,

$$\int_Y \big(c_2(T\mathbb{C}P^3|_Y) + (c_1 \nu Y)^2\big) = 6\boldsymbol{d} + \boldsymbol{d}^3,$$

$$\int_Y \big(c_2TY + c_1\nu Y \cdot c_1(T\mathbb{C}P^3|_Y)\big) = \chi(Y) + 4\boldsymbol{d}^2.$$

Let M be a smooth real submanifold of $\mathbb{C}P^3$ isotopic to Y, that is, homotopic through a family of smooth embeddings. Then the Chern classes of Y pull back to classes on M, and if M is in general position, then

$$\begin{aligned}
\int_Y \big(c_2(T\mathbb{C}P^3|_Y) + (c_1\nu Y)^2\big) &= \int_M \big(c_2(T\mathbb{C}P^3|_M) + \widetilde{\Omega}^2\big)\\
&= 6\boldsymbol{d} + \boldsymbol{d}^3 = I_+ - I_-,\\
\int_Y \big(c_2TY + c_1\nu Y \cdot c_1(T\mathbb{C}P^3|_Y)\big) &= \int_M \big(\Omega + \widetilde{\Omega}\cdot c_1(T\mathbb{C}P^3|_M)\big)\\
&= \chi(M) + 4\boldsymbol{d}^2 = I_+ + I_-
\end{aligned}$$

by the formulas from Example 3.5. These numbers are always positive; there is no CR generic submanifold of $\mathbb{C}P^3$ isotopic to a smooth complex hypersurface. One also expects that sufficiently nearby perturbations of Y would have only positively oriented complex tangents ($N_2^- = \emptyset$), so $I_- = 0$ and we recover the formula $\chi(Y) = \chi(M) = \boldsymbol{d}^3 - 4\boldsymbol{d}^2 + 6\boldsymbol{d}$ ([GH], p. 601).

In particular, consider the holomorphic embedding given in homogeneous coordinates by $\mathbf{c} : \mathbb{C}P^2 \to \mathbb{C}P^3 : [z_0 : z_1 : z_2] \to [z_0 : z_1 : z_2 : 0]$, so the image Y is a complex hyperplane with degree $\boldsymbol{d} = 1$. $\mathbb{C}P^2$ considered only as a smooth, oriented 4-manifold (forgetting its complex structure) has $\chi(\mathbb{C}P^2) = 3$. An embedding of $M = \mathbb{C}P^2$ as an oriented real submanifold in general position and isotopic to $\mathbf{c}$ has seven complex tangents, counted as an index sum with multiplicity according to either the $I_+ + I_-$ or $I_+ - I_-$ sign convention. In Section 6, we give a concrete example of such an isotopy from $\mathbf{c}$ to a smooth embedding $\mathbb{C}P^2 \to \mathbb{C}P^3$ in general position with exactly seven CR singular points in N_2^+, each with index $= +1$: $I_+ = e_+ = 7$, $h_+ = I_- = e_- = h_- = 0$.

Remark 1. If $\iota : \mathbb{C}P^2 \to \mathbb{C}P^3$ is any immersion (or any continuous map homotopic to an immersion), then ι is homotopic to either the embedding $\mathbf{c}$ from the above example, or the composite $\mathbf{c} \circ \kappa$, where κ is the orientation-preserving involution $[z_0 : z_1 : z_2] \mapsto [\bar{z}_0 : \bar{z}_1 : \bar{z}_2]$ ([Thomas], [LP]). Equivalently, either ι or $\iota \circ \kappa$ is homotopic to $\mathbf{c}$.

3.4. Local coordinates for the Grassmannian. For $0 < m < 2n$, each element $v \in G(m, \mathbb{R}^{2n})$ is the image of some linear map $\mathbb{R}^m \to \mathbb{R}^{2n}$ with standard matrix representation $X_{2n\times m}$ of rank m, and any two linear maps with the same image are right-equivalent ($X \sim XY$ for invertible $Y_{m\times m}$). If v^0 is the m-plane $\{(v_1, \ldots, v_m, 0, \ldots, 0)^T\}$ in $\mathbb{R}^{2n}$, then it is the image of $X^0 = \binom{\mathbb{1}_{m\times m}}{0}_{2n\times m}$, and any elements sufficiently near v^0 are the image of some linear map whose matrix representation can be column-reduced to the form

$$\begin{pmatrix} \mathbb{1}_{m\times m} \\ V_{(2n-m)\times m} \end{pmatrix}_{2n\times m}. \tag{3.3}$$

The matrices V in a neighborhood of the $(2n-m)\times m$ zero matrix form a local coordinate chart around v^0 in $G(m,\mathbb{R}^{2n})$. The inverse image of a sufficiently small chart under $\mathcal{F}: SG(m,\mathbb{R}^{2n}) \to G(m,\mathbb{R}^{2n})$ gives a pair of charts in $SG(m,\mathbb{R}^{2n})$, one around each element of $\mathcal{F}^{-1}(v^0)$, the m-plane with its two possible orientations.

In the case where m is even and $\mathbb{R}^{2n}$ has coordinates $(x_1,y_1,\ldots,x_n,y_n)^T$ and a complex structure operator

$$J_{2n\times 2n} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix},$$

consider the m-plane

$$v^0 = \{(x_1,y_1,\ldots,x_{m/2},y_{m/2},0,\ldots,0)^T\}.$$

Since v^0 is J-invariant, $v^0 \in \mathcal{C}$. Any element v of the intersection of $\mathcal{C}$ with the local coordinate chart around v^0 would have matrix representation $X = \binom{\mathbb{1}}{V}_{2n\times m}$, such that $J\cdot X \sim X$. The equivalence

$$\begin{aligned} J\cdot\begin{pmatrix}\mathbb{1}\\ V\end{pmatrix} &= \begin{pmatrix} J_{m\times m} \\ J_{(2n-m)\times(2n-m)}\cdot V\end{pmatrix} \\ &\sim \begin{pmatrix} J \\ J\cdot V\end{pmatrix}\cdot(-J)_{m\times m} = \begin{pmatrix}\mathbb{1} \\ -J\cdot V\cdot J\end{pmatrix} \end{aligned}$$

shows $v\in\mathcal{C} \iff V = -J\cdot V\cdot J \iff V\cdot J = J\cdot V$, that is, V is complex linear with respect to $J_{m\times m}$ and $J_{(2n-m)\times(2n-m)}$.

For example, in the $m=4$, $n=3$ case, the coordinates near v^0 in the 8-dimensional space $G(4,\mathbb{R}^6)$ are of the form $V = \left(\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4\end{smallmatrix}\right)$, and the elements of $\mathcal{C}$ have coordinates V satisfying

$$\begin{aligned} &\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4\end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4\end{pmatrix}\cdot\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0\end{pmatrix}. \end{aligned}$$

This linear condition on $a_1,\ldots,b_4$ has a 4-dimensional solution subspace

$$\{b_1 = -a_2, b_2 = a_1, b_3 = -a_4, b_4 = a_3\}.$$

Now consider v^0 with its orientation as a complex subspace of $(\mathbb{R}^6, J)$, so $v^0 \in \mathcal{C}^+ \subseteq SG(4, \mathbb{R}^6)$. Also, instead of 2×4 matrices, we put the eight coordinate functions for $SG(4,\mathbb{R}^6)$ near v^0 in a column vector format, $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)^T$. Then, in this chart around v^0, $\mathcal{C}^+$ is the image of the linear map $\mathbb{R}^4 \to \mathbb{R}^8$ with matrix representation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{4\times 4} \\ J_{4\times 4} \end{pmatrix}.$$

Returning to the general situation of codimension 2 in $\mathbb{C}^n$, the above pattern still holds, so that $\mathcal{C}^+$ is the column space of $\binom{\mathbb{1}}{J}_{2(2n-2)\times(2n-2)}$ in the coordinate chart around v^0 in $SG(2n-2, \mathbb{R}^{2n})$.

4. A reformulation of Garrity's transversality criterion

Given a real $(2n-2)$-submanifold of $\mathbb{C}^n$ with a CR singular point, we now consider the problem of determining from the local defining equation whether M is in general position, as defined in Section 3, and if so, how the local defining equation determines the intersection index. The transversality problem was also considered by [Garrity] for real $(2n-2)$-submanifolds of $\mathbb{C}^n$, using different methods but arriving at an equivalent result. Our result (Theorem 4.1) relates transversality to an expression in terms of the coefficient matrix notation from Section 2.

