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LINEAR MAPS PRESERVING REGULARITY IN C^*-ALGEBRAS

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ABSTRACT. Let A and B be unital C^*-algebras such that at least one of them is of real rank zero. We investigate surjective linear maps from A to B preserving the conorm, the (von Neumann) regularity, the generalized spectrum, and their essential versions. As a consequence, we recover results of Mbekhta, and Mbekhta and Šemrl for $\mathcal{L}(H)$ when H is an infinite-dimensional complex Hilbert space.

1. Introduction

Linear preserver problems deal with characterizations of linear maps on matrix algebras, on operator algebras or more generally on Banach algebras, that leave invariant certain functions, subsets, properties, or relations. In particular, a substantial attention has been paid to the Kaplansky's problem of characterization of linear maps preserving invertibility, [20], and also to the related question concerning spectrum preserving linear maps (see for instance [2], [3], [7], [8], [13], [16], [21] and the references therein).

In [27], Mbekhta, Rodman, and Šemrl, treated the problem of describing unital surjective linear maps on the algebra $\mathcal{L}(H)$, for an infinite-dimensional separable complex Hilbert space, preserving generalized invertibility in both directions. They showed that the ideal of all compact operators is invariant under such a map and that the induced mapping into the Calkin algebra is either an automorphism or an anti-automorphism. Later, in [28], Mbekhta and Šemrl improved this result and provided a new proof relying on the fact that

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a unital surjective linear map on $\mathcal{L}(H)$, preserving generalized invertibility in both directions, preserves semi-Fredholm operators in both directions.

Finally in [25], Mbekhta characterized surjective linear maps on $\mathcal{L}(H)$ preserving the generalized spectrum, and described those unital that preserve the reduced minimum modulus. He furthermore conjectured for the nonunital case that a surjective linear map $\varphi : \mathcal{L}(H) \to \mathcal{L}(H)$ preserves the reduced minimum modulus if and only if there are unitary operators $U, V \in \mathcal{L}(H)$ such that φ takes either the form $\varphi(T) = UTV$, $(T \in \mathcal{L}(H))$ or $\varphi(T) = UT^{tr}V$, $(T \in \mathcal{L}(H))$, where T^{tr} denotes the transpose of T with respect to an arbitrary fixed orthonormal basis of H. It is well known that these forms describe exactly surjective linear isometries on $\mathcal{L}(H)$. Thus, Mbekhta's conjecture can be rephrased by saying that a surjective linear map on $\mathcal{L}(H)$ preserves the reduced minimum modulus if and only if it is an isometry.

The aim of the present paper is to extend results of Mbekhta, [25], and Mbekhta and Šemrl, [28], by characterizing linear maps preserving generalized invertibility, the generalized spectrum and the conorm on unital C^*-algebras of real rank zero. Section 2 gathers all the preliminary results on regular, Atkinson and Fredholm elements in unital C^*-algebras needed for the rest of the paper. In Section 3, we study surjective linear maps between unital C^*-algebras, one of them having real rank zero, that preserve the generalized spectrum and the conorm. Section 4 is concerned with surjective linear maps $\varphi : A \to B$ preserving generalized invertibility in both directions from a unital C^*-algebra A of real rank zero to a unital C^*-algebra B. We perform this task in two opposite settings: when A and B are prime with nonzero socle, generalizing in this way [28, Theorem 1.1], and when A and B have zero socle. The last part of this manuscript analyzes the essential version of the results appearing in the preceding two sections.

2. Preliminaries and notation

Throughout this paper, the term Banach algebra means a unital complex associative Banach algebra with unit $\mathbf{1}$, and a C^*-algebra means a unital complex associative C^*-algebra.

Let A and B be Banach algebras. A linear map $\varphi : A \to B$ is called *unital* if $\varphi(\mathbf{1}) = \mathbf{1}$, and is said to be a *Jordan homomorphism* if $\varphi(a^2) = \varphi(a)^2$ for all $a \in A$. Equivalently, the map φ is a Jordan homomorphism if and only if $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$ for all a and b in A. It is called a *Jordan isomorphism* provided that it is a bijective Jordan homomorphism. Clearly, every homomorphism and every anti-homomorphism is a Jordan homomorphism. It is well known that if $\varphi : A \to B$ is a Jordan homomorphism, then

$$\varphi(aba) = \varphi(a)\varphi(b)\varphi(a) \tag{2.1}$$

for all $a, b \in A$. Moreover, if φ is a Jordan isomorphism, then φ strongly preserves invertibility, that is

$$\varphi(a^{-1}) = \varphi(a)^{-1} \tag{2.2}$$

for all invertible elements a in A. If A and B are C^*-algebras, then the map φ is said to be *self-adjoint* provided that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

Let A be a Banach algebra, and let A^{-1} denote the group of all invertible elements of A. For an element a in A, let $\sigma(a)$, $\partial\sigma(a)$ and $\mathrm{r}(a)$ denote the spectrum, the boundary of the spectrum and the spectral radius of a, respectively. A map Λ from A to the closed subsets of $\mathbb{C}$ is called a *∂-spectrum* if $\partial\sigma(a) \subseteq \Lambda(a) \subseteq \sigma(a)$ for all $a \in A$.

The next result shows that a surjective ∂-spectra preserving linear maps between two semisimple Banach algebras, one of them is a C^*-algebra of real rank zero, is a Jordan isomorphism. Recall that a C^*-algebra A has *real rank zero* if the set of all real linear combinations of orthogonal projections is dense in the set of all hermitian elements of A (see [9]). Notice that every von Neumann algebra and, in particular, the algebra $\mathcal{L}(H)$ of all bounded linear operators on a complex Hilbert space H, have real rank zero. Other examples of this kind of algebra include Bunce-Deddens algebras, Cuntz algebras, AF-algebras, and irrational rotation algebras (see [14]).

