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POINCARÉ SERIES AND EICHLER INTEGRALS

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Dedicated to the memory of Atle Selberg: a brilliant mind—fine company for Gauss and Riemann, Poincaré and Ramanujan.

Abstract. By employing work concerning Selberg's Kloosterman zeta-function, we carry out the decomposition of a special value of a nonanalytic Poincaré series of nonpositive even weight, with a nonsingular multiplier system, on the full modular group. The summands that emerge are connected meaningfully to each other as well as to classical expressions for Eichler integrals and modular forms.

In previous work [23] the author reveals how the study of nonanalytic Eisenstein series of negative even weight on the full modular group leads without difficulty to the discovery of Eichler integrals of Eisenstein series of positive weight. Furthermore, in [12] Knopp and the author demonstrate how a special value of a nonanalytic Poincaré series of weight zero decomposes naturally into three pieces involving generalized Abelian integrals. The present paper provides an extension of this result to all nonpositive even weights. We establish that for negative even weights the analogous Poincaré series breaks up nicely into four interesting pieces. The first three offer no surprise (the aforementioned generalized Abelian integrals now become Eichler integrals), whereas the fourth one is a nonanalytic modular form. The proof of its transformation behavior requires the use of weight-changing operators as well as the verification of a combinatorial identity. The latter is included, for the sake of completeness, in the Appendix. In the course of our work, we arrive at some new expressions pertaining to modular forms and Eichler integrals, and we

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also recover some well-known results that underpin Eichler cohomology. All of our findings are summarized toward the end of the paper.

Let us consider, for ν any integer, the *nonanalytic Poincaré series*

$$P_\nu(\tau|s) = P_\nu(\tau|s;k,v) = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{e^{2\pi i(\nu+\kappa)M\tau}}{v(M)(c\tau+d)^k|c\tau+d|^s}, \tag{1}$$

where $M = \left[\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right] \in \Gamma(1)$, $\tau \in \mathcal{H}$, and $\mathrm{Re}(s) > 2-k$. As usual, $\Gamma(1) = SL(2,\mathbb{Z})$ denotes the full modular group, and $\mathcal{H}$ symbolizes the complex upper half-plane. Here v is a multiplier system (MS) for $\Gamma(1)$ in (real) weight k, and κ is a parameter determined by $v(S) = e^{2\pi i\kappa}$, $0 \le \kappa < 1$, where $S = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. A well-known argument using absolute convergence of (1) shows that, for $\mathrm{Re}(s) > 2-k$, $P_\nu(\tau|s)$ obeys the transformation law

$$P_\nu(L\tau|s) = v(L)(\gamma\tau+\delta)^k|\gamma\tau+\delta|^s P_\nu(\tau|s), \tag{2}$$

for all $L = \left[\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right] \in \Gamma(1)$ and $\tau \in \mathcal{H}$. In a beautiful *tour de force* that exploited spectral theory, Selberg [26] established by the middle of the 1960s that $P_\nu(\tau|s)$ has a meromorphic continuation to the whole s-plane. Although he explained the result for the case $\nu+\kappa > 0$, the case $\nu+\kappa < 0$ follows without much difficulty from his profound work. (Of course, the case $\nu+\kappa = 0$ corresponds to nonanalytic Eisenstein series and is rather well known.) For simplicity we assume from now on that κ is nonzero and also that k is an even integer. (The latter implies that both v and its complex conjugate MS $\overline{v}$ are even Abelian characters for $\Gamma(1)$.) From (1) it follows through standard techniques (see [9] or [21]) that $P_\nu(\tau|s)$ has the Fourier expansion

$$\begin{aligned}
P_\nu(\tau|s) = e^{2\pi i(\nu+\kappa)\tau} &+ i^{-k}\frac{(2\pi)^{k+s}}{\Gamma(s/2)} \\
&\times \Bigg\{\sum_{n=0}^{\infty}(n+\kappa)^{k+s-1}e^{2\pi i(n+\kappa)\tau}\sum_{p=0}^{\infty}\frac{[-4\pi^2(n+\kappa)(\nu+\kappa)]^p}{p!\Gamma(k+p+s/2)} \\
&\times \sigma(4\pi(n+\kappa)y, k+p+s/2, s/2)Z_{\nu,n}(s/2+k/2+p) \\
&+ \sum_{n=1}^{\infty}(n-\kappa)^{k+s-1}e^{-2\pi i(n-\kappa)\overline{\tau}}\sum_{p=0}^{\infty}\frac{[-4\pi^2(n-\kappa)(\nu+\kappa)]^p}{p!\Gamma(k+p+s/2)} \\
&\times \sigma\big(4\pi(n-\kappa)y, s/2, k+p+s/2\big)Z_{\nu,-n}(s/2+k/2+p)\Bigg\}.
\end{aligned} \tag{3}$$

Here $y = \mathrm{Im}(\tau)$ and $\sigma(\eta,\alpha,\beta)$ is the special function which has the representation

$$\sigma(\eta,\alpha,\beta) = \int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}e^{-\eta u}\,du, \tag{4}$$

for $\mathrm{Re}(\eta)>0$, $\alpha\in\mathbb{C}$, and $\mathrm{Re}(\beta)>0$. Note that the function $\sigma(\eta,\alpha,\beta)/\Gamma(\beta)$ is entire in α and β. Additionally, for any ν, $n\in\mathbb{Z}$ and $\mathrm{Re}(w)>1$,

$$Z_{\nu,n}(w)=Z_{\nu,n}(w;v)=\sum_{c=1}^{\infty}\frac{A_c(\nu,n)}{c^{2w}} \tag{5}$$

is *Selberg's Kloosterman zeta-function*, where

$$A_c(\nu,n)=A_c(\nu,n;v)=\sum_{\substack{-d=0\\(c,d)=1}}^{c-1}\overline{v}(M)e^{\frac{2\pi i}{c}[(\nu+\kappa)a+(n+\kappa)d]} \tag{6}$$