4.1. A determinantal formula. We begin by assuming M is in standard position, given an orientation agreeing with the orientation of its complex tangent space at the origin. M can be described by a complex implicit equation (2.1) in a neighborhood Δ of $\vec{0}$:

$$0 = z_n - h(z,\bar{z}) = z_n - \left(z^T Q z + \bar{z}^T R z + \bar{z}^T S \bar{z} + e(z,\bar{z})\right). \tag{4.1}$$

If we now consider two real functions, $f^1(x_1, y_1, \dots, x_{n-1}, y_{n-1}) = \operatorname{Re}(h(z,\bar{z}))$ and $f^2(x_1,\dots,y_{n-1}) = \operatorname{Im}(h(z,\bar{z}))$, then the real $(2n-2)$-manifold M has a local parametrization π with domain $\mathcal{D} \subseteq \mathbb{R}^{2n-2}$ and embedding target $\mathbb{R}^{2n}$:

$$\pi : (x_1, \dots, y_{n-1})^T \mapsto (x_1, \dots, y_{n-1}, f^1, f^2)^T. \tag{4.2}$$

The differential of this map assigns to each point $z \in \mathcal{D}$ a linear map from $\mathbb{R}^{2n-2}$ to $T_{\pi(z)}M \subseteq \mathbb{R}^{2n}$; this linear map has matrix representation:

$$\begin{pmatrix} & \mathbb{1}_{(2n-2)\times(2n-2)} & \\ & & \\ \frac{df^1}{dx_1} & \cdots & \frac{df^1}{dy_{n-1}} \\ \frac{df^2}{dx_1} & \cdots & \frac{df^2}{dy_{n-1}} \end{pmatrix}_{(2n)\times(2n-2)}.$$

This matrix is already in the form (3.3), so in the local coordinate systems π for M and V for $SG(2n-2,\mathbb{R}^{2n})$, the oriented Gauss map has the form

$$\boldsymbol{\gamma}_s : (x_1, y_1, \ldots, x_{n-1}, y_{n-1})^T \mapsto \begin{pmatrix} \frac{df^1}{dx_1} & \cdots & \frac{df^1}{dy_{n-1}} \\ \frac{df^2}{dx_1} & \cdots & \frac{df^2}{dy_{n-1}} \end{pmatrix}_{2\times(2n-2)}.$$

This map takes the CR singular point $\vec{0} \in M \subseteq (\mathbb{R}^{2n}, J)$ to the complex hyperplane $v^0 \in \mathcal{C}^+ \subseteq SG(2n-2,\mathbb{R}^{2n})$, with coordinates $V = 0_{2\times(2n-2)}$. If we arrange the above two rows into column vector format (as in the end of Section 3.4), then the differential of the Gauss map at the origin has matrix representation:

$$\left.\begin{pmatrix}
\frac{d}{dx_1}(\frac{df^1}{dx_1}) & \frac{d}{dy_1}(\frac{df^1}{dx_1}) & \cdots & \frac{d}{dx_{n-1}}(\frac{df^1}{dx_1}) & \frac{d}{dy_{n-1}}(\frac{df^1}{dx_1}) \\
\frac{d}{dx_1}(\frac{df^1}{dy_1}) & \frac{d}{dy_1}(\frac{df^1}{dy_1}) & \cdots & \frac{d}{dx_{n-1}}(\frac{df^1}{dy_1}) & \frac{d}{dy_{n-1}}(\frac{df^1}{dy_1}) \\
\vdots & & & & \vdots \\
\frac{d}{dx_1}(\frac{df^1}{dx_{n-1}}) & \frac{d}{dy_1}(\frac{df^1}{dx_{n-1}}) & \cdots & \frac{d}{dx_{n-1}}(\frac{df^1}{dx_{n-1}}) & \frac{d}{dy_{n-1}}(\frac{df^1}{dx_{n-1}}) \\
\frac{d}{dx_1}(\frac{df^1}{dy_{n-1}}) & \frac{d}{dy_1}(\frac{df^1}{dy_{n-1}}) & \cdots & \frac{d}{dx_{n-1}}(\frac{df^1}{dy_{n-1}}) & \frac{d}{dy_{n-1}}(\frac{df^1}{dy_{n-1}}) \\
\frac{d}{dx_1}(\frac{df^2}{dx_1}) & \frac{d}{dy_1}(\frac{df^2}{dx_1}) & \cdots & \frac{d}{dx_{n-1}}(\frac{df^2}{dx_1}) & \frac{d}{dy_{n-1}}(\frac{df^2}{dx_1}) \\
\frac{d}{dx_1}(\frac{df^2}{dy_1}) & \frac{d}{dy_1}(\frac{df^2}{dy_1}) & \cdots & \frac{d}{dx_{n-1}}(\frac{df^2}{dy_1}) & \frac{d}{dy_{n-1}}(\frac{df^2}{dy_1}) \\
\vdots & & & & \vdots \\
\frac{d}{dx_1}(\frac{df^2}{dx_{n-1}}) & \frac{d}{dy_1}(\frac{df^2}{dx_{n-1}}) & \cdots & \frac{d}{dx_{n-1}}(\frac{df^2}{dx_{n-1}}) & \frac{d}{dy_{n-1}}(\frac{df^2}{dx_{n-1}}) \\
\frac{d}{dx_1}(\frac{df^2}{dy_{n-1}}) & \frac{d}{dy_1}(\frac{df^2}{dy_{n-1}}) & \cdots & \frac{d}{dx_{n-1}}(\frac{df^2}{dy_{n-1}}) & \frac{d}{dy_{n-1}}(\frac{df^2}{dy_{n-1}})
\end{pmatrix}\right|_{\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}}$$

$$= \begin{pmatrix} Hf^1 \\ Hf^2 \end{pmatrix}_{2(2n-2)\times(2n-2)},$$

where Hf^1 and Hf^2 are the real $(2n-2)\times(2n-2)$ real Hessian matrices of second derivatives, evaluated at $(x_1, \ldots, y_{n-1})^T = (0, \ldots, 0)^T$. The tangent space of the image $\boldsymbol{\gamma}_s(M)$ at $\boldsymbol{\gamma}_s(\vec{0}) = v^0$ is spanned by the columns of this matrix, so in the coordinate chart around v^0, it is a $(2n-2)$-dimensional subspace that meets $\mathcal{C}^+$ transversely if the columns of this matrix are independent:

$$\begin{pmatrix} \mathbb{1} & Hf^1 \\ J & Hf^2 \end{pmatrix}_{2(2n-2)\times 2(2n-2)}. \tag{4.3}$$

So, M has a CR singularity in general position if and only if the determinant of the above matrix is nonzero. The intersection index at the origin is related to the sign of the determinant: $ind = +1$ for $\det > 0$, $ind = -1$ for $\det < 0$.

This product has the same determinant:

$$\begin{pmatrix} \mathbb{1} & 0 \\ \mathbb{1} & J \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1} & Hf^1 \\ J & Hf^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & Hf^1 \\ 0 & Hf^1 + J \cdot Hf^2 \end{pmatrix},$$

so calculating the determinant reduces to a smaller, $(2n-2)\times(2n-2)$ real determinant, $\det(Hf^1 + J\cdot Hf^2)$.

In the $m=4$, $n=3$ case, the 4×4 matrix $Hf^1 + J\cdot Hf^2$ is:

$$\begin{pmatrix} f^1_{x_1x_1} - f^2_{y_1x_1} & f^1_{x_1y_1} - f^2_{y_1y_1} & f^1_{x_1x_2} - f^2_{y_1x_2} & f^1_{x_1y_2} - f^2_{y_1y_2} \\ f^1_{y_1x_1} + f^2_{x_1x_1} & f^1_{y_1y_1} + f^2_{x_1y_1} & f^1_{y_1x_2} + f^2_{x_1x_2} & f^1_{y_1y_2} + f^2_{x_1y_2} \\ f^1_{x_2x_1} - f^2_{y_2x_1} & f^1_{x_2y_1} - f^2_{y_2y_1} & f^1_{x_2x_2} - f^2_{y_2x_2} & f^1_{x_2y_2} - f^2_{y_2y_2} \\ f^1_{y_2x_1} + f^2_{x_2x_1} & f^1_{y_2y_1} + f^2_{x_2y_1} & f^1_{y_2x_2} + f^2_{x_2x_2} & f^1_{y_2y_2} + f^2_{x_2y_2} \end{pmatrix}, \tag{4.4}$$

evaluated at $(x_1, y_1, x_2, y_2)^T = (0,0,0,0)^T$. These real number entries can be expressed in terms of the derivatives at the origin of the original function $h(z,\bar{z})$:

$$2\begin{pmatrix} \mathrm{Re}(h_{z_1\bar{z}_1} + h_{\bar{z}_1\bar{z}_1}) & \mathrm{Im}(-h_{z_1\bar{z}_1} + h_{\bar{z}_1\bar{z}_1}) & \mathrm{Re}(h_{\bar{z}_1z_2} + h_{\bar{z}_1\bar{z}_2}) & \mathrm{Im}(-h_{\bar{z}_1z_2} + h_{\bar{z}_1\bar{z}_2}) \\ \mathrm{Im}(h_{z_1\bar{z}_1} + h_{\bar{z}_1\bar{z}_1}) & \mathrm{Re}(h_{z_1\bar{z}_1} - h_{\bar{z}_1\bar{z}_1}) & \mathrm{Im}(h_{\bar{z}_1z_2} + h_{\bar{z}_1\bar{z}_2}) & \mathrm{Re}(h_{\bar{z}_1z_2} - h_{\bar{z}_1\bar{z}_2}) \\ \mathrm{Re}(h_{z_1\bar{z}_2} + h_{\bar{z}_1\bar{z}_2}) & \mathrm{Im}(-h_{z_1\bar{z}_2} + h_{\bar{z}_1\bar{z}_2}) & \mathrm{Re}(h_{\bar{z}_2z_2} + h_{\bar{z}_2\bar{z}_2}) & \mathrm{Im}(-h_{\bar{z}_2z_2} + h_{\bar{z}_2\bar{z}_2}) \\ \mathrm{Im}(h_{z_1\bar{z}_2} + h_{\bar{z}_1\bar{z}_2}) & \mathrm{Re}(h_{z_1\bar{z}_2} - h_{\bar{z}_1\bar{z}_2}) & \mathrm{Im}(h_{\bar{z}_2z_2} + h_{\bar{z}_2\bar{z}_2}) & \mathrm{Re}(h_{\bar{z}_2z_2} - h_{\bar{z}_2\bar{z}_2}) \end{pmatrix},$$

evaluated at $z = (0,0)^T$. The $(3,1)$ entry can be checked, for example, by the following calculation which is similar to the derivation of all the other entries:

$$\begin{aligned} &2\,\mathrm{Re}(h_{z_1\bar{z}_2} + h_{\bar{z}_1\bar{z}_2}) \\ &\quad = 2\,\mathrm{Re}\bigg(\frac{1}{2}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_2}\right)\frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}\right)(f^1 + if^2) \\ &\qquad + \frac{1}{2}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_2}\right)\frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}\right)(f^1 + if^2)\bigg) \\ &\quad = \frac{1}{2}(f^1_{x_1x_2} + f^1_{y_1y_2} - f^2_{x_1y_2} + f^2_{y_1x_2} + f^1_{x_1x_2} - f^1_{y_1y_2} - f^2_{x_1y_2} - f^2_{y_1x_2}) \\ &\quad = f^1_{x_1x_2} - f^2_{x_1y_2}. \end{aligned}$$

Note that the entries in the above matrix do not depend on the second z-derivatives $h_{z_jz_k}$, which are determined by the coefficient matrix Q in (4.1). This agrees with the notion that transversality and the index should not depend on the local holomorphic coordinate system, since it was shown in Section 2 how the coefficients Q could be arbitrarily altered by holomorphic coordinate changes. However, it is not as easy to see from the form of (4.3) or (4.4) that they do not depend on the Q coefficients. We also see in the above matrix that the entries depend only on the coefficients from R and S in the expression (4.1) for $h(z,\bar{z})$, and not on $e(z,\bar{z}) = O(3)$.

By Lemma 4.3 (the proof of which is left to Section 4.3), the determinant of the above matrix is equal to

$$2^4 \det \begin{pmatrix} R & 2S \\ \overline{2S} & \overline{R} \end{pmatrix}.$$

The above calculations (including the lemma) generalize to other dimensions n, and the index formula is even simpler with the defining equation in the form (2.12), with $(n-1)\times(n-1)$ complex symmetric coefficient matrix $P = 2\bar{S}$:

THEOREM 4.1. *Given a real* $(2n-2)$*-submanifold* M *in* $\mathbb{C}^n$ *with a CR singular point in standard position and local defining equation*:

$$z_n = \bar{z}^T R z + \mathrm{Re}(z^T P z) + e(z,\bar{z}), \tag{4.5}$$

then M *is in general position if and only if the matrix*

$$\Gamma = \begin{pmatrix} R & \overline{P} \\ P & \overline{R} \end{pmatrix}$$

is nonsingular. Further, if M *is given an orientation agreeing with the orientation of the complex* $(z_1,\ldots,z_{n-1})$*-hyperplane tangent to* M *at* $\vec{0}$*, then the intersection index* (± 1) *is the sign of the determinant* $\det(\Gamma)$.

EXAMPLE 4.2. In the $n=2$ case, if M is in standard position, oriented to agree with the orientation of the z_1-axis near the origin, and has defining equation

$$z_2 = z_1\bar{z}_1 + \gamma_1 z_1^2 + \bar{\gamma}_1\bar{z}_1^2 + e(z_1,\bar{z}_1) = z_1\bar{z}_1 + \mathrm{Re}(2\gamma_1 z_1^2) + e(z_1,\bar{z}_1), \quad \gamma_1 \in \mathbb{C},$$

then the index is the sign of $\det\left(\begin{smallmatrix} 1 & 2\bar{\gamma}_1 \\ 2\gamma_1 & 1 \end{smallmatrix}\right) = 1 - 4\gamma_1\bar{\gamma}_1$, which is $+1$ for $0 \le |\gamma_1| < \frac{1}{2}$ and -1 for $|\gamma_1| > \frac{1}{2}$. M is not in general position for $|\gamma_1| = \frac{1}{2}$ (the "parabolic" case).

For n in general, the real number value of $\det(\Gamma)$ is not an invariant under holomorphic transformations, only its sign is invariant. Generalizing the action of the group from (2.13) to n dimensions,

$$(R,P) \mapsto (c\bar{A}^T R A, \bar{c}A^T P A), \tag{4.6}$$

for $A = A_{(n-1)\times(n-1)}$, $c = c_{n,n}$. The matrix Γ and its determinant transform as:

$$\begin{aligned}\Gamma \mapsto &\begin{pmatrix} c\bar{A}^T R A & \overline{\bar{c}A^T P A} \\ \bar{c}A^T P A & \overline{c\bar{A}^T R A} \end{pmatrix} \\ &= \begin{pmatrix} c\mathbb{1} & 0 \\ 0 & \bar{c}\mathbb{1} \end{pmatrix} \begin{pmatrix} \bar{A}^T & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} R & \overline{P} \\ P & \overline{R} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix},\end{aligned} \tag{4.7}$$

$$\det(\Gamma)\mapsto |c|^{2(n-1)}|\det A|^4\det\begin{pmatrix} R & \overline{P}\\ P & \overline{R}\end{pmatrix}.$$

Even if the determinant is zero, (4.7) shows that the rank of Γ is a biholomorphic invariant.

It also follows from the transformation formula that the vector

$$(|\det(R)|^2, |\det(P)|^2, \det(\Gamma))\in\mathbb{R}^3$$

has an invariant direction (the expression is invariant modulo positive scalar multiplication), and ratios such as $\frac{\det(\Gamma)}{|\det(R)|^2}$ are numerical invariants (when well-defined).

4.2. Invariants at flat points. In the holomorphically flat case (as in Sections 2.2, 2.5), where in some local coordinates $h(z,\bar z)$ is real valued and so M is a real $(2n-2)$-hypersurface inside $\mathbb{R}^{2n-1}=\{y_n=0\}$, and $f^2=0$ in (4.2), the $(2n-2)\times(2n-2)$ matrix $Hf^1+J\cdot Hf^2$ (4.4) is just the Hessian Hf^1. Since Hf^1 is a real symmetric matrix, it has all real eigenvalues, and it follows from the above construction and the proof of Lemma 4.3 that Γ is Hermitian symmetric, with real eigenvalues equal to $\frac{1}{2}$ times the eigenvalues of Hf^1. When M is oriented as in Theorem 4.1, so the normal vector at the origin is in the positive x_n direction, the Hessian is exactly the matrix representation of the Weingarten shape operator at the origin [Thorpe]. Its determinant is the Gauss–Kronecker curvature (the product of the $2n-2$ real eigenvalues, which are the principal curvatures). So, when M is a hypersurface in $\mathbb{R}^{2n-1}$, in general position, and positively oriented at a CR singular point, the index and the curvature have the same sign, which is [L], Lemma 4.11. If we consider holomorphic coordinate changes that preserve the property that M is contained in $\mathbb{R}^{2n-1}$ (as in Section 2.5), then the matrix A in the transformations (4.6), (4.7) may be arbitrary, but if $R\neq 0$, then c must be real. The A part acts as a Hermitian congruence transformation on Γ, preserving its rank $\rho(\Gamma)$ and signature (p,q), but if c is negative, then the signature is switched to (q,p). For $R=0$, the transformation (4.7) is a Hermitian congruence transformation of Γ for any complex $c\neq 0$: instead of using the matrix factorization as in (4.7), let $c=\rho e^{i\theta}$, $\rho>0$, $0\le\theta<2\pi$, in order to choose a square root, and then use the matrix $\begin{pmatrix}\sqrt{\rho}e^{-i\theta/2}A & 0\\ 0 & \sqrt{\rho}e^{i\theta/2}\bar{A}\end{pmatrix}$ on the right.

