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HERMITIAN MORITA EQUIVALENCES BETWEEN MAXIMAL ORDERS IN CENTRAL SIMPLE ALGEBRAS

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Abstract. Let R be a Dedekind domain with quotient field K. That every maximal order in a finite dimensional central simple K-algebra A, (the algebra of nxn matrices over D), where D is separable over K, is Morita equivalent to every maximal order in D is a well known linear result. Hahn defined the notion of Hermitian Morita equivalence (HME) for algebras with anti-structure, generalizing previous work by Frohlich and McEvett. The question this paper investigates is the hermitian analogue of the above linear result. Specifically, when are maximal orders with anti-structure in A, HME to maximal orders with anti-structure in D in the sense of Hahn? Two sets of necessary and sufficient conditions are obtained with an application which provides the hermitian analogue under some conditions.

1. Introduction

Morita theory essentially consists of Morita I and Morita II. The first derived the consequences of a Morita context and the latter concluded that every equivalence between categories is induced by a Morita context. Frohlich and McEvett [4] formulated a hermitian Morita theory for algebras with involution. There have been applications in K-theory of forms, Wall [9], and computation of surgery obstruction groups in Bak [1]. The standard for evaluating a successful hermitian Morita theory for algebras with anti-structure is how close it comes to the original Morita theory in terms of having analogues of Morita I and Morita II as well as other analogues. Hahn [5] formulated a far superior hermitian Morita theory than the one in Frohlich and McEvett [4]. It has a complete hermitian Morita I theorem and has had successful applications in the isomorphism theory of hyperbolic classical groups in Hahn

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and Zun-Xian [6], recently in the isomorphism theory of unitary groups over semisimple rings in Dasgupta [3] and in a forthcoming paper we apply it to that of hyperbolic quadratic modules over maximal orders in central simple algebras. These applications come up with the exact relationship between the underlying modules and rings of the unitary groups. These developments furnish positive evidence that Hahn's formulation is correct.

Let R be a Dedekind domain with quotient field $K \neq R$. Let Λ and Δ be rings. Λ and Δ are said to be Morita equivalent(ME) if there is an aggregate

$$(\Lambda, \Delta, {}_{\Lambda}M_{\Delta}, {}_{\Delta}(M^{*})_{\Lambda}\mu, \tau)$$

with μ and τ satisfying certain associativity properties. Now let Λ and Δ be maximal R-orders in a central simple K-algebra $A = M_n(D)$ and in D respectively, where D is separable over K. Then every Λ is ME to Δ by 21.6 and 21.7 of Reiner [7]. This is a well known linear result. For example, see also Chapter 5 of Swan [8].

Now suppose that $\Lambda = (\Lambda, \alpha, \varepsilon)$ and $\Delta = (\Delta, \beta, \delta)$ are algebras with anti-structure. Λ and Δ are said to be hermitian Morita equivalent(HME) if they are ME and if there is a map $\theta : {}_{\Lambda}M_{\Delta} \longrightarrow {}_{\Delta}(M^{*})_{\Lambda}$, satisfying certain properties Hahn [5].

We looked at the following problem: Assume further that $\alpha|_K = id_K$ and $\beta|_K = id_K$. Let $\Lambda = (\Lambda, \alpha, \varepsilon)$ and $\Delta = (\Delta, \beta, \delta)$ be maximal R-orders with anti-structure in A and D, respectively. Then are Λ and Δ, HME in the sense of Hahn [5]? An affirmative answer will be an interesting hermitian analogue of Morita equivalences between maximal R-orders and will be further validation of Hahn's formulation. While we cannot prove this to be true in general, we can come up with some useful necessary and sufficient conditions for them to be HME.

Let $(a^{\beta})_{ij} = a^{\beta}_{ji}$ for $a = (a_{ij}) \in A$ and let $a^{\alpha} = y_1 a^{\beta} y_1^{-1}$. Note that y_1 exists since $\beta^{-1}\alpha$ fixes each element of K, and hence is an inner automorphism by the Skolem–Noether theorem, Reiner [7]. Further, y_1 is unique up to multiplication by non-zero elements of K. A direct computation shows that $\delta^{-1} y_1^{\beta} y_1^{-1} \varepsilon = \pm 1$. If $V = KM$, assume that $\dim V_D \geq 2$. We show by an application of the necessary and sufficient conditions listed below that if y_1 can be chosen to be a unit of Λ, then Λ is HME to either Δ or $\Delta_1 = (\Delta, \beta, -\delta)$.

Let $v = x_1 d_1 + x_2 d_2 + \cdots + x_n d_n$ and let $v^{\beta} = d_1^{\beta} x_1^{*} + d_2^{\beta} x_2^{*} + \cdots + d_n^{\beta} x_n^{*}$ where $\{x_i^{*}\}$ is a dual basis of $\{x_i\}$, a basis of a vector space V. Assume $A = (A, \alpha, \varepsilon)$ and $D = (D, \beta, \delta)$ are HME via the aggregate,

$$\big(A, D, \theta : {}_{A}V_{D} \longrightarrow {}_{D}(V^{*})_{A}, \mu, \tau\big),$$

where V and V^{*} are represented by matrices and μ and τ are given by matrix multiplication. We come up with a new formulation or definition of θ which does not exist in the literature. We show that $\theta(v) = k v^{\beta} y_1^{-1}$ for some $k \in K$.

In fact, if

$$\big(\Lambda, \Delta, \theta : {}_{\Lambda}M_{\Delta} \longrightarrow {}_{\Delta}(M^*)_{\Lambda}, \mu, \tau\big)$$

is a set of hermitian equivalence data, where M and M^* are represented by matrices and μ and τ are given by multiplication of matrices, then we prove that $\theta(m) = km^{\beta}y_1^{-1}$. Conversely, if θ is a bijection given by $\theta(m) = km^{\beta}y_1^{-1}$, then either Λ and Δ or Λ and Δ_1 are HME.

