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CHEEGER CONSTANTS OF ARITHMETIC HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. We study the Cheeger constants of certain infinite families of arithmetic hyperbolic three-manifolds, as well as certain graphs associated to these manifolds. We derive computable bounds on the Cheeger constants, and therefore bounds on the first eigenvalue of the Laplacian, by adapting discrete methods due to Brooks, Perry and Petersen. We then modify probabilistic methods due to Brooks and Zuk to obtain sharper, asymptotic bounds. A consequence is that the Cheeger constants are quite small, implying that Cheeger's inequality is generally insufficient to prove Selberg's eigenvalue conjecture.

1. Introduction

Let M be a Riemannian manifold of dimension n and finite volume. The Cheeger constant of M is defined to be

$$h(M) = \inf_N \frac{a(N)}{\min(v(A), v(B))},$$

where N runs over codimension one submanifolds of M that disconnect M into manifolds A and B. The expressions $a(\cdot)$, $v(\cdot)$ refer to $(n-1)$-dimensional and n-dimensional volume, respectively, where the measures are with respect to the Riemannian metric. Denote by $\lambda_1(M)$ the first eigenvalue of the Laplacian on M. In [6], Cheeger showed that

$$\lambda_1(M) \geq \frac{1}{4} h(M)^2. \qquad (1)$$

This, coupled with Buser's inequality [5], $\lambda_1(M) \leq 2a(n-1)h(M) + 10h(M)^2$, where $-(n-1)a^2 \leq R(M)$ for some $a \geq 0$ and $R(M)$ is the Ricci curvature of M, show that $h(M)$ and $\lambda_1(M)$ are qualitatively identical. Let $1_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$

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and let $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\langle \pm 1_2 \rangle$. For convenience, we treat the elements of $PSL_2(\mathbb{Z})$ as matrices instead of two element cosets. Let $\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid y > 0\}$ be the complex upper-half plane, and for $N \in \mathbb{Z}_{\geq 2}$ define

$$\Gamma(N) = \{\gamma \in PSL_2(\mathbb{Z}) \mid \gamma \equiv 1_2 \pmod{N}\}.$$

Set $X(N) = \mathbb{H}^2/\Gamma(N)$. It was suggested in [2] that one might prove Selberg's eigenvalue conjecture $\lambda_1(X(N)) \geq 1/4$, see [17], by establishing that $h(X(N)) \geq 1$. However, in [3], the following was shown:

THEOREM 1 (Brooks, Perry and Petersen [3]). *Let p be a prime satisfying $p \equiv 1 \pmod 4$. Then*

$$h(X(p)) \leq \frac{3\log(3)}{2\pi}\left(\frac{p-1}{p+1}\right).$$

It was further shown in [11] that $h(X(p)) \leq 3\log(3)/2\pi$ for $p \equiv 3 \pmod 4$. Thus, $h(X(p)) \leq 0.52455\ldots$ for odd primes p and it follows that Cheeger's inequality (1) is insufficient to obtain even Selberg's 3/16 bound. The value of $h(X(N))$ was further studied in [4] and the following result was shown:

THEOREM 2 (Brooks and Zuk [4]). *There is a constant $C < 1/2$ so that*

$$h(X(N)) < C$$

for sufficiently large N.

A value of approximately $0.4402\ldots$ suffices for C. The analogue of the Selberg conjecture for Cheeger constants is $h(X(N)) \geq 1/2$, see [4], and thus the above result shows that the Selberg conjecture for Cheeger constants is not true.

Let $d \in \mathbb{Z}_{>0}$ and set $K_d = \mathbb{Q}(\sqrt{-d})$. Let $\mathcal{O}_d$ be the ring of integers of K_d. Let

$$\mathbb{H}^3 = \{(z, r) \in \mathbb{C} \times \mathbb{R} \mid r > 0\}$$

be the upper-half 3-space with $z = x + iy$ and volume form $r^{-3}\,dx\,dy\,dr$. Let $PSL_2(\mathcal{O}_d) = SL_2(\mathcal{O}_d)/\langle \pm 1_2 \rangle$ as in [8]. The groups $\widetilde{\Gamma}_d = PSL_2(\mathcal{O}_d)$ are known as the Bianchi groups [1]. Let $\mathfrak{n}$ be a nonzero ideal in $\mathcal{O}_d$ and let

$$\widetilde{\Gamma}_d(\mathfrak{n}) = \{\gamma \in PSL_2(\mathcal{O}_d) \mid \gamma \equiv 1_2 \pmod{\mathfrak{n}}\}.$$

A congruence subgroup of $PSL_2(\mathbb{C})$ is a discrete group which is $SL_2(\mathbb{C})$-conjugate to a group containing $\widetilde{\Gamma}_d(\mathfrak{n})$ for some nonzero ideal $\mathfrak{n} \subset \mathcal{O}_d$. Given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{C})$, we have the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z, r) = \left(\frac{(az+b)(\overline{cz}+\overline{d}) + a\overline{c}r^2}{|cz+d|^2 + |c|^2 r^2}, \frac{r}{|cz+d|^2 + |c|^2 r^2}\right).$$

Set $\widetilde{X}_d = \mathbb{H}^3/PSL_2(\mathcal{O}_d)$ and $\widetilde{X}_d(\mathfrak{n}) = \mathbb{H}^3/\widetilde{\Gamma}_d(\mathfrak{n})$. The Bianchi groups are cofinite and it follows that the Cheeger constants $h(\widetilde{X}_d(\mathfrak{n}))$ are well defined. We denote the first eigenvalue of the Laplacian

$$\Delta = r^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial r^2}\right) - r\frac{\partial}{\partial r}$$

on $\mathbb{H}^3/\widetilde{\Gamma}$ by $\lambda_1(\mathbb{H}^3/\widetilde{\Gamma})$, and note that Selberg's conjecture for $\mathbb{H}^3$ asserts that

$$\lambda_1(\mathbb{H}^3/\widetilde{\Gamma}) \geq 1,$$

see [7]. It is known, see [9] and [16], that for any congruence subgroup $\widetilde{\Gamma}$ of $PSL_2(\mathcal{O}_d)$ we have $\lambda_1(\mathbb{H}^3/\widetilde{\Gamma}) \geq 3/4$.

The goal of this paper is to extend Theorems 1 and 2 to $h(\widetilde{X}_d(\mathfrak{n}))$. Our results confirm the intuition that the Cheeger constants for $\widetilde{X}_d(\mathfrak{n})$ are too small to give a good estimate for $\lambda_1(\mathbb{H}^3/\widetilde{\Gamma}_d)$. Note that Selberg's conjecture for the Cheeger constant of these manifolds is $h(\widetilde{X}_d(\mathfrak{n})) \geq h(\mathbb{H}^3)/2 = 1$.

