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SPECTRAL DECIMATION ON HAMBLY'S HOMOGENEOUS HIERARCHICAL GASKETS

SHAWN DRENNING AND ROBERT S. STRICHARTZ

Abstract. We give a complete description of the Dirichlet and Neumann spectra of the Laplacian on a class of homogeneous hierarchical fractals introduced by Hambly. These fractals are finitely ramified but not self-similar. We use the method of spectral decimation. As applications, we show that these spectra always have infinitely many large spectral gaps, allowing for nice convergence results for eigenfunction expansions, and under certain restrictions we give a computer-assisted proof that the set of ratios of eigenvalues has gaps, implying the existence of quasielliptic PDE's on the product of two such fractals. The computer programs used in this paper and more detailed explanations of the algorithms can be found at www.math.cornell.edu/~sld32/FractalAnalysis.html.

1. Introduction

Analysis on fractals has been highly developed in the context of finitely ramified self-similar fractals (see the books [3], [13], [20]) and the expository article [17]). In an attempt to extend the theory beyond this context, Hambly ([9], [10]) introduced various examples of finitely ramified hierarchical fractals that are not self-similar. Roughly speaking, a *hierarchical fractal* is a set that may be written as a union of cells of level m, for each m, so that each cell of level m subdivides into cells of level $m+1$ in a specified manner. It is *finitely ramified* if the intersections of distinct cells of a given level consist of

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finite sets. It is *homogeneous* if the subdivision scheme is the same for all cells of a given level (but it may vary from level to level). In [9], the subdivision scheme is a variant of the one for the usual Sierpinski gasket. Each cell of level m is contained in a triangle, and that triangle is split into triangles of sides $1/b_{m+1}$ times the side of the original triangle for $b_{m+1} \in 2, 3, \ldots$, the upside-down triangles are deleted and the ones with the same orientation as the original contain the cells of level $m+1$. The resulting gasket, which we denote $\mathrm{HH}(b)$ for $b = (b_1, b_2, \ldots)$ is homogeneous (in [10] the subdivision scheme is allowed to vary from cell to cell). In this paper, we will study these gaskets in the case where each b_j equals 2 or 3. Although Hambly's original motivation was to look at these gaskets from a probablistic point of view, allowing the choice of the b_j's to be governed by random processes that produce "statistically self-similar fractals," there are interesting statements that can be made about the entire class of fractals. Most important, the definition of a standard Laplacian following the lines of Kigami's definition for SG [12] is quite straight forward. In this work, we study the spectrum (Dirichlet or Neumann) of this Laplacian.

Here, we briefly recap the definition. The fractal will be realized as the limit of a sequence of graphs $\Gamma_0, \Gamma_1, \ldots$ with vertices $V_0 \subseteq V_1 \subseteq \cdots$. The initial graph Γ_0 is just the complete graph on $V_0 = \{q_0, q_1, q_2\}$, the vertices of a triangle, which is considered the boundary of $\mathrm{HH}(b)$. The entire fractal is the only 0-cell, which has V_0 as its boundary. At stage m of the construction, all the cells of level $m-1$ lie in triangles whose vertices make up V_{m-1}. If $b_m = 2$, then each cell of level $m-1$ splits into three cells of level m, adding three new vertices to V_m, connected exactly as in the SG_2 (the usual Sierpinski gasket) construction. If $b_m = 3$, then each cell splits into six cells of level m, of side length $1/3$ that of the $(m-1)$-cell, adding seven vertices in V_m, connected as shown in Figure 1 (Section 2). The choice of all $b_j = 2$ leads to the ordinary SG_2, and the choice of all $b_j = 3$ leads to a self-similar fractal SG_3 that is well studied [20].

We define the unrenormalized energy of a function on V_m by

$$E_m(u) = \sum_{x \underset{m}{\sim} y} \big(u(x) - u(y)\big)^2. \tag{1.1}$$

The energy renormalization factors are $r_2 = 3/5$ and $r_3 = 7/15$ for SG_2 and SG_3, respectively. That means we want to define

$$\mathcal{E}_m(u) = (3/5)^{-m_2}(7/15)^{-m_3} E_m(u) \tag{1.2}$$

as the renormalized energy, where $m = m_2 + m_3$ and $m_2 = \#\{j \le m : b_j = 2\}$ and $m_3 = \#\{j \le m : b_j = 3\}$. Then $\mathcal{E}_m(u)$ is always nondecreasing, and is constant if u is harmonic, so we may define energy on $\mathrm{HH}(b)$ by

$$\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u) \tag{1.3}$$

with $\operatorname{dom}\mathcal{E}$ (always a subspace of the continuous functions) defined as the space of functions with $\mathcal{E}(u) < \infty$. Then $\mathcal{E}$ extends by polarization to a bilinear form $\mathcal{E}(u,v)$ which serves as an inner product on the Hilbert space $\operatorname{dom}\mathcal{E}/\text{constants}$. We let μ denote the probability measure assigning weight $(1/3)^{m_2}(1/6)^{m_3}$ to each m-cell. The standard Laplacian may then be defined using the weak formulation: $u \in \operatorname{dom}\Delta$ with $\Delta u = f$ if f is continuous, $u \in \operatorname{dom}\mathcal{E}$, and

$$\mathcal{E}(u,v) = -\int f v \, d\mu \tag{1.4}$$

for all $v \in \operatorname{dom}_0\mathcal{E}$ (the functions in $\operatorname{dom}\mathcal{E}$ vanishing on V_0). There is also a pointwise formula for any point in $V_* = \bigcup_m V_m$ (not in V_0):

$$\Delta u(x) = 3/2 \lim_{m\to\infty} 5^{m_2}(90/7)^{m_3}\Delta_m u(x), \tag{1.5}$$

where Δ_m is a discrete Laplacian associated to the graph Γ_m. If x is a point with 4 neighbors in Γ_m, then

$$\Delta_m u(x) = \sum_{y \underset{m}{\sim} x} \big(u(y) - u(x)\big) \tag{1.6}$$

(with 4 terms in sum), while if x has 6 neighbors in Γ_m then

$$\Delta_m u(x) = 2/3 \sum_{y \underset{m}{\sim} x} \big(u(y) - u(x)\big) \tag{1.7}$$

(with 6 terms in sum). Note that the renormalization factor 5 is the reciprocal of the product of the factors 3/5 and 1/3 associated to energy and measure for $b_j = 2$, while 90/7 is the reciprocal of the product of the factors 7/15 and 1/6 for $b_j = 3$. The derivation of (1.6) and (1.7) from (1.4) is obtained by taking v to be the piecewise harmonic function at level m which satisfies $v(x) = 1$ and $v(z) = 0$ for all $z \in V_m$, $z \neq x$, and then taking the limit as $m \to \infty$. The exact statement of the converse is that if the limit (1.5) is uniform and converges to a continuous function f, then (1.4) holds.

The method of spectral decimation, first described for SG_2 and related examples by Fukushima and Shima ([8], [15]), describes a connection between the eigenfunctions and eigenvalues of the discrete Laplacians Δ_m for different values of m, and the Laplacian Δ. It also yields a complete description of the Dirichlet and Neumann spectra of those Laplacians. (The Neumann spectrum may be defined in terms of normal derivatives, but is easiest to understand by considering the double cover $\widetilde{\mathrm{HH}}(b)$, extending functions from $\mathrm{HH}(b)$ by even reflection, and imposing the pointwise eigenfunction equation at the boundary points in V_0, which now have 4 neighbors.) It is known from [16] that SG_3 also enjoys spectral decimation. We present the explicit details in Sections 2 and 3. These results have also been obtained independently in [2] and [21].