THEOREM 2.1 ([6]). *Let A be a C^*-algebra of real rank zero and B be a semisimple Banach algebra. Let Λ be a ∂-spectrum in A and B, and let $\varphi : A \to B$ be a surjective linear map. If $\Lambda(\varphi(a)) \subseteq \Lambda(a)$ for all $a \in A$, then φ is a continuous unital Jordan homomorphism. Moreover, if $\Lambda(\varphi(a)) = \Lambda(a)$, for all $a \in A$, then φ is a Jordan isomorphism.*

2.1. Von Neumann regularity in Banach algebras. Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on an infinite dimensional complex Banach space X. For an operator $T \in \mathcal{L}(X)$, we denote its kernel and its range by $\ker(T)$ and $\mathrm{ran}(T)$, respectively. The *reduced minimum modulus* of T is given by

$$\gamma(T) := \begin{cases} \inf\{\|T(x)\| : \operatorname{dist}(x, \ker(T)) \geq 1\}, & \text{if } T \neq 0, \\ \infty, & \text{if } T = 0. \end{cases}$$

It is well known that $\gamma(T) > 0$ if and only if $\mathrm{ran}(T)$ is closed.

Let A be a Banach algebra. An element $a \in A$ is called (*von Neumann*) *regular* if it has a generalized inverse, that is, if there exists $b \in A$ such that $a = aba$ and $b = bab$. Observe that the first equality $a = aba$ is enough to ensure that a is regular because if $a = aba$ then $b' = bab$ is a generalized inverse of a. In particular, the generalized inverse of a regular element a is not unique. In fact, if $a = aba$ and $b = bab$, for every $x \in A$, choosing $y = b + x - baxab$, we get $a = aya$, and thus yay is a generalized inverse of a.

Also note that, if a has a generalized inverse b, then $p=ab$ and $q=ba$ are idempotents in A satisfying $aA=pA$ and $Aa=Aq$.

Obviously, if a is regular, then so are the left and right multiplication operators by a, $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$. Hence, their ranges $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed.

Let us denote by A^r the set of all regular elements in A. Following [12], we say that an element $a \in A$ is *persistently regular* (or *of persistently closed range*) if all the elements in a neighborhood of a are regular. The set of persistently regular elements in A will be denoted by A^{pr}.

The *conorm* or the *reduced minimum modulus* of an element $a \in A$ is given by

$$\gamma(a) := \gamma(L_a) = \begin{cases} \inf\{\|ax\| : \operatorname{dist}(x, \ker(L_a)) \geq 1\}, & \text{if } a \neq 0, \\ \infty, & \text{if } a = 0. \end{cases}$$

If b is a generalized inverse of a, with $a \neq 0$, then

$$\|b\|^{-1} \leq \gamma(a) \leq \|ba\|\|ab\|\|b\|^{-1} \tag{2.3}$$

(see [17, Theorem 2]). Regular elements in C^*-algebras were studied by Harte and Mbekhta in [17] and [18]. They proved that a is a regular element in a C^*-algebra A if and only if aA is closed, or equivalently $\gamma(a) > 0$, and that

$$\gamma(a)^2 = \gamma(a^*a) = \inf\{\lambda : \lambda \in \sigma(a^*a) \setminus \{0\}\} = \gamma(a^*)^2. \tag{2.4}$$

Furthermore, they showed that if a is a regular element, then

$$\gamma(a) = \|a^\dagger\|^{-1}, \tag{2.5}$$

where $a^\dagger$ is the *Moore–Penrose inverse* of a, that is, the unique element $b \in A$ for which $a = aba$, $b = bab$ and the associated idempotents ab and ba are self-adjoint (see [18, Theorem 2]).

For an element a in a Banach algebra A, denote by $\operatorname{reg}(a)$ the *regular set* of a, that is, the set of all $\lambda \in \mathbb{C}$ such that there exists a neighborhood $\mathcal{U}_\lambda$ of λ, and an analytic function $b : \mathcal{U}_\lambda \to A$, such that $b(\mu)$ is a generalized inverse of $a - \mu$ for any $\mu \in \mathcal{U}_\lambda$. The *generalized spectrum* (also called *Saphar spectrum*) of a is given by $\sigma_g(a) := \mathbb{C} \setminus \operatorname{reg}(a)$, and the *Kato spectrum* of a is defined as

$$\sigma_K(a) := \Big\{\lambda \in \mathbb{C} : \lim_{\mu \to \lambda} \gamma(a - \mu) = 0\Big\}.$$

The following properties of the generalized spectrum and the Kato spectrum are well known (see [24], [26] and [29, Sections 12 and 13]):

(1) $0 \notin \sigma_g(a)$ if and only if a is regular and $\ker(L_a) \subseteq \bigcap_{n\geq 1} a^n A$.
(2) $0 \notin \sigma_K(a)$ if and only if aA is closed and $\ker(L_a) \subseteq \bigcap_{n\geq 1} a^n A$.
(3) $\partial\sigma(a) \subseteq \sigma_K(a) \subseteq \sigma_g(a) \subseteq \sigma(a)$.
(4) If A is a C^*-algebra, then $\sigma_g(a^*) = \overline{\sigma_g(a)}$, and $\sigma_g(a) = \sigma_K(a)$ for all $a \in A$.

2.2. Fredholm theory. Let A be a C^*-algebra. An element x of A is said to be *finite*, respectively *compact*, in A, if the wedge operator $x \wedge x : A \to A$, given by $x \wedge x(a) = xax$ $(a \in A)$, is a finite rank, respectively compact, operator on A. It is known that the ideal $\mathcal{F}(A)$ of finite rank elements in A coincides with the *socle* of A, $\mathrm{soc}(A)$, that is, the sum of all minimal right (equivalently left) ideals of A, and that $\mathcal{K}(A) = \overline{\mathrm{soc}(A)}$ is the ideal of all compact elements in A (see [4]). We call $\mathcal{C}(A) := \frac{A}{\mathcal{K}(A)}$ the *generalized Calkin algebra* of A.

An element $a \in A$ is called *Fredholm* if it is invertible modulo $\mathcal{F}(A)$ and is called *Atkinson* if it is left or right invertible modulo $\mathcal{F}(A)$. Denote by $\Phi(A)$ and $\mathcal{A}(A)$ the set of Fredholm and Atkinson elements in A, respectively.