Let $d \in \{1,2,3,7,11\}$. In these cases we have that $\mathcal{O}_d$ is Euclidean. Let $(\mathfrak{p})$ be a prime ideal in $\mathcal{O}_d$ such that $(2,\mathfrak{p})$ is a unit. The methods we employ for estimating $h(\widetilde{X}_d(\mathfrak{p}))$ are modifications of the discrete arguments from [3], [11], and [12], and the probabilistic arguments from [4]. Note that $N(\mathfrak{n}) = |\mathcal{O}_d/(\mathfrak{n})|$ denotes the norm of $\mathfrak{n} \subset \mathcal{O}_d$. Our main results are the following.

THEOREM 3. *Let* $d \in \{1,2,3,7,11\}$ *and let* $(\mathfrak{p})$ *be a prime ideal in* $\mathcal{O}_d$ *where* $(2,\mathfrak{p})$ *is a unit in* $\mathcal{O}_d$. *Then*

$$h(\widetilde{X}_d(\mathfrak{p})) \leq C_d \frac{N(\mathfrak{p})-1}{N(\mathfrak{p})+1}$$

for some $C_d < 1.7$.

It follows that the Cheeger constants of these manifolds are all less than $\sqrt{3}$. Consequently, we cannot use (1) to obtain even the 3/4 estimate from [9] and [16]. From Section 3, we have the values

$$C_1 \approx 1.11304\ldots, \qquad C_2 \approx 1.22624\ldots, \qquad C_3 \approx 1.08634\ldots,$$
$$C_7 \approx 1.17806\ldots, \qquad C_{11} \approx 1.63876\ldots.$$

THEOREM 4. *For* d *and* $(\mathfrak{p})$ *as in Theorem* 3, *we have*

$$\limsup_{N(\mathfrak{p})\to\infty} h(\widetilde{X}_d(\mathfrak{p})) < C'_d$$

for some $C'_d < 1$.

Thus, as in the 2-dimensional case, the analogue of Selberg's conjecture for Cheeger constants is not true. From Section 4, we have the values

$$C'_1 \approx 0.82758\ldots, \qquad C'_2 \approx 0.52031\ldots, \qquad C'_3 \approx 0.76147\ldots,$$
$$C'_7 \approx 0.83640\ldots, \qquad C'_{11} \approx 0.98846\ldots.$$

2. Cheeger constants of Cayley graphs

Throughout the paper, we take $d \in \{1, 2, 3, 7, 11\}$. The respective groups $\widetilde{\Gamma}_d$ are referred to as the Euclidean Bianchi groups, see [8]. The respective discriminants are $D_{K_d} = -4, -8, -3, -7, -11$. We denote the group of units in $\mathcal{O}_d$ by $\mathcal{O}_d^\times$.

If v_1, v_2 are adjacent vertices of a graph, then we write $v_1 \sim v_2$. Let Γ be a finite group and let Ω be a generating set for Γ. If $\Omega = \Omega^{-1}$, then we say that Ω is symmetric. The Cayley graph of Γ with respect to a symmetric generating set Ω, denoted $G(\Gamma, \Omega)$, is defined as follows: The vertex set of $G = G(\Gamma, \Omega)$ is denoted $V(G)$ and consists of the elements of Γ. Distinct vertices γ_1 and γ_2 are adjacent if and only if $\gamma_1 = \omega\gamma_2$ for some $\omega \in \Omega$. Cayley graphs are $|\Omega|$-regular. Let $A \subset V(G)$. The boundary of A, denoted ∂A, is defined to be the collection of edges in G with exactly one endpoint in A. The Cheeger constant, or isoperimetric number, of a finite graph G is then defined to be

$$\iota(G) = \inf_A \frac{|\partial A|}{|A|},$$

where A ranges over all subsets of $V(G)$ satisfying $|A| \le |V(G)|/2$. Since the rings $\mathcal{O}_d$ are Euclidean, we can follow the methods of [8] to compute

$$[\widetilde{\Gamma}_d : \widetilde{\Gamma}_d(\mathfrak{n})] = N(\mathfrak{n})^3 \prod_{\mathfrak{p}|\mathfrak{n}} \left(1 - \frac{1}{N(\mathfrak{p})^2}\right),$$

where $(2, \mathfrak{n}) \in \mathcal{O}_d^\times$. Let $(\mathfrak{p})$ be a prime ideal in $\mathcal{O}_d$ such that $(2, \mathfrak{p}) \in \mathcal{O}_d^\times$. Let $\widetilde{\Gamma}_{d,\mathfrak{p}} = PSL_2(\mathcal{O}_d/(\mathfrak{p}))$. It follows that $|\widetilde{\Gamma}_{d,\mathfrak{p}}| = [\widetilde{\Gamma}_d : \widetilde{\Gamma}_d(\mathfrak{p})] = (N(\mathfrak{p})^3 - N(\mathfrak{p}))/2$. Let

$$\Omega_d = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_0^{\pm 1} \\ -\omega_0^{\mp 1} & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm\omega_d \\ 0 & 1 \end{pmatrix} \right\},$$

where $\omega_d = i, i\sqrt{2}, \frac{1+i\sqrt{3}}{2}, \frac{1+i\sqrt{7}}{2}, \frac{1+i\sqrt{11}}{2}$ for $d = 1, 2, 3, 7, 11$ respectively, and where $\omega_0 = \omega_d$ for $d = 1$ or 3 and $\omega_0 = -1$ otherwise. It follows from [8] that Ω_d is a symmetric generating set for $\widetilde{\Gamma}_{d,\mathfrak{p}}$. Note that $|\Omega_d| = 5$ for $d = 2, 7, 11$, $|\Omega_1| = 6$, and $|\Omega_3| = 7$. Let $G_{d,\mathfrak{p}} = G(\widetilde{\Gamma}_{d,\mathfrak{p}}, \Omega_d)$. The main result of this section is the following theorem.