We can conclude that for $M\subseteq\mathbb{R}^{2n-1}$, the rank $\rho(\Gamma)$ and the quantity $\sigma(\Gamma)=|p-q|$ (from which one can recover the signature, modulo switching, of Γ) are invariant under holomorphic transformations preserving the real valued property of $h(z,\bar z)$. Since the action of (c,A) on Γ is not affected by the higher degree terms, $\rho(\Gamma)$ and $\sigma(\Gamma)$ are also invariants of a defining function $h(z,\bar z)$ in a quadratically flat normal form, under holomorphic transformations preserving the property of being in a quadratically flat normal form.

For Hermitian R, if we think of the quadratic part of (4.5),

$$\bar{z}^T Rz + \mathrm{Re}(z^T Pz),$$

as a real valued quadratic form on $\mathbb{R}^{2n-2}$, then its zero locus is a real algebraic variety. The dimension of this variety, and whether it is reducible or irreducible, are properties that are invariant under scalar multiplication of the form, and also under real linear coordinate changes of the domain $\mathbb{R}^{2n-2}$. In particular, they are also invariants under complex linear transformations $z \mapsto Az$ of $\mathbb{C}^{n-1}$. Real valued quadratic forms on $\mathbb{C}^{n-1}$ are considered by [CS], where the zero set is called a "quadratic cone" if it is irreducible and has dimension $2n-3$. The action of scalar multiplication and complex linear transformations A on the set of equations of quadratic cones in $\mathbb{C}^{n-1}$ is the same as the above action (4.6) on the set of matrix pairs (R,P) appearing in a quadratically flat normal form.

In the $n=3$ case, a list of equivalence classes of real valued quadratic forms defining quadratic cones in $\mathbb{C}^2$ is given by [CS], and each type of cone corresponds to one of the normal forms for 2×2 matrix pairs (R,P) computed in Section 2.4.1. Our list of normal forms for pairs (R,P), with R Hermitian, (summarized in Examples 7.3–7.7) is longer since [CS] excludes the reducible and lower-dimensional varieties.

4.3. Some matrix calculations. Let $R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $S = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}$ be 2×2 matrices with arbitrary complex entries. Consider the following 4×4 matrices with real entries:

$$R' = \begin{pmatrix} \mathrm{Re}(\alpha) & -\mathrm{Im}(\alpha) & \mathrm{Re}(\beta) & -\mathrm{Im}(\beta) \\ \mathrm{Im}(\alpha) & \mathrm{Re}(\alpha) & \mathrm{Im}(\beta) & \mathrm{Re}(\beta) \\ \mathrm{Re}(\gamma) & -\mathrm{Im}(\gamma) & \mathrm{Re}(\delta) & -\mathrm{Im}(\delta) \\ \mathrm{Im}(\gamma) & \mathrm{Re}(\gamma) & \mathrm{Im}(\delta) & \mathrm{Re}(\delta) \end{pmatrix},$$

$$S' = \begin{pmatrix} \mathrm{Re}(\mathrm{a}) & \mathrm{Im}(\mathrm{a}) & \mathrm{Re}(\mathrm{b}) & \mathrm{Im}(\mathrm{b}) \\ \mathrm{Im}(\mathrm{a}) & -\mathrm{Re}(\mathrm{a}) & \mathrm{Im}(\mathrm{b}) & -\mathrm{Re}(\mathrm{b}) \\ \mathrm{Re}(\mathrm{c}) & \mathrm{Im}(\mathrm{c}) & \mathrm{Re}(\mathrm{d}) & \mathrm{Im}(\mathrm{d}) \\ \mathrm{Im}(\mathrm{c}) & -\mathrm{Re}(\mathrm{c}) & \mathrm{Im}(\mathrm{d}) & -\mathrm{Re}(\mathrm{d}) \end{pmatrix}.$$

LEMMA 4.3. $\det(R'+S')_{4\times 4} = \det\begin{pmatrix} R & S \\ \bar{S} & \bar{R} \end{pmatrix}_{4\times 4}$.

Proof. Let

$$K = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}.$$

Then K is a unitary matrix with $\det(K)=1$. A calculation shows that

$$KR'\bar{K}^T = \begin{pmatrix} R & 0 \\ 0 & \bar{R} \end{pmatrix},$$

$$KS'\bar{K}^T = \begin{pmatrix} 0 & S \\ \overline{S} & 0 \end{pmatrix}.$$

Since $\det(R'+S') = \det(K(R'+S')\bar{K}^T)$, the claimed formula follows. □

The lemma generalizes to complex $\mathsf{n}\times\mathsf{n}$ matrices and the corresponding $2\mathsf{n}\times 2\mathsf{n}$ matrices following the same pattern. In the $\mathsf{n}=1$ case, the analogous matrix K is $K=\frac{\sqrt{2}}{2}\left(\begin{smallmatrix}1 & i\\ 1 & -i\end{smallmatrix}\right)$, for $\mathsf{n}=3$,

$$K = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i \end{pmatrix},$$

etc. Even without the result of the lemma, it is easy to see that elementary identities imply $\left(\begin{smallmatrix}R & S\\ \overline{S} & \overline{R}\end{smallmatrix}\right)_{2\mathsf{n}\times 2\mathsf{n}}$ has a real determinant. In the application of the lemma in Section 4.1, the S block is assumed to be symmetric—however, the lemma does not need that assumption.

5. Complexification

We return to the description of $M\subseteq\mathbb{R}^{2n}=\mathbb{C}^n$ as the image of a real analytic parametric map, with domain $\mathcal{D}\subset\mathbb{R}^{2n-2}$ and target $\mathbb{C}^n$. Rewriting equation (4.2) in complex form gives:

$$z\mapsto(z,h(z,\bar{z}))=(z_1,\dots,z_{n-1},h(z_1,\bar{z}_1,\dots,z_{n-1},\bar{z}_{n-1})).$$

The following complex analytic map, with domain $\mathcal{D}_c\subseteq\mathbb{C}^{2n-2}$ [as in (2.3)] and target $\mathbb{C}^n$, is a complexification of the above parametric map:

$$(z_1,\dots,z_{n-1},w_1,\dots,w_{n-1})\mapsto(z_1,\dots,z_{n-1},h(z_1,w_1,\dots,z_{n-1},w_{n-1})).$$

The coordinates $w=(w_1,\dots,w_{n-1})$ are new complex variables; the new map restricted to $w=\bar{z}$ [by substitution in the series expansion of $h(z,\bar{z})$, as in (2.2)] is exactly the original map. A geometric interpretation of such a complexification construction (as the composite of a holomorphic embedding $\mathcal{D}_c\to\mathbb{C}^{2n}$ and a linear projection $\mathbb{C}^{2n}\to\mathbb{C}^n$) is given in [$\mathrm{C}_3$], Section 4; the $n=2$ case is used in [MW]. The complex map is singular at the origin [its complex Jacobian drops rank there since $h(z,w)$ has no linear terms].

The origin-preserving local biholomorphic transformations of $\mathbb{C}^n$, $\tilde{\vec{z}}=C\vec{z}+\vec{p}(\vec{z})$, as in (2.9), act on the function $h(z,\bar{z})$; this induces an action on the complex map $(z,w)\mapsto(z,h(z,w))$. This group of transformations is a subgroup of a larger transformation group, which acts on the set of (germs at the origin of) maps $\mathbb{C}^{2n-2}\to\mathbb{C}^n$, by composition with origin-preserving biholomorphic transformations of both the domain and the target. The algebraic interpretation of the complexification construction is that the larger group allows the transformation of the z and w (formerly $\bar{z}$) variables independently. Any invariants under the larger group action are also invariants under the action of the subgroup.

In the case $n=2$, the map $(z_1, w_1) \mapsto (z_1, h(z_1, w_1))$ can be put into one of three normal forms under the larger group: $(z_1, w_1^2 + O(3))$, $(z_1, z_1 w_1 + O(3))$, $(z_1, O(3))$. As described in [C3], the first two cases correspond, respectively, to Whitney's fold and cusp singularities of maps $\mathbb{C}^2 \to \mathbb{C}^2$. In the fold case, a point near the origin in the target $\mathbb{C}^2$ has two inverse image points.