is the *generalized Kloosterman sum* and $M=\left[\begin{smallmatrix}a&*\\c&d\end{smallmatrix}\right]\in\Gamma(1)$. In the aforementioned work [26] Selberg proved that $Z_{\nu,n}(w)$ possesses a meromorphic continuation to the whole w-plane. Later on Goldfeld and Sarnak [3] and Pribitkin [22] provided estimates on $Z_{\nu,n}(w)$. These results on Selberg's Kloosterman zeta-function combined with basic facts concerning the σ-function (see, for example, [23]) imply that the expansion (3), at first computed for $\mathrm{Re}(s)>2-k$, remains valid throughout the whole s-plane wherever $P_\nu(\tau|s)$ is analytic. Furthermore, it is known from Selberg theory [26] (see also [20, Section 2.3]) that $P_\nu(\tau|s)$ is analytic, in s, at $s=0$. So let $P_\nu(\tau)=P_\nu(\tau;k,v)=P_\nu(\tau|0)$. From (2) it follows that

$$P_\nu(L\tau)=v(L)(\gamma\tau+\delta)^kP_\nu(\tau), \tag{7}$$

for all $L=\left[\begin{smallmatrix}*&*\\\gamma&\delta\end{smallmatrix}\right]\in\Gamma(1)$ and $\tau\in\mathcal{H}$. Hence $P_\nu(\tau)$ transforms like a modular form of weight k and multiplier system v on $\Gamma(1)$. But $P_\nu(\tau)$ need not be analytic in τ. Of course, if $k\geq 2$, then $P_\nu(\tau)$ is a modular form of weight k and MS v on $\Gamma(1)$. (This was largely known to Petersson [17], [18].) But what happens when $k\leq 0$? The answer to this question is provided below, where we see that it is fruitful to look at the family of *all* even weights simultaneously.

Initially we decompose $P_\nu(\tau)$ as follows:

$$P_\nu(\tau)=A_\nu(\tau)+R_\nu(\tau),$$

where

$$\begin{aligned}A_\nu(\tau)=e^{2\pi i(\nu+\kappa)\tau}&\\&+i^{-k}(2\pi)^k\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{[-4\pi^2(n+\kappa)(\nu+\kappa)]^p}{p!}\\&\times\left.\frac{Z_{\nu,n}(s/2+k/2+p)}{\Gamma(k+p+s/2)}\right|_{s=0}(n+\kappa)^{k-1}e^{2\pi i(n+\kappa)\tau}\end{aligned} \tag{8}$$

is the analytic piece and

$$R_\nu(\tau)=-2\pi i\sum_{n=1}^{\infty}\sum_{p=0}^{-k/2}\frac{[-2\pi i(\nu+\kappa)]^p}{p!}\cdot\mathrm{Res}\big(Z_{\nu,-n}(w);k/2+p\big) \tag{9}$$

$$\times \int_{-\overline{\tau}}^{i\infty} (z+\tau)^{-k-p} e^{2\pi i(n-\kappa)z}\, dz$$

is the remaining piece. Here $A_\nu(\tau) = A_\nu(\tau; k, v)$ and $R_\nu(\tau) = R_\nu(\tau; k, v)$. (*Nota bene* that throughout this paper we adhere to the convention that a finite sum is considered empty if the upper limit of summation is less than the lower limit of summation. In such cases the value of the sum is taken to be zero.) To obtain (8) and (9), we started with (3) and used the fact that

$$\left.\frac{\sigma(\eta, \alpha, \beta)}{\Gamma(\beta)}\right|_{\beta=0} = 1$$

(which follows quickly from (4) by integration by parts), as well as the relation

$$\eta^{\beta} \frac{\sigma(\eta, \alpha, \beta)}{\Gamma(\beta)} = \eta^{1-\alpha} \frac{\sigma(\eta, 1-\beta, 1-\alpha)}{\Gamma(1-\alpha)}$$

(which likewise is not difficult to derive directly from (4)). Additionally, we invoked the deep fact (see [26] and [20, Section 2.3]) that $Z_{\nu,n}(w)$ has at most simple poles at nonpositive integers, and the easier fact that it is analytic at positive integers. (From (5) and (6) we see instantly that $Z_{\nu,n}(w)$ is analytic for $\mathrm{Re}(w) > 1$; the analyticity (in fact absolute convergence) at $w = 1$ follows from any nontrivial estimate (see [14]) on generalized Kloosterman sums.) Obviously, if $k \geq 2$, then $R_\nu(\tau) \equiv 0$ and so $A_\nu(\tau) = P_\nu(\tau)$ is a modular form of weight k and MS v on $\Gamma(1)$. But what happens to $A_\nu(\tau)$ and $R_\nu(\tau)$ when $k \leq 0$? In order to resolve this question, we recall some basic facts.

Let $F(z)$ be *any* complex-valued function of the complex variable $z = x + iy$ and assume that its rth order complex partial derivative exists. Here r is a nonnegative integer, and as usual the first order complex partial differential operator is defined by $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. Then it is easy to verify (by induction on r) that

$$\frac{\partial^r F}{\partial z^r}(Lz) = \sum_{j=0}^{r} \binom{r}{j} \frac{\Gamma(w+r)}{\Gamma(w+j)} \gamma^{r-j} (\gamma z + \delta)^{w+r+j} \frac{\partial^j}{\partial z^j}\{(\gamma z+\delta)^{-w} F(Lz)\},$$

where $L = \left[\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right] \in SL(2, \mathbb{C})$ and w is a complex number. Setting $w = 1 - r$, we obtain the important special case

$$(\gamma z+\delta)^{-r-1} \frac{\partial^r F}{\partial z^r}(Lz) = \frac{\partial^r}{\partial z^r}\{(\gamma z+\delta)^{r-1} F(Lz)\}. \tag{10}$$