THEOREM 5. *Let $d \in \{1, 2, 3, 7, 11\}$ and let $(\mathfrak{p}) \subset \mathcal{O}_d$ be a prime ideal such that $(2, \mathfrak{p}) \in \mathcal{O}_d^\times$. Then*

$$\iota(G_{d,\mathfrak{p}}) \le \frac{|\mathcal{O}_d^\times|}{4} \left(\frac{N(\mathfrak{p}) - 1}{N(\mathfrak{p}) + 1}\right). \tag{2}$$

Since $G_{d,\mathfrak{p}}$ is $|\Omega_d|$-regular and $|V(G_{d,\mathfrak{p}})| = (N(\mathfrak{p})^3 - N(\mathfrak{p}))/2$, we have that $|E(G_{d,\mathfrak{p}})| = |\Omega_d|(N(\mathfrak{p})^3 - N(\mathfrak{p}))/4$. Consider the unipotent subgroup of $\widetilde{\Gamma}_{d,\mathfrak{p}}$,

$$U_{d,\mathfrak{p}} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathcal{O}_d/(\mathfrak{p}) \right\}.$$

Then $|U_{d,\mathfrak{p}}| = N(\mathfrak{p})$. Consider the quotient $\widetilde{\Gamma}'_{d,\mathfrak{p}} = U_{d,\mathfrak{p}}\backslash\widetilde{\Gamma}_{d,\mathfrak{p}}$ and the quotient map $\phi : \widetilde{\Gamma}_{d,\mathfrak{p}} \to \widetilde{\Gamma}'_{d,\mathfrak{p}}$. Define a graph $G'_{d,\mathfrak{p}}$ with respect to the quotient by the following equivalence relation: For $v, w \in V(G_{d,\mathfrak{p}})$, we set $v \sim_U w$ if and only if $v = \omega w$ for $\omega \in U_{d,\mathfrak{p}} \cap \Omega_d$. Then we have $V(G'_{d,\mathfrak{p}}) = V(G_{d,\mathfrak{p}})/\sim_U$. Thus, the quotient map induces a map on the vertex sets $\phi : V(G_{d,\mathfrak{p}}) \to V(G'_{d,\mathfrak{p}})$. We say that v' is adjacent to w' in $G'_{d,\mathfrak{p}}$ if $v' \not\sim_U w'$ and there exists some $v \in \phi^{-1}(v')$ and some $w \in \phi^{-1}(w')$ so that v is adjacent to w in $G_{d,\mathfrak{p}}$. Therefore, we have an onto graph homomorphism $\phi : G_{d,\mathfrak{p}} \to G'_{d,\mathfrak{p}}$. Since left multiplication of a matrix by an element of $U_{d,\mathfrak{p}}$ leaves its bottom row unchanged, elements of $G'_{d,\mathfrak{p}}$ can be indexed by ordered pairs representing the bottom rows of matrices. That is, we have

$$\widetilde{\Gamma}'_{d,\mathfrak{p}} \cong \langle \pm 1_2\rangle\backslash\{(\alpha,\beta) \mid \alpha \text{ or } \beta \text{ is a unit in } \mathcal{O}_d/(\mathfrak{p})\}.$$

We can index $V(G'_{d,\mathfrak{p}})$ by the above set of ordered pairs. Further, $\widetilde{\Gamma}_{d,\mathfrak{p}}$ acts by multiplication on the right of $\widetilde{\Gamma}'_{d\mathfrak{p}}$ and so on the set of ordered pairs above, and this action commutes with the above isomorphism.

Note that by [9] and [16], the groups $\widetilde{\Gamma}_d$ have property (τ) with respect to the congruence subgroups $\widetilde{\Gamma}_d(\mathfrak{n})$. It follows, from [14] for example, that the graphs $G'_{d,\mathfrak{p}}$ form families of expanders in the sense of [13].

Lemma 1. *Let* $(\alpha,\beta),(\gamma,\delta) \in V(G'_{d,\mathfrak{p}})$. *Then* (α,β) *is adjacent to* (γ,δ) *if and only if*

$$\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{O}_d^\times.$$

Proof. Let $v', w' \in V(G'_{d,\mathfrak{p}})$ be adjacent vertices. Then there exists $v, w \in V(G_{d,\mathfrak{p}})$ so that $v \in \phi^{-1}(v')$, $w \in \phi^{-1}(w')$, and $v = xw$ with $x \in \Omega_d$ and $x \notin U_{d,\mathfrak{p}}$. It follows that $v = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) w$ or $v = \left(\begin{smallmatrix}0 & \omega_0^{\pm 1}\\ -\omega_0^{\mp 1} & 0\end{smallmatrix}\right) w$. Thus, $\det(v) = \det(w)$. Let $v = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. Taking ϕ of both sides of $v = xw$ gives

$$(c,d) = \phi(v) = \phi(xw) = \varepsilon(\alpha,\beta),$$

where $\varepsilon \in \{\pm 1, \pm\omega_0, \pm\omega_0^{\pm 1}\} \subseteq \mathcal{O}_d^\times$. It follows that

$$\det\begin{pmatrix}c & d\\ \gamma & \delta\end{pmatrix} = \det\begin{pmatrix}\varepsilon\alpha & \varepsilon\beta\\ \gamma & \delta\end{pmatrix} = \varepsilon \in \mathcal{O}_d^\times.$$

For the converse, note that $\det\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathcal{O}_d^\times$ implies $\left(\begin{smallmatrix}\varepsilon\alpha & \varepsilon\beta\\ \gamma & \delta\end{smallmatrix}\right) \in \widetilde{\Gamma}_{d,\mathfrak{p}}$ for some $\varepsilon \in \mathcal{O}_d^\times$. It follows that multiplication by $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0 & \omega_0^{\pm 1}\\ -\omega_0^{\mp 1} & 0\end{smallmatrix}\right)$ takes (γ,δ) to (α,β) in $\widetilde{\Gamma}'_{d,\mathfrak{p}}$. Thus, (α,β) and (γ,δ) are adjacent in $G'_{d,\mathfrak{p}}$. □

Note that $|V(G'_{d,\mathfrak{p}})| = (N(\mathfrak{p})^2 - 1)/2$ and that $G'_{d,\mathfrak{p}}$ is $|\mathcal{O}_d^\times|N(\mathfrak{p})/2$-regular. Thus, we have

$$|E(G'_{d,\mathfrak{p}})| = \frac{|\mathcal{O}_d^\times|N(\mathfrak{p})(N(\mathfrak{p})^2 - 1)}{8}.$$

We now prove a decomposition theorem for $G'_{d,\mathfrak{p}}$ that allows us to obtain a good estimate for $\iota(G'_{d,\mathfrak{p}})$. Let $\alpha \in (\mathcal{O}_d/(\mathfrak{p}))^\times$, and define $V_\alpha \subseteq V(G'_{d,\mathfrak{p}})$ by

$$V_\alpha = \{(0, u\alpha), (u'\alpha^{-1}, \beta) \mid \beta \in \mathcal{O}_d/(\mathfrak{p}), u, u' \in \mathcal{O}_d^\times\}.$$

Note that since $V_\alpha \subset \widetilde{\Gamma}'_{d,\mathfrak{p}}$, we have $|V_\alpha| = |\mathcal{O}_d^\times|(N(\mathfrak{p}) + 1)/2$. Let H_α denote the subgraph of $G'_{d,\mathfrak{p}}$ induced by V_α. Denote the vertices $\{(0, u\alpha) \mid u \in \mathcal{O}_d^\times/\langle \pm 1\rangle\}$ as the *center* of H_α, and all other vertices in V_α as the *crown* of H_α. Further note that every $v' \in V(G'_{d,\mathfrak{p}})$ is in V_α for some $\alpha \in (\mathcal{O}_d/(\mathfrak{p}))^\times$.