In the case $n=3$, the map $\mathcal{D}_c \to \mathbb{C}^3$:

$$(z_1, z_2, w_1, w_2) \mapsto (z_1, z_2, h(z_1, w_1, z_2, w_2))$$

can be transformed to $(\tilde{z}_1, \tilde{z}_2, \tilde{\mathbf{q}}(\tilde{z}, \tilde{w}) + O(3))$, where $\tilde{\mathbf{q}}(\tilde{z}, \tilde{w})$ is the quadratic part, falling into one of the following six normal forms (the calculation is omitted):

$$\tilde{w}_1^2 + \tilde{w}_2^2, \qquad \tilde{z}_2\tilde{w}_2 + \tilde{w}_1^2, \qquad \tilde{w}_1^2, \qquad \tilde{z}_1\tilde{w}_1 + \tilde{z}_2\tilde{w}_2, \qquad \tilde{z}_1\tilde{w}_1, \qquad 0.$$

With $h(z, \bar{z})$ of the form (2.1), the rank of the coefficient matrix S is an invariant of the complexification under the large group, and so is the rank of $(R|S)_{2\times 4}$. These two numbers uniquely determine the equivalence class of the quadratic part under the larger group; the rank of R is not an invariant. The first of the above six cases is the generic one, where $\rho(S) = 2$ and the inverse image of a point near the origin in the target $\mathbb{C}^3$ is a complex analytic curve in $\mathcal{D}_c \subseteq \mathbb{C}^4$.

6. Various global examples

Here, we collect some examples of compact real 4-manifolds embedded in complex 3-manifolds.

Example 6.1. The 4-sphere S^4 has a real algebraic embedding in a real hyperplane $\mathbb{R}^5 \subseteq \mathbb{C}^3$, which is (globally) holomorphically flat. For positive coefficients d_1, d_2, d_3, d_4, d_5, the implicit equation

$$\mathsf{d}_1 x_1^2 + \mathsf{d}_2 y_1^2 + \mathsf{d}_3 x_2^2 + \mathsf{d}_4 y_2^2 + \mathsf{d}_5 x_3^2 = 1$$

defines an ellipsoidal hypersurface in the real hyperplane $\{y_3 = 0\}$. There are exactly two CR singularities, where the tangent space is parallel to the (z_1, z_2)-subspace. The two points are holomorphically equivalent to each other, and the two local real defining equations can be put into a complex normal form (2.12) with $N = \mathbb{1}$, and P real diagonal. The Hessian at each point is definite, with the entries of P in the interval $[0,1)$, depending on $\mathsf{d}_1, \ldots, \mathsf{d}_5$. Putting an orientation on S^4 induces opposite orientations at the two points (one complex, the other anticomplex), so $I_+ = I_- = 1$, consistent with the characteristic class formulas from Section 3.3: for any immersion of S^4 in general position in $\mathbb{C}^3$, $I_+ + I_- = \chi(S^4) = 2$, and $I_+ - I_- = -p_1 S^4 = 0$. For $n \geq 3$, spheres S^{2n-2} in $\mathbb{C}^n$ are considered by [DTZ2].

EXAMPLE 6.2. Every compact, oriented, three-dimensional, smooth manifold M^3 admits a smooth immersion in $\mathbb{R}^4$, $\tau_1 : M^3 \to \mathbb{R}^4$ ([Hirsch], [JL]). Not every such 3-manifold M^3 admits an embedding, but the manifolds S^3, $S^2 \times S^1$, and $S^1 \times S^1 \times S^1$ all can be embedded as hypersurfaces of revolution in $\mathbb{R}^4$. There is an immersion (but not an embedding) of $\mathbb{R}P^3$ in $\mathbb{R}^4$ ([Hirsch], [M]). Also consider any oriented immersion of the circle, $\tau_2 : S^1 \to \mathbb{R}^2$. For any (almost) complex structures on $\mathbb{R}^4$ and $\mathbb{R}^2$, τ_1 is a CR regular immersion, τ_2 is a totally real immersion, and the product $\tau_1 \times \tau_2 : M^3 \times S^1 \to \mathbb{R}^4 \times \mathbb{R}^2$ is an oriented, CR regular immersion (with respect to the product complex structure). The index sums $I_+ = I_- = 0$ are consistent with the topological formulas from Section 3.3, since $\chi(M^3 \times S^1)$ and $p_1(M^3 \times S^1)$ are both zero.

EXAMPLE 6.3. The 4-manifold $S^2 \times S^2$ does not admit any CR regular immersion in $\mathbb{C}^3$; an immersion in general position has $I_+ + I_- = \chi(S^2 \times S^2) = 4$ and $I_+ - I_- = -p_1(S^2 \times S^2) = 0$, so $I_+ = I_- = 2$. We consider a real algebraic embedding in $\mathbb{R}^3 \times \mathbb{R}^3$, given by a product of ellipsoids:

$$\{\mathsf{a}x_1^2 + \mathsf{b}y_1^2 + \mathsf{c}x_3^2 = 1, \mathsf{d}x_2^2 + \mathsf{e}y_2^2 + \mathsf{f}y_3^2 = 1\}, \tag{6.1}$$

with positive coefficients $\mathsf{a}, \ldots, \mathsf{f}$. There are exactly four CR singularities, at the points with $z_1 = z_2 = 0$. Solving the real defining equations for x_3 and y_3, setting $z_3 = x_3 + iy_3$, and translating the CR singularities into standard position, the local equation at each point is real analytic:

$$\begin{aligned} x_3 + iy_3 &= \frac{\eta_1}{\sqrt{\mathsf{c}}}\left((-1) + \sqrt{1 - \mathsf{a}x_1^2 - \mathsf{b}y_1^2}\right) \\ &\quad + i\frac{\eta_2}{\sqrt{\mathsf{f}}}\left((-1) + \sqrt{1 - \mathsf{d}x_2^2 - \mathsf{e}y_2^2}\right), \\ z_3 &= \frac{\eta_1}{2\sqrt{\mathsf{c}}}\left(\mathsf{a} \cdot \left(\frac{z_1 + \bar{z}_1}{2}\right)^2 + \mathsf{b} \cdot \left(\frac{z_1 - \bar{z}_1}{2i}\right)^2\right) \\ &\quad + i\frac{\eta_2}{2\sqrt{\mathsf{f}}}\left(\mathsf{d} \cdot \left(\frac{z_2 + \bar{z}_2}{2}\right)^2 + \mathsf{e} \cdot \left(\frac{z_2 - \bar{z}_2}{2i}\right)^2\right) + O(4), \end{aligned}$$

where the four CR singular points correspond to the four sign choices $\eta_1 = \pm 1$, $\eta_2 = \pm 1$. After a transformation (2.5), the equation can be written in the form (2.12), with diagonal coefficient matrices (R, P):

$$\begin{aligned} z_3 &= (\bar{z}_1, \bar{z}_2)\begin{pmatrix} \eta_1 \frac{\mathsf{a}+\mathsf{b}}{4\sqrt{\mathsf{c}}} & 0 \\ 0 & i\eta_2 \frac{\mathsf{d}+\mathsf{e}}{4\sqrt{\mathsf{f}}} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &\quad + \mathrm{Re}\left((z_1, z_2)\begin{pmatrix} \eta_1 \frac{\mathsf{a}-\mathsf{b}}{4\sqrt{\mathsf{c}}} & 0 \\ 0 & \eta_2 i \frac{\mathsf{d}-\mathsf{e}}{4\sqrt{\mathsf{f}}} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) + O(4). \end{aligned}$$

The normal form for the coefficient matrix pair, from Case (1a) of Section 2.4.1, has the same form for all four points: for $\eta_1 = \eta_2$,

$$(R,P) \sim \left(\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} \frac{|\mathsf{a}-\mathsf{b}|}{\mathsf{a}+\mathsf{b}} & 0 \\ 0 & \frac{|\mathsf{d}-\mathsf{e}|}{\mathsf{d}+\mathsf{e}} \end{pmatrix} \right),$$

and for $\eta_1 = -\eta_2$,

$$(R,P) \sim \left(\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} \frac{|\mathsf{d}-\mathsf{e}|}{\mathsf{d}+\mathsf{e}} & 0 \\ 0 & \frac{|\mathsf{a}-\mathsf{b}|}{\mathsf{a}+\mathsf{b}} \end{pmatrix} \right),$$

not depending on c, f. The nonnegative diagonal entries are local biholomorphic invariants. For any values of a, b, d, e, η_1, η_2, the matrix pair satisfies $\det(\Gamma) > 0$, so the index is $+1$ for each CR singular point; there are two complex points and two anticomplex points.

We also note that this particular embedding of $S^2 \times S^2$ is contained in the (Levi nondegenerate) real hypersurface

$$\{\mathsf{a}x_1^2 + \mathsf{b}y_1^2 + \mathsf{c}x_3^2 + \mathsf{d}x_2^2 + \mathsf{e}y_2^2 + \mathsf{f}y_3^2 = 2\},$$

an ellipsoid in $\mathbb{C}^3$. So, the 4-manifold $S^2 \times S^2$ admits some topological embedding as a hypersurface in $\mathbb{R}^5$, but the extrinsic geometry of this product embedding (6.1) is that globally, it is not contained in a Levi flat hypersurface, and locally, it is not quadratically flat at its CR singular points.