If additionally the rth order complex derivative of $F(z)$ exists, then rather evidently (since $\frac{\partial}{\partial z}$ and $\frac{d}{dz}$ act equivalently) we obtain

$$(\gamma z+\delta)^{-r-1} F^{(r)}(Lz) = \frac{d^r}{dz^r}\{(\gamma z+\delta)^{r-1} F(Lz)\}, \tag{11}$$

an identity established by Bol [1] in 1949. (Note that (11) can also be proved by using Cauchy's integral formula.) It follows from (10) that if $F(\tau)$, $\tau =$

$x+iy \in \mathcal{H}$, transforms like an automorphic form of integer weight $k \le 0$ and MS v on a suitable group Γ, then $\frac{\partial^{1-k} F(\tau)}{\partial \tau^{1-k}}$ transforms like an automorphic form of weight $2-k$ and MS v on Γ. We emphasize that $F(\tau)$ need not be an analytic function. (In fact, v need not be a unitary MS, but we shall not deal with this scenario.) Furthermore, if $f(\tau)$ transforms like an automorphic form of weight $2-k$ and MS v on Γ and $F(\tau)$ is any function such that $\frac{\partial^{1-k} F(\tau)}{\partial \tau^{1-k}} = f(\tau)$, then (10) implies that $F(\tau)$ satisfies the functional equation

$$\overline{v}(L)(\gamma\tau+\delta)^{-k} F(L\tau) - F(\tau) = r_L(\tau),$$

for all $L = \left[\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right] \in \Gamma$ and $\tau \in \mathcal{H}$, where $r_L(\tau)$ is some function annihilated by $\frac{\partial^{1-k}}{\partial \tau^{1-k}}$. Finally, recall that if $F(\tau)$ is in fact an analytic function throughout $\mathcal{H}$ such that its $(1-k)$th derivative is an automorphic form of weight $2-k$ and MS v on Γ, then $F(\tau)$ is called an *Eichler integral of weight k and MS v on* Γ, *with period polynomials* $r_L(\tau)$. Observe that these polynomials are necessarily of degree $\le -k$.

We return now to our study of $A_\nu(\tau)$ and $R_\nu(\tau)$. First we claim that if $k \le 0$, then

$$\frac{\partial^{1-k} R_\nu(\tau)}{\partial \tau^{1-k}} \equiv 0. \tag{12}$$

This stems from the observation that the integral in (9) is a polynomial in τ of degree $-k-p$ with coefficients that are entire functions of $-\overline{\tau}$. (In fact, these coefficients can be found by applying the binomial theorem to $(z+\tau)^{-k-p}$ and then integrating by parts sufficiently often.) Hence $R_\nu(\tau)$ is itself a polynomial in τ of degree not exceeding $-k$ with coefficients that are analytic in $-\overline{\tau}$ throughout $\mathcal{H}$. Since $\frac{\partial}{\partial\tau}$ obviously annihilates analytic functions of $\overline{\tau}$, it is clear that $\frac{\partial^{1-k}}{\partial \tau^{1-k}}$ annihilates $R_\nu(\tau)$. Next we assert that if $k \le 0$, then $A_\nu(\tau)$ is an Eichler integral of weight k and MS v on $\Gamma(1)$. By design $A_\nu(\tau)$ is analytic throughout $\mathcal{H}$. Its behavior under modular transformations follows from the fact that

$$A_\nu^{(1-k)}(\tau) = \frac{\partial^{1-k} A_\nu(\tau)}{\partial \tau^{1-k}} = \frac{\partial^{1-k} (P_\nu(\tau) - R_\nu(\tau))}{\partial \tau^{1-k}} = \frac{\partial^{1-k} P_\nu(\tau)}{\partial \tau^{1-k}}$$

by (12). But from (7) we know that $P_\nu(\tau)$ transforms like a modular form of weight k and MS v on $\Gamma(1)$, and therefore (based upon remarks of the previous paragraph) we see that $\frac{\partial^{1-k} P_\nu(\tau)}{\partial \tau^{1-k}}$ transforms like a modular form of weight $2-k$ and MS v on $\Gamma(1)$. Since $A_\nu^{(1-k)}(\tau)$ is clearly analytic throughout $\mathcal{H}$ and meromorphic at $i\infty$, we conclude that $A_\nu^{(1-k)}(\tau)$ is a modular form of weight $2-k$ and MS v on $\Gamma(1)$. And so, for $k \le 0$, $A_\nu(\tau)$ is an Eichler integral as asserted. But we still want to decompose $P_\nu(\tau)$ into more recognizable parts!

So we write $A_\nu(\tau)$ as

$$A_\nu(\tau) = F_\nu(\tau) + \mathcal{Z}_\nu(\tau),$$

where

$$F_\nu(\tau) = e^{2\pi i(\nu+\kappa)\tau} + i^{-k}(2\pi)^k \sum_{n=0}^{\infty} \sum_{\substack{p=1-k/2 \\ p\geq 0}}^{\infty} \frac{[-4\pi^2(n+\kappa)(\nu+\kappa)]^p}{p!\Gamma(k+p)} \times Z_{\nu,n}(k/2+p)(n+\kappa)^{k-1} e^{2\pi i(n+\kappa)\tau} \tag{13}$$

and

$$\mathcal{Z}_\nu(\tau) = i^{-k}(2\pi)^k \sum_{n=0}^{\infty} \sum_{p=0}^{-k/2} \frac{[4\pi^2(n+\kappa)(\nu+\kappa)]^p}{p!}(-k-p)! \times \operatorname{Res}\big(Z_{\nu,n}(w); k/2+p\big)(n+\kappa)^{k-1} e^{2\pi i(n+\kappa)\tau}. \tag{14}$$