LEMMA 2. *The graph $G'_{d,\mathfrak{p}}$ decomposes into $(N(\mathfrak{p})-1)/|\mathcal{O}_d^\times|$ distinct copies of H_α, with $N(\mathfrak{p})|\mathcal{O}_d^\times|^3/4$ edges between any pair of distinct H_α's.*

Proof. For $\alpha, \alpha' \in (\mathcal{O}_d/(\mathfrak{p}))^\times$, we have $V_\alpha \cap V_{\alpha'} = \emptyset$ or $V_\alpha = V_{\alpha'}$ and so we can find distinct elements $\alpha_i \in (\mathcal{O}_d/(\mathfrak{p}))^\times$ such that

$$V(G'_{d,\mathfrak{p}}) = \bigsqcup_i V_{\alpha_i}.$$

Therefore, $V(G'_{d,\mathfrak{p}})$ decomposes into

$$\frac{N(\mathfrak{p})^2 - 1}{|\mathcal{O}_d^\times|(N(\mathfrak{p}) + 1)} = \frac{N(\mathfrak{p}) - 1}{|\mathcal{O}_d^\times|}$$

distinct copies of V_α. It follows that $G'_{d,\mathfrak{p}}$ decomposes into $(N(\mathfrak{p}) - 1)/|\mathcal{O}_d^\times|$ copies of H_α, plus the edges between pairs of H_α's. Note that this implies $(N(\mathfrak{p})-1)/|\mathcal{O}_d^\times| \in \mathbb{Z}$. By Lemma 1, every vertex in the center of H_α is adjacent to every vertex in the crown of H_α. This accounts for $|\mathcal{O}_d^\times|N(\mathfrak{p})/2$ edges for each vertex in the center.

For $u' \in \mathcal{O}_d^\times$, we have that

$$\det\begin{pmatrix} \alpha & \beta \\ u'\alpha & x \end{pmatrix} = u \in \mathcal{O}_d^\times$$

can be solved for $x \in \mathcal{O}_d/(\mathfrak{p})$ given any $u, u' \in \mathcal{O}_d^\times$. Since the solutions $(u'\alpha, x)$ are in $\widetilde{\Gamma}'_{d,\mathfrak{p}}$ we can choose any $u, u' \in \mathcal{O}_d^\times/\langle \pm 1\rangle$. Thus, there are $|\mathcal{O}_d^\times|^2/4$ solutions for each α. Since we also have solutions for $(-u'\alpha, x)$, it follows from Lemma 1 that (α, β) is adjacent to $|\mathcal{O}_d^\times|^2/2$ other vertices in the crown of H_α. Counting the edges connecting vertices in the crown plus the edges from the crown to the center, we obtain

$$|E(H_\alpha)| = \frac{|\mathcal{O}_d^\times|^3 N(\mathfrak{p})}{8} + \frac{|\mathcal{O}_d^\times|^2 N(\mathfrak{p})}{4} = \frac{N(\mathfrak{p})|\mathcal{O}_d^\times|^2(|\mathcal{O}_d^\times| + 2)}{8}.$$

As there are $(N(\mathfrak{p})-1)/|\mathcal{O}_d^\times|$ copies of H_α in $G'_{d,\mathfrak{p}}$, this gives a total of $N(\mathfrak{p})(N(\mathfrak{p})-1)|\mathcal{O}_d^\times|(|\mathcal{O}_d^\times|+2)/8$ edges in all of the H_α's. Thus, there are

$$\begin{aligned}\frac{N(\mathfrak{p})(N(\mathfrak{p})^2-1)|\mathcal{O}_d^\times|}{8} &- \frac{N(\mathfrak{p})(N(\mathfrak{p})-1)|\mathcal{O}_d^\times|(|\mathcal{O}_d^\times|+2)}{8}\\ &= \frac{N(\mathfrak{p})(N(\mathfrak{p})-1)|\mathcal{O}_d^\times|}{8}\big(N(\mathfrak{p})-|\mathcal{O}_d^\times|-1\big)\end{aligned}$$

edges in $G'_{d,\mathfrak{p}}$ connecting vertices in different H_α's. Since the number of edges between any distinct pair H_α and $H_{\alpha'}$ does not depend on α and α', there are

$$\frac{N(\mathfrak{p})(N(\mathfrak{p})-1)|\mathcal{O}_d^\times|(N(\mathfrak{p})-|\mathcal{O}_d^\times|-1)/8}{\binom{\frac{N(\mathfrak{p})-1}{|\mathcal{O}_d^\times|}}{2}} = \frac{N(\mathfrak{p})|\mathcal{O}_d^\times|^3}{4}$$

edges between any pair of H_α's. □

Proof of Theorem 5. From Lemma 2, we can view $G'_{d,\mathfrak{p}}$ as a complete multigraph on $(N(\mathfrak{p})-1)/|\mathcal{O}_d^\times|$ vertices, with $N(\mathfrak{p})|\mathcal{O}_d^\times|^3/4$ edges between any pair of vertices and where each vertex can be viewed as a copy of H_α. The isoperimetric number for the complete graph K_n from [15] is $\mathrm{iso}(K_n) = [(n+1)/2]$. For our set S, we take $(N(\mathfrak{p})-1)/(2|\mathcal{O}_d^\times|)$ copies of H_α. We compute that

$$\iota(G'_{d,\mathfrak{p}}) \le \frac{N(\mathfrak{p})-1}{2|\mathcal{O}_d^\times|}\frac{N(\mathfrak{p})|\mathcal{O}_d^\times|^3}{2|\mathcal{O}_d^\times|(N(\mathfrak{p})+1)} = N(\mathfrak{p})\frac{|\mathcal{O}_d^\times|}{4}\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})+1}.$$