We show that $y_1 M^{*\beta} P_1 = M$ for a $(\Delta - \Delta)$-bimodule P_1. We show that for Λ and Δ to be HME, it is necessary that $kP_1 = \Delta$ for some $k \in K$. Conversely, if $kP_1 = \Delta$ for some $k \in K$, either Λ and Δ or Λ and Δ_1 are HME.

2. Preliminaries

The only prerequisites for this paper are Hahn [5] and some basic facts about maximal orders in a central simple algebra as set out below. All Λ-modules M are assumed to be unitary i.e., $1_{\Lambda}m = m$ for all $m \in M$.

Let A and B be rings. Let

$$(A, B, {}_A P_B, {}_B Q_A, \mu, \tau)$$

be a Morita context, i.e., it is an aggregate such that

(a) there exists an $(A - A)$-bimodule isomorphism $\mu : P \otimes_B Q \longrightarrow A$,
(b) there exists $(B - B)$-bimodule isomorphism $\tau : Q \otimes_A P \longrightarrow B$,
(c) $\mu(p \otimes q)p' = p\tau(q \otimes p')$ for all $p, p' \in P$ and $q \in Q$,
(d) $\tau(q \otimes p)q' = q\mu(p \otimes q')$ for all $p \in P$ and $q, q' \in Q$,

so that ${}_A P_B$ is a bimodule over R, i.e. $(r.1)p = p(r.1)$. The rings A and B are then said to be Morita equivalent (ME).

Let R be a commutative ring with a involution $^{-}$. An R-algebra with anti-automorphism is a pair (A, α) with A an R-algebra, α an anti-automorphism and $\bar{r}.1 = (r.1)^{\alpha}$ for all $r \in R$. If $\alpha^2 = id_A$, then (A, α) is an R-algebra with involution. An R-algebra with anti-structure is a triple $A = (A, \alpha, \varepsilon)$ consisting of an R-algebra A with anti-automorphism α and a unit ε in A such that $\varepsilon^{\alpha} = \varepsilon^{-1}$ and $a^{\alpha^2} = \varepsilon a \varepsilon^{-1}$.

Assumption 1. In the rest of the paper, the anti-automorphism of the R-algebra with anti-structure is an R-anti-automorphism, i.e., $r^{\alpha} = r$ for $r \in R$.

Let $A = (A, \alpha, \varepsilon)$ and $B = (B, \beta, \delta)$ be two rings with anti-structure and regard both A and B as R-algebras with anti-structure. Recall from Hahn [5] that a set of hermitian equivalence data is a Morita context along with a map θ which satisfies items (e)–(g) below.

(e) $\theta : P \longrightarrow Q$ which satisfies $\theta(apb) = b^{\beta}\theta(p)a^{\alpha}$, and
(f) $\mu(p \otimes \theta(p_1)) = \mu(p_1 \otimes \theta(\varepsilon^{-1}p\delta))^{\alpha}$, and
(g) $\tau(\theta(p) \otimes p_1) = \tau(\theta(p_1\delta) \otimes \varepsilon p)^{\beta}$,

i.e., a hermitian equivalence data is an aggregate,

$$\left(A, B, \theta : {}_A P_B \longrightarrow {}_B Q_A, \mu, \tau\right)$$

such that items (a)–(g) above are satisfied. The algebras A and B are then said to hermitian Morita equivalent (HME).

ASSUMPTION 2. In the rest of this section, let R be a Dedekind domain with quotient field K, $R \neq K$, and let A be a separable K-algebra. In particular, assume that A is a finite dimensional central simple K-algebra, $A = M_n(D)$, where D is a separable K-algebra.

The following facts and definitions hold under these conditions.

(1) An R-lattice is a finitely generated R-torsion free R-module. If M is a R-torsion free R module, then $M \otimes_R K = M.K = \{\sum_{\text{finite}} m_i k_i | m_i \in M, k_i \in K\}$ (p. 44 of Reiner [7]). An R-lattice is a R-submodule of a finite dimensional vector space $V = KM$ over K.
(2) A full R-lattice in finite dimensional V, a D-vector space, is a finitely generated R-submodule M such that $KM = V$ (p. 108 of Reiner [7]).
(3) If L is a left Λ-module with $1 \in \Lambda$, $L \subseteq \Lambda L$ since L is unitary. Since L is a Λ-module, $\Lambda L \subseteq L$. Hence, $L = \Lambda L$.
(4) An R-order in the K-algebra A is a subring Λ of A with the same unity as A, and such that Λ is a full R-lattice in A (p. 108 of Reiner [7]).
(5) A maximal R-order R-order in A is an R-order which is not properly contained in any other R-order in A (p. 110 of Reiner [7]).
(6) Further, if $\Lambda = (\Lambda, \alpha, \varepsilon)$ is a maximal R-order with anti-structure in A, since it is assumed that α is an R-anti-automorphism, the anti-structure on Λ extends uniquely to A via $a^\alpha = (\lambda r^{-1})^\alpha = (\lambda)^\alpha r^{-1}$ and then (A, α, ε) is a K-algebra and α is a K-anti-automorphism.

If L is a right R-lattice in A, then define

(i) the left order of $L = O_l(L) = \{x \in A | xL \subseteq L\}$ is an R-order in A (p. 109 of Reiner [7]),
(ii) the right order of $L = O_r(L) = \{x \in A | Lx \subseteq L\}$ is an R-order in A (p. 109 of Reiner [7]) and
(ii) $L^{-1} = \{x \in A | Lx \subseteq O_l(L)\} = \{x \in A | xL \subseteq O_r(L)\}$ (p. 192 of Reiner [7]).

Let Λ be an R-order in A. The following hold.