Here is an outline of spectral decimation for both cases SG_2 and SG_3. In each case there is a finite set of *forbidden eigenvalues*, and a rational function denoted R_2 or R_3. First, there is a local extension algorithm that tells you how to extend (uniquely) a function u defined on V_m to a function defined on V_{m+1} so as to satisfy the λ-eigenvalue equations on points of $V_{m+1}\backslash V_m$, provided λ is not a forbidden eigenvalue. Then, if it is assumed that u satisfies a λ_m-eigenvalue equation on V_m, the extended function will satisfy the λ_{m+1}-eigenvalue equation on V_{m+1} provided $\lambda_m = R_j(\lambda_{m+1})$ and λ_{m+1} is not a forbidden eigenvalue. Every eigenfunction u on SG_j has a generation of birth m_0 and a sequence $\{\lambda_m\}$ of discrete eigenvalues for $m \geq m_0$ (related as above) such that u restricted to V_m is a λ_m-eigenfunction, and λ_m is not a forbidden eigenvalue for any $m > m_0$ (it may be, and often is, for $m = m_0$). For SG_2, $R_2(x) = x(5-x)$ and the forbidden eigenvalues are 2, 5, 6. For SG_3, we will show that

$$R_3(x) = \frac{3x(5-x)(4-x)(3-x)}{14-3x} \tag{1.8}$$

and the forbidden eigenvalues are $3 \pm \sqrt{5}$, 3, 5, and 6. For SG_2 the function R_2 has two inverses $R_{21}^{-1}(x) = \frac{5-\sqrt{25-4x}}{2}$ and $R_{22}^{-1}(x) = \frac{5+\sqrt{25-4x}}{2}$, so we have either $\lambda_{m+1} = R_{21}^{-1}(\lambda_m)$ or $\lambda_{m+1} = R_{22}^{-1}(\lambda_m)$ for $m \geq m_0$. In order for the limit to exist in

$$\lambda = 3/2 \lim_{m\to\infty} 5^m \lambda_m, \tag{1.9}$$

which gives the eigenvalues of Δ in terms of the eigenvalues of Δ_m, it is necessary that $\lambda_m \to 0$, and hence $\lambda_{m+1} = R_{21}^{-1}(\lambda_m)$ for all but a finite number of m's. For SG_3 there are 4 inverses $R_{31}^{-1}, R_{32}^{-1}, R_{33}^{-1}$, and R_{34}^{-1} of R_3, and similar conditions hold.

To describe the explicit Dirichlet and Neumann spectra, we have to describe all possible generations of birth and values for λ_{m_0}, and describe the multiplicity of the eigenvalue by giving an explicit basis for the λ_{m_0}-eigenspace of Δ_{m_0}. For each m, we have to add up the dimensions of eigenspaces with generation of birth $m_0 \leq m$, extended to Γ_m in all allowable ways. This total must be $\#V_m$ (Neumann) or $\#V_m - 3$ (Dirichlet), the dimension of the space on which the symmetric operator Δ_m acts. The details are known for SG_2 ([7], [20]). We present the details for SG_3 in Section 3.

In Section 4, we show that spectral decimation is also valid on HH(b). Essentially, in passing from level m to level $m+1$, we use the method of spectral decimation associated to $SG_{b_{m+1}}$. As a consequence, we are able to give an order of magnitude estimate for the eigenvalue counting function

$$\rho(x) = \#\{j : \lambda_j \leq x\}. \tag{1.10}$$

For x between $\#V_{m-1}$ and $\#V_m$ (on the order of $3^{m_2} \cdot 6^{m_3}$), we will have $\rho(x)$ on the order of $5^{m_2}(90/7)^{m_3}$, which means $x^{\alpha(x)}$ for

$$\alpha(x) = \frac{m_2 \log 5 + m_3 \log 90/7}{m_2 \log 3 + m_3 \log 6}. \tag{1.11}$$

Then the Weyl ratio

$$W(x) = \frac{\rho(x)}{x^{\alpha(x)}} \tag{1.12}$$

will be bounded, and bounded away from zero. In fact the log–log graph of $W(x)$ shows a characteristic pattern that allows us to read off the sequence $(b_1, b_2, \ldots)$. We believe that this is an important clue for the existence of a spectral segment heuristic that should have consequences in many other contexts (see the discussion at the end of Section 4).

In Section 5, we give some applications involving the gap structure of the spectra. We define a *c-gap* with constant $c > 1$ to be a consecutive pair of eigenvalues λ, λ' such that $\frac{\lambda'}{\lambda} \geq c$. We show that there are infinitely many c-gaps, and that c may be chosen greater than 2. This implies the uniform convergence of eigenfunction expansions if you take the partial sums up to a gap, both on $\mathrm{HH}(b)$ and any product $\mathrm{HH}(b) \times \mathrm{HH}(b')$, by the methods of [18] and [19]. We also show that there are gaps in the set of ratios $\frac{\lambda}{\lambda'}$ where λ and λ' are any eigenvalues of $\mathrm{HH}(b)$, provided the sequence b is composed of only $(2,3)$ or $(3,2)$ pairs. These gaps are very small. We give a computer assisted proof. Our method also allows us to find many gaps in the ratios of eigenvalues for SG_2 and SG_3. The significance of these gaps is that it enables you to construct quasielliptic PDE's on the product $\mathrm{HH}(b) \times \mathrm{HH}(b')$. These ideas were introduced in [5] for SG_2. Other recent works establishing such gaps for Vicsek type fractals are [6] and [11].

2. Spectral decimation on SG_3

The goal of this paper is to extend some of the fractal analysis that has been developed on the Sierpinski gasket (SG_2). First, we define SG_3. The first level of SG_3 is shown in Figure 1. SG_3 is the unique solution of

$$K = \bigcup_{i=1}^{6} F_i K \tag{2.1}$$

among nonempty compact sets, where each F_i is a similarity mapping the large triangle into the six smaller triangles in the same orientation.

Just as with SG_2 it is possible to define a Laplacian on each approximation to SG_3 (and in the process define energy) and derive a pointwise formula for it. The pointwise formula for the Laplacian will be the same as the one for SG_2 except for vertices where three cells meet. Here, it is necessary to scale everything by 2/3. This factor arises because when a function is integrated

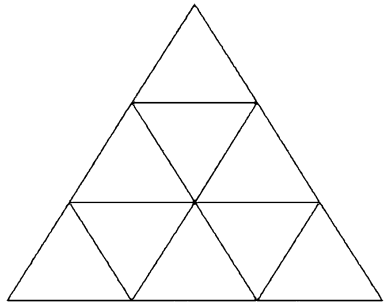

FIGURE 1. Building block for SG_3.

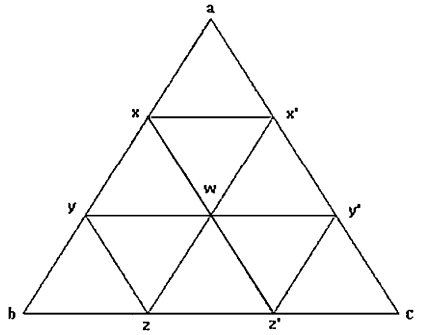

FIGURE 2. A general function on one cell.

over SG_3, a vertex neighboring three cells is given 3/2 as much weight in the integral as a vertex neighboring only two cells. An easy computation shows that the energy renormalization factor is 7/15. We also have a measure renormalizaton factor of 1/6. Together, these will give a Laplacian renormalization factor of 90/7. Finally, a Laplacian on SG_3 can be defined as the renormalized limit of the Laplacian on level m. We want to study eigenfunctions for this Laplacian. Our first step will be to derive a spectral decimation process similar to the one already defined on SG_2.

Given a function on V_m, we want to extend it to V_{m+1} so that the λ-eigenvalue equation holds at all points of $V_{m+1}\backslash V_m$. This is a local process and it is only necessary to solve the problem for one cell. If we can come up with an extension algorithm for Figure 2, we can extend it by linearity to get an algorithm for a cell with any values on the boundary. The explicit eigenvalue equations are

$$(4-\lambda)x = a + x' + y + w,$$

$$(4-\lambda)w = \frac{2}{3}(x + x' + y + y' + z + z')$$

and similarly for other vertices. It is a straightforward calculation to solve these equations.

LEMMA 2.1. *Let u be defined on V_m and take values on V_{m+1} in one V_m cell shown in Figure 2. Then u satisfies the λ-eigenvalue equation on the points of V_{m+1} for $\lambda \neq 3 \pm \sqrt{5}$, 3, 5 if and only if*

$$w = \frac{4}{3(4-6\lambda+\lambda^2)}(a+b+c),$$
$$x = \left(\frac{1}{3-\lambda} + \frac{36-7\lambda}{3(3-\lambda)(5-\lambda)(4-6\lambda+\lambda^2)}\right)a + \left(\frac{16-3\lambda}{3(5-\lambda)(4-6\lambda+\lambda^2)}\right)b + \left(\frac{36-7\lambda}{3(3-\lambda)(5-\lambda)(4-6\lambda+\lambda^2)}\right)c$$

and similarly for the other vertices.