It is known that (left, right) invertibility modulo $\mathcal{F}(A)$ is equivalent to (left, right) invertibility modulo $\mathcal{K}(A)$. As the latter is a closed ideal of A, $\Phi(A)$ and $\mathcal{A}(A)$ are open multiplicative semigroups of A that are stable under compact perturbations. Note that for a complex Hilbert space H, $\mathcal{F}(\mathcal{L}(H)) = \mathcal{F}(H)$ is the ideal of all finite rank operators on H, $\mathcal{K}(\mathcal{L}(H)) = \mathcal{K}(H)$ is the closed ideal of all compact operators on H, $\Phi(\mathcal{L}(H)) = \Phi(H)$ is the set of Fredholm operators on H, and $\mathcal{A}(\mathcal{L}(H)) = \mathcal{SF}(H)$ is the set of semi-Fredholm operators on H.

Finally, let us recall that if A is a primitive C^*-algebra with nonzero socle and e is a minimal projection in A, the minimal left ideal Ae can be endowed with an inner product given by $\langle x, y\rangle e = y^*x$ for all $x, y \in Ae$, under which Ae becomes a Hilbert space in the algebra norm and the left regular representation $\rho \colon A \to \mathcal{L}(Ae)$, defined as $\rho(a)(x) = ax$ for all $x \in Ae$, is an isometric irreducible $*$-representation, satisfying

$$\rho(\mathcal{F}(A)) = \mathcal{F}(Ae), \tag{2.6}$$

$$\rho(\mathcal{K}(A)) = \mathcal{K}(Ae), \tag{2.7}$$

$$\rho(\Phi(A)) = \Phi(Ae) \cap \rho(A), \tag{2.8}$$

$$\rho((A)) = \mathcal{SF}(Ae) \cap \rho(A). \tag{2.9}$$

(See [4] and [32, Theorem 7.2]). Atkinson and Fredholm theory in general Banach algebras is developed in [4], [22], and [32].

3. Linear maps preserving the reduced minimum modulus

In this section, we consider the extension of Mbekhta's conjecture to the setting of C^*-algebras and we provide a positive answer under some additional conditions. The arguments of the proofs are inspired in the arguments used in [6].

CONJECTURE 3.1. *Let A and B be C^*-algebras, and let φ be a linear map from A onto B. The following statements are equivalent.*

(i) *The equality $\gamma(\varphi(a)) = \gamma(a)$ holds for all $a \in A$.*

(ii) *The map φ is an isometry.*

In his celebrated paper [19], Kadison proved that a surjective linear map between two C^*-algebras A and B is an isometry if and only it is a selfadjoint Jordan isomorphism multiplied by a unitary element in B. This together with the following lemma show that the implication (ii) $\Rightarrow$ (i) always holds. Notice that, if u is a unitary element in B, then $\gamma(ux) = \gamma(x)$ for all $x \in B$.

LEMMA 3.2. *Let A and B be two C^*-algebras. If $\varphi : A \to B$ is a Jordan isomorphism, then*

$$\|\varphi\|^{-1}\gamma(a) \leq \gamma(\varphi(a)) \leq \|\varphi^{-1}\|\gamma(a) \tag{3.1}$$

for all $a \in A$.

Proof. Since the first inequality of (3.1) applied to φ^{-1} yields the second, we only need to show that $\gamma(a) \leq \|\varphi\|\gamma(\varphi(a))$ for all $a \in A$.

By (2.1), we know that an element $x \in A$ has a generalized inverse y if and only if $\varphi(y)$ is a generalized inverse of $\varphi(x)$. This implies, in particular, that for an element $x \in A$, $\gamma(x) = 0$ if and only if $\gamma(\varphi(x)) = 0$. Hence, we only need to prove that the above inequality holds for all regular elements $a \in A$. So take a regular element $a \in A$, and note that $\varphi(a^\dagger)$ is a generalized inverse of $\varphi(a)$. By the left inequality of (2.3), we have

$$\gamma(\varphi(a))^{-1} \leq \|\varphi(a^\dagger)\| \leq \|\varphi\|\|a^\dagger\| = \|\varphi\|\gamma(a)^{-1}.$$

Thus, $\gamma(a) \leq \|\varphi\|\gamma(\varphi(a))$; as desired. □

Before stating the main results of this section, we need to recall some concepts from nonassociative algebras. Following [34], a *Jordan algebra* J is a commutative algebra satisfying the *Jordan identity* $(xy)x^2 = x(yx^2)$ for all $x,\ y \in J$. For an element x in a Jordan algebra J, denote by U_x the mapping given by $U_x(y) := 2x(xy) - x^2y$ for all $y \in J$. If A is an associative algebra, then the algebra A^+, consisting on the underlying vector space of A and the product

$$x \circ y := \frac{1}{2}(xy + yx) \quad (x, y \in A),$$

becomes a Jordan algebra. Clearly, a linear map $\varphi : A \to B$ between Banach algebras is a Jordan homomorphism if and only if $\varphi : A^+ \to B^+$ is a homomorphism.

By a *JB^*-algebra* we mean a complete normed complex Jordan algebra, J, endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every $x \in J$ (see [31], [35], [36]). It is easy to prove that, if A is a C^*-algebra, then A^+, with the norm and involution of A, becomes a JB^*-algebra. In [31, Theorem 2], Rodríguez showed that the converse is also true, that is, if A is an associative complex algebra such that A^+ is a JB^*-algebra for some norm and involution, then A with the same norm and involution is a C^*-algebra.

THEOREM 3.3. *Let A be a C^*-algebra with real rank zero and B be a semi-simple Banach algebra. For a unital surjective linear map $\varphi\colon A\to B$, the following statements are equivalent.*

(i) $\gamma(\varphi(x))=\gamma(x)$ *for all* $x\in A$.

(ii) $\gamma(x)\le\gamma(\varphi(x))$ *for all* $x\in A$, *and* φ *is injective.*

(iii) B (*with its norm and some involution*) *is a* C^*-*algebra, and* φ *is an isometric Jordan isomorphism.*

Proof. It is clear that (iii) ⇒ (i).

If $\gamma(\varphi(x))=\gamma(x)$ for all $x\in\mathcal{A}$, then $\sigma_K(\varphi(x))=\sigma_K(x)$ for all $x\in A$. From Theorem 2.1 we deduce, in particular, that φ is injective. This shows that (i) ⇒ (ii).