Since $\iota(G_{d,\mathfrak{p}}) \le \iota(G'_{d,\mathfrak{p}})/N(\mathfrak{p})$, we obtain the estimate

$$\iota(G_{d,\mathfrak{p}}) \le \frac{|\mathcal{O}_d^\times|}{4}\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})+1}$$

as was to be shown. □

3. Cheeger constants of 3-manifolds

The fundamental domains $\mathcal{F}_d$ for the action of $\widetilde{\Gamma}_d$ on $\mathbb{H}^3$ were computed for many values of d by Bianchi in [1], and a systematic method of constructing such domains was developed by Swan in [19]. Let I_d be the Wigner–Seitz cell of $\mathcal{O}_d$ at the origin, see [18]. In particular, for $z = x+iy$ we have from [18],

$$\begin{aligned}(3)\qquad I_1 &= \{z\in\mathbb{C} \mid x\in[-1/2,1/2], y\in[0,1/2]\},\\ I_2 &= \big\{z\in\mathbb{C} \mid x\in[-1/2,1/2], y\in\big[-\sqrt{2}/2,\sqrt{2}/2\big]\big\},\\ I_3 &= \big\{z\in\mathbb{C} \mid x\in[0,1/2], y\in\big[-x/\sqrt{3},(1-x)/\sqrt{3}\big]\big\},\\ I_7 &= \big\{z\in\mathbb{C} \mid x\in[-1/2,1/2], y\in\big[(|x|-2)/\sqrt{7},(-|x|+2)/\sqrt{7}\big]\big\},\\ I_{11} &= \big\{z\in\mathbb{C} \mid x\in[-1/2,1/2], y\in\big[(|x|-3)/\sqrt{11},(-|x|+3)/\sqrt{11}\big]\big\}.\end{aligned}$$

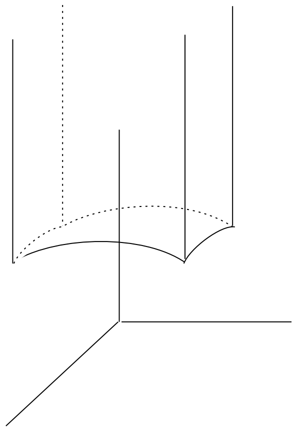

FIGURE 1. The fundamental domain $\mathcal{F}_1$.

Explicit fundamental domains $\mathcal{F}_d$ are therefore given by the following, see for example [1], [18], and [19].

LEMMA 3 (Bianchi, Steil, Swan). *For $d \in \{1,2,3,7,11\}$, a fundamental domain for the action of $\widetilde{\Gamma}_d$ on $\mathbb{H}^3$ is*

$$\mathcal{F}_d = \{(z,r) \in \mathbb{H}^3 \mid z \in I_d, |z|^2 + r^2 > 1\}. \tag{4}$$

The fundamental domain for the case $d=1$ is shown in Figure 1.

The manifold $\widetilde{X}_d(\mathfrak{p})$ is tiled by copies of $\mathcal{F}_d$ and this extends to a tiling of all of $\mathbb{H}^3$. Denote the collection of these tiles in $\widetilde{X}_d(\mathfrak{p})$ by $\widetilde{X}_d(\mathfrak{p})^*$, and denote the collection of the faces of these tiles by $\partial\widetilde{X}_d(\mathfrak{p})^*$. Two tiles are adjacent if and only if they share a common face. Any $\mathcal{F}_d \in \widetilde{X}_d(\mathfrak{p})^*$ is isometrically mapped by the action of $\widetilde{\Gamma}_d$ onto the particular domain given in (4). Further, any face in $\partial\widetilde{X}_d(\mathfrak{p})^*$ is isometrically mapped by $\widetilde{\Gamma}_d$ onto one of the faces of (4). Thus, the action of $\widetilde{\Gamma}_d$ defines an equivalence relation on $\partial\widetilde{X}_d(\mathfrak{p})^*$ whereby two faces $F_d, F'_d \in \partial\widetilde{X}_d(\mathfrak{p})^*$ are equivalent if and only if there is some $\gamma \in \widetilde{\Gamma}_d$ so that $F'_d = \gamma F_d$. Note that since the action is isometric, $a(F_d) = a(F'_d)$ for equivalent faces F_d and F'_d. The graphs $G_{d,\mathfrak{p}}$ can be obtained from $\widetilde{X}_d(\mathfrak{p})$ by associating one vertex to each tile in $\widetilde{X}_d(\mathfrak{p})^*$. Adjacent vertices in the graph correspond to tiles sharing a face. Therefore, each edge in $G_{d,\mathfrak{p}}$ corresponds to an element in $\partial\widetilde{X}_d(\mathfrak{p})^*$. Let $v, v' \in V(G_{d,\mathfrak{p}})$ correspond to tiles $\mathcal{F}_d, \mathcal{F}'_d$, respectively. Then $v' \sim v$ if and only if $\mathcal{F}'_d = \omega\mathcal{F}_d$ for some $\omega \in \Omega_d$. This gives a simple relation between $h(\widetilde{X}_d(\mathfrak{p}))$ and $\iota(G_{d,\mathfrak{p}})$. For $A \subset V(G_{d,\mathfrak{p}})$, let A^* denote the corresponding set of tiles in $\widetilde{X}_d(\mathfrak{p})^*$ and similarly, for $\partial A \subset E(G_{d,\mathfrak{p}})$ let ∂A^* denote the corresponding set of faces in $\partial\widetilde{X}_d(\mathfrak{p})^*$. Therefore, for $A^* \subset \widetilde{X}_d(\mathfrak{p})^*$

with $|A^*| \le |\widetilde{X}_d(\mathfrak{p})^*|/2$ we can estimate the Cheeger constant by

$$h(\widetilde{X}_d(\mathfrak{p})) \le \frac{\sum_{F_d \in \partial A^*} a(F_d)}{\sum_{\mathcal{F}_d \in A^*} v(\mathcal{F}_d)}.$$

Let $m(F_d) = \max(a(F_d))$ where F_d ranges over the faces of the domain in (4). As $v(\mathcal{F}_d)$ is the same for any tile, this gives

$$h(\widetilde{X}_d(\mathfrak{p})) \le \frac{|\partial A^*|}{|A^*|}\frac{m(F_d)}{v(\mathcal{F}_d)} = \frac{|\partial A|}{|A|}\frac{m(F_d)}{v(\mathcal{F}_d)}$$

for the corresponding $A \subset V(G_{d,\mathfrak{p}})$ with $|A^*| = |A| \le |V(G_{d,\mathfrak{p}})|/2$. As A^* is an arbitrary set in $\widetilde{X}_d(\mathfrak{p})^*$ with $|A^*| \le |V(G_{d,\mathfrak{p}})|/2$, it follows that

$$h(\widetilde{X}_d(\mathfrak{p})) \le \iota(G_{d,\mathfrak{p}})\frac{m(F_d)}{v(\mathcal{F}_d)}.$$