(v) L is called a (full) right Λ-lattice (in A) to indicate that L is a right Λ-module which is a (full) R-lattice (in A) (pp. 129 and 192 of Reiner [7]).
(w) If L is a full right R-lattice in A, then the left order of L is maximal if and only if the right order of L is maximal (21.2 of Reiner [7]).
(x) If L is a full right Λ lattice in A, where Λ is a maximal order in A, then $LL^{-1} = O_l(L) = O_r(L^{-1})$, $L^{-1}L = \Lambda = O_r(L) = O_l(L^{-1})$ and $(L^{-1})^{-1} = L$ (pp. 192–193 of Reiner [7]).

(y) If Λ is a maximal R-order of A and Δ is a maximal R-order of D, $\Lambda = \operatorname{End}(M_\Delta)$ for some full right Δ-lattice M in V. Further, every maximal R-order in A is Morita equivalent(ME) to any maximal R-order in D. This follows from 21.6 and 21.7 of Reiner [7].
(z) For the structure of Λ in terms of Δ, refer to 27.6 of Reiner [7].

Let β be an anti-automorphism of D. If $A = \operatorname{End}(V_D)$ and if for $v \in V$ and $d_i \in D$,

$$v = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}, \tag{1}$$

then define

$$v^\beta = \begin{bmatrix} d_1^\beta & d_2^\beta & \cdots & d_n^\beta \end{bmatrix}. \tag{2}$$

If for $v^* \in V^*$ and $d_i' \in D$,

$$v^* = \begin{bmatrix} d_1' & d_2' & \cdots & d_n' \end{bmatrix}, \tag{3}$$

then define

$$(v^*)^\beta = \begin{bmatrix} d_1'^\beta \\ d_2'^\beta \\ \vdots \\ d_n'^\beta \end{bmatrix}. \tag{4}$$

Hence, $v^{\beta^2} = \delta v \delta^{-1}$ and $v^{*\beta^2} = \delta v^* \delta^{-1}$ where for $d \in D$, $d^{\beta^2} = \delta d \delta^{-1}$.
For a set V_1 of V or V^* define $V_1^\beta = \{v^\beta | v \in V_1\}$.
Extend β to A thus: For $a = (a_{ij}) \in A$, define a^β by $(a^\beta)_{ij} = a_{ji}^\beta$.

3. The structure of $\theta : {}_AV_D \longrightarrow {}_D(V^*)_A$

Assumption 3. Throughout this section, let $A = (A = M_n(D), \alpha, \varepsilon)$ and $D = (D, \beta, \delta)$ be central simple K-algebras with anti-structure.

So α and β are K-anti-automorphisms. If $a = (a_{ij}) \in A$, extend β to A as in Section 2. Then β is a K-anti-automorphism of A. Note $\beta^{-1}\alpha$ determines a K-automorphism of the central-simple algebra A and hence is an inner automorphism of A by the Skolem–Noether theorem, Reiner [7]. Hence, $a^\alpha = y_1 a^\beta y_1^{-1}$ for some y_1 of A where y_1 is unique up to multiplication by a nonzero scalar. This section comes up with a definition for θ of a hermitian equivalence data between A and D. Let $u(\Lambda)$ be the units of Λ.

Proposition 1. *Let $a^\alpha = y_1 a^\beta y_1^{-1}$. Then $\delta^{-1} y_1^\beta y_1^{-1} \varepsilon = \pm 1$.*

Proof. Observe for $\lambda \in \Lambda$, $\lambda^\alpha = y_1 \lambda^\beta y_1^{-1}$ and

$$\varepsilon\lambda\varepsilon^{-1} = \lambda^{\alpha^2} = y_1(y_1^{-1})^\beta \lambda^{\beta^2} y_1^\beta y_1^{-1} = y_1(y_1^{-1})^\beta \delta\lambda\delta^{-1} y_1^\beta y_1^{-1}.$$

So

$$\delta^{-1} y_1^\beta y_1^{-1} \varepsilon\lambda = \lambda\delta^{-1} y_1^\beta y_1^{-1}\varepsilon$$

and

$$\delta^{-1} y_1^\beta y_1^{-1}\varepsilon = r,$$

for some r is in the centralizer of Λ in A. Since Λ is a full lattice in A, centralizer of Λ in $A \subseteq \text{Cen } A$. But since A is a central simple K-algebra, Cen $A = K$. So $r \in K$. So

$$\varepsilon = y_1(y_1^{-1})^\beta \delta r.$$

But

$$\varepsilon^\alpha = \varepsilon^{-1}$$

and

$$\begin{aligned}(y_1(y_1^{-1})^\beta\delta)^\alpha &= y_1(y_1(y_1^{-1})^\beta\delta)^\beta y_1^{-1}\\ &= y_1\delta^{-1}\delta y_1^{-1}\delta^{-1}y_1^\beta y_1^{-1}\\ &= \delta^{-1}y_1^\beta y_1^{-1}\\ &= (y_1(y_1^{-1})^\beta\delta)^{-1}.\end{aligned}$$

So $r^\alpha = r^{-1}$. Since $\alpha|_K = id$, $r^2 = 1$, and since K is an integral domain $r = \pm 1$. ☐

Lemma 2. *If B, $B(v)$, E, $E(v)$ are $n \times 1$ matrices and C, $C(v')$, F, $F(v')$ are $1 \times n$ matrices, then the following hold.*

(a) *If $BC = EF$, then $B = Cd$ and $E = d^{-1}F$ for $d \in D$ and*
(b) *if $B(v)C(v') = E(v)F(v')$ $\forall v, v' \in V$, then $B(v) = E(v)d_1$ and $C(v') = d_1^{-1}F(v')$ for a constant $d_1 \in D$.*

Proof. (a) Let $B = (b_{i1})$, $E = (e_{i1})$, $C = (c_{1j})$ and $F = (f_{1j})$, and observe that since $b_{i1}c_{1j} = e_{i1}f_{1j}$, if $b_{11} = e_{11}d$, $c_{1j} = d^{-1}f_{1j}$ for $j = 1, \ldots, n$. Hence, $b_{i1} = e_{i1}d$ for $i = 1, \ldots, n$, $B = Ed$ and $C = d^{-1}F$.