Assume that $\gamma(x)\le\gamma(\varphi(x))$ for all $x\in A$, and that φ is injective. It follows that $\sigma_K(\varphi(x))\subseteq\sigma_K(x)$ for all $x\in A$. By Theorem 2.1, the mapping φ is a continuous Jordan isomorphism, and by (2.2) and (2.5), $\|\varphi(x)\|\le\|x\|$ for all invertible elements $x\in A$. This together with [33, Corollary 1] proves that $\|\varphi\|=1$, and that $\|\varphi(x)\|\le\|x\|$ for all $x\in A$. Now, as the mapping $x\mapsto\|\varphi(x)\|$ is an algebra norm on the JB^*-algebra A^+, and every JB^*-algebra has minimality of the norm (see [30, Proposition 11]), we deduce that φ is in fact isometric. Since φ is a Jordan isomorphism,

$$\varphi\big((x\circ y)^*\big)=\varphi(x^*\circ y^*)=\varphi(x^*)\circ\varphi(y^*)$$

holds for all $x,y\in A$. By (2.1)

$$\begin{aligned}\big\|U_{\varphi(x)}(\varphi(x^*))\big\|&=\|\varphi(x)\varphi(x^*)\varphi(x)\|=\|\varphi(xx^*x)\|\\&=\|xx^*x\|=\|x\|^3=\|\varphi(x)\|^3\end{aligned}$$

for all $x\in A$. Hence, the mapping $\varphi(x)\mapsto\varphi(x^*)$ defines a JB^*-involution in B^+. By [31, Theorem 2], B with its norm and this involution is a C^*-algebra. Clearly, $\varphi: A\to B$ is selfadjoint, and the implication (ii) ⇒ (iii) holds. □

One can switch the conditions on A and B in the last theorem. However, the resulting assertion is slightly different mainly because the injectivity is already implicit in (ii).

THEOREM 3.4. *Let B be a C^*-algebra with real rank zero and A be a semi-simple Banach algebra. For a unital surjective linear map φ from A to B, the following statements are equivalent.*

(i) $\gamma(\varphi(x))=\gamma(x)$ *for all* $x\in A$.

(ii) $\gamma(\varphi(x))\le\gamma(x)$ *for all* $x\in A$.

(iii) A (*with its norm and some involution*) *is a* C^*-*algebra, and* φ *is an isometric Jordan isomorphism.*

Proof. It is clear that (iii) ⇒ (i) ⇒ (ii). Suppose that $\gamma(\varphi(x))\le\gamma(x)$ for all $x\in A$. We have $\sigma_K(x)\subseteq\sigma_K(\varphi(x))$ for all $x\in A$. As the Kato spectrum is a ∂-spectrum, it follows that $\mathrm{r}(x) \leq \mathrm{r}(\varphi(x))$, for all $x \in A$. Following the arguments in [6], we show next that φ is injective. Let $a_0 \in A$ be such that $\varphi(a_0) = 0$, and pick $a \in A$. For every $\lambda \in \mathbb{C}$, we have

$$\mathrm{r}(\lambda a_0 + a) \leq \mathrm{r}\big(\varphi(\lambda a_0 + a)\big) = \mathrm{r}(\varphi(a)).$$

As $\lambda \mapsto \mathrm{r}(\lambda a_0 + a)$ is a subharmonic function on $\mathbb{C}$, Liouville's theorem implies that $\mathrm{r}(\lambda a_0 + a) = \mathrm{r}(a)$ for all $\lambda \in \mathbb{C}$. Because a is an arbitrary element of A, the spectral characterization of the radical, together with the semisimplicity of A imply that $a_0 = 0$, and hence φ is injective. The proof now concludes by applying (ii) $\Rightarrow$ (iii) in the preceding theorem to the mapping φ^{-1}. $\square$

A similar proof to the one of Theorems 3.3 and 3.4 yields the next result that generalizes [25, Theorem 3.1]. The details are left to the reader.

THEOREM 3.5. *Let A and B be two C^*-algebras such that at least one of them is of real rank zero. For a unital surjective linear map $\varphi : A \to B$, the following conditions are equivalent.*

(i) *There exists $\alpha > 0$ such that $\alpha\gamma(x) \leq \gamma(\varphi(x))$ for all $x \in A$, and φ is injective.*
(ii) *There exists $\beta > 0$ such that $\gamma(\varphi(x)) \leq \beta\gamma(x)$ for all $x \in A$.*
(iii) *There exist $\alpha, \beta > 0$ such that $\alpha\gamma(x) \leq \gamma(\varphi(x)) \leq \beta\gamma(x)$ for all $x \in A$.*
(iv) *$\sigma_g(\varphi(x)) = \sigma_g(x)$ for all $x \in A$.*
(v) *φ is a continuous Jordan isomorphism.*

We close this section with a characterization of inner maps that preserve the reduced minimum modulus.

THEOREM 3.6. *Let A be a C^*-algebra, and let $a, b \in A^{-1}$. If φ is the map that takes either the form $\varphi(x) := axb$ for all $x \in A$, or $\varphi(x) := ax^*b$ for all $x \in A$, then the following statements are equivalent.*

(i) *The element ab is unitary, and $|a|$ is central in A.*
(ii) *The map φ is an isometry.*
(iii) *The equality $\gamma(x) = \gamma(\varphi(x))$ holds for all $x \in A$.*

Proof. We first treat the case $\varphi(x) = axb$, for all $x \in A$.

Assume that ab is a unitary element and that $|a|$ is central in A. For every $x \in A$, we have

$$\|\varphi(x)\| = \|axa^{-1}\| = \||a|x|a|^{-1}\| = \|x\|.$$

Hence, φ is an isometry, and (i) $\Rightarrow$ (ii) holds.

The implication (ii) $\Rightarrow$ (iii) follows directly from Lemma 3.2.