Note that $F_\nu(\tau) = F_\nu(\tau; k, v)$ and $\mathcal{Z}_\nu(\tau) = \mathcal{Z}_\nu(\tau; k, v)$. It is clear that (13) and (14) follow readily from (8). In particular, to obtain (14) we used the fact that, for nonpositive even k and $0 \leq p \leq -k/2$,

$$\left.\frac{Z_{\nu,n}(s/2+k/2+p)}{\Gamma(k+p+s/2)}\right|_{s=0} = (-1)^p(-k-p)! \cdot \operatorname{Res}\big(Z_{\nu,n}(w); k/2+p\big).$$

Evidently, if $k \geq 2$, then $\mathcal{Z}_\nu(\tau) \equiv 0$ and so $F_\nu(\tau) = A_\nu(\tau) = P_\nu(\tau)$ is a modular form of weight k and MS v on $\Gamma(1)$. Next, by the usual reasoning (see, for example, [21]), we transform (13) into

$$F_\nu(\tau) = e^{2\pi i(\nu+\kappa)\tau} + \sum_{n=0}^{\infty} a_n(\nu; k, v) e^{2\pi i(n+\kappa)\tau}, \tag{15}$$

where

$$a_n(\nu; k, v) = 2\pi i^{-k} \left(\frac{n+\kappa}{|\nu+\kappa|}\right)^{(k-1)/2} \sum_{c=1}^{\infty} \frac{A_c(\nu, n)}{c} \times \begin{cases} I_{k-1}\big(\frac{4\pi\sqrt{(n+\kappa)|\nu+\kappa|}}{c}\big) & \text{if } \nu < 0, \\ J_{k-1}\big(\frac{4\pi\sqrt{(n+\kappa)(\nu+\kappa)}}{c}\big) & \text{if } \nu \geq 0. \end{cases} \tag{16}$$

This is valid for all $k \in 2\mathbb{Z}$! Here I_{k-1} and J_{k-1} are the modified and regular Bessel functions of the first kind, respectively. We now recall the rather well-known identities (see, for example, [29, Chapter 17]) $I_{-m} = I_m$ and $J_{-m} = (-1)^m J_m$, valid for $m \in \mathbb{Z}$. Not only do they expose the connection between (16), $\nu < 0$, and the classical work pertaining to modular forms of nonpositive even weight (see Rademacher and Zuckerman [25] for $k < 0$ and [12] for citations concerning $k = 0$), but also they reveal that

$$\frac{a_n(\nu; k, v)}{(n+\kappa)^{k-1}} = \frac{a_n(\nu; 2-k, v)}{(\nu+\kappa)^{k-1}}, \tag{17}$$

again for all $\nu \in \mathbb{Z}$ and $k \in 2\mathbb{Z}$. This beautiful duality relation (which is analogous to $\sigma_{k-1}(n)/n^{k-1} = \sigma_{1-k}(n)$—the relation involving the divisor function which arises with Eichler integrals of Eisenstein series in [23]) allows us to deduce that, for $k \le 0$,

$$F_\nu^{(1-k)}(\tau; k, v) = [2\pi i(\nu+\kappa)]^{1-k} F_\nu(\tau; 2-k, v). \tag{18}$$

However, if $k \le 0$, then $F_\nu(\tau; 2-k, v)$ is a modular form of weight $2-k$ and MS v on $\Gamma(1)$. And so (18) tells us that, for $k \le 0$, $F_\nu(\tau; k, v)$ is an Eichler integral of weight k and MS v on $\Gamma(1)$. Denote the period polynomials of $F_\nu(\tau)$ by $c_L(\tau) = c_L(\tau; \nu, k, v)$. Therefore

$$\overline{v}(L)(\gamma\tau+\delta)^{-k} F_\nu(L\tau; k, v) - F_\nu(\tau; k, v) = c_L(\tau; \nu, k, v), \tag{19}$$

for all $L = \left[\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right] \in \Gamma(1)$ and $\tau \in \mathcal{H}$. Note that the degree of $c_L(\tau)$ is at most $-k$. Since $A_\nu(\tau)$ is also an Eichler integral (assuming $k \le 0$), we ascertain that $\mathcal{Z}_\nu(\tau) = A_\nu(\tau) - F_\nu(\tau)$ must be an Eichler integral of weight k and MS v on $\Gamma(1)$. Denote the period polynomials of $\mathcal{Z}_\nu(\tau)$ by $p_L(\tau) = p_L(\tau; \nu, k, v)$. So

$$\overline{v}(L)(\gamma\tau+\delta)^{-k} \mathcal{Z}_\nu(L\tau; k, v) - \mathcal{Z}_\nu(\tau; k, v) = p_L(\tau; \nu, k, v), \tag{20}$$

for all $L = \left[\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right] \in \Gamma(1)$ and $\tau \in \mathcal{H}$. Let $C^0(k, v)$ be the space of cusp forms on $\Gamma(1)$ of weight k and MS v. Because of the key assumption $\kappa > 0$, it is evident from (14) that $\mathcal{Z}_\nu^{(1-k)}(\tau) \in C^0(2-k, v)$ for all $\nu \in \mathbb{Z}$ and $k \le 0$, $k \in 2\mathbb{Z}$. Under these conditions it follows immediately from (14) that

$$\begin{aligned} \mathcal{Z}_\nu^{(1-k)}(\tau) = 2\pi i \sum_{n=0}^{\infty} \sum_{p=0}^{-k/2} & \frac{[4\pi^2(n+\kappa)(\nu+\kappa)]^p}{p!} (-k-p)! \\ & \times \operatorname{Res}\bigl(Z_{\nu,n}(w); k/2+p\bigr) e^{2\pi i(n+\kappa)\tau}. \end{aligned} \tag{21}$$

Note that

$$\mathcal{Z}_\nu(\tau) = \frac{1}{(-k)!} \int_{i\infty}^{\tau} \mathcal{Z}_\nu^{(1-k)}(z)(z-\tau)^{-k}\, dz \tag{22}$$

and so through traditional arguments

$$p_L(\tau) = \frac{1}{(-k)!} \int_{L^{-1}(i\infty)}^{i\infty} \mathcal{Z}_\nu^{(1-k)}(z)(z-\tau)^{-k}\, dz. \tag{23}$$

It is transparent from (23) that the degree of $p_L(\tau)$ is at most $-k$. We still want to dissect $R_\nu(\tau)$.