EXAMPLE 6.4. Consider $M = \mathbb{C}P^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$ and $\mathcal{A} = \mathbb{C}P^3$ with coordinates $[Z_0 : Z_1 : Z_2 : Z_3]$. For each $t \in \mathbb{R}$, let

$$\iota_t : \mathbb{C}P^2 \to \mathbb{C}P^3 : \qquad (6.2)$$
$$[z_0 : z_1 : z_2] \mapsto [z_0 \cdot \mathcal{P} : z_1 \cdot \mathcal{P} : z_2 \cdot \mathcal{P} : t \cdot \mathcal{Q}],$$

where $\mathcal{P}$ is the polynomial expression

$$\mathcal{P} = \mathcal{P}(z_0, z_1, z_2) = 6z_0\bar{z}_0 + z_1\bar{z}_1 + 6z_2\bar{z}_2$$

and $\mathcal{Q}$ is the polynomial expression

$$\mathcal{Q} = \mathcal{Q}(z_0, z_1, z_2) = 2z_0^2\bar{z}_1 + 2z_0z_1\bar{z}_2 - z_0z_2\bar{z}_1 + 2z_1z_2\bar{z}_0.$$

Note that for any $[z_0 : z_1 : z_2]$, $\mathcal{P} \neq 0$, and the first three components in the RHS of (6.2) have no common zeros. The formula (6.2) also has a homogeneity property:

$$\iota_t([\lambda \cdot z_0 : \lambda \cdot z_1 : \lambda \cdot z_2]) = [\lambda^2\bar{\lambda} \cdot z_0 \cdot \mathcal{P} : \lambda^2\bar{\lambda} \cdot z_1 \cdot \mathcal{P} : \lambda^2\bar{\lambda} \cdot z_2 \cdot \mathcal{P} : \lambda^2\bar{\lambda} \cdot t \cdot \mathcal{Q}],$$

so ι_t is well defined. Observe that for $t = 0$,

$$\iota_0([z_0 : z_1 : z_2]) = [z_0 \cdot \mathcal{P} : z_1 \cdot \mathcal{P} : z_2 \cdot \mathcal{P} : 0 \cdot \mathcal{Q}] = [z_0 : z_1 : z_2 : 0]$$

is exactly the embedding $\mathbf{c}$ from Example 3.7. To show that for each t, ι_t is a real analytic embedding, and that the family ι_t is real analytic in t (so that this construction is an isotopy as in Example 3.7, and an "unfolding" as in [C3]), we view ι_t in local affine coordinate charts.

The restriction of ι_t to the $\{z_0 = 1\}$ neighborhood has image contained in the $\{Z_0 \neq 0\}$ neighborhood of the target $\mathbb{C}P^3$, and is given by the formula:

$$(6.3) \qquad [1 : z_1 : z_2] \mapsto \left[1 : z_1 : z_2 : \frac{t \cdot \mathcal{Q}(1, z_1, z_2)}{\mathcal{P}(1, z_1, z_2)}\right],$$
$$(z_1, z_2) \mapsto \left(z_1, z_2, \frac{t \cdot (2\bar{z}_1 + 2z_1\bar{z}_2 - z_2\bar{z}_1 + 2z_1z_2)}{6 + z_1\bar{z}_1 + 6z_2\bar{z}_2}\right).$$

This is a graph over the (z_1, z_2)-hyperplane of a rational (in z, $\bar{z}$) function $F_t(z_1, \bar{z}_1, z_2, \bar{z}_2) = t \cdot \mathcal{Q}/\mathcal{P}$ with a nonvanishing denominator, so this restriction of ι_t is a real analytic embedding. The dependence on t is real analytic, where ι_0 is just the graph of the constant function 0. For each $t \neq 0$, the graph is a smooth real submanifold M_t in $\mathbb{C}^3$, which is not a complex submanifold but which inherits its orientation from the (z_1, z_2)-hyperplane. Its CR singularities can be located by solving the system of two complex equations

$$(6.4) \qquad \frac{d}{d\bar{z}_1} F_t = 0, \qquad \frac{d}{d\bar{z}_2} F_t = 0.$$

The solution set does not depend on t (for $t \neq 0$), and it is easy to check that each of the following five points (ζ_1, ζ_2) in the domain is a zero of the $\bar{z}$-derivatives:

$$\left\{(0, 2), \left(\sqrt{3}, 1\right), \left(-\sqrt{3}, 1\right), (3i, -1), (-3i, -1)\right\}.$$

The points $(\zeta_1, \zeta_2, F_t(\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2))$ are the CR singularities of M_t, each with a positively oriented complex tangent space. Given F_t, calculations with the assistance of [Maple] found the above five points by solving four real equations in four real unknowns, and (omitting the details) verified that these five points are the only solutions of (6.4) in this neighborhood.

The challenge in constructing this example by making a good choice for the above $\mathcal{P}$ and $\mathcal{Q}$ is to find coefficients which are simple and sparse enough so that solving the system is a tractable computation with a numerically exact solution set, but not so simple that M_t is not in general position and has a CR singularity with a degenerate normal form ($ind \neq \pm 1$).

To calculate the index of the CR singular point at (ζ_1, ζ_2, F_t), we use Theorem 4.1 and find the sign of $\det(\Gamma)$—this is where we need the exact coordinates of the CR singularities. By the form of (6.3), it is enough to check the $t = 1$ case, since for $t \neq 0$, M_t is related to M_1 by an invertible complex linear transformation of $\mathbb{C}^3$.

Corresponding to the solution $(0, 2)$, translating the CR singular point to the origin in $\mathbb{C}^3$ gives a multivariable Taylor expansion:

$$F_1(z_1, \bar{z}_1, z_2 + 2, \overline{z_2 + 2}) = \frac{4}{15} z_1 - \frac{1}{25} z_1 z_2 - \frac{1}{25} z_1 \bar{z}_2 - \frac{1}{30} z_2 \bar{z}_1 + O(3).$$

The linear term can be eliminated by a complex linear transformation $\tilde{z}_3 = z_3 + c_{31} z_1$, which does not change the quadratic or higher degree terms and brings M_1 into standard position (2.1). The (R,S) coefficient matrix pair satisfies $S=0_{2\times 2}$ and $\det(R)\neq 0$, so $\Gamma=\left(\begin{smallmatrix} R & 0\\ 0 & \bar R\end{smallmatrix}\right)$ and $\det(\Gamma)=|\det(R)|^2>0$. This CR singularity has index $+1$.

Similarly, corresponding to the solution $(\sqrt{3},1)$, translating to the origin gives the series expansion:

$$\begin{aligned}
&F_1\big(z_1+\sqrt{3},\overline{z_1+\sqrt{3}},z_2+1,\overline{z_2+1}\big)-\frac{\sqrt{3}}{3}\\
&\quad=\frac{1}{5}z_1-\frac{\sqrt{3}}{15}z_2-\frac{\sqrt{3}}{75}z_1^2+\frac{1}{15}z_1z_2+\frac{2\sqrt{3}}{75}z_2^2\\
&\qquad-\frac{8\sqrt{3}}{225}z_1\bar z_1+\frac{4}{75}z_1\bar z_2-\frac{4}{75}z_2\bar z_1-\frac{8\sqrt{3}}{75}z_2\bar z_2+O(3).
\end{aligned}$$

Again in this case, the holomorphic terms are irrelevant, $S=0$, and $\det(R)\neq 0$, so the CR singularity has index $+1$. Each of the remaining three points, by a similar (but omitted) calculation, also has index $+1$, so M_1 is in general position.

It remains to check the points "at infinity," where $z_0=0$ and ι_t restricts to a holomorphic linear embedding

$$[0:z_1:z_2]\mapsto[0:z_1\cdot\mathcal{P}:z_2\cdot\mathcal{P}:t\cdot 0]=[0:z_1:z_2:0].$$

This restriction is one-to-one and misses the image of F_t in the $\{Z_0=1\}$ affine neighborhood, which shows that ι_t is one-to-one for each t.