Now, suppose that $\gamma(x) = \gamma(\varphi(x))$ for all $x \in A$. For every $x \in A^{-1}$, we have

$$\|b^{-1}x^{-1}a^{-1}\|^{-1} = \|\varphi(x)^{-1}\|^{-1} = \gamma(\varphi(x)) = \gamma(x) = \|x^{-1}\|^{-1}.$$

This shows that the map $\psi : A \to A$, defined by $\psi(x) := b^{-1}xa^{-1}$, is an isometry on the group of invertible elements of A. Again [33, Corollary 1] shows that ψ is an isometry on A. Thus, $\psi(\mathbf{1}) = b^{-1}a^{-1}$ is unitary (and hence ab is unitary as well), and the linear map $\phi := ab\psi$, is an isometry. In particular, ϕ is selfadjoint, and thus

$$axa^{-1} = \phi(x) = \phi(x^*)^* = a^{*-1}xa^*$$

for all $x \in A$. This proves that $|a|^2x = x|a|^2$ for all $x \in A$. As $|a|$ can be approximated by polynomials in $|a|^2$, it follows that $|a|x = x|a|$ for all $x \in A$. Therefore, (iii) $\Rightarrow$ (i) holds.

Finally, assume that φ is given by $\varphi(x) = ax^*b$ for all $x \in A$. The first case just proved applied to the linear map $\chi : A \to A$ defined by $\chi(x) := \varphi(x)^* = b^*xa^*$, together with (2.4), give the desired conclusion. □

As an immediate consequence of the previous theorem, we obtain the following characterization of unitary elements in C^*-algebras with trivial center.

COROLLARY 3.7. *Let A be a C^*-algebra with trivial center, and let $a \in A^{-1}$. The following statements are equivalent.*

(i) *The element a is a scalar multiple of a unitary element of A.*
(ii) *The identity $\gamma(x) = \gamma(axa^{-1})$ holds for all $x \in A$.*

4. Linear maps preserving regularity

Let A and B be C^*-algebras. A linear map $\varphi : A \to B$ *preserves regularity* if $\varphi(a) \in B^r$ whenever $a \in A^r$. We say that $\varphi : A \to B$ *preserves regularity in both directions* if $a \in A^r$ if and only if $\varphi(a) \in B^r$. The map φ is called *surjective up to finite rank* (respectively, *compact*) *elements* if $B = \varphi(A) + \mathcal{F}(B)$ (respectively, $B = \varphi(A) + \mathcal{K}(B)$).

The maps considered in the previous section preserve regularity in one or both directions, and satisfy some additional conditions on norm preserving. In this section, we study those surjective up to finite rank elements linear maps, between C^*-algebras of real rank zero, that preserve regularity in one or both directions. Notice that, if A and B are C^*-algebras with nonzero socle, and $\psi : A \to \mathcal{F}(B)$ is a linear map, every linear map $\varphi : A \to B$ preserves generalized invertibility in both directions if and only if $\varphi + \psi$ does. We show in the next result that when A and B are prime, and A is of real rank zero, every surjective up to finite rank elements linear map $\varphi : A \to B$ preserving generalized invertibility in both directions factorizes as a Jordan homomorphism through the generalized Calkin algebras.

THEOREM 4.1. *Let A and B be prime C^*-algebras with nonzero socle such that at least one of them has real rank zero. Let $\varphi : A \to B$ be a surjective up to finite rank elements linear map. If φ preserves regularity in both directions, then $\varphi(\mathcal{F}(A)) \subseteq \mathcal{F}(B)$, $\varphi(\mathcal{K}(A)) \subseteq \mathcal{K}(B)$ and the induced mapping $\hat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$, defined by $\hat{\varphi}(a + \mathcal{K}(A)) = \varphi(a) + \mathcal{K}(B)$, is a Jordan isomorphism multiplied by an invertible element in $\mathcal{C}(B)$.*

The proof of this result uses some auxiliary lemmas. The first of them provides an interesting characterization of persistently regular elements in a prime C^*-algebra with nonzero socle.

LEMMA 4.2. *Let A be a prime C^*-algebra with nonzero socle. For an element $a \in A$, the following assertions are equivalent.*

(i) *a is persistently regular.*
(ii) *For every $b \in A$, there exists $\delta > 0$ such that $a + \lambda b \in A^r$ for every $\lambda \in \mathbb{C}$, satisfying $|\lambda| < \delta$.*
(iii) *a is Atkinson.*
(iv) *$a + k \in A^r$ for all $k \in \mathcal{K}(A)$.*

Proof. It is obvious that (i) $\Rightarrow$ (ii). Assume that the second statement holds, and let us show that a is Atkinson. For $b = 0$, the hypothesis implies that $a \in A^r$. Now, suppose to the contrary that a is not Atkinson. Notice that, since A is a prime C^*-algebra with nonzero socle, A is primitive (see [23]). Let e be a minimal projection in A and $\rho : A \to Ae$ the left regular representation introduced in Section 2.2. By (2.9), $\rho(a)$ is regular and not semi-Fredholm on Ae. In view of [15, Theorem V.2.6], there exists $K \in \mathcal{K}(Ae)$ such that $\rho(a) + \lambda K \notin \mathcal{L}(Ae)^r$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. By (2.6), there exists $k \in \mathcal{K}(A)$ ($\rho(k) = K$) such that $a + \lambda k \notin A^r$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. This contradicts the second statement and establishes (ii) $\Rightarrow$ (i).

The implication (iii) $\Rightarrow$ (i) also holds trivially taking into account that the set $\mathcal{A}(A)$ of Atkinson elements is an open semigroup contained in A^r. Moreover, if a is Atkinson and k is compact, it is clear that $a + k$ is also Atkinson and, in particular, has a generalized inverse. Hence, we get that (iii) $\Rightarrow$ (iv).

Finally, assume that fourth statement holds, and note that $a + \lambda k \in A^r$ for all $k \in \mathcal{K}(A)$ and all $\lambda \in \mathbb{C}$. The same argument used to prove (ii) $\Rightarrow$ (iii) shows that a is necessarily Atkinson, and hence the implication (iv) $\Rightarrow$ (iii) holds. □

By particularizing [5, Lemma 3.2] to the setting of C^*-algebras, we obtain the following result.

LEMMA 4.3. *If A is a C^*-algebra with nonzero socle, then*

$$\mathcal{K}(A) = \{a \in \mathcal{A} : a + \mathcal{A}(A) \subseteq \mathcal{A}(A)\}.$$

It is well known that every element in the socle of a semisimple Banach algebra has generalized inverse. In the next result, we show that for a prime C^*-algebra its socle is exactly the perturbation class of its set of regular elements.