(b) Note that if $B(v)C(v') = E(v)F(v')$, then for a fixed $v \in V$, $B(v) = E(v)x_v$ and $\forall v' \in V$, $C(v') = x_{v'}^{-1}F(v') = x_v^{-1}F(v')$. Here the latter follows from (a). Hence, $x_v = x_{v'}$. By symmetry $x_v = x_{v'}$, $\forall v \in V$ and a fixed $v' \in V$. Thus, $d_1 = x_v = x_{v'}$ $\forall v, v' \in V$. ☐

Proposition 3. *Let*

$$(A, D, {}_AV_D, {}_D(V^*)_A, \mu, \tau)$$

be a Morita context where V is the set of $n \times 1$ matrices over D and V^* is the set of $1 \times n$ matrices over D and μ and τ are given by multiplication of matrices, i.e., $\mu(v \otimes f) = vf$ and $\tau(f \otimes v) = fv$ where $v \in V$ and $f \in V^*$.

Let y_1 satisfy $a^\alpha = y_1 a^\beta y_1^{-1}$. If $A=(A=M_n(D),\alpha,\varepsilon)$ is hermitian Morita equivalent to $D=(D,\beta,\delta)$ via

$$\big(A, D, \theta : {}_AV_D \longrightarrow {}_D(V^*)_A, \mu, \tau\big),$$

then $\theta: V \longrightarrow V^*$ is a bijection defined by $\theta(v)=kv^\beta y_1^{-1}$, for some $k\in K$. If $\theta: V \longrightarrow V^*$ is a bijection defined by $\theta(v)=kv^\beta y_1^{-1}$, for some $k\in K$,

(I) *then (A,α,ε) is HME to (D,β,δ) when $\delta^{-1}y_1^\beta y_1^{-1}\varepsilon=1$ via*

$$\big(A, D, \theta : {}_AV_D \longrightarrow {}_D(V^*)_A, \mu, \tau\big),$$

(II) *then (A,α,ε) is HME to $D_1=(D,\beta,-\delta)$ when $\delta^{-1}y_1^\beta y_1^{-1}\varepsilon=-1$ via*

$$\big(A, D_1, \theta : {}_AV_{D_1} \longrightarrow {}_{D_1}(V^*)_A, \mu, \tau\big).$$

Proof. β extends to A. Let $\dim V_D = n$. As above $a^\alpha = y_1 a^\beta y_1^{-1}$ for some y_1 in A and $a\in A$. By Proposition 1, $\delta^{-1}y_1^\beta y_1^{-1}\varepsilon=\pm 1$. Let

$$(A, D, {}_AV_D, {}_D(V^*)_A, \mu, \tau)$$

be a Morita context as stated in the proposition. Assume that θ and μ satisfy items (a)–(g) of an hermitian equivalence data. In particular, $\theta: V \longrightarrow V^*$ is a bijection satisfying $\theta(avd)=d^\beta\theta(v)a^\alpha$ and that $\mu(v\otimes\theta(v')=\mu(v'\otimes\theta(\varepsilon^{-1}v\delta))^\alpha$. We will try to solve for θ(v). Since $a^\alpha = y_1 a^\beta y_1^{-1}$,

$$\begin{aligned}\mu\big(v\otimes\theta(v')\big) &= y_1\mu\big(v'\otimes\theta(\varepsilon^{-1}v\delta)\big)^\beta y_1^{-1},\\ (v\theta(v')) &= y_1(v'\theta(\varepsilon^{-1}v\delta))^\beta y_1^{-1},\\ v\theta(v') &= y_1\theta(\varepsilon^{-1}v\delta)^\beta v'^\beta y_1^{-1}.\end{aligned}$$

Now v, v' are $n\times 1$ matrices since $v,v'\in V$, v^β, v'^β are $1\times n$ matrices by the definition of v^β for $v\in V$ in Section 2, $\theta(v)$, $\theta(v')$ are $1\times n$ matrices since $\theta: V\longrightarrow V^*$, $\theta(v)^\beta$, $\theta(v')^\beta$ are $n\times 1$ matrices by the definition of $v^{*\beta}$ for $v^*\in V^*$ in Section 2, and y_1 is a $n\times n$ matrix. By Lemma 2,

$$v = y_1\theta(\varepsilon^{-1}v\delta)^\beta x \quad\text{and}\quad \theta(v') = x^{-1}v'^\beta y_1^{-1} \quad \text{for an } x\in D.$$

It is easily checked that these two equations form a consistent set of equations having the solution $\theta(v)=x^{-1}v^\beta y_1^{-1}$. Since

$$\theta(vd) = d^\beta\theta(v), \quad x^{-1}d^\beta v^\beta y_1^{-1} = d^\beta x^{-1} v^\beta y_1^{-1},$$

x^{-1} commutes with every $d\in D$, so $x^{-1}\in K$. Thus, $\theta: V\longrightarrow V^*$, a bijection is defined by $\theta(v)=kv^\beta y_1^{-1}$ for some $k=x^{-1}\in K$.

Conversely, if $\theta: V\longrightarrow V^*$ is a bijection defined by $\theta(v)=kv^\beta y_1^{-1}$ for some $k\in K$, we will show (I) and (II) below.