Let $d = 2, 7$ or 11. Consider an edge in $G_{d,\mathfrak{p}}$ incident with v and v' above that is obtained from multiplication by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Omega_d$. For simplicity, we denote this generator by J. Thus, $v' = Jv$. The face in $\partial\widetilde{X}_d(\mathfrak{p})^*$ that corresponds to this edge, $\mathcal{F}_d \cap \mathcal{F}'_d$ for example, is equivalent by the action of $\widetilde{\Gamma}_d$ to the face of $\mathcal{F}_d$ in (4) that borders the unit sphere centered at the origin. Now, for d above, the edges cut for estimate (2) of $\iota(G_{d,\mathfrak{p}})$ arose solely through multiplication by the generator J. So for $d = 2, 7$, or 11 a cut-set for $G_{d,\mathfrak{p}}$ corresponds to a cut-set of $\widetilde{X}_d(\mathfrak{p})$ where the cut is along the faces of tiles in $\widetilde{X}_d(\mathfrak{p})^*$. The cut-set used to determine (2) only removed those edges of $G_{d,\mathfrak{p}}$ that correspond to the action of J. The corresponding faces in $\partial\widetilde{X}_d(\mathfrak{p})$ that consist of the border of the bi-partition of $\widetilde{X}_d(\mathfrak{p})$ are all equivalent to the face of (4) that borders the unit sphere centered at the origin. Let $F_{d,0}$ be the face of $\mathcal{F}_d$ that corresponds to J. The faces above then all have area $a(F_{d,0})$. The volumes of the domains $\mathcal{F}_d$ are well known, see [7] for example, and can be expressed as

$$v(\mathcal{F}_d) = \frac{|D_{K_d}|^{3/2}\zeta_{K_d}(2)}{4\pi^2},$$

where $\zeta_{K_d}(s) = \frac{1}{4}\sum_{\mathfrak{n}} N(\mathfrak{n})^{-s}$ is the Dedekind zeta function of K_d and the sum is over the nonzero ideals of $\mathcal{O}_d$. It follows from Theorem 5 that we have

$$h(\widetilde{X}_d(\mathfrak{p})) \le \frac{1}{2}\frac{a(F_{d,0})}{v(\mathcal{F}_d)}\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})+1} = \frac{a(F_{d,0})2\pi^2}{|D_{K_d}|^{3/2}\zeta_{K_d}(2)}\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})+1}. \tag{5}$$

In the case $d = 1$, there is the complication that the graph $G_{d,\mathfrak{p}}$ contains edges corresponding to the generator $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. Nevertheless, formula (5) remains valid in this case. To simplify the notation, set $\omega = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. Let $d = 1$, and let $\mathcal{F}_1$ be the particular domain from (4). Note that the domains $\mathcal{F}_1$ and $J\omega\mathcal{F}_1$ share a face, and that the latter domain can be expressed as $\left(\begin{smallmatrix} 1 & -i/2 \\ 0 & 1 \end{smallmatrix}\right)\mathcal{F}_1$. Likewise, the domains $J\mathcal{F}_1$ and $\omega\mathcal{F}_1$ also share a face. Let

$v_1 \in V(G_{d,\mathfrak{p}})$ be the vertex that corresponds to $\mathcal{F}_1$. Every edge cut for the estimate of $\iota(G_{d,\mathfrak{p}})$ in Theorem 5 connects vertices in different sets $V(\alpha)$. However, it is easy to see that the pairs $\{v_1, J\omega v_1\}$ and $\{Jv_1, \omega v_1\}$ always reside in the same $V(\alpha)$. It follows that both elements of each of these pairs correspond to vertices on the same side of the cut. As a consequence, though there are four edges in the graph connecting the pair $\{\mathcal{F}_1, J\omega\mathcal{F}_1\}$ to $\{J\mathcal{F}_1, \omega\mathcal{F}_1\}$, these correspond to only two faces that need to be cut to separate them in the manifold. This shows that formula (5) remains valid. Thus, cutting along faces isometric to $F_{1,0}$ means that we only need to consider those edges from $G_{d,\mathfrak{p}}$ that correspond to the action of the generator J. In a similar way, we obtain the formula above for the $d=3$ case, and so (5) holds for $d \in \{1,2,3,7,11\}$. (Note that Lackenby in [10] obtained an estimate for the Cheeger constants of more general 3-manifolds in terms of the Heegaard genus.)

Since

$$F_{d,0} = \{(z,r) \in \mathcal{F}_d \mid z \in I_d \subset \mathbb{R}^2, |z|^2 + r^2 = 1\},$$

the hyperbolic area is

$$\begin{aligned} a(F_{d,0}) &= \iint_{I_d} \sqrt{1 + r_x^2 + r_y^2}\,\frac{dx\,dy}{|r|^2} \\ &= \iint_{I_d} \frac{1}{(1-x^2-y^2)^{3/2}}\,dx\,dy, \end{aligned}$$

where I_d is given by (3). This gives

$$\begin{aligned} a(F_{1,0}) &= 4\arcsin\bigl(\sqrt{2/3}\bigr) - \pi = 0.679673818\ldots, \\ a(F_{2,0}) &= 4\arcsin\bigl(2\sqrt{2/3}\bigr) + 4\arcsin\bigl(\sqrt{2/3}\bigr) - 2\pi = 2.461918835\ldots, \\ a(F_{3,0}) &= 4\arcsin\bigl(1/\sqrt{3}\bigr) - \arctan\bigl(1/\sqrt{3}\bigr) - \pi = 0.367523734\ldots, \\ a(F_{7,0}) &= 8\arcsin\bigl(1/2\bigr) + 4\arcsin\bigl(3/2\sqrt{3}\bigr) - 2\pi = 2.094395102\ldots, \\ a(F_{11,0}) &= 8\arcsin\bigl(1/\sqrt{3}\bigr) + 4\arcsin\bigl(5/3\sqrt{3}\bigr) - 2\pi = 3.821266473\ldots. \end{aligned}$$

From page 313 of [7], we have

$$v(\mathcal{F}_d) = 0.305321\ldots, 1.003841\ldots, 0.169156\ldots, 0.888914\ldots, 1.165895\ldots$$

for $d = 1,2,3,7,11$, respectively. Putting together the values of $a(F_{d,0})$ and $v(\mathcal{F}_d)$ into (5) gives Theorem 3.