To look for more CR singularities on the line at infinity, we consider the restriction of ι_t to another affine coordinate chart. The restriction of ι_t to the $\{z_1=1\}$ neighborhood has image contained in the $\{Z_1\neq 0\}$ neighborhood of the target $\mathbb{C}P^3$, and is given by the formula:

$$[z_0:1:z_2]\mapsto\left[z_0:1:z_2:\frac{t\cdot\mathcal{Q}(z_0,1,z_2)}{\mathcal{P}(z_0,1,z_2)}\right],$$

$$(z_0,z_2)\mapsto\left(z_0,z_2,\frac{t\cdot(2z_0^2+2z_0\bar z_2-z_0z_2+2z_2\bar z_0)}{6z_0\bar z_0+1+6z_2\bar z_2}\right). \tag{6.5}$$

This is another graph of a rational function $G_t(z_0,\bar z_0,z_2,\bar z_2)=t\cdot\mathcal{Q}/\mathcal{P}$ with a nonvanishing denominator, which is real analytic in z, $\bar z$ and t. Since we have already found all the CR singularities of the form $\iota_t([1:z_1:z_2])$, we can simplify the computational problem of finding new CR singularities in the graph of G_t by adding the equation $z_0=0$ to get this system:

$$\frac{d}{d\bar z_0}G_t=0,\qquad \frac{d}{d\bar z_2}G_t=0,\quad z_0=0. \tag{6.6}$$

The solution set does not depend on t (for $t\neq 0$), and a calculation with [Maple] shows that $(\zeta_0,\zeta_2)=(0,0)$ is the only solution of (6.6). The graph of G_t is already in standard position; at $t=1$, the quadratic part of the defining function is the numerator from (6.5):

$$G_1(z_0,\bar{z}_0,z_2,\bar{z}_2)=2z_0^2+2z_0\bar{z}_2-z_0z_2+2z_2\bar{z}_0+O(3).$$

This is a CR singularity with index $+1$; the graph is in general position.

There is one last point to check, $\iota_t([0:0:1])=[0:0:1:0]$. The restriction of ι_t to the $\{z_2=1\}$ neighborhood is given by the formula:

$$\begin{aligned}[z_0:z_1:1]&\mapsto\left[z_0:z_1:1:\frac{t\cdot\mathcal{Q}(z_0,z_1,1)}{\mathcal{P}(z_0,z_1,1)}\right],\\ (z_0,z_1)&\mapsto\left(z_0,z_1,\frac{t\cdot(2z_0^2\bar{z}_1+2z_0z_1-z_0\bar{z}_1+2z_1\bar{z}_0)}{6z_0\bar{z}_0+z_1\bar{z}_1+6}\right).\end{aligned}$$

Since this image is also a real analytic graph, ι_t is a (global) real analytic embedding depending real analytically on t. The origin in this neighborhood is another CR singularity of the image; the graph is already in standard position and, for $t\neq 0$, in general position, with a CR singular point of index $+1$.

The conclusion is that for $t\neq 0$, $\iota_t:\mathbb{C}P^2\to\mathbb{C}P^3$ has exactly seven CR singular points, at the image of

$$\big\{[1:0:2],\big[1:\pm\sqrt{3}:1\big],[1:\pm 3i:-1],[0:1:0],[0:0:1]\big\}.$$

Every image point is in N_2^+ (positively oriented tangent space), with index $+1$, consistent with the characteristic class calculations of Example 3.7.

7. Summary

Theorem 7.1. *Given a real analytic* 4*-dimensional submanifold M in $\mathbb{C}^3$, for any CR singular point there is a local holomorphic coordinate neighborhood so that the CR singular point is at the origin, the tangent space is the* (z_1,z_2)*-hyperplane, and the local defining equation for M is given by*

$$\begin{aligned}z_3&=h(z,\bar{z})\\&=(\bar{z}_1,\bar{z}_2)N\begin{pmatrix}z_1\\z_2\end{pmatrix}+\mathrm{Re}\left((z_1,z_2)P\begin{pmatrix}z_1\\z_2\end{pmatrix}\right)+e(z_1,\bar{z}_1,z_2,\bar{z}_2),\end{aligned}\tag{7.1}$$

where $e(z,\bar{z})=O(3)$ is real analytic. The coefficient matrices N, P fall into one of the cases from Table 1, and exactly one (*modulo equivalences as indicated*).

In Table 1, the third column is the number of real moduli for each case. The last column indicates the sign of $\det(\Gamma)$ that can occur for various values of the entries of N and P: positive, negative, or zero $(+,-,0)$.

Theorem 7.2. *Given a real analytic* 4*-dimensional submanifold M in $\mathbb{C}^3$ with local defining equation* (7.1) *in one of the normal forms from Theo-*

TABLE 1. Normal forms for Theorem 7.1

N	P			
$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ $0<\theta<\pi$	$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$	5	$a>0, d>0, b\sim -b\in\mathbb{C}$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$	3	$b\geq 0, d\geq 0$	$+-0$
	$\begin{pmatrix} a & b \\ b & 0 \end{pmatrix}$	3	$a>0, b\geq 0$	$+-0$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	2	$0\leq a\leq d$	$+-0$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	2	$0\leq a\leq d$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+$
	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	0		$+$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & b \\ b & 1 \end{pmatrix}$	1	$b>0$	$+0$
	$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	2	$\mathrm{Im}(d)>0$	$+$
$\begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix}$ $0<\tau<1$	$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$	5	$b>0, \|a\|=1, (a,d)\sim(-a,-d)$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & d \end{pmatrix}$	3	$b>0, \|d\|=1, d\sim -d$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	2	$b>0$	$+-0$
	$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	3	$d\in\mathbb{C}$	$+0$
	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	1		$+$
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1		$+$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b \\ b & 1 \end{pmatrix}$	3	$b>0, a\in\mathbb{C}$	$+-0$
	$\begin{pmatrix} 1 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+-0$
	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	1	$a\geq 0$	$+0$
	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
$\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	3	$a>0, d\in\mathbb{C}$	$+-0$
	$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	1	$b>0$	$+0$
	$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	1	$d\geq 0$	$+$
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	1	$a\geq 0$	$+-0$
	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	0		$+$
	$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$	1	$a\geq 0$	0
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0		$+$
	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0		0
	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0		0

TABLE 2. Normal forms for Theorem 7.2, Example 7.3

P		$\rho(P)$	$\rho(N\|P)$	$\rho(\Gamma)$	$\sigma(\Gamma)$	$\mathrm{sign}(\det(\Gamma))$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$		0	2	4	4	$+$
$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	$0<d<1$	1	2	4	4	$+$
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$		1	2	3	3	0
$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	$1<d$	1	2	4	2	$-$
$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	$0<a\leq d<1$	2	2	4	4	$+$
$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	$0<a<1$	2	2	3	3	0
$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	$0<a<1<d$	2	2	4	2	$-$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		2	2	2	2	0
$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	$1<d$	2	2	3	1	0
$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	$1<a\leq d$	2	2	4	0	$+$

rem 7.1, if the quadratic part of $h(z,\bar{z})$ in (7.1) is real valued, then the coefficient matrices N, P fall into exactly one of the cases from the following Examples 7.3–7.7.

The whole numbers in the middle columns are invariants of the defining function $h(z,\bar{z})$ under local biholomorphic coordinate changes that preserve the property of being in a quadratically flat normal form.

As previously mentioned, related lists of normal forms for pairs have appeared in [I], [E], and [CS].

EXAMPLE 7.3. For $N=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\rho(N)=2$, $\sigma(N)=2$ (see Table 2). The diagonal elements of P in the above table are the previously mentioned "generalized Bishop invariants." This is the only example where cases with $\sigma(\Gamma)=4$ occur, that is, where Γ is definite, or, equivalently, where the real Hessian is definite, as discussed in Section 4.1. These are the points called "flat elliptic" points by [DTZ$_1$], [DTZ$_2$], and [Dolbeault], and they appeared in the flat embedding of the ellipsoid in Example 6.1. The definiteness of Γ characterizes this elliptic property among all the equivalence classes from Theorem 7.2 and Examples 7.3–7.7. The first line in the above chart, where $P=0$, represents the case considered by [HY].

We also see a difference from the real surface $M\subseteq\mathbb{C}^2$ case, where a complex point (in N_1^+) has the elliptic property if and only if $ind=+1$. That characterization does not generalize to dimensions $m=4$, $n=3$; even in Table 2, flat elliptic complex points have $ind=+1$, but so do some quadratically flat, nonelliptic points.

The normal forms in Table 2 with full rank $\rho(\Gamma)=4$ and indefinite signature $\sigma(\Gamma)=2$ or 0 represent the equivalence classes of points called "hyperbolic" by [Dolbeault].