$R_\nu(\tau)$ is somewhat inscrutable, but it can be decomposed as follows:

$$R_\nu(\tau) = \mathcal{Z}_\nu^*(\tau) + N_\nu(\tau),$$

where

$$\overline{\mathcal{Z}_\nu^*(\tau)} = -2\pi i \sum_{n=0}^{\infty} \sum_{p=0}^{-k/2} \frac{\{4\pi^2[n+(1-\kappa)][(-\nu-1)+(1-\kappa)]\}^p}{p!} \tag{24}$$

$$\times \frac{(-k-p)!}{(-k)!} \cdot \operatorname{Res}\big(Z_{-\nu-1,n}(w;\overline{v}); k/2+p\big)$$
$$\times \int_{\tau}^{i\infty} e^{2\pi i[n+(1-\kappa)]z}(z-\overline{\tau})^{-k}\,dz$$

and

$$\overline{N_\nu(\tau)} = -2\pi i \sum_{n=0}^{\infty}\sum_{p=1}^{-k/2}\sum_{j=1}^{p} \frac{\{2\pi i[(-\nu-1)+(1-\kappa)]\}^p}{p!} \times \frac{\{2\pi i[n+(1-\kappa)]\}^{j-1}(-1)^j(2yi)^{-k-p+j}}{(-k-p+1)(-k-p+2)\cdots(-k-p+j)} \times \operatorname{Res}\big(Z_{-\nu-1,n}(w;\overline{v}); k/2+p\big)e^{2\pi i[n+(1-\kappa)]\tau}. \tag{25}$$

Of course, $\mathcal{Z}_\nu^*(\tau) = \mathcal{Z}_\nu^*(\tau;k,v)$ and $N_\nu(\tau) = N_\nu(\tau;k,v)$. Note that (24) and (25) follow naturally from (9). Along the way we used the simple property

$$\overline{Z_{\nu,n}(w;v)} = Z_{-\nu-1,-n-1}(\overline{w};\overline{v}),$$

which holds for any ν, $n \in \mathbb{Z}$ and $0<\kappa<1$, applied the straightforward identity

$$\left[\int_\alpha^\beta f(z)\,dz\right]^{-} = -\int_{-\overline{\alpha}}^{-\overline{\beta}} \overline{f(-\overline{z})}\,dz,$$

as well as integrated by parts precisely p times. (Here $[\cdot]^-$ denotes complex conjugation.) Observe by comparing (21) and (24) that

$$\overline{\mathcal{Z}_\nu^*(\tau;k,v)} = \frac{1}{(-k)!}\int_{i\infty}^{\tau} \mathcal{Z}_{-\nu-1}^{(1-k)}(z;k,\overline{v})(z-\overline{\tau})^{-k}\,dz, \tag{26}$$

for $k \le 0$ and $\nu \in \mathbb{Z}$. This allows us to establish the modular behavior of $\mathcal{Z}_\nu^*(\tau)$. Since $\mathcal{Z}_{-\nu-1}^{(1-k)}(z;k,\overline{v}) \in C^0(2-k,\overline{v})$, we get that

$$\begin{aligned}&\overline{v}(L)(\gamma\tau+\delta)^{-k}\mathcal{Z}_\nu^*(L\tau;k,v) - \mathcal{Z}_\nu^*(\tau;k,v)\\ &\qquad = \left[\frac{1}{(-k)!}\int_{L^{-1}(i\infty)}^{i\infty} \mathcal{Z}_{-\nu-1}^{(1-k)}(z;k,\overline{v})(z-\overline{\tau})^{-k}\,dz\right]^{-},\end{aligned} \tag{27}$$

for all $L = \left[\begin{smallmatrix} * & * \\ \gamma & \delta\end{smallmatrix}\right] \in \Gamma(1)$ and $\tau \in \mathcal{H}$. And by recalling (23) we see from (27) that

$$\overline{v}(L)(\gamma\tau+\delta)^{-k}\mathcal{Z}_\nu^*(L\tau;k,v) - \mathcal{Z}_\nu^*(\tau;k,v) = [p_L(\overline{\tau};-\nu-1,k,\overline{v})]^{-}. \tag{28}$$

We remark that, modulo a constant factor, $\mathcal{Z}_\nu^*(\tau;k,v)$ is the "auxiliary integral" (as defined in [24]) of the cusp form $\mathcal{Z}_{-\nu-1}^{(1-k)}(z;k,\overline{v})$.

It remains to decipher the modular behavior of $N_\nu(\tau)$. We claim that, in fact, $N_\nu(\tau)$ *transforms exactly like a modular form of weight k and MS v on* $\Gamma(1)$! That is,

$$\overline{v}(L)(\gamma\tau+\delta)^{-k}N_\nu(L\tau;k,v) - N_\nu(\tau;k,v) = 0, \tag{29}$$

for all $L = \left[\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right] \in \Gamma(1)$ and $\tau \in \mathcal{H}$. But $P_\nu(\tau) = F_\nu(\tau) + \mathcal{Z}_\nu(\tau) + \mathcal{Z}^*_\nu(\tau) + N_\nu(\tau)$ transforms like a modular form of weight k and MS v on $\Gamma(1)$, and hence we can deduce from (19), (20), (28), and (29) that

$$c_L(\tau;\nu,k,v) + p_L(\tau;\nu,k,v) + [p_L(\overline{\tau};-\nu-1,k,\overline{v})]^- = 0. \tag{30}$$

From this identity we recapture Knopp's relation [7] (see [24] for yet another proof)

$$c_L(\tau;\nu,k,v) = [c_L(\overline{\tau};-\nu-1,k,\overline{v})]^-, \tag{31}$$

for all $\nu \in \mathbb{Z}$. By analytic continuation it is quite clear that (30) and (31) hold for $\tau \in \mathbb{C}$.