Eigenvalues associated with the eigenfunction extension algorithm satisfy the eigenvalue equation at the vertices of V_m. Specifically, let f_m be an eigenfunction on level m with eigenvalue λ_m, we want to find an eigenvalue λ_{m+1} such that when we extend f_m to f_{m+1} using λ_{m+1} in the eigenfunction extension algorithm, f_{m+1} is an eigenfunction for λ_{m+1}. The only vertices where f_{m+1} could possibly not satisfy the eigenfunction equations are vertices where two or three cells meet. We will solve the problem in the case of two cells meeting. It is easy to check that our solution also works for where three cells meet.

Figure 3 shows the values of f_{m+1} in a neighborhood of a junction point in V_m. At the point e, we have that f_m and f_{m+1} satisfy

$$(4-\lambda_m)e = a+b+c+d,$$
$$(4-\lambda_{m+1})e = w+x+y+z,$$

respectively. Using Lemma 2.1, it is now straightforward to verify that when $\lambda_{m+1} \neq 3 \pm \sqrt{5}$, 3, 5, or 6 we can write λ_m in terms of λ_{m+1} as follows:

$$\lambda_m = \frac{3(\lambda_{m+1}-5)(\lambda_{m+1}-4)(\lambda_{m+1}-3)\lambda_{m+1}}{3\lambda_{m+1}-14}. \tag{2.2}$$

THEOREM 2.2. *Suppose λ_m and λ_{m+1} are related by (2.2), and $\lambda_{m+1} \neq 3 \pm \sqrt{5}$, 3, 5, or 6. Then every λ_m-eigenfunction on V_m extends uniquely to a λ_{m+1}-eigenfunction on V_{m+1} by Lemma 2.1, and every λ_{m+1}-eigenfunction on V_{m+1} restricts to a λ_m-eigenfunction on V_m.*

Proof. If $\lambda \neq 3 \pm \sqrt{5}$, 3, or 5, the above argument shows the existence of an extension. If $\lambda_{m+1} \neq 6$, we can run the argument backward and obtain uniqueness. □

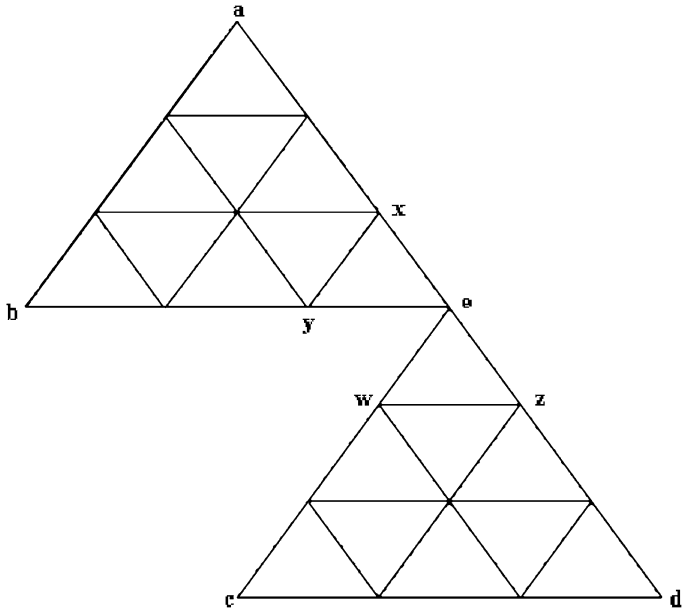

FIGURE 3. Two cells meeting.

Recall that for the five forbidden eigenvalues 3, 5, $3 \pm \sqrt{5}$, and 6 that the eigenfunction extension algorithm will not work. If f is an eigenfunction on level m corresponding to eigenvalue λ, then using the inverse to Equation (2.2) (this inverse can be explicitly computed), we get four potential eigenvalues to use with the eigenfunction extension formula in order to extend f to an eigenfunction on level $m+1$. For each of these four potential eigenvalues that is not a forbidden value, we can use the eigenfunction extension algorithm to get an eigenfunction on level $m+1$. From the second graph in Figure 4, it is easy to see that in the interval $[0,6]$ only the eigenvalues 0 and 6 are mapped to forbidden values. The eigenvalue 0 maps to 0, 3, 4, and 5. Since 3 and 5 are forbidden eigenvalues, if $\lambda = 0$, f can only be extended in two ways. Similarly, 6 maps to $3 \pm \sqrt{2}$ and $3 \pm \sqrt{5}$ and since $3 \pm \sqrt{5}$ are forbidden values, if $\lambda = 6$, f can only be extended in two ways. Finally, for any other value of λ, f can be extended in four ways.

3. Dirichlet and Neumann spectra for SG_3

For eigenvalues $\lambda = 3$, 5, $3 \pm \sqrt{5}$, and 6, we saw that the eigenfunction extension algorithm did not work. However, there are eigenfunctions with these eigenvalues—they just cannot be obtained using the eigenfunction extension algorithm. We can still completely describe all eigenfunctions with these

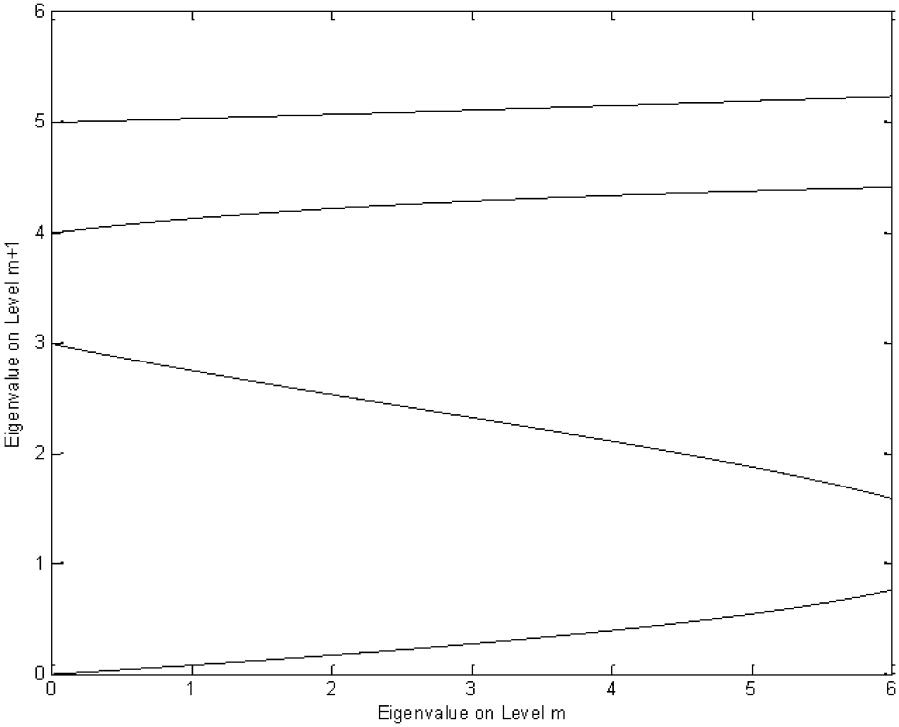

FIGURE 4. The graph of the inverse of the function in Equation (2.2).

forbidden eigenvalues. We will do this for the particular cases of Dirichlet and Neumann eigenfunctions. First, we describe the Dirichlet eigenfunctions appearing on level 1. If the necessary equations are solved on level 1, the Dirichlet eigenfunctions illustrated in Figure 5 are found. These eigenfunctions can be rotated to give additional eigenfunctions. Doing this gives us 1, 1, 2 and 2 linear independent eigenfunctions for eigenvalues 6, $3 \pm \sqrt{5}$, 5, and 3, respectively.