(I) If $\delta^{-1}y_1^\beta y_1^{-1}\varepsilon=1$, we will prove that (A,α,ε) is HME to (D,β,δ). All that is required is to show that (e)–(g) below of a hermitian equivalence data (refer to Section 2 for the definition of a hermitian equivalence data) are satisfied since by assumption a Morita context between A and D already exists.

$$\begin{aligned}
&\text{(e)} & \theta(avd) &= kd^\beta v^\beta a^\beta y_1^{-1}\\
&& &= d^\beta kv^\beta y_1^{-1} a^\alpha \quad \text{since } a^\alpha = y_1 a^\beta y_1^{-1}\\
&& &= d^\beta \theta(v) a^\alpha,\\
&\text{(f)} & \mu\big(v' \otimes \theta(\varepsilon^{-1} v\delta)\big)^\alpha &= y_1 \mu\big(v' \otimes \theta(\varepsilon^{-1} v\delta)\big)^\beta y_1^{-1}\\
&& &= y_1 (v' k\delta^\beta v^\beta \varepsilon^{-1\beta} y_1^{-1})^\beta y_1^{-1}\\
&& &= y_1 y_1^{-1\beta} \delta\varepsilon^{-1}\delta^{-1}\delta v\delta^{-1}\delta\delta\delta^{-1} k^\beta v'^\beta y_1^{-1}\\
&& &\qquad\qquad \text{since } d^{\beta^2} = \delta d\delta^{-1}\\
&& &= vkv'^\beta y_1^{-1} \quad \text{since } y_1 y_1^{-1\beta}\delta\varepsilon^{-1} = 1 \text{ and } \beta|_K = id_K\\
&& &= \mu\big(v \otimes \theta(v')\big),\\
&\text{(g)} & \tau\big(\theta(v'\delta) \otimes \varepsilon v\big)^\beta &= (k\delta^\beta v'^\beta y_1^{-1}\varepsilon v)^\beta\\
&& &= (\delta^{-1} v'^\beta y_1^{-1} \varepsilon v)^\beta k^\beta \quad \text{since } \delta^\beta = \delta^{-1}\\
&& &= (\delta^{-1} v'^\beta (y_1^{-1})^\beta \delta v)^\beta k \quad \text{since } \delta^{-1} y_1^\beta y_1^{-1}\varepsilon = 1\\
&& &\qquad\qquad \text{and } \beta|_K = id_K\\
&& &= v^\beta \delta^\beta \delta y_1^{-1} \delta^{-1} \delta v' \delta^{-1} \delta k\\
&& &= kv^\beta y_1^{-1} v' \quad \text{since } \delta^\beta = \delta^{-1}\\
&& &= \tau\big(\theta(v) \otimes v'\big).
\end{aligned}$$

(II) If $\delta^{-1} y_1^\beta y_1^{-1}\varepsilon = -1$, then we will prove that (A, α, ε) is HME to $(D, \beta, -\delta)$. We will do so by replacing δ by $-\delta$ in (I) above and repeating the steps. Note then that $(-\delta)^\beta = -\delta^{-1}$, $(-\delta)^{-1} = -\delta^{-1}$ and $\delta^{-1} y_1^\beta y_1^{-1}\varepsilon = 1$ gets replaced by by $-\delta^{-1} y_1^\beta y_1^{-1}\varepsilon = 1$. □

4. Hermitian Morita equivalences between maximal orders

ASSUMPTION 4. Throughout this section, $\Lambda = (\Lambda = \mathrm{End}(M_\Delta), \alpha, \varepsilon)$ is a maximal R-order in $A = \mathrm{End}((KM)_D) = M_n(D)$, a central simple K-algebra, and $\Delta = (\Delta, \beta, \delta)$ is a maximal R-order with anti-structure in D where α and β are R-anti-automorphisms. Further, D is separable over K.

So by 27.6 of Reiner [7], for J a right Δ-lattice in D, $\{x_i\}$ a basis of $V = KM$, and $\{x_i^*\}$ the dual basis of $\{x_i\}$,

$$\begin{aligned}
M &= x_1\Delta + x_2\Delta + \cdots + x_{n-1}\Delta + x_n J,\\
M^* &= \Delta x_1^* + \cdots + \Delta x_{n-1}^* + J^{-1} x_n^*
\end{aligned}$$

and

$$\Lambda = \mathrm{End}(M_\Delta) = \begin{bmatrix} \Delta & \Delta & \cdots & \Delta & J^{-1} \\ \Delta & \Delta & \cdots & \Delta & J^{-1} \\ \vdots & \vdots & \vdots & \vdots & \\ \Delta & \Delta & \cdots & \Delta & J^{-1} \\ J & J & \cdots & J & \Delta' \end{bmatrix}, \tag{5}$$

where $\Delta' = O_l(J)$ by 27.6 of Reiner [7].

Consider M as a subset of $n \times 1$ matrices, so

$$M = \begin{bmatrix} \Delta \\ \Delta \\ \vdots \\ \Delta \\ J \end{bmatrix} \tag{6}$$

and

$$M^* = \begin{bmatrix} \Delta & \Delta & \cdots & \Delta & J^{-1} \end{bmatrix}. \tag{7}$$

The module M_Δ is a progenerator of $\mathcal{M}_\Delta$. Hence, Λ and Δ are ME by 21.7 of Reiner [7], and by 16.9 of Reiner [7], via the Morita context below derived from M,

$$\big(\Lambda, \Delta, {}_\Lambda M_\Delta \longrightarrow {}_\Delta(M^*)_\Lambda, \mu, \tau\big),$$

where μ and τ are given by multiplication of matrices.

Note that by matrix multiplication it follows that M is a $\Lambda - \Delta$ bimodule, M^* is a $\Delta - \Lambda$ bimodule. Recall the definition of $M^{*\beta}$ and M^β from Section 2. Then M^β is a $\Delta^\beta - \Lambda^\beta$ bimodule and $M^{*\beta}$ is a $\Lambda^\beta - \Delta^\beta$ bimodule. Both Λ^β and Δ^β are maximal orders.

Further $A = K\Lambda = M_n(D)$ and, as in Section 3, α extends to a K-anti-automorphism of A. Moreover, β extends to K-anti-automorphisms of D and A and $a^\alpha = y_1 a^\beta y_1^{-1}$ for some $y_1 \in A$.