We now substantiate our claim concerning $N_\nu(\tau)$. If $k \geq 0$, then $N_\nu(\tau) \equiv 0$, and there is nothing to prove. So, assume that $k \leq -2$, $k \in 2\mathbb{Z}$. We can rewrite (25) as follows:

$$\begin{aligned} \overline{N_\nu(\tau)} = {} & 2\pi i(2yi)^{-k} \sum_{d=0}^{-k/2-1} \frac{1}{(2yi)^d} \\ & \times \sum_{n=0}^{\infty} \sum_{j=0}^{-k/2-d-1} \frac{\{2\pi i[(-\nu-1)+(1-\kappa)]\}^{d+1}}{(j+d+1)!} \\ & \times \frac{\{4\pi^2[n+(1-\kappa)][(-\nu-1)+(1-\kappa)]\}^j}{(-k-j-d)(-k-j-d+1)\cdots(-k-d)} \\ & \times \operatorname{Res}(Z_{-\nu-1,n}(w;\overline{v}); k/2+j+d+1) e^{2\pi i[n+(1-\kappa)]\tau}. \end{aligned} \tag{32}$$

Let

$$N^*_\nu(\tau) = (2yi)^k \overline{N_\nu(\tau)} \tag{33}$$

and observe that $N_\nu(\tau)$ transforms like a modular form of weight k and MS v if and only if $N^*_\nu(\tau)$ transforms like a modular form of weight $-k$ and MS $\overline{v}$. Next we recall that, for nonpositive even k and ν any integer, $\mathcal{Z}_\nu^{(1-k)}(\tau;k,v) \in C^0(2-k,v)$. So, for $k \leq -2$, $k \in 2\mathbb{Z}$, we have that $\mathcal{Z}_{-\nu-1}^{(-k-2q-1)}(\tau;k+2q+2,\overline{v}) \in C^0(-k-2q,\overline{v})$, where $q = 0,1,\ldots,-k/2-1$. For the sake of notational brevity, put

$$h_q(\tau) = \mathcal{Z}_{-\nu-1}^{(-k-2q-1)}(\tau;k+2q+2,\overline{v}). \tag{34}$$

We now invoke weight-changing operators. Specifically, consider the operator δ^r_w, defined by

$$\delta^r_w = \sum_{j=0}^{r} \binom{r}{p} \frac{\Gamma(w+r)}{\Gamma(w+j)} (2yi)^{j-r} \frac{\partial^j}{\partial \tau^j}, \tag{35}$$

where w is any complex number and r is any nonnegative integer. This operator maps a form of weight w to one of weight $w+2r$. A special case of it reconfirms the remark (made soon after (10)) that, for any nonpositive integer k, $\delta_k^{1-k} = \frac{\partial^{1-k}}{\partial \tau^{1-k}}$ maps a form to a form. (Here, a "form" means a function in τ which transforms like an automorphic form. We assume sufficiently many partial derivatives exist. Obviously, the MS remains unchanged by δ_w^r. Observe that δ_w^0 equals the identity and $\delta_w^r = \delta_{w+2r-2} \circ \cdots \circ \delta_{w+2} \circ \delta_w$, where $r > 0$ and $\delta_w = \frac{\partial}{\partial \tau} + \frac{w}{2yi}$ is the well-known weight-raising operator introduced by Maass [16].) Returning to the matter at hand, we consider

$$h_q^*(\tau) = \delta_{-k-2q}^q(h_q(\tau)), \tag{36}$$

where $k \le -2$, $k \in 2\mathbb{Z}$, and $q = 0, \dots, -k/2 - 1$. Note that $h_q^*(\tau)$ transforms like a modular form of weight $-k$ and MS $\overline{v}$. (Of course, $h_0^*(\tau) = h_0(\tau)$ is a cusp form of weight $-k$ and MS $\overline{v}$.) As an aside, we remark that $h_q^*(\tau)$ is "nearly holomorphic" in the sense of Shimura [28] (also see, for example, [5, Section 10.1]). To prove our claim, it suffices to show that N_ν^* is in the space spanned by $\{h_q^*\}_{q=0}^{-k/2-1}$. That is, we want to show that the equation

$$N_\nu^*(\tau) = \sum_{q=0}^{-k/2-1} C_q h_q^*(\tau) \tag{37}$$

can be solved for $C_0, \dots, C_{-k/2-1}$. After a good bit of manipulation involving (33) and (32) on the one hand, as well as (36), (35), (34), and (21) on the other hand, it turns out that equation (37) is solved by

$$C_q = \frac{\{2\pi i[(-\nu - 1) + (1 - \kappa)]\}^{q+1}}{(q+1)!} \cdot \frac{-k - 2q - 1}{(-k - q)!}.$$

The proof of this requires verification of the following combinatorial identity:

$$\begin{aligned} &\sum_{p=0}^{j} (-1)^p \binom{j}{p} \\ &\qquad \times \frac{-k - 2d - 2p - 1}{(-k - p - 2d - 1)(-k - p - 2d - 2) \cdots (-k - p - 2d - j - 1)} \\ &\qquad \times \frac{1}{(p + d + 1)(-k - d - p)} \\ &= \frac{d!j!}{(j + d + 1)!(-k - j - d)(-k - j - d + 1) \cdots (-k - d)}, \end{aligned} \tag{38}$$

where $k \le -2$, $k \in 2\mathbb{Z}$, $d = 0, \dots, -k/2 - 1$, and $j = 0, \dots, -k/2 - d - 1$. Nowadays, an identity such as (38) can be checked by computer using the revolutionary method of Wilf–Zeilberger [30] (see also [19]). It can also be proved by hand—a fun exercise which we undertake in the Appendix. At any rate, this concludes our demonstration of why $N_\nu(\tau)$ transforms like a modular form of weight k and MS v.