LEMMA 4.4. *If A is a prime C^*-algebra with nonzero socle, then*

$$\mathcal{F}(A) = \{a \in A : a + A^r \subseteq A^r\}.$$

Proof. First, let $a \in A$ and note that if $aba - a$ is regular, for some b in A, then a is itself regular. Indeed, if $(aba-a)x(aba-a) = aba-a$, then $a = a(b+(1-ba)x(1-ab))a$, which shows that a is regular. Thus, for $a \in \mathcal{F}(A)$ and $x \in A^r$ with generalized inverse y, the element $(a+x)-(a+x)y(a+x)$ lies in $\mathcal{F}(A)$, and hence it is regular. Therefore, $a+x$ is also regular. Conversely, we need to prove that every element $a \in A$ satisfying $a + A^r \subseteq A^r$ is of finite rank. As such an element is regular, it suffices to show that it is compact (see [22, Theorem 6]), or equivalently, by the preceding lemma, that $a+y \in \mathcal{A}(A)$, for every $y \in \mathcal{A}(A)$. Given $y \in \mathcal{A}(A)$ and $z \in \mathcal{K}(A)$, from (iii) $\Rightarrow$ (iv) of Lemma 4.2, $y+z \in A^r$, and thus, by hypothesis, $a+y+z \in A^r$. Finally, (iv) $\Rightarrow$ (iii) of Lemma 4.2, shows that $a+y \in \mathcal{A}(A)$, as desired. □

Proof of Theorem 4.1. Lemma 4.4 shows that $\varphi(\mathcal{F}(A)) \subseteq \mathcal{F}(B)$. From Lemmas 4.2 and 4.4 it follows that φ preserves the set of Atkinson elements in both directions, that is, $a \in A$ is Atkinson if and only if $\varphi(a)$ so is. Now, [5, Corollary 3.5] entails the desired conclusion. □

Remark 4.5. By considering $A = \mathcal{L}(H)$ and $B = \mathcal{L}(K)$, with H and K infinite dimensional complex Hilbert spaces, from Theorem 4.1, we obtain [28, Theorem 1.1] and [27, Theorem 3.1]. Observe that, in this last paper, we can replace the separability of the Hilbert space by the assumption that the map is unital (see [28, Theorem 4.1]).

In view of Theorem 4.1, one might wonder what happens if the algebras A and B have zero socle. We focus now on this problem. First, let us recall some additional concepts and results.

Let A be a C^*-algebra. Let B_A denote the closed unit ball of A and $\mathcal{E}(A)$ the set of extreme points of B_A, that is, the partial isometries $v \in A$ such that $(\mathbf{1}-v^*v)A(\mathbf{1}-vv^*) = \{0\}$. In [10], [11], the elements belonging to the open set $A_q^{-1} = A^{-1}\mathcal{E}(A)A^{-1}$ are called *quasi-invertible*. Note that in the particular case where A is a prime C^*-algebra, A_q^{-1} coincides with the set of one-sided invertible elements.

For an element $a \in A$, the set

$$\sigma_q(a) = \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \text{ is not quasi-invertible}\}$$

is the *quasi-spectrum* of a (see [11]).

Every quasi-invertible element is persistently regular and moreover from [12, Theorem 7.7] it follows that $A^{pr} = A_q^{-1} + \mathcal{F}(A)$. In particular, if $\mathcal{F}(A) = \{0\}$ we get that the only persistently regular elements in A are the quasi-invertible elements.

Theorem 4.6. *Let A and B be C^*-algebras. Suppose that A has real rank zero and that $\mathcal{F}(B) = \{0\}$. Let $\varphi : A \to B$ be a unital surjective linear map. Then, φ preserves regularity if and only if φ is a continuous Jordan homomorphism.*

Proof. The sufficiency follows from (2.1). We only have to prove that if φ preserves regularity, then φ is a continuous Jordan homomorphism. So, assume that φ preserves regularity, and note that $\varphi(a)$ is persistently regular whenever a is. Moreover, since $\mathcal{F}(B)=\{0\}$, we have $B^{pr}=B_q^{-1}$, and thus $\varphi(a)$ is quasi-invertible, for all persistently regular elements $a\in A$. This shows, in particular, that $\sigma_q(\varphi(a))\subseteq\sigma_q(a)$, for all $a\in A$. From Theorem 2.1 and [11, Theorem 1.4], φ is a continuous Jordan homomorphism. □

COROLLARY 4.7. *Let A and B be C^*-algebras having zero socle. Suppose that A or B has real rank zero. Let $\varphi: A\to B$ be a unital surjective linear map. Then φ preserves regularity in both directions if and only if φ is a continuous Jordan isomorphism.*

5. Essential regularity and generalized essential spectrum

In this section, we analyze the essential version of the results appearing in the last two sections.

All C^*-algebras appearing here are supposed to have nonzero socle. Let A be a C^*-algebra, and let $\pi: A\to\mathcal{C}(A)$ be the natural quotient homomorphism. An element a in A is said to be *essentially regular* if $\pi(a)$ is regular. Note that a is essentially regular if and only if its *essential conorm*, $\gamma_e(a):=\gamma(\pi(a))$, is positive. Let A^r_{ess} denote the set of essentially regular elements of A. Observe that for $k\in\mathcal{K}(A)$, $a\in\mathcal{A}^r_{ess}$ if and only if $a+k\in\mathcal{A}^r_{ess}$. In the following lemma we show that, in fact, this property characterizes the compact elements in a (von Neumann) factor.