There are no eigenfunctions of eigenvalue $3 \pm \sqrt{5}$ on any level other than level 1. For eigenvalues 3, 5, and 6, there are Dirichlet eigenfunctions that are "born" on each level. We can completely describe these eigenfunctions. In order to do this, first, we need to consider the Neumann eigenfunctions on level 1. The best way to compute these is to separately find the symmetric and anti-symmetric eigenfunctions. First, observe that the Dirichlet-6 eigenfunction on level 1 shown in Figure 5 is also a Neumann anti-symmetric eigenfunction. Additionally, we find the Neumann anti-symmetric eigenfunction shown in Figure 6. The Neumann symmetric eigenfunctions on level 1 are shown in Figure 7. After considering rotations, we see that together these

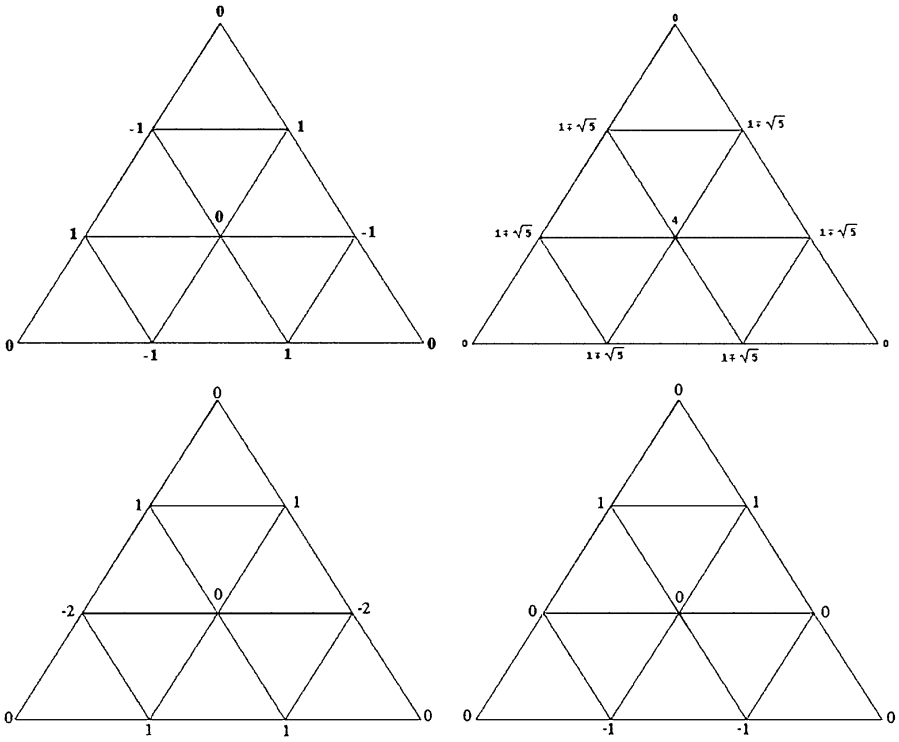

FIGURE 5. Dirichlet eigenfunctions on level 1 of SG_3 for eigenvalues 6, $3 \pm \sqrt{5}$, 5, and 3.

Neumann eigenfunctions generate a 10-dimensional space as desired. The multiplicities for eigenvalues 6, 0, 4, and $3 \pm \sqrt{2}$ respectively, are 4, 1, 1, and 4 (2 for each).

We can use the eigenfunctions we derived on level 1 to construct eigenfunctions with forbidden eigenvalues on higher levels. The constructions we will give are general; that is they work for any level of our approximation to SG_3. Furthermore, each eigenfunction will be supported in only a small part of the domain (this will be made more explicit later). For this reason, we can use the same procedure to construct Neumann and Dirichlet eigenfunctions. We just need to be careful near the boundary.

For eigenvalue 6, we can construct an eigenfunction that is equal to the Dirichlet-6 eigenfunction on one cell of level m and zero elsewhere. This will be both a Dirichlet and Neumann eigenfunction. It is clear that all of these eigenfunctions are linearly independent. Next, notice that the symmetric Neumann-6 eigenfunction can be rotated to give three linearly independent Neumann-6 eigenfunctions. Around any vertex v on level m, we can construct

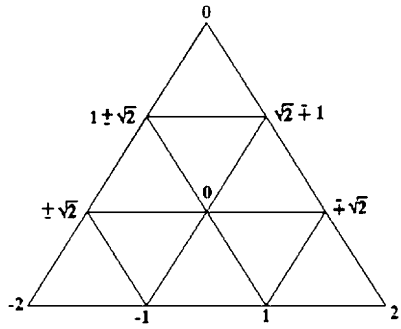

FIGURE 6. Neumann anti-symmetric eigenfunction for eigenvalues $3 \pm \sqrt{2}$.

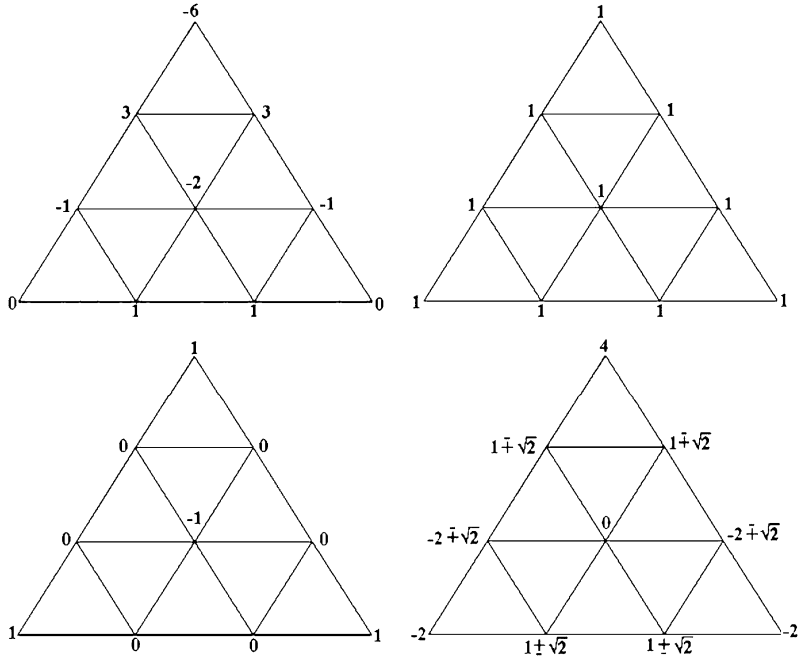

FIGURE 7. Neumann symmetric eigenfunctions for eigenvalues 6, 0, 4, and $3 \pm \sqrt{2}$.

an eigenfunction on level $m+1$ that is equal to one of the symmetric Neumann-6 eigenfunctions on each cell containing v and equal to zero on each cell not containing v. Notice that each one of these eigenfunctions is nonzero on exactly one vertex of level m. Thus, it is easy to see that these eigenfunctions

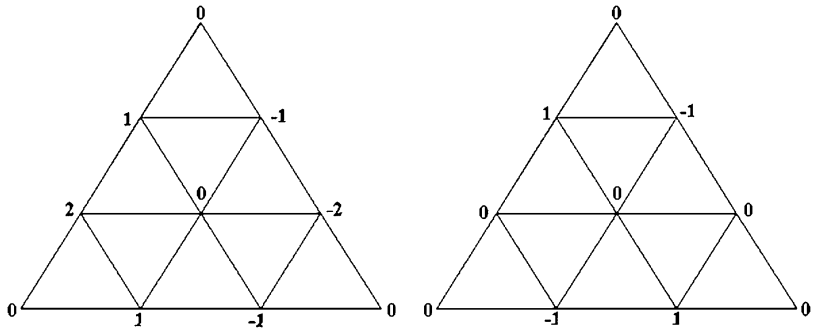

FIGURE 8. Anti-symmetric Dirichlet eigenfunctions for eigenvalues 3 and 5.

together with the first set are linearly independent. Also, each one of these eigenfunctions will be Neumann and all but the three that are nonzero on the boundary will be Dirichlet. Thus, on level $m+1$ we have $\#T_m + \#V_m - 3$ Neumann and $\#T_m + \#V_m$ Dirichlet eigenfunctions where T_m and V_m are respectively the set of cells and vertices on level m.

For eigenvalues 3 and 5, the first thing we need is an anti-symmetric Dirichlet eigenfunction on level 1. We already have symmetric Dirichlet eigenfunctions on level 1; if we reflect these eigenfunctions about an axis of nonsymmetry and subtract the result, we get anti-symmetric eigenfunctions. The result is exhibited in Figure 8. Around any cycle of level m we can connect these eigenfunctions like batteries to obtain a new eigenfunction. This concept is illustrated in Figure 9.

Thus, if C_m is the set of cycles on level m, on level $m+1$ we get $\#C_m$ eigenfunctions each for eigenvalues 3 and 5. It can be checked that these eigenfunctions are linearly independent. Additionally, in the Dirichlet case, we can string the eigenfunctions in Figure 8 along the sides of the fractal to obtain two additional eigenfunctions. The multiplicities of all these new eigenfunctions are summarized in Table 1.