Proposition 4 below shows that there always exists a $(\Delta - \Delta)$-bimodule P_1, such that $y_1(M^*)^\beta P_1 = M$.

Proposition 4. *Let $a^\alpha = y_1 a^\beta y_1^{-1}$ for some $y_1 \in A$. Then there exists a $(\Delta - \Delta)$-bimodule P_1 which is a full R-lattice in D such that $y_1(M^*)^\beta P_1 = M$ and $P_1^{-1} M^\beta y_1^{-1} = (y_1 M^{*\beta} P_1)^* = M^*$.*

Proof. Let $V = KM$ and $\dim V_D \geq 2$. Let $y_1 P = M$. Since M is a finitely generated R-module, so is P. P is a right Δ-module which is a full R-lattice in V since $Ky_1P = KM = V$. Let

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}. \tag{8}$$

Since each P_i is a summand of P it is a finitely generated R-module. Since $KP = y_1^{-1}V$, $KP_i = D$ and P_i is a full right Δ-lattice in D for each i. Hence, as in 27.6 of Reiner [7],

$$\operatorname{End}(P_\Delta) = \begin{bmatrix} Hom_\Delta(P_1,P_1) & Hom_\Delta(P_2,P_1) & \cdots & Hom_\Delta(P_n,P_1) \\ Hom_\Delta(P_1,P_2) & Hom_\Delta(P_2,P_2) & \cdots & Hom_\Delta(P_n,P_2) \\ \vdots & \vdots & \vdots & \vdots \\ Hom_\Delta(P_1,P_n) & Hom_\Delta(P_2,P_n) & \cdots & Hom_\Delta(P_n,P_n) \end{bmatrix}. \tag{9}$$

Since $P_i^{-1}P_i = O_r(P_i) = \Delta$, $P_iP_i^{-1} = O_l(P_i)$ and $P_i\Delta = P_i$ by 22.7 of Reiner [7],

$$\operatorname{End}(P_\Delta) = \begin{bmatrix} P_1P_1^{-1} & P_1P_2^{-1} & \cdots & P_1P_n^{-1} \\ P_2P_1^{-1} & P_2P_2^{-1} & \cdots & P_2P_n^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ P_nP_1^{-1} & P_nP_2^{-1} & \cdots & P_nP_n^{-1} \end{bmatrix}. \tag{10}$$

Now, $\operatorname{End}(P_\Delta) = \Lambda^\beta$, as $y_1 \operatorname{End}(P_\Delta)y_1^{-1} = \operatorname{End}((y_1P)_\Delta)) = \operatorname{End}(M_\Delta) = \Lambda = \Lambda^\alpha = y_1\Lambda^\beta y_1^{-1}$. Let $\operatorname{End}(P_\Delta)$ act on P on the left in the natural way, so P is a left Λ^β-module.

Now, by the structure of Λ as at the beginning of Section 4, and the definition of a^β for $a \in A$,

$$\Lambda^\beta = \begin{bmatrix} \Delta^\beta & \Delta^\beta & \cdots & \Delta^\beta & J^\beta \\ \Delta^\beta & \Delta^\beta & \cdots & \Delta^\beta & J^\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta^\beta & \Delta^\beta & \cdots & \Delta^\beta & J^\beta \\ J^{-1\beta} & J^{-1\beta} & \cdots & J^{-1\beta} & \Delta'^\beta \end{bmatrix}. \tag{11}$$

We will show by comparing matrices $(M^*)^\beta P_1$ and P that $(M^*)^\beta P_1 = P$, and hence $y_1(M^*)^\beta P_1 = y_1P = M$. Compare the two matrices Λ^β and $\operatorname{End}(P_\Delta)$ to get $P_1P_1^{-1} = P_2P_1^{-1} = \cdots = P_{n-1}P_1^{-1} = \Delta^\beta$ and $P_nP_1^{-1} = (J^{-1})^\beta$. Since, by 22.7 of Reiner [7] $P_1^{-1}P_1 = O_r(P_1) = \Delta$, and since P_i are right Δ-modules, $P_1 = P_2 = P_{n-1} = \Delta^\beta P_1$ and $P_n = (J^{-1})^\beta P_1$. Thus, P_1 is a left Δ^β-module and

$$P = \begin{bmatrix} \Delta^{\beta} P_1 \\ \Delta^{\beta} P_1 \\ \vdots \\ \Delta^{\beta} P_1 \\ (J^{-1})^{\beta} P_1 \end{bmatrix}. \tag{12}$$

Recall the definition of V_1^{β} for $V_1 \subseteq V^*$ from Section 2, so

$$(M^*)^{\beta} = \begin{bmatrix} \Delta^{\beta} \\ \Delta^{\beta} \\ \vdots \\ \Delta^{\beta} \\ (J^{-1})^{\beta} \end{bmatrix}. \tag{13}$$

Now compare matrices to obtain $(M^*)^{\beta} P_1 = P$ and $y_1 (M^*)^{\beta} P_1 = M$.