We recapitulate the main findings of this paper. We begin with our decomposition result.

THEOREM 1. *Let $P_\nu(\tau|s;k,v)$ be defined by* (1), *where ν is any integer, and suppose that v is a nonsingular MS in even integer weight k. Recall that $P_\nu(\tau|s;k,v)$ has a meromorphic continuation to the whole s-plane, where it is analytic at $s=0$, and $P_\nu(\tau;k,v)=P_\nu(\tau|0;k,v)$ transforms like a modular form of weight k and MS v on $\Gamma(1)$. Then for $\tau\in\mathcal{H}$,*

$$P_\nu(\tau;k,v)=F_\nu(\tau;k,v)+\mathcal{Z}_\nu(\tau;k,v)+\mathcal{Z}^*_\nu(\tau;k,v)+N_\nu(\tau;k,v),$$

where $F_\nu(\tau;k,v)$, $\mathcal{Z}_\nu(\tau;k,v)$, $\mathcal{Z}^*_\nu(\tau;k,v)$, and $N_\nu(\tau;k,v)$ are defined by (13), (14), (24), *and* (25), *respectively.*

We continue with some basic consequences.

COROLLARY 2. *Let $F_\nu(\tau;k,v)$, $\mathcal{Z}_\nu(\tau;k,v)$, $\mathcal{Z}^*_\nu(\tau;k,v)$, and $N_\nu(\tau;k,v)$ be as in Theorem* 1, *where ν is any integer and v is a nonsingular MS in even weight k. Then the following hold*:

(i) *$F_\nu(\tau;k,v)$ is given by* (15), (16), *and* (6). *Moreover, the duality relation* (17) *is true for all integers $n\geq 0$.*

(ii) *$\mathcal{Z}_\nu(\tau;k,v)\equiv 0$ if $k\geq 2$, and if $k\leq 0$, then $\mathcal{Z}_\nu(\tau;k,v)$ is given by* (22) *and* (21).

(iii) *$\mathcal{Z}^*_\nu(\tau;k,v)\equiv 0$ if $k\geq 2$, and if $k\leq 0$, then $\mathcal{Z}^*_\nu(\tau;k,v)$ is given by* (26) *and* (21).

(iv) *$N_\nu(\tau;k,v)\equiv 0$ if $k\geq 0$, and if $k\leq -2$, then $N_\nu(\tau;k,v)$ is given by* (32).

Next we describe the modular behavior of our summands. For succinctness we call an Eichler integral of weight k "polar" (respectively "cuspidal") if its $(1-k)$th derivative is a "pole form" (respectively cusp form) of weight $2-k$. (At $i\infty$ a pole form is meromorphic, but not holomorphic, in the uniformizing variable.)

COROLLARY 3. *Let $F_\nu(\tau;k,v)$, $\mathcal{Z}_\nu(\tau;k,v)$, $\mathcal{Z}^*_\nu(\tau;k,v)$, and $N_\nu(\tau;k,v)$ be as before, where ν is any integer and v is a nonsingular MS in even weight k. Recall that if $k\geq 2$, then $F_\nu(\tau;k,v)$ is a modular form of weight k and MS v on $\Gamma(1)$. If $k\leq 0$, then the following hold*:

(i) *$F_\nu(\tau;k,v)$ is an Eichler integral of weight k and MS v on $\Gamma(1)$. Moreover, $F_\nu(\tau;k,v)$ is polar if $\nu<0$, and if $\nu\geq 0$, then $F_\nu(\tau;k,v)$ is cuspidal.*

(ii) *$\mathcal{Z}_\nu(\tau;k,v)$ is a cuspidal Eichler integral of weight k and MS v on $\Gamma(1)$.*

(iii) *$\mathcal{Z}^*_\nu(\tau;k,v)$ vanishes at infinity and transforms like an Eichler integral of weight k and MS v on $\Gamma(1)$, but is nonanalytic* (*unless it vanishes identically*).

(iv) *$N_\nu(\tau;k,v)$ vanishes at infinity and transforms like a modular form of weight k and MS v on $\Gamma(1)$, but is nonanalytic* (*unless it vanishes identically*).

Lastly we record some fundamental interrelationships. Note that the first part of (iv) below (due to Knopp [7]) follows instantly from (iii) and the nonexistence of nontrivial cusp forms of nonpositive weight. Furthermore, the second part of (iv) (due to Knopp [7] as well as Knopp and Lehner [11]) follows readily from (15) and (17). Similarly, (v) is a consequence of (ii) and (26).

COROLLARY 4. *Let all notation be as before, where ν is any integer and v is a nonsingular MS in even weight $k \le 0$. If we denote the period polynomials of $F_\nu(\tau;k,v)$, $\mathcal{Z}_\nu(\tau;k,v)$, and $\mathcal{Z}^*_\nu(\tau;k,v)$ by $c_L(\tau;\nu,k,v)$, $p_L(\tau;\nu,k,v)$, and $p^*_L(\tau;\nu,k,v)$, respectively, then the following are true*:

(i) $c_L(\tau;\nu,k,v) + p_L(\tau;\nu,k,v) + p^*_L(\tau;\nu,k,v) = 0,\ L \in \Gamma(1)$.

(ii) $p^*_L(\tau;\nu,k,v) = [p_L(\overline{\tau};-\nu-1,k,\overline{v})]^-,\ L \in \Gamma(1)$.

(iii) $c_L(\tau;\nu,k,v) = [c_L(\overline{\tau};-\nu-1,k,\overline{v})]^-,\ L \in \Gamma(1)$.