We know that on level m we should have $\#V_m$ Neumann and $\#V_m - 3$ Dirchlet eigenfunctions. Now we want to check that our spectral decimation process actually gives this many eigenfunctions. The first step is to give some simple recursive formulas for $\#T_m$, $\#C_m$, and $\#V_m$. We easily see that:

$$\#T_m = 6 \cdot \#T_{m-1}, \tag{3.1}$$

$$\#C_m = \#C_{m-1} + 3 \cdot \#T_{m-1}, \tag{3.2}$$

$$\#V_m = \#V_{m-1} + 7 \cdot \#T_{m-1}. \tag{3.3}$$

Now we will verify that the spectral decimation gives all of the Dirchlet eigenfunctions. We will do this by induction on the level m. On level 1, we

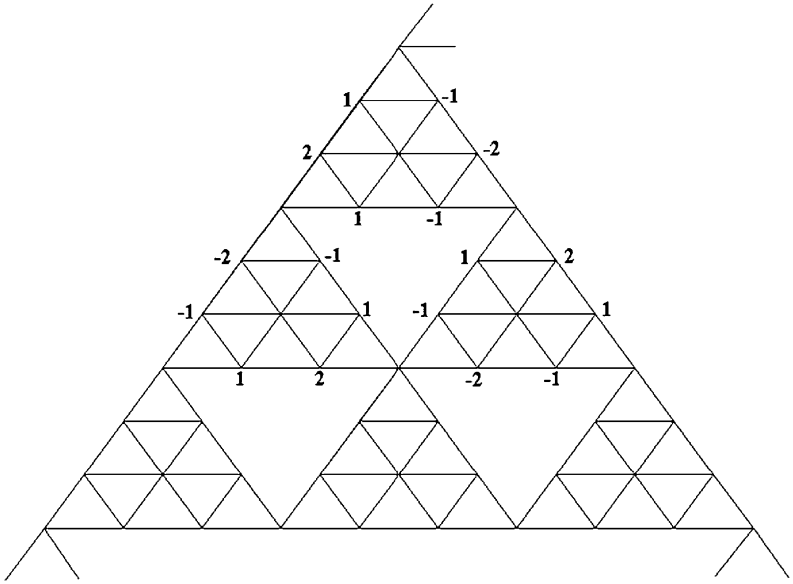

FIGURE 9. An eigenfunction for eigenvalue 3 obtained by connecting eigenfunctions like batteries.

TABLE 1. Multiplicities of eigenvalues not coming from the eigenfunction extension algorithm

λ_{m+1}	Dirichlet multiplicity	Neumann multiplicty
3 or 5	$\#C_m + 2$	$\#C_m$
6	$\#T_m + \#V_m - 3$	$\#T_m + \#V_m$

have given all 7 (counting multiplicities) eigenfunctions. On level 2, we get 26 eigenfunctions by extending the 7 eigenfunctions on level 1 with the eigenfunction extension algorithm. Using Table 1, we see that we also get 5 eigenfunctions each for eigenvalues 3 and 5 and 13 eigenfunctions for eigenvalue 6. Together this gives us 49 Dirichlet eigenfunctions for level 2. It is easy to see that $V_m - 3$ is indeed 49. Now assume that the spectral decimation process gives us $\#V_m - 3$ Dirichlet eigenfunctions on level m (with $m > 3$). We want to show that we get $V_{m+1} - 3$ eigenfunctions on level $m+1$. From Table 1, we know that on level $m+1$ we will have, in total, $2 \cdot \#C_m + \#T_m + \#V_m + 1$ eigenfunctions of eigenvalues 3, 5, or 6. Next note that there will never be any Dirichlet eigenfunctions with eigenvalue 0. Thus, for every eigenfunction on level m not of eigenvalue 6, we will get four eigenfunctions on level $m+1$ from the eigenfunction extension algorithm. For each eigenfunction of eigenvalue 6, we will get two eigenfunctions on level $m+1$. Since there

are $\#T_{m-1} + \#V_{m-1} - 3$ eigenfunctions of eigenvalue 6 on level m, we get $4 \cdot \#V_m - 12 - 2(\#T_{m-1} + \#V_{m-1} - 3) = 4 \cdot \#V_m - 2 \cdot \#T_{m-1} - 2 \cdot \#V_{m-1} - 6$ eigenfunctions on level $m+1$ from the eigenfunction extension algorithm. Thus, we want to show that:

$$\begin{aligned}\#V_{m+1} - 3 = (&2 \cdot \#C_m + \#T_m + \#V_m + 1)\\ &+ (4 \cdot \#V_m - 2 \cdot \#T_{m-1} - 2 \cdot \#V_{m-1} - 6).\end{aligned}$$

Replacing $\#V_{m+1}$ with $\#V_m + 7 \cdot \#T_m$ and collecting terms, we get

$$6 \cdot \#T_m = 4 \cdot \#V_m + 2 \cdot \#C_m - 2 \cdot \#V_{m-1} - 2 \cdot \#T_{m-1} - 2.$$

Now use the recursive formulas for $\#T_m$, $\#V_m$, and $\#C_m$ to rewrite this as

$$\begin{aligned}36 \cdot \#T_{m-1} = &4 \cdot \#V_{m-1} + 28 \cdot \#T_{m-1} + 2 \cdot \#C_{m-1}\\ &+ 6 \cdot \#T_{m-1} - 2 \cdot \#V_{m-1} - 2 \cdot \#T_{m-1} - 2.\end{aligned}$$

Simplified this becomes

$$2 \cdot \#T_{m-1} = \#V_{m-1} + \#C_{m-1} - 1.$$

This final formula is easy to prove for all m by induction. This completes the proof and proves the following theorem.

THEOREM 3.1. *If $m \geq 2$, then the Dirchlet "born" eigenfunctions (the multiplicities of which are summarized in Table 1) together with the eigenfunctions obtained by applying the eigenfunction extension algorithm to Dirichlet eigenfunctions of Δ_{m-1} form a basis for the Dirichlet eigenspace of Δ_m.*

A similar theorem can be formulated for Neumann eigenfunctions.

4. Spectral decimation on homogeneous hierarchical gaskets

It is a straightforward matter to extend the spectral decimation method from SG_2 and SG_3 to HH(b) for any b. If we have any eigenfunction u,

$$-\Delta_{m-1} u = \lambda^{(m-1)} u \tag{4.1}$$

on the graph Γ_{m-1}, then u extends to an eigenfunction $\widetilde{u}$ on the graph Γ_m,

$$-\Delta_m \widetilde{u} = \lambda^{(m)} u, \tag{4.2}$$

via the SG_{b_m} extension algorithm, provided $R_{b_m}(\lambda^{(m)}) = \lambda^{(m-1)}$ and $\lambda^{(m)}$ is not a forbidden eigenvalue for SG_{b_m}. Moreover, we obtain all Dirichlet (or Neumann) eigenfunctions on Γ_m by starting with a Dirichlet (or Neumann) eigenfunction with a forbidden eigenvalue on Γ_{m_0} for some $m_0 \leq m$ and iterating the above extension. The first statement holds because the extension algorithm is local, and the second follows by the same counting arguments for SG_2 and SG_3. By passing to the limit as $m \to \infty$ in (1.5), we obtain a complete set of eigenfunctions for Δ on HH(b). We summarize the results as follows.

THEOREM 4.1. *If $m \geq 2$, then the Dirichlet "born" eigenfunctions corresponding to SG_{b_m} together with the eigenfunctions obtained by applying the eigenfunction extension algorithm associated to b_m to Dirichlet eigenfunctions of Δ_{m-1} form a basis for the Dirichlet eigenspace of Δ_m. Every Dirichlet eigenfunction on* HH(b) *arises in this fashion.*

A similar theorem can be formulated for Neumann eigenfunctions.

Now we want to define a Weyl ratio for an arbitrary HH(b). Let $\{\lambda_j\}$ denote the eigenvalues of Δ on HH(b) and $\{\lambda_j^{(m)}\}$ denote the eigenvalues of Δ_m. We define

$$g_m(x) = \lim_{n\to\infty} 5^{N_2(n)}(90/7)^{N_3(n)} R^{-1}_{b_{m+n}1} \circ \cdots \circ R^{-1}_{b_{m+1}1}(x),$$

where $N_k(n) = \#\{j : m < j \leq n \text{ and } b_j = k\}$.