Now let $\dim V_D = 1$. We will show in this case too that $y_1 (M^*)^{\beta} P_1 = M$. Let $\Lambda = \mathrm{End}(J_{\Delta})$ for a right Δ-lattice J in D and assume that $\lambda^{\alpha} = y_1 \lambda^{\beta} y_1^{-1}$ and that $y_1^{-1} J = J_1$ for a right Δ-lattice J_1. Obtain $\mathrm{End}((J_1)_{\Delta}) = (\mathrm{End}(J_{\Delta}))^{\beta}$ since $y_1 \mathrm{End}((J_1)_{\Delta}) y_1^{-1} = \mathrm{End}(J_{\Delta}) = \Lambda = \Lambda^{\alpha} = (\mathrm{End}(J_{\Delta}))^{\alpha} = y_1 (\mathrm{End}(J_{\Delta}))^{\beta} y_1^{-1}$. J is a left $\mathrm{End}(J_{\Delta})$-lattice and $\mathrm{End}(J_{\Delta})$ is a maximal order (21.2 of Reiner [7]). Hence, $O_l(J) \subseteq \mathrm{End}(J_{\Delta}) \subseteq O_l(J)$. Thus, obtain $O_l(J) = \mathrm{End}(J_{\Delta})$. Similarly obtain $O_l(J_1) = \mathrm{End}((J_1)_{\Delta})$, to give $O_l(J_1) = (O_l(J))^{\beta}$. Let $P_1 = J^{\beta} J_1$. Clearly, P_1 is a $\Delta^{\beta} - \Delta$-module, since $J\Delta \subseteq J$ and $\Delta^{\beta} J^{\beta} \subseteq J^{\beta}$. By 22.7 of Reiner [7], $JJ^{-1} = O_l(J)$. By setting $M = J$ and $P = J_1$, obtain $y_1 (M^*)^{\beta} P_1 = y_1 (J^{-1})^{\beta} P_1 = y_1 J^{-1\beta} J^{\beta} J_1 = y_1 (JJ^{-1})^{\beta} J_1 = y_1 (O_l(J))^{\beta} J_1 = y_1 O_l(J_1) J_1 = y_1 J_1 = J = M$.

Since $\Delta^{\beta} = \Delta$, P_1 is a $(\Delta - \Delta)$-bimodule and the first result follows.

We will now show that $P_1^{-1} M^{\beta} y_1^{-1} = M^*$ if $y_1 M^{*\beta} P_1 = M$. Since $J^{-1} J = \Delta$ and $JJ^{-1} = O_l(J)$ by 22.7 of Reiner [7], by matrix multiplication $M^* M = \Delta$ and $MM^* = \Lambda$. By 22.7 of Reiner [7], $P_1 P_1^{-1} = \Delta = \Delta^{\beta}$. Recall that M^{β} is a left Δ^{β}-module and $M^{*\beta}$ is a right Δ^{β}-module. Since

$$MM^* = \Lambda = \Lambda^{\alpha} = y_1 (MM^*)^{\beta} y_1^{-1} = y_1 M^{*\beta} P_1 P_1^{-1} M^{\beta} y_1^{-1} = M P_1^{-1} M^{\beta} y_1^{-1}.$$

Now $P_1^{-1} M^{\beta} y_1^{-1} M = \Delta$, so $P_1^{-1} M^{\beta} y_1^{-1} \subseteq M^*$. If $P_1^{-1} M^{\beta} y_1^{-1} \subset M^*$, then

$$M^* \subseteq \Delta M^* = M^*(MM^*) = M^* M P_1^{-1} M^{\beta} y_1^{-1} = \Delta P_1^{-1} M^{\beta} y_1^{-1} = P_1^{-1} M^{\beta} y_1^{-1}$$

leading to a contradiction. Hence, $P_1^{-1} M^{\beta} y_1^{-1} = M^*$. □

4.1. Necessary and sufficient conditions for hermitian Morita equivalences between maximal orders. Given below are previously proved necessary conditions for Λ and Δ to be HME in terms of nonsingular forms on M, Hahn [5]. If Λ is hermitian Morita equivalent Δ, then there exists a nonsingular β-sesquilinear form, $\Phi_1 : M_\Delta \times M_\Delta \longrightarrow \Delta$, as well as a nonsingular α^{-1}-sesquilinear form $\Phi_1' : {}_\Lambda M \times {}_\Lambda M \longrightarrow \Lambda$ by the Morita theorem of Hahn [5]. So this is an obvious necessary condition for Λ to be HME to Δ. Conversely, if $\Delta = (\Delta, \beta, \delta)$ and if there exists a nonsingular β-sesquilinear form, $\Phi_1 : M_\Delta \times M_\Delta \longrightarrow \Delta$, then by 1.8 of Hahn [5], (M_Δ, Φ_1) defines an hermitian equivalence data so that $\Lambda_1 = (\mathrm{End}(M_\Delta), \alpha_1, \varepsilon_1)$ and Δ are HME for some α_1 and ε_1. Analogously, if $\Lambda = (\mathrm{End}(M_\Delta), \alpha, \varepsilon)$ and if, there exists a nonsingular α^{-1}-sesquilinear form, $\Phi_1' : {}_\Lambda M \times {}_\Lambda M \longrightarrow \Lambda$, then by 1.9 of Hahn [5] and Dasgupta [2], $({}_\Lambda M, \Phi_1')$ defines an hermitian equivalence data so that Λ and $\Delta_1 = (\mathrm{End}({}_\Lambda M), \beta_1, \delta_1)$ are HME for some β_1 and δ_1.

As at the beginning of Section 4, there exists a Morita context derived from M,

$$(\Lambda, \Delta, {}_\Lambda M_\Delta, {}_\Delta M^*_\Lambda, \mu, \tau),$$

where μ and τ are given by the multiplication of matrices.

Given below is a necessary and sufficient condition for Λ and Δ to be HME in terms of the structure of θ of the hermitian equivalence data and in terms of y_1 such that $a^\alpha = y_1 a^\beta y_1^{-1}$. Note that Assumption 4 holds.