TABLE 3. Normal forms for Theorem 7.2, Example 7.4

P		$\rho(P)$	$\rho(N\|P)$	$\rho(\Gamma)$	$\sigma(\Gamma)$	sign(det(Γ))
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$		0	2	4	0	+
$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	$0 < d < 1$	1	2	4	0	+
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$		1	2	3	1	0
$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	$1 < d$	1	2	4	2	−
$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	$0 < a \leq d < 1$	2	2	4	0	+
$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	$0 < a < 1$	2	2	3	1	0
$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	$0 < a < 1 < d$	2	2	4	2	−
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		2	2	2	0	0
$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	$1 < d$	2	2	3	1	0
$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$	$1 < a \leq d$	2	2	4	0	+
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$		1	2	4	0	+
$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$	$0 < b$	2	2	4	0	+

TABLE 4. Normal forms for Theorem 7.2, Example 7.5

P		$\rho(P)$	$\rho(N\|P)$	$\rho(\Gamma)$	$\sigma(\Gamma)$	sign(det(Γ))
$\begin{pmatrix} 0 & b \\ b & 1 \end{pmatrix}$	$0 < b \neq 1$	2	2	4	0	+
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$		2	2	3	1	0
$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	$0 < \mathrm{Im}(d)$	2	2	4	0	+

EXAMPLE 7.4. For $N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho(N) = 2$, $\sigma(N) = 0$ (see Table 3). In Table 3, we see that the discrete invariants in the last 5 columns are repeated in a few cases, so they are not enough to distinguish inequivalent matrix normal forms of different shapes (such as diagonalizable or not).

EXAMPLE 7.5. For $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho(N) = 2$, $\sigma(N) = 0$ (see Table 4). Some of the rows in Table 4 have the same ρ and σ data as rows from the previous example.

EXAMPLE 7.6. For $N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\rho(N) = 1$, $\sigma(N) = 1$ (see Table 5).

EXAMPLE 7.7. For $N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\rho(N) = 0$, $\sigma(N) = 0$ (see Table 6).

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TABLE 5. Normal forms for Theorem 7.2, Example 7.6

P		$\rho(P)$	$\rho(N\|P)$	$\rho(\Gamma)$	$\sigma(\Gamma)$	sign(det(Γ))
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$		1	2	4	2	$-$
$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	$0 < a < 1$	2	2	4	2	$-$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		2	2	3	1	0
$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	$1 < a$	2	2	4	0	$+$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		2	2	4	0	$+$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$		0	1	2	2	0
$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$	$0 < a < 1$	1	1	2	2	0
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$		1	1	1	1	0
$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$	$1 < a$	1	1	2	0	0

TABLE 6. Normal forms for Theorem 7.2, Example 7.7

P	$\rho(P)$	$\rho(N\|P)$	$\rho(\Gamma)$	$\sigma(\Gamma)$	sign(det(Γ))
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	2	4	0	$+$
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	1	1	2	0	0
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0	0	0	0	0

REFERENCES

[ABKLR] B. Aebischer, M. Borer, M. Kälin, C. Leuenberger and H. Reimann, *Symplectic geometry*, PiM, vol. 124, Birkhäuser, Basel, 1994. MR 1296462

[BF] T. Banchoff and F. Farris, *Tangential and normal Euler numbers, complex points, and singularities of projections for oriented surfaces in four-space*, Pacific J. Math. **161** (1993), 1–24. MR 1237138

[Bishop] E. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–21. MR 0200476

[Bohr] C. Bohr, *Immersions of surfaces in almost-complex 4-manifolds*, Proc. Amer. Math. Soc. **130** (2002), 1523–1532. MR 1879979

[CS] D. Chakrabarti and R. Shafikov, *Holomorphic extension of CR functions from quadratic cones*, Math. Ann. **341** (2008), 543–573; and *Erratum*, Math. Ann. **345** (2009), 491–492. MR 2399158

[C_1] A. Coffman, *Enumeration and normal forms of singularities in Cauchy–Riemann structures*, dissertation, University of Chicago, 1997.

[C_2] A. Coffman, *CR singular immersions of complex projective spaces*, Beiträge zur Algebra und Geometrie **43** (2002), 451–477. MR 1957752

[C_3] A. Coffman, *Unfolding CR singularities*, to appear in Mem. Amer. Math. Soc. **205** (2010).

[Dolbeault] P. Dolbeault, *On Levi-flat hypersurfaces with given boundary in $\mathbb{C}^n$*, Sci. China Ser. A **51** (2008), 541–552. MR 2395404

[DTZ_1] P. Dolbeault, G. Tomassini and D. Zaitsev, *On boundaries of Levi-flat hypersurfaces in $\mathbb{C}^n$*, C. R. Math. Acad. Sci. Paris **341** (2005), 343–348. MR 2169149

[DTZ$_2$] P. Dolbeault, G. Tomassini and D. Zaitsev, *On Levi-flat hypersurfaces with prescribed boundary*, preprint; available at arXiv:0904.0481.

[Domrin] A. V. Domrin, *A description of characteristic classes of real submanifolds in complex manifolds via RC-singularities*, Izv. Ross. Akad. Nauk Ser. Mat. (English transl.) **59** (1995), 899–918. MR 1360632

[E] Ju. B. Ermolaev, *The simultaneous reduction of symmetric and Hermitian forms*, Izv. Vysš. Učebn. Zaved. Matematika **21** (1961), 10–23. MR 0155836

[F] F. Forstnerič, *Complex tangents of real surfaces in complex surfaces*, Duke Math. J. **67** (1992), 353–376. MR 1177310

[Garrity] T. Garrity, *Global structures on CR manifolds via Nash blow-ups*, W. Fulton Birthday Volume, Michigan Math. J. **48** (2000), 281–294. MR 1786491

[Gong] X. Gong, *Existence of real analytic surfaces with hyperbolic complex tangent that are formally but not holomorphically equivalent to quadrics*, Indiana Univ. Math. J. **53** (2004), 83–95. MR 2048184

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978. MR 0507725

[HL$_1$] F. R. Harvey and H. B. Lawson, Jr., *A theory of characteristic currents associated with a singular connection*, Astérisque **213** (1993). MR 1230025

[HL$_2$] F. R. Harvey and H. B. Lawson, Jr., *Geometric residue theorems*, Amer. J. Math. **117** (1995), 829–873. MR 1342833

[Herbert] R. Herbert, *Multiple points of immersed manifolds*, Mem. Amer. Math. Soc. **34** (1981). MR 0634340

[Hirsch] M. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276. MR 0119214

[HJ] R. Horn and C. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985. MR 0832183

[HS] R. Horn and V. Sergeichuk, *Canonical forms for complex matrix congruence and *congruence*, Linear Algebra Appl. **416** (2006), 1010–1032. MR 2242477

[Hua] L.-K. Hua, *On the theory of automorphic functions of a matrix variable I—geometrical basis*, Amer. J. Math. **66** (1944), 470–488. MR 0011133

[HK] X. Huang and S. Krantz, *On a problem of Moser*, Duke Math. J. **78** (1995), 213–228. MR 1328757

[HY] X. Huang and W. Yin, *A codimension two CR singular submanifold that is formally equivalent to a symmetric quadric*, Int. Math. Res. Notices **2009** (2009), 2789–2828. MR 2525841

[I] G. E. Izotov, *Simultaneous reduction of a quadratic and a Hermitian form*, Izv. Vysš. Učebn. Zaved. Matematika **1957** (1957), 143–159. MR 0125852

[JL] H. Jacobowitz and P. Landweber, *Manifolds admitting generic immersions into* $\mathbb{C}^N$, Asian J. Math. **11** (2007), 151–165. MR 2304588

[L] H.-F. Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. Amer. Math. Soc. **172** (1972), 1–33. MR 0314066

[LR] P. Lancaster and L. Rodman, *Canonical forms for Hermitian matrix pairs under strict equivalence and congruence*, SIAM Review **47** (2005), 407–443. MR 2178635

[LP] B. H. Li and F. Peterson, *Immersions of n-manifolds into $(2n-2)$-manifolds*, Proc. Amer. Math. Soc. **97** (1986), 531–538. MR 0840641

[Maple] Maple 12, Waterloo Maple Inc., 2008. www.maplesoft.com.

[M] J. Milnor, *On the immersion of n-manifolds in $(n+1)$-space*, Comment. Math. Helv. **30** (1956), 275–284. MR 0079268

[MW] J. Moser and S. Webster, *Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations*, Acta Math. **150** (1983), 255–296. MR 0709143

[Thomas] E. Thomas, *Submersions and immersions with codimension one or two*, Proc. Amer. Math. Soc. **19** (1968), 859–863. MR 0229246

[Thorpe] J. Thorpe, *Elementary topics in differential geometry*, UTM, Springer, New York, 1979. MR 0528129

[Webster] S. Webster, *On the relation between Chern and Pontrjagin numbers*, Contemp. Math. **49** (1986), 135–143. MR 0833810

[Wells] R. Wells, Jr., *Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles*, Math. Ann. **179** (1969), 123–129. MR 0237823

Adam Coffman, Department of Mathematical Sciences, Indiana University–Purdue University Fort Wayne, 2101 E. Coliseum Blvd., Fort Wayne, IN 46805-1499, USA

E-mail address: CoffmanA@ipfw.edu