(iv) *If $\nu < 0$, then $F_\nu(\tau;k,v)$ is a modular form of weight k and MS v on $\Gamma(1)$ if and only if $F_{-\nu-1}(\tau;k,\overline{v})$ vanishes identically. Moreover, this happens if and only if the cusp form $F_{-\nu-1}(\tau;2-k,\overline{v})$ vanishes identically.*

(v) *$\mathcal{Z}^*_\nu(\tau;k,v)$ transforms like a modular form of weight k and MS v on $\Gamma(1)$ if and only if it vanishes identically. Moreover, this happens if and only if $\mathcal{Z}_{-\nu-1}(\tau;k,\overline{v})$ vanishes identically. (Evidently, this occurs if and only if the cusp form $\mathcal{Z}^{(1-k)}_{-\nu-1}(\tau;k,\overline{v})$ vanishes identically.)*

The previous two corollaries prompt the question: Under what conditions on ν, k, and v does any one of the functions $F_\nu(\tau;k,v)$, $\mathcal{Z}_\nu(\tau;k,v)$, $\mathcal{Z}^*_\nu(\tau;k,v)$, and $N_\nu(\tau;k,v)$ vanish identically? In fact, it can be shown that in *all* cases at least one of these functions must be identically zero! The proof of this depends upon a deeper probe of Selberg's Kloosterman zeta-function. What is more, for special values of k it is possible to work out explicitly (by exploiting the setting of $\Gamma(1)$) the nature of these constituent functions. We hope to substantiate these claims in a future article.

We finish by pointing out several references. For information on multiplier systems, consult the texts [10] and [15]. And for a thorough explanation of Eichler integrals (including cohomology theory), read the seminal papers of Eichler [2] and Shimura [27], as well as the work of Gunning [4], Husseini and Knopp [6], Knopp [8], and Kohnen and Zagier [13].

Appendix: Proof of combinatorial identity

There are certainly a couple different ways for a human being to establish the crucial identity (38). We choose to evaluate the left-hand side, which we call S. Because $-k-2d-2p-1 = (-k-d-p)-(p+d+1)$, we have that

$$S = \sum_{p=0}^{j} (-1)^p \binom{j}{p} \frac{1}{(-k-p-2d-1)\cdots(-k-p-2d-j-1)(p+d+1)}$$

$$-\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{(-k-p-2d-1)\cdots(-k-p-2d-j-1)(-k-d-p)}.$$

Write $S=S_1-S_2$, where S_1 and S_2 are the first and second sums above, respectively. We shall now analyze S_1 by making repeated use of the simple partial fraction decomposition

$$\frac{1}{z(z-1)\cdots(z-r)}=\sum_{\ell=0}^{r}\frac{1}{(-1)^{r-\ell}\ell!(r-\ell)!}\cdot\frac{1}{z-\ell}, \tag{39}$$

valid for any nonnegative integer r and any (suitably restricted) complex number z. Applying (39), with $r=j$ and $z=-k-p-2d-1$, to the pertinent product in S_1 gives us that

$$\begin{aligned}
S_1&=\sum_{p=0}^{j}(-1)^p\binom{j}{p}\sum_{\ell=0}^{j}\frac{1}{(-1)^{j-\ell}\ell!(j-\ell)!}\cdot\frac{1}{-k-p-2d-1-\ell}\cdot\frac{1}{p+d+1}\\
&=\sum_{\ell=0}^{j}\frac{1}{(-1)^{j-\ell}\ell!(j-\ell)!}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{(-k-p-2d-1-\ell)(p+d+1)}\\
&=\sum_{\ell=0}^{j}\frac{1}{(-1)^{j-\ell}\ell!(j-\ell)!}\cdot\frac{1}{-k-d-\ell}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{-k-p-2d-1-\ell}\\
&\quad+\sum_{\ell=0}^{j}\frac{1}{(-1)^{j-\ell}\ell!(j-\ell)!}\cdot\frac{1}{-k-d-\ell}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{p+d+1}\\
&=\sum_{\ell=0}^{j}\frac{(-1)^\ell}{\ell!(j-\ell)!}\cdot\frac{1}{-k-d-\ell}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{p-k-2d-1-\ell-j}\\
&\quad+\frac{1}{(-k-d)\cdots(-k-d-j)}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{p+d+1}.
\end{aligned}$$

Along the way we interchanged sums and employed partial fractions to decompose $[(-k-p-2d-1-\ell)(p+d+1)]^{-1}$. We then replaced p with $j-p$ in the first sum over p, as well as applied (39), with $r=j$ and $z=-k-d$, to the second sum over ℓ. To simplify S_1, we shall invoke the basic combinatorial equality

$$\sum_{p=0}^{r}(-1)^p\binom{r}{p}\frac{1}{p+m+1}=\frac{m!r!}{(r+m+1)!}, \tag{40}$$

valid for any nonnegative integers r and m. Observe that (40) is in fact an immediate consequence of a special case of (39). (Alternatively, it can be proved from scratch by starting with the binomial theorem.) A double dose of (40), with $r=j$ and $m=-k-2d-1-\ell-j-1$ to transform the first sum over p, as well as $r=j$ and $m=d$ to handle the second sum over p, produces

$$S_1=\sum_{\ell=0}^{j}(-1)^{\ell}\binom{j}{\ell}\frac{(-k-2d-1-\ell-j-1)!}{(-k-2d-1-\ell)!}\cdot\frac{1}{-k-d-\ell}$$
$$+\frac{1}{(-k-d)\cdots(-k-d-j)}\cdot\frac{d!j!}{(j+d+1)!}.$$

Now, note that the first term above equals S_2. This implies that

$$S=S_1-S_2=\frac{d!j!}{(j+d+1)!(-k-d)\cdots(-k-d-j)},$$

and secures the desired result.

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