4. Asymptotics of the Cheeger constants

Let $\mathcal{F}_d, \mathcal{F}'_d \in \widetilde{X}_d(\mathfrak{p})^*$ correspond to the elements $g, g' \in \widetilde{\Gamma}_{d,\mathfrak{p}}$, respectively. Consider the unipotent subgroup of $SL_2(\mathbb{C})$,

$$U = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \,\middle|\, z \in \mathbb{C} \right\}.$$

We define an equivalence relation $\sim_U$ on $\widetilde{X}_d(\mathfrak{p})^*$ by setting $\mathcal{F}_d \sim_U \mathcal{F}_d'$ if and only if $g = ug'$ for some $u \in U$. Denote the set of equivalence classes by $\widetilde{X}_d(\mathfrak{p})^U$. For $d = 1$, we can choose a different fundamental domain than (4) from Lemma 3. In particular, we choose the domain

$$\mathcal{F}_1 = \big\{(z,r) \in \mathbb{H}^3 \mid x \in [0,1], y \in [0,1/2], \\ r \geq \max\big(\sqrt{1-x^2-y^2}, \sqrt{1-(x-1)^2-y^2}\big)\big\}. \tag{6}$$

For this section, we tile $\widetilde{X}_1(\mathfrak{p})$ by copies, via the action of $\widetilde{\Gamma}_1(\mathfrak{p})$, of the above domain. We abuse notation and as in Section 3 denote the set of these tiles by $\widetilde{X}_1(\mathfrak{p})^*$. Note that any $\mathcal{F}_1 \in \widetilde{X}_1(\mathfrak{p})^*$ has a face F_1 that is equivalent to the face from (6) that consists of the union of the surfaces of two distinct unit spheres, but of the same hyperbolic area as the corresponding face from the domain given in (4). We now give a new tiling of $\widetilde{X}_d(\mathfrak{p})$ by altering these domains in the following way: Consider $F_1 = F_{1,A} \cup F_{1,B}$ where $F_{1,A}$ and $F_{1,B}$ are on the surfaces of distinct unit spheres. Replace these two faces with the single face F_1' which is on the surface of the sphere

$$\left(x - \frac{1}{2}\right)^2 + y^2 + r^2 = \frac{5}{4}.$$

That is, F_1' is the intersection of this sphere with the domain $\mathcal{F}_1$ from (6). The altered region is then defined to be

$$\mathcal{F}_1' = \{(z,r) \in \mathbb{H}^3 \mid x \in [0,1], y \in [0,1/2], |z - 1/2|^2 + r^2 \geq 5/4\}.$$

Note that in the example above, $a(F_1') < a(F_{1,A}) + a(F_{1,B}) = a(F_1)$. The new choice for fundamental domain, coupled with its alteration, is shown in Figure 2.

More generally, let $y_1 = y_7 = y_{11} = 0$, $y_2 = \sqrt{2}/2$, and $y_3 = \sqrt{3}/6$. Let $r_d^2 = 5/4 + y_d^2$ and

$$I_d' = \{z + 1/2 + iy_d \mid z \in I_d\}. \tag{7}$$

Thus, for $d \in \{1,2,3,7,11\}$ we take the altered region to be

$$\mathcal{F}_d' = \{(z,r) \in \mathbb{H}^3 \mid z \in I_d', |z - (1/2 + iy_d)|^2 + r^2 \geq r_d^2\}.$$

Let $\mathcal{F}_d(1), \mathcal{F}_d(2)$ be tiles in $\widetilde{X}_d(\mathfrak{p})^*$ so that $\mathcal{F}_d(1) \sim_U \mathcal{F}_d(2)$. Assume further that they share a common face, and note that their common face must correspond to an element of U. That is, there exists a unique $u \in U$ so that $\mathcal{F}_d(2) = u\mathcal{F}_d(1)$. It follows that their respective altered domains $\mathcal{F}_d(1)'$ and $\mathcal{F}_d(2)'$ are also adjacent and altered in the same way as above, so that their volumes are either both increased or both decreased. Therefore, tiles in the same equivalence class of $\widetilde{X}_d(\mathfrak{p})^*$ are altered in the same way. That is, if $\mathcal{F}_d(1) \sim_U \mathcal{F}_d(2)$ then $v(\mathcal{F}_d(1)') = v(\mathcal{F}_d(2)')$.

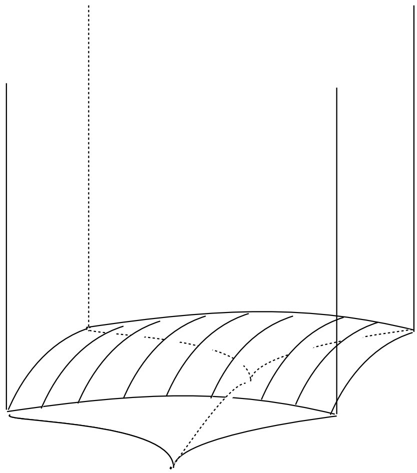

FIGURE 2. The modified domain for $d=1$.

Altering all of the tiles in $\widetilde{X}_d(\mathfrak{p})^U$ in this way gives a new tiling of $\widetilde{X}_d(\mathfrak{p})$. In this new tiling, 1/2 of the new tiles have volume strictly larger than that of $\mathcal{F}_d$ and 1/2 have volume strictly smaller than that of $\mathcal{F}_d$. For $n \in \mathbb{Z}$, denote by $\varepsilon(n)$ any positive function satisfying $\lim \varepsilon(n) = 0$ as $n \to \infty$. Consider the collection of equivalence classes $\{\widetilde{X}_d(\mathfrak{p})^U\}_{(\mathfrak{p})}$. Note that it is easy to show that $|\widetilde{X}_d(\mathfrak{p})^U| \to \infty$ as $N(\mathfrak{p}) \to \infty$. Therefore, we can divide each set of equivalence classes $\widetilde{X}_d(\mathfrak{p})^U$ randomly into two sets, $A^* = A^*(\mathfrak{p}) \subset \widetilde{X}_d(\mathfrak{p})^U$ and $B^* = B^*(\mathfrak{p}) \subset \widetilde{X}_d(\mathfrak{p})^U$, so that $|A^*|/|B^*| = 1 - \varepsilon(N(\mathfrak{p}))$. That is, A^* and B^* are asymptotically of equal size but without loss of generality we can choose A^* to be no larger than B^*. Furthermore, we can choose the sets A^* and B^* so that asymptotically, one half of the tiles in A^* have volume strictly larger than $\mathcal{F}_d$ and one half have strictly smaller volume, and similarly for B^*. Let

$$A^{*\pm} = \{\mathcal{F}'_d \in A^* \mid v(\mathcal{F}'_d) \gtrless v(\mathcal{F}_d)\}.$$