Then for small j (corresponding to eigenfunctions born on level $m_0 \leq m$) we will have

$$\lambda_j = (3/2)5^{mp_m}(90/7)^{m(1-p_m)} g_m\big(\lambda_j^{(m)}\big),$$

where $p_m = m_2/m$. Now let ρ be the eigenvalue counting function for HH(b) defined by (1.10) and ρ_m be the eigenvalue counting function for Δ_m. We then have the following relationship

$$\rho(x) = \rho_m\big(g_m^{-1}\big(5^{-mp_m}(90/7)^{-m(1-p_m)}x\big)\big).$$

Note that for small enough x that we can ignore the g_m^{-1} since $g_m(x) = x + O(x^2)$ and we can bound the error independent of the choice of b. Let

$$t_m = m\big(p_m \log 5 + (1-p_m)\log 90/7\big).$$

Since t_m increases with m, it follows that for every x that there is an m so that $t_m \leq \log x < t_{m+1}$. Now we want to define p_x as a linear interpolation of p_m and p_{m+1}

$$p_x = \frac{\log x - t_m}{t_{m+1} - t_m} p_{m+1} + \frac{t_{m+1} - \log x}{t_{m+1} - t_m} p_m.$$

We now define the *Weyl ratio* as

$$W(x) = \frac{\rho(x)}{x^{\alpha(x)}},$$

where $\alpha(x)$ is defined as

$$\alpha(x) = \frac{p_x \log 3 + (1-p_x)\log 6}{p_x \log 5 + (1-p_x)\log(90/7)}.$$

The next theorem explains why $W(x)$ is a reasonable definition for the Weyl ratio.

THEOREM 4.2. *There exists an M so that*

$$1/M < W(x) < M.$$

Proof. In this proof, we will use the notation $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$. First assume that $x = e^{t_m}$ for some m. We then have

$$x = 5^{mp_m}(90/7)^{m(1-p_m)}$$

and thus

$$\rho(x) = \rho_m(g_m^{-1}(1)).$$

We easily compute that

$$x^{\alpha(x)} = 3^{mp_m}6^{m(1-p_m)} = \Theta(\rho_m(6)).$$

It's clear that $\rho(x)$ is bounded above by $\rho_m(6)$. From Figure 10 we can see that $1/2$ is a lower bound for $g_m^{-1}(1)$. In particular, any eigenvalue of Δ_m where R_{31}^{-1} and R_{21}^{-1} were applied at step $m-1$ and m of the spectral decimation process will be less than $g_m^{-1}(1)$. We conclude that $\rho(x) > c\rho_m(6)$ for some constant c.

We now have

$$W(x) = \rho(x)/x^{\alpha(x)} = \Theta(\rho_m(6))/\Theta(\rho_m(6)) = \Theta(1). \qquad \square$$

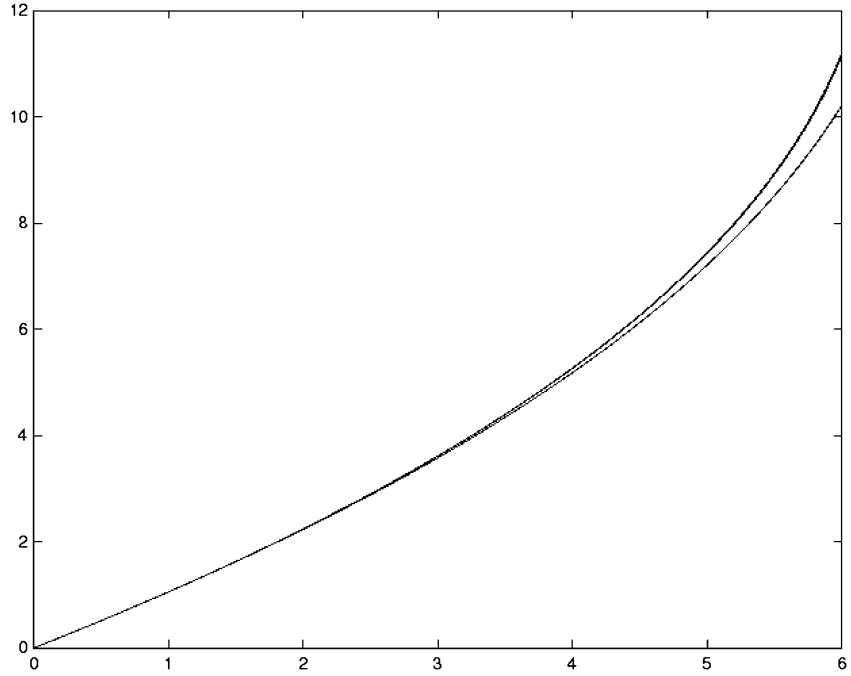

FIGURE 10. A plot of the 32 different 5th level approximations to $g_m(x)$.

An interesting observation is that in the figures above one can read off the b_i's from the graph of the Weyl ratio plotted on a log–log scale. The reason for this is that b_i has a major influence on the spectrum over the interval where $t_i \leq \log x < t_{i+1}$, and a relatively small influence elsewhere (except that t_i depends on all values of b_j for $j \leq i$). The interval $[t_i, t_{i+1}]$ in the log–log plot of $W(x)$ will be bounded by two long nearly vertical curves, corresponding to spectral gaps. These curves are immediately apparent visually. The length of these intervals will depend on the value of b_i, and the graph over this interval will closely resemble the graph for SG_2 when $b_i = 2$ and SG_3 when $b_i = 3$, as seen in Figures 11, 12, and 13.

The fact that the spectrum of HH(b) determines the sequence b is not at all surprising. What is more significant is that the spectrum can be divided into segments, each of which determines a particular value of b_j. We see this as a piece of evidence in favor of a *spectral segment heuristic*, which says that the spectrum of Laplacians in many different contexts (graphs, manifolds, fractals, etc.) may be divided into segments in a way that each segment may be explained by the geometry of the underlying space at a certain scale. Note that this is a heuristic, not a conjecture, and it is unlikely to be true

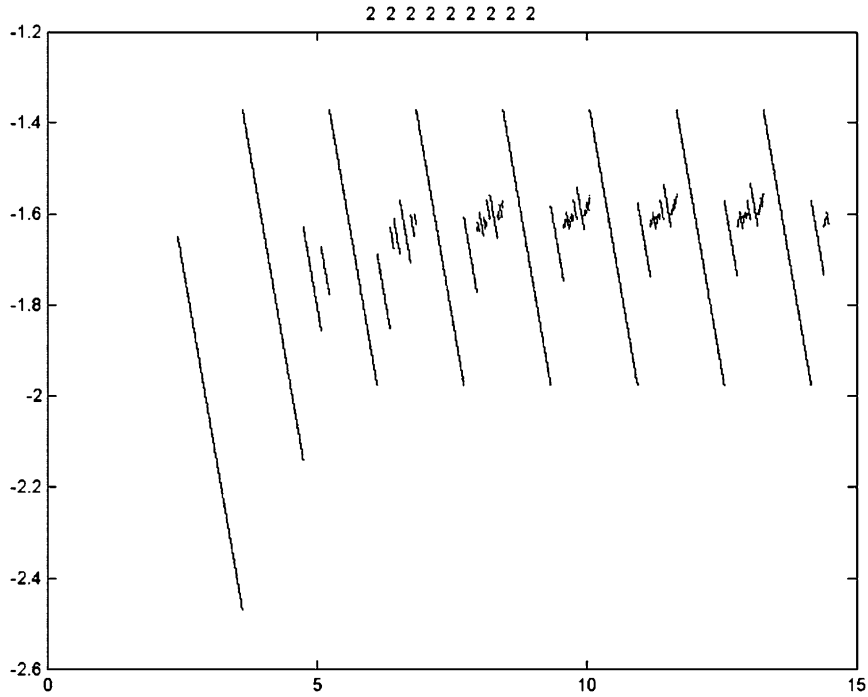

FIGURE 11. A graph of the Weyl ratio when $b = (2, 2, 2, 2, \ldots)$.