LEMMA 5.1. *Let A be a factor. Then*

$$\mathcal{K}(A)=\{k\in A: k+A^r_{ess}\subseteq A^r_{ess}\}.$$

Proof. It is clear that

$$\mathcal{K}(A)\subseteq\{k\in A: k+A^r_{ess}\subseteq A^r_{ess}\}.$$

Conversely, let $k\in A$ be such that $k+A^r_{ess}\subseteq A^r_{ess}$. This means that, $\pi(k)+\mathcal{C}(A)^r\subseteq\mathcal{C}(A)^r$, and thus, $\pi(k)+\mathcal{C}(A)^{pr}\subseteq\mathcal{C}(A)^{pr}$. As $\mathcal{F}(\mathcal{C}(A))=\{0\}$ (see [23, Proposition 2.3]), $\mathcal{C}(A)^{pr}=\mathcal{C}(A)_q^{-1}$. Hence,

$$\pi(k)+\mathcal{C}(A)_q^{-1}\subseteq\mathcal{C}(A)_q^{-1},$$

which shows that

$$\sigma_q\big(\pi(k)+\pi(x)\big)\subseteq\sigma_q(\pi(x))$$

for all $x\in A$. Finally, as $\sigma_q(.)$ is a ∂-spectrum, for every quasinilpotent element $\pi(x)\in\mathcal{C}(A)$, we have

$$\partial\sigma\big(\pi(k)+\pi(x)\big)\subseteq\sigma_q\big(\pi(k)+\pi(x)\big)\subseteq\sigma_q(\pi(x))\subseteq\sigma(\pi(x))=\{0\}.$$

This proves that $\sigma(\pi(k)+\pi(x))=\{0\}$, for all quasinilpotent element $\pi(x)\in\mathcal{C}(A)$. By the Zémanek spectral characterization of the radical, [1, Theorem 5.3.1], and the semisimplicity of $\mathcal{C}(A)$, it follows that $k\in\mathcal{K}(A)$. □

For A and B prime C^*-algebras, one of them of real rank zero, we deduce from Theorem 4.1 that a unital surjective up to finite rank elements linear map, $\varphi: A \to B$, preserving regularity in both directions induces a continuous Jordan isomorphism between their generalized Calkin algebras. In the next result, we shows that the same holds for surjective up to compact elements linear maps between factors preserving essential regularity in both directions.

THEOREM 5.2. *Let A and B be factors, and let $\varphi: A \to B$ be a surjective up to compact elements linear map such that $\varphi(\mathbf{1}) - \mathbf{1} \in \mathcal{K}(B)$. The map φ preserves essential regularity in both directions if and only if $\varphi(\mathcal{K}(A)) \subseteq \mathcal{K}(B)$ and the induced mapping $\hat{\varphi}: \mathcal{C}(A) \to \mathcal{C}(B)$ is a continuous Jordan isomorphism.*

Proof. The sufficiency condition is trivially true since every Jordan isomorphism strongly preserves regularity. Now assume that φ preserves essential regularity in both directions, and let us first show that $\varphi(\mathcal{K}(A)) \subseteq \mathcal{K}(B)$. Fix an arbitrary element $b \in B^r_{ess}$. As φ is surjective up to compact elements, there exist $x \in A$ and $k \in \mathcal{K}(B)$ such that $b = \varphi(x) + k$. Then $\varphi(x) = b - k \in \mathcal{B}^r_{ess}$, and thus, by hypothesis, $x \in A^r_{ess}$. Given $a \in \mathcal{K}(A)$, we have that $a + x \in \mathcal{A}^r_{ess}$. Therefore, $\varphi(a) + b = \varphi(a+x) + k \in \mathcal{B}^r_{ess}$. By the preceding lemma, $\varphi(a) \in \mathcal{K}(B)$.

The induced mapping $\hat{\varphi}: \mathcal{C}(A) \to \mathcal{C}(B)$, is a unital surjective linear map that preserves regularity in both directions. Moreover $\mathcal{C}(A)$ and $\mathcal{C}(B)$ have zero socle (because A and B are factors) and thus, by Theorem 4.7, $\hat{\varphi}$ is a continuous Jordan isomorphism. $\square$

For an element a in a C^*-algebra A, the *essential spectrum* of a, denoted by $\sigma_e(a)$, is defined as

$$\sigma_e(a) := \sigma(\pi(a)) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Fredholm}\},$$

and the *generalized essential spectrum* is given by $\sigma_{ge}(a) := \sigma_g(\pi(a))$. It is clear that $\sigma_{ge}(a) = \{\lambda \in \mathbb{C} : \lim_{\mu\to\lambda} \gamma_e(a-\mu) = 0\}$, and that $\partial\sigma_e(a) \subseteq \sigma_{ge}(a) \subseteq \sigma_e(a)$. For a surjective up to compact elements linear map $\varphi: A \to B$ between C^*-algebras A and B, let us show that if φ is σ_{ge}-preserving, then $\varphi(\mathcal{K}(A)) \subseteq \mathcal{K}(B)$. Choose $a \in \mathcal{K}(A)$ and $y \in B$. There exist $x \in A$ and $k \in \mathcal{K}(B)$ such that $y = \varphi(x) + k$. Then

$$\begin{aligned} \sigma_g\big(\pi(\varphi(a)) + \pi(y)\big) &= \sigma_{ge}\big(\varphi(a) + y\big) = \sigma_{ge}\big(\varphi(a+x)\big) \\ &= \sigma_{ge}(a+x) = \sigma_{ge}(x) = \sigma_{ge}(\varphi(x)) \\ &= \sigma_{ge}(y) = \sigma_g(\pi(y)). \end{aligned}$$

Since, $\sigma_g(\cdot)$ is a ∂-spectrum, arguing as in the proof of Lemma 5.1, we conclude that $\varphi(a) \in \mathcal{K}(B)$.

Taking into account the above comments and Theorems 3.3, 3.4, and 3.5, the following corollaries are straightforward to prove.

Corollary 5.3. *Let A and B be C^*-algebras, one of them having real rank zero. Let $\varphi : A \to B$ be a surjective up to compact elements linear map such that $\mathbf{1} - \varphi(\mathbf{1}) \in \mathcal{K}(B)$. The following conditions are equivalent.*

(i) *There exist $\alpha, \beta > 0$ such that $\alpha\gamma_e(x) \leq \gamma_e(\varphi(x)) \leq \beta\gamma_e(x)$, for all $x \in \mathcal{A}$.*
(ii) *$\sigma_{ge}(\varphi(x)) = \sigma_{ge}(x)$, for all $x \in A$.*
(iii) *$\varphi(\mathcal{K}(A)) \subseteq \varphi(\mathcal{K}(B))$, and the induced mapping $\hat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is a continuous Jordan isomorphism.*

Corollary 5.4. *Let A be C^*-algebra of real rank zero and let B be a factor. Let $\varphi : A \to B$ be a surjective up to compact elements linear map such that $\mathbf{1} - \varphi(\mathbf{1}) \in \mathcal{K}(B)$. The following conditions are equivalent.*

(i) *$\gamma_e(\varphi(a)) = \gamma_e(a)$, for all $a \in A$.*
(ii) *$\varphi(\mathcal{K}(A)) \subseteq \varphi(\mathcal{K}(B))$, and the induced mapping $\hat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is an isometric Jordan isomorphism.*

Note added in proof

The Conjecture 3.1 has been affirmatively solved in [A. Bourhim, M. Burgos and V.S. Shulman, Linear maps preserving the minimum and reduced minimum moduli, *J. Functional Analysis*, **258** (2010) 50–66].