Thus, we have

$$|A^{*+}| = |A^*|/2 \pm \varepsilon(N(\mathfrak{p})),$$
$$|A^{*-}| = |A^*|/2 \mp \varepsilon(N(\mathfrak{p})),$$
$$A^* = A^{*+} \sqcup A^{*-}.$$

Note that if $\mathcal{F}_d^+ \in A^{*+}$ and $\mathcal{F}_d^- \in A^{*-}$ then $v(\mathcal{F}_d^+) + v(\mathcal{F}_d^-) = 2v(\mathcal{F}_d)$. As in the previous section, let $m(F'_d)$ denote the maximum of $a(F'_d)$, where F'_d ranges over the faces of the domain $\mathcal{F}'_d$. Therefore, for nontrivial $A^* \subset \widetilde{X}_d(\mathfrak{p})^*$

so that $|A^*| \le |\widetilde{X}_d(\mathfrak{p})^*|/2$ and corresponding sets $A \subset G_{d,\mathfrak{p}}$, we have

$$\begin{aligned} h(\widetilde{X}_d(\mathfrak{p})) &\le \frac{\sum_{F'_d \in \partial A^*} a(F'_d)}{\sum_{\mathcal{F}'_d \in A^*} v(\mathcal{F}'_d)} \\ &\le \frac{|\partial A^*| m(F'_d)}{\sum_{\mathcal{F}^+_d \in A^{*+}} v(\mathcal{F}^+_d) + \sum_{\mathcal{F}^-_d \in A^{*-}} v(\mathcal{F}^-_d)} \\ &= \frac{|\partial A^*| m(F'_d)}{|A^{*+}| v(\mathcal{F}^+_d) + |A^{*-}| v(\mathcal{F}^-_d)} \\ &\le \frac{|\partial A| m(F'_d)}{\frac{|A|}{2} v(\mathcal{F}^+_d) + \frac{|A|}{2} v(\mathcal{F}^-_d) - \varepsilon(N(\mathfrak{p}))(v(\mathcal{F}^+_d) + v(\mathcal{F}^-_d))} \\ &= \frac{|\partial A| m(F'_d)}{|A| v(\mathcal{F}_d) - \varepsilon(N(\mathfrak{p})) 2 v(\mathcal{F}_d)}. \end{aligned}$$

As $|A| \ge 1$, this is

$$\frac{|\partial A|}{|A|} \frac{m(F'_d)}{v(\mathcal{F}_d) - \varepsilon(N(\mathfrak{p}))2v(\mathcal{F}_d)/|A|} \le \frac{|\partial A|}{|A|} \frac{m(F'_d)}{v(\mathcal{F}_d) - \varepsilon(N(\mathfrak{p}))2v(\mathcal{F}_d)}.$$

As there is a 1–1 correspondence between $\widetilde{X}_d(\mathfrak{p})^*$ and $V(G_{d,\mathfrak{p}})$, this holds for any $A \subset V(G_{d,\mathfrak{p}})$ with $|A| \le |V(G_{d,\mathfrak{p}})|/2$. Thus, we have

$$h(\widetilde{X}_d(\mathfrak{p})) \le \iota(G_{d,\mathfrak{p}}) \frac{m(F'_d)}{v(\mathcal{F}_d) - \varepsilon(N(\mathfrak{p}))2v(\mathcal{F}_d)}.$$

As in Section 3, the edges removed for the cut-set in Theorem 5 are all equivalent to $F_{d,0}$, and thus are all altered in the same way in the new tiling above. Denote the altered face by

$$F'_{d,0} = \{(z,r) \in \mathbb{H}^3 \mid z \in I'_d, |z - (1/2 + iy_d)|^2 + r^2 = r_d^2\}. \tag{8}$$

Therefore, from Theorem 5, the arguments from Section 3, and the previous paragraph we have

$$h(\widetilde{X}_d(\mathfrak{p})) \le \frac{1}{2}\left(\frac{N(\mathfrak{p}) - 1}{N(\mathfrak{p}) + 1}\right) \frac{a(F'_{d,0})}{v(\mathcal{F}_d) - \varepsilon(N(\mathfrak{p}))2v(\mathcal{F}_d)}.$$

Taking the limit, we get the asymptotic version of (5),

$$\limsup_{N(\mathfrak{p}) \to \infty} h(\widetilde{X}_d(\mathfrak{p})) \le \frac{1}{2} \frac{a(F'_{d,0})}{v(\mathcal{F}_d)} = \frac{a(F'_{d,0}) 2\pi^2}{|D_{K_d}|^{3/2} \zeta_{K_d}(2)}. \tag{9}$$

To prove Theorem 4, we determine $a(F'_{d,0})$ and apply (9). From (7) and (8), we have

$$a(F'_{d,0}) = \iint_{I_d} \frac{r_d}{(r_d^2 - x^2 - y^2)^{3/2}} \, dx \, dy,$$

where I_d and r_d are given by (3) and (7). This gives us

$$a(F'_{1,0}) = 4\arcsin\big(\sqrt{10}/4\big) - \pi = 0.5053605\ldots,$$
$$a(F'_{2,0}) = 4\arcsin\big(\sqrt{7}/3\big) + 4\arcsin\big(\sqrt{7/15}\big) - 2\pi = 1.0446296\ldots,$$
$$a(F'_{3,0}) = 4\arcsin\big(2/\sqrt{13}\big) - \arctan(1/\sqrt{15}) - \pi/2 = 0.2576153\ldots,$$
$$a(F'_{7,0}) = 4\arcsin\big(3\sqrt{5}/8\big) + 8\arcsin\big(\sqrt{5}/2\sqrt{6}\big) - 2\pi = 1.4869790\ldots,$$
$$a(F'_{11,0}) = 4\arcsin\big(5\sqrt{5}/12\big) + 8\arcsin\big(\sqrt{5/24}\big) - 2\pi = 2.3048998\ldots.$$

The above results, the known values of $v(\mathcal{F}_d)$, and (9) prove Theorem 4.

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