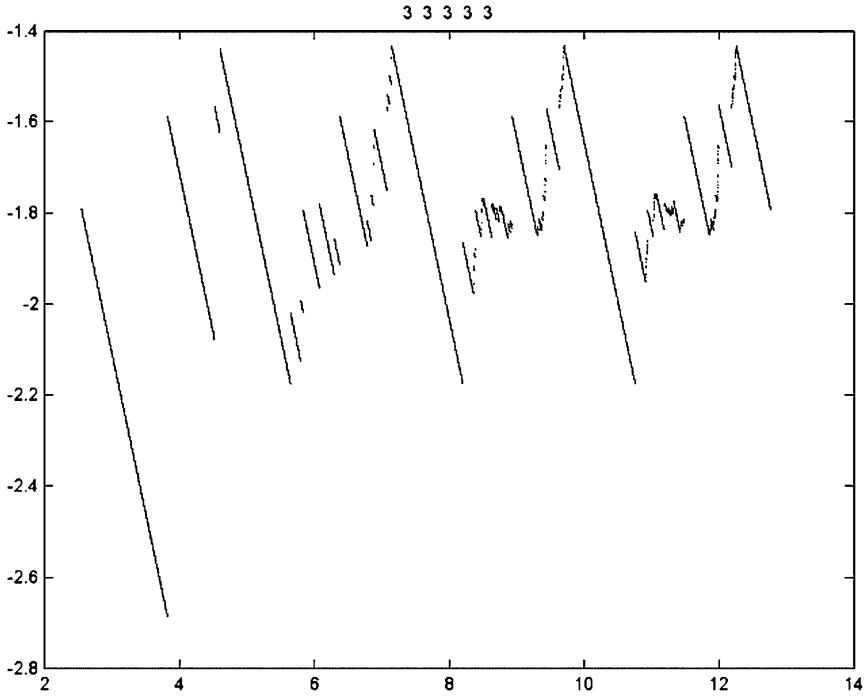

FIGURE 12. A graph of the Weyl ratio when $b = (3, 3, 3, 3, \ldots)$.

in complete generality. Our hope is that it will spur investigations that may lead to specific conjectures and perhaps theorems. We mention just a couple of examples below.

EXAMPLE 1. Spectral gaps are known to occur for certain fractals, and we prove their existence for HH(b) in the next section. There are other fractals, such as the pentagasket, for which there is experimental evidence for spectral gaps [1] but, as yet, no proof, and there are other fractals, such as the Sierpinski carpet, where the experimental evidence points to nonexistence of spectral gaps [4]. Is it possible to relate the existence of spectral gaps to "geometric gaps" in the fractal structure?

EXAMPLE 2. For domains in Euclidean space (or more generally Riemannian manifolds) with fractals boundary, there is a relationship between the Minkowski dimension of the boundary and the remainder term in the Weyl ratio for the Dirichlet or Neumann Laplacian due to Lapidus [14]. This involves a global statement about the fractal nature of the boundary across all scales on the one hand, and a global statement about the remainder in the spectral asymptotics on the other hand. Is there a refinement of this result

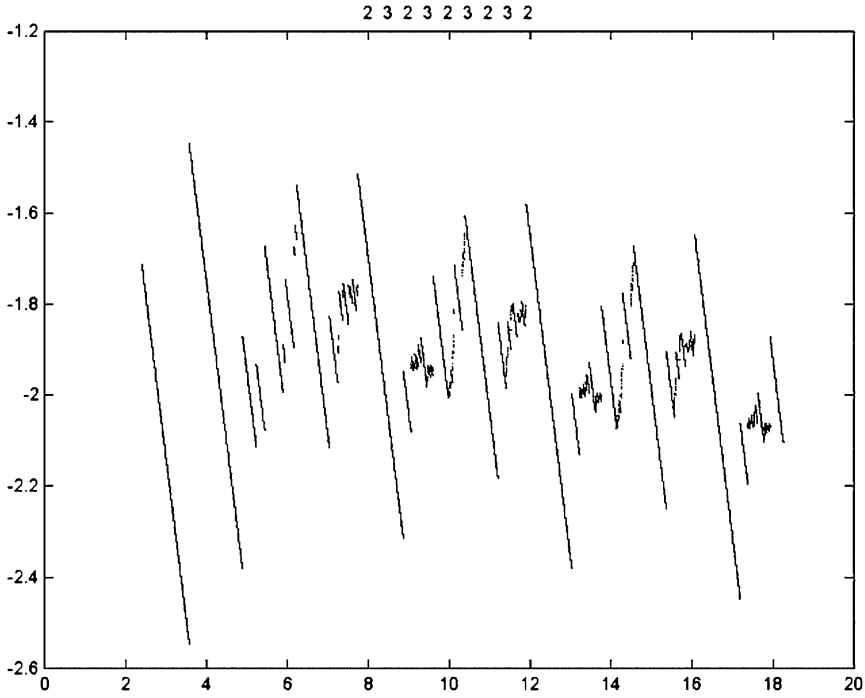

FIGURE 13. A graph of the Weyl ratio when $b = (2, 3, 2, 3, \ldots)$.

that relates the fractal nature of the boundary on a specified scale with the remainder in the Weyl ratio on a related segment of the spectrum?

5. Spectral gaps

Now we look at gaps in the spectrum of the Laplacian.

DEFINITION 5.1. We define a *c-gap* for any constant $c > 1$ to be a consecutive pair of eigenvalues λ, λ' such that $\frac{\lambda'}{\lambda} \geq c$.

We will prove a theorem about the existence of *c-gaps* for $c > 2$. First, we need the following lemma.

LEMMA 5.1. *The function $g_m(x)/x$ is an increasing function. Thus, $g_m(x)$ magnifies gaps.*

Proof. Let $f_n(x) = R^{-1}_{b_{m+n}1} \circ \cdots \circ R^{-1}_{b_{m+1}1}(x)$. It is clear from the definition of $g_m(x)$ that it is enough to prove that $f_n(x)/x$ is increasing for all n. For $n = 1$, this reduces to showing

$$R_{b_{m+1}}(f_n(x)) > R'_{b_{m+1}}(f_n(x))f_n(x),$$

which is easily verified. The result now follows from induction and the fact that if $\alpha(x), \beta(x)$ are differentiable functions with $\alpha(x)/x$ and $\beta(x)/x$ increasing, then $\alpha \circ \beta(x)/x$ is increasing. □

THEOREM 5.2. *For any choice of b, there exist infinitely many c-gaps in the Dirichlet spectrum of* HH(b). *Furthermore, we may take $c > 2$. That is, we can find infinitely many pairs of consecutive eigenvalues λ, λ' with $\frac{\lambda'}{\lambda} \geq c$.*

Proof. First, we show that there is such a gap in the spectrum of Δ_m. For $b_m = 2$, it is easy to see from spectral decimation that there will be a gap (with $c > 2$) between $R_{21}^{-1}(3+\sqrt{5})$ and $R_{22}^{-1}(6)$. Similarly, for $b_m = 3$, there will be a c-gap (with $c > 2$) between $R_{31}^{-1}(6)$ and $R_{32}^{-1}(6)$. The result now follows from the lemma. □

THEOREM 5.3 (Computer assisted proof). *There exist gaps in the ratios of eigenvalues from the Dirichlet spectrum of SG_3.*

Proof. The computer program used in the proof of this theorem and the next one can be found at www.math.cornell.edu/~sld32/FractalAnalysis.html.

For every eigenvalue λ of Δ, there is some smallest m so that only the function R_{31}^{-1} is applied after the mth step of the spectral decimation algorithm. We will say that λ comes from step m if this is the case. We will show that there are gaps in the ratios of all eigenvalues coming from the same or consectutive steps. If λ and λ' come from (say) level m and level $m+2$ respectively, then λ_{m+2} is at most $R_{b_{m+1}1}^{-1}(R_{b_{m+2}1}^{-1}(6))$ and λ'_{m+2} is at least $3-\sqrt{2}$. Since g_{m+2} magnifies gaps, it is now easy to verify that all gaps in the ratio of eigenvalues not coming from the same or consecutive levels are sufficiently large (or small) so as not to interfere with the gaps we find.

First, we construct a covering U_m of the spectrum of Δ_m (minus the eigenvalue 6). For Δ_1, the interval $[0, 3+\sqrt{5}]$ provides a cover. Now assume we have a cover consisting of a finite union of closed intervals for Δ_m. For each interval I in U_m, we add the interval $R_{3i}^{-1}(I)$ for $i = 1, 2, 3, 4$ to U_{m+1}. We also add the points $3, 5$ and $3 \pm \sqrt{2}$. Since $R_{3i}^{-1}[0, 3+\sqrt{5}] \subset [0, 3+\sqrt{5}]$ for $i = 1, 2, 3, 4$, we have that $U_1 \supset U_2 \supset \cdots$.