References

[1] B. Aupetit, *A primer on spectral theory*, Springer-Verlag, New York, 1991. MR 1083349

[2] B. Aupetit, *Spectrum-preserving linear mappings between Banach algebras or Jordan–Banach algebras*, J. London Math. Soc. **62** (2000), 917–924. MR 1794294

[3] B. Aupetit and H. du T. Mouton, *Spectrum preserving linear mappings in Banach algebras*, Studia Math. **109** (1994), 91–100. MR 1267714

[4] B. A. Barnes, G. J. Murphy, M. R. F. Smyth and T. T. West, *Riesz and Fredholm theory in Banach algebras*, Research Notes in Math., vol. 67, Pitman, Boston, MA, 1982. MR 0668516

[5] M. Bendaoud, A. Bourhim, M. Burgos and M. Sarih, *Linear maps preserving Fredholm and Atkinson elements of C^*-algebras*, Linear Multilinear Algebra **57** (2009), 823–838. MR 2571856

[6] A. Bourhim and M. Burgos, *Linear maps preserving the minimum modulus*, Operators and Matrices **4** (2010), 245–256.

[7] M. Brešar and P. Šemrl, *Linear maps preserving the spectral radius*, J. Funct. Anal. **142** (1996), 360–368. MR 1423038

[8] M. Brešar and P. Šemrl, *Invertibility preserving maps preserve idempotents*, Michigan Math. J. **45** (1998), 483–488. MR 1653251

[9] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149. MR 1120918

[10] L. G. Brown and G. K. Pedersen, *On the geometry of the unit ball of a C^*-algebra*, J. Reine Angew. Math. **469** (1995), 113–147. MR 1363827

[11] L. G. Brown and G. K. Pedersen, *Approximation and convex decomposition by extremals in a C^*-algebra*, Math. Scand. **81** (1997), 69–85. MR 1490775

[12] L. G. Brown and G. K. Pedersen, *Extremal K-theory and index for C^*-algebras*, K-Theory **20** (2000), 201–241. MR 1798827

[13] J. L. Cui and J. C. Hou, *Linear maps between Banach algebras compressing certain spectral functions*, Rocky Mountain J. Math. **34** (2004), 565–585. MR 2072795

[14] K. R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, Amer. Math. Soc., Providence, RI, 1996. MR 1402012

[15] S. Goldberg, *Unbounded linear operators: Theory and applications*, McGraw-Hill Book Co., New York, 1966. MR 0200692

[16] A. Guterman, C. K. Li and P. Šemrl, *Some general techniques on linear preserver problems*, Linear Algebra Appl. **315** (2000), 61–81. MR 1774960

[17] R. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. **103** (1992), 71–77. MR 1184103

[18] R. Harte and M. Mbekhta, *Generalized inverses in C^*-algebras II*, Studia Math. **106** (1993), 129–138. MR 1240309

[19] R. V. Kadison, *Isometries of operator algebras*, Ann. Math. **54** (1951), 325-338. MR 0043392

[20] I. Kaplansky, *Algebraic and analytic aspect of operator algebras*, Amer. Math. Soc., Providence, RI, 1970. MR 0312283

[21] A. A. Jafarian and A. R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal. **66** (1986), 255–261. MR 0832991

[22] D. Männle and C. Schmoeger, *Generalized Fredholm theory in semisimple algebras*, Rend. Mat. Appl. **19** (1999), 583–613. MR 1789488

[23] M. Mathieu, *Elementary operators on prime C^*-algebras II*, Glasg. Math. J. **30** (1988), 275–284. MR 0964574

[24] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasg. Math. J. **29** (1987), 159–175. MR 0901662

[25] M. Mbekhta, *Linear maps preserving the generalized spectrum*, Extracta Math. **22** (2007), 45–54. MR 2368700

[26] M. Mbekhta and A. Ouahab, *Opérateur s-régulier dans un espace de Banach et théorie spectrale*, Acta Sci. Math. (Szeged) **59** (1994), 525–543. MR 1317171

[27] M. Mbekhta, L. Rodman and P. Šemrl, *Linear maps preserving generalized invertibility*, Integral Equations Operator Theory **55** (2006), 93–109. MR 2226639

[28] M. Mbekhta and P. Šemrl, *Linear maps preserving semi-Fredholm operators and generalized invertibility*, Linear Multilinear Algebra **57** (2009), 55–64. MR 2536067

[29] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Operator Theory: Advances and Applications, vol. 139, Birkhäuser Verlag, Basel, 2003. MR 1975356

[30] J. Pérez, L. Rico and A. Rodríguez, *Full subalgebras of Jordan–Banach algebras and algebra norms on JB^*-algebras*, Proc. Amer. Math. Soc. **121** (1994), 1133–1143. MR 1195486

[31] A. Rodríguez, *Jordan axioms for C^*-algebras*, Manuscripta Math. **61** (1988), 297–314. MR 0949820

[32] J. W. Rowell, *Unilateral Fredholm theory and unilateral spectra*, Proc. Roy. Irish. Acad. **84A** (1984), 69–85. MR 0771647

[33] B. Russo and H. A. Dye, *A note on unitary operators in C^*-algebras*, Duke Math. J. **33** (1966), 413–416. MR 0193530

[34] R. D. Schaefer, *An introduction to nonassociative algebras*, Pure and Applied Mathematics, vol. 22, Academic Press, New York, 1966. MR 0210757

[35] J. D. M. Wright, *Jordan C^*-algebras*, Michigan Math. J. **24** (1977), 291–302. MR 0487478

[36] M. A. Youngson, *Nonunital Banach Jordan algebras and C^*-triple systems*, Proc. Edinburgh Math. Soc. **24** (1981), 19–29. MR 0621449

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