Now we can compute a cover R_m for the ratios of all eigenvalues coming from steps $m-1$ and m. To each U_n, we let V_n consist of all the intervals not obtained by applying R_{31}^{-1} to an interval of U_{n-1}. Let $W_m = V_m \cup R_{31}^{-1}(V_{m-1})$. For each pair of intervals $[a, b]$ and $[c, d]$ in W_m with $d > a$, we add the interval $[\frac{g_m(c)}{g_m(b)}, \frac{g_m(d)}{g_m(a)}]$ to R_m. The set R_m will contain all possible ratios of eigenvalues coming from steps $m-1$ and m. We know that for any m_1, m_2, the functions $g_{m_1}(x)$ and $g_{m_2}(x)$ are the same. Since we also know $U_1 \supset U_2 \supset \cdots$, it follows that $R_1 \supset R_2 \supset \cdots$. Therefore, for any $m' > m$, R_m will contain all possible ratios of eigenvalues coming from steps $m'-1$ and m'. Since there are only

TABLE 2. Gaps in the ratios of eigenvalues from the Dirichlet spectrum of SG_3. The largest gap is $(4.34572567, 4.35469114)$ of length 0.01707184

(2.93907568, 2.94842483)	(2.95248083, 2.96642935)
(3.01181320, 3.02032781)	(3.02032781, 3.02359946)
(3.09070006, 3.09456100)	(3.72506327, 3.72517052)
(3.72517052, 3.72665657)	(3.72665657, 3.72711091)
(4.15475473, 4.15887569)	(4.28486874, 4.29647225)
(4.34572567, 4.35469114)	(4.37775352, 4.37846477)
(4.38966354, 4.39183030)	(4.40998448, 4.41087665)

finitely many eigenvalues coming from steps prior to step m, it follows that R_m union a finite set forms a cover for all ratios of eigenvalues coming from consecutive steps.

Implementing this algorithm on a computer with $m = 4$, we find that there actually exist gaps in the ratios of eigenvalues (see Table 2). It is possible that carrying out the algorithm with a larger m would reveal more gaps, but most likely those gaps would be smaller than the ones we found. □

Now we adapt this algorithm so it can be applied to HH(b) for more general choices of b. As with the case of SG_3, it is enough to compute all ratios of eigenvalues coming from consecutive levels. First for an $m \in \mathbb{N}$, we want to construct a suitable cover for the spectrum of Δ_m. Fix an $n \in \mathbb{N}$. For an arbitrary m, we do not know in general the values of $b_{m-n+1}, \ldots, b_m$, however we do know there are only 2^n possibilities. For each of these 2^n possibilities, we can compute as set of intervals $W_{m,i}$ analogous to the set W_m in the proof of Theorem 5.3. Now for each i we want to compute all ratios of eigenvalues coming from the set of interval $W_{m,i}$. The ratios we get will depend on the sequence $b_{m+1}, b_{m+2}, \ldots$. In theory this means that for each i we have to consider an infinite number of possibilities. In practice the approximate eigenvalues converge fast enough so that we only have to know $b_{m+1}, \ldots, b_{m+k}$ for k sufficiently large. So for each $W_{m,i}$ we get 2^k coverings of all ratios of eigenvalues. In total this gives us 2^{k+n} coverings $R_{m,1}, \ldots, R_{m,2^{k+n}}$. Now any interval in the complement of $\bigcup_{i=1}^{2^{k+n}} R_{m,i}$ will be a gap in ratios of eigenvalues of the Laplacian associated to HH(b). Unfortunately this approach does not yield any gaps in the ratio of eigenvalues from the Dirichlet spectrum of an arbitrary HH(b). However, if (b_{2m+1}, b_{2m+2}) is either $(2,3)$ or $(3,2)$ for all $m \geq 0$, then this method does yield gaps in the ratio of eigenvalues from the Dirichlet spectrum of HH(b) [see Table 3].

TABLE 3. Gaps in the ratios of eigenvalues from the Dirichlet spectrum for those HH(b) described in Theorem 5.4. The largest gap is $(1.99513995, 1.99595190)$ of length 0.00081194

(1.94662032, 1.94673709)	(1.95519566, 1.95526996)
(1.95526996, 1.95527236)	(1.95527236, 1.95529896)
(1.95529896, 1.95530136)	(1.95530136, 1.95540053)
(1.95873916, 1.95880628)	(1.96461768, 1.96462662)
(1.97676087, 1.97679670)	(1.97679670, 1.97679854)
(1.97679854, 1.97679863)	(1.97679863, 1.97680047)
(1.97680047, 1.97680419)	(1.99485083, 1.99496915)
(1.99496915, 1.99497172)	(1.99497172, 1.99513738)
(1.99513738, 1.99513995)	(1.99513995, 1.99595190)
(2.00719160, 2.00749815)	(2.01347351, 2.01360832)

THEOREM 5.4 (Computer assisted proof). *Let* $b = (b_1, b_2, \ldots)$ *and suppose* (b_{2m+1}, b_{2m+2}) *is either* $(2,3)$ *or* $(3,2)$ *for all* $m \geq 0$. *Then there are gaps in the ratios of eigenvalues coming from the Dirichlet spectrum of* HH(b).

REFERENCES

[1] B. Adams, S. A. Smith, R. S. Strichartz and A. Teplyaev, *The spectrum of the Laplacian on the pentagasket*, Fractals in Graz 2001, Trends Math., Birkhäuser, Basel, 2003, pp. 1–24. MR 2091699

[2] N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst and A. Teplyaev, *Vibration modes of 3n-gaskets and other fractals*, J. Phys. A **41** (2008), 015101. MR 2450694

[3] M. T. Barlow, *Diffusions on fractals*, Lectures on probability theory and statistics (Saint-Flour, 1995), Lecture Notes in Math., vol. 1690, Springer, Berlin, 1998, pp. 1–121. MR 1668115

[4] T. Berry, S. Heilman and R. Strichartz, *Outer approximation of the spectrum of a fractal Laplacian*, to appear in Experiment. Math. MR 2583544

[5] B. Bockelman and R. S. Strichartz, *Partial differential equations on products of Sierpinski gaskets*, Indiana Univ. Math. J. **56** (2007), 1361–1375. MR 2333476

[6] S. Constantin, R. Strichartz and M. Wheeler, *Analysis of the Laplacian and spectral operators on the Vicsek set*, to appear in Comm. Pure Appl. Anal.

[7] K. Dalrymple, R. S. Strichartz and J. P. Vinson, *Fractal differential equations on the Sierpinski gasket*, J. Fourier Anal. Appl. **5** (1999), 203–284. MR 1683211

[8] M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket*, Potential Anal. **1** (1992), 1–35. MR 1245223

[9] B. M. Hambly, *Brownian motion on a homogeneous random fractal*, Probab. Theory Related Fields **94** (1992), 1–38. MR 1189083

[10] B. M. Hambly, *Brownian motion on a random recursive Sierpinski gasket*, Ann. Probab. **25** (1997), 1059–1102. MR 1457612

[11] K. Hare and D. Zhou, *Gaps in the ratios of the spectrum of Laplacians on fractals*, to appear in Fractals.

[12] J. Kigami, *A harmonic calculus on the Sierpiński spaces*, Japan J. Appl. Math. **6** (1989), 259–290. MR 1001286

[13] J. Kigami, *Analysis on fractals*, Cambridge Tracts in Mathematics, vol. 143, Cambridge Univ. Press, Cambridge, 2001. MR 1840042

[14] M. L. Lapidus, *Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl–Berry conjecture*, Trans. Amer. Math. Soc. **325** (1991), 465–529. MR 0994168

[15] T. Shima, *On eigenvalue problems for the random walks on the Sierpiński pre-gaskets*, Japan J. Indust. Appl. Math. **8** (1991), 127–141. MR 1093832

[16] T. Shima, *On eigenvalue problems for Laplacians on p.c.f. self-similar sets*, Japan J. Indust. Appl. Math. **13** (1996), 1–23. MR 1377456

[17] R. S. Strichartz, *Analysis on fractals*, Notices Amer. Math. Soc. **46** (1999), 1199–1208. MR 1715511

[18] R. S. Strichartz, *Analysis on products of fractals*, Trans. Amer. Math. Soc. **357** (2005), 571–615 (electronic). MR 2095624

[19] R. S. Strichartz, *Laplacians on fractals with spectral gaps have nicer Fourier series*, Math. Res. Lett. **12** (2005), 269–274. MR 2150883

[20] R. S. Strichartz, *Differential equations on fractals: A tutorial*, Princeton Univ. Press, Princeton, NJ, 2006. MR 2246975

[21] D. Zhou, *Criteria for spectral gaps of Laplacians on fractals*, to appear in J. Fourier Anal. App.

Shawn Drenning, Math Department, Malott Hall, Cornell University, Ithaca, NY 14853, USA

Current address: Math Department, Eckhart Hall, University of Chicago, Chicago, IL 60637, USA

E-mail address: shawn@math.uchicago.edu

Robert S. Strichartz, Math Department, Malott Hall, Cornell University, Ithaca, NY 14853, USA

E-mail address: str@math.cornell.edu