THEOREM 1. *If $\Lambda = (\Lambda, \alpha, \varepsilon)$ is hermitian Morita equivalent to $\Delta = (\Delta, \beta, \delta)$ via the hermitian equivalence data,*

$$(\Lambda, \Delta, \theta : {}_\Lambda M_\Delta \longrightarrow {}_\Delta M^*_\Lambda, \mu, \tau),$$

where μ and τ are given by multiplication of matrices, then $\theta : {}_\Lambda M_\Delta \longrightarrow {}_\Delta M^*_\Lambda$ is a bijection given by $\theta(m) = km^\beta y_1^{-1}$ for some $k \in K$. If $\theta : {}_\Lambda M_\Delta \longrightarrow {}_\Delta M^*_\Lambda$ is a bijection defined by $\theta(m) = km^\beta y_1^{-1}$ for some $k \in K$, then if μ and τ are given by multiplication of matrices,

(a) *$(\Lambda, \alpha, \varepsilon)$ is HME to (Δ, β, δ) when $\delta^{-1} y_1^\beta y_1^{-1} \varepsilon = 1$ via*

$$\big(\Lambda, \Delta, \theta : {}_\Lambda M_\Delta \longrightarrow {}_\Delta(M^*)_\Lambda, \mu, \tau\big).$$

(b) *$(\Lambda, \alpha, \varepsilon)$ is HME to $\Delta_1 = (\Delta, \beta, -\delta)$ when $\delta^{-1} y_1^\beta y_1^{-1} \varepsilon = -1$ via*

$$\big(\Lambda, \Delta_1, \theta : {}_\Lambda M_{\Delta_1} \longrightarrow {}_{\Delta_1}(M^*)_\Lambda, \mu, \tau\big).$$

Proof. Since M is a full Δ-lattice in V, $KM = V$. If there exists an hermitian equivalence data,

$$(\Lambda, \Delta, \theta : {}_\Lambda M_\Delta \longrightarrow {}_\Delta M^*_\Lambda, \mu, \tau),$$

then it extends to the following hermitian equivalence data,

$$\big(A, D, \bar\theta : {}_A V_D \longrightarrow {}_D(V_D)^*_A, \bar\mu, \bar\tau\big),$$

where $\bar\theta(\bar k m) = \bar k \theta(m)$ for any $\bar k \in K$ and $m \in M$ and $\bar\mu$ and $\bar\tau$ are the natural extensions of μ and τ. Note that $\bar\theta$ is well defined and a bijection. Since $\bar\mu$ and $\bar\tau$ are given by multiplication of matrices if μ and τ are, $\bar\theta(\bar k m) = k\bar k m^\beta y_1^{-1}$ by Proposition 3 and $\bar\theta(m) = \theta(m) = km^\beta y_1^{-1}$.

Conversely, if $\theta : {}_\Lambda M_\Delta \longrightarrow {}_\Delta M^*_\Lambda$ given by $\theta(m) = km^\beta y_1^{-1}$ is a bijection, it is clear from the proof of Proposition 3 that θ, that μ and τ satisfy the properties of the hermitian equivalence data listed below,

(a)

$$(\Lambda, \Delta, \theta : {}_{\Lambda}M_{\Delta} \longrightarrow {}_{\Delta}M^*_{\Lambda}, \mu, \tau),$$

when $\delta^{-1}y_1^{\beta}y_1^{-1}\varepsilon = 1$,

(b)

$$\big(\Lambda, \Delta_1 = (\Delta, \beta, -\delta), \theta : {}_{\Lambda}M_{\Delta_1} \longrightarrow {}_{\Delta_1}M^*_{\Lambda}, \mu, \tau\big),$$

when $\delta^{-1}y_1^{\beta}y_1^{-1}\varepsilon = -1$. □

By Proposition 4, $y_1(M^*)^{\beta}P_1 = M$ for some two sided full Δ-lattice P_1 in D. Given below is a necessary and sufficient condition in terms of P_1 for Λ and Δ to be hermitian Morita equivalent under Assumption 4.

Theorem 2. *If $\Lambda = (\Lambda, \alpha, \varepsilon)$ is hermitian Morita equivalent $\Delta = (\Delta, \beta, \delta)$ via the hermitian equivalence data,*

$$(\Lambda, \Delta, \theta : {}_{\Lambda}M_{\Delta} \longrightarrow {}_{\Delta}M^*_{\Lambda}, \mu, \tau),$$

where μ and τ are given by multiplication of matrices, then for some $k \in K$, $kP_1 = \Delta$. If $kP_1 = \Delta$ for some $k \in K$ then

(a) *$(\Lambda, \alpha, \varepsilon)$ is HME to (Δ, β, δ) when $\delta^{-1}y_1^{\beta}y_1^{-1}\varepsilon = 1$ and*

(b) *$(\Lambda, \alpha, \varepsilon)$ is HME to $(\Delta, \beta, -\delta)$ when $\delta^{-1}y_1^{\beta}y_1^{-1}\varepsilon = -1$.*

Proof. If there exists an hermitian equivalence data,

$$\big(\Lambda, \Delta, \theta : {}_{\Lambda}M_{\Delta} \longrightarrow {}_{\Delta}(M^*)_{\Lambda}, \mu, \tau\big),$$

where μ and τ are given by multiplication of matrices, then by Theorem 1, θ is given by $\theta(m) = km^{\beta}y_1^{-1}$ for $m \in M$. Since θ is a bijection, $kM^{\beta}y_1^{-1} = M^*$. Now by Proposition 4, $y_1M^{*\beta}P_1 = M$ for a $(\Delta - \Delta)$-bimodule P_1, so $\Delta = M^*M = kM^{\beta}y_1^{-1}y_1M^{*\beta}P_1 = k(M^*M)^{\beta}P_1 = k\Delta^{\beta}P_1 = k\Delta P_1 = kP_1$. Thus $kP_1 = \Delta$.

Conversely, assume that for some $k \in K$, $kP_1 = \Delta$. Now $y_1(M^*)^{\beta}P_1 = M$, so by Proposition 4, $P_1^{-1}M^{\beta}y_1^{-1} = M^*$ and $k^{-1}P_1^{-1}M^{\beta}y_1^{-1}k = M^*$. Since $kP_1 = \Delta$, $k^{-1}P_1^{-1} = \Delta = \Delta^{\beta}$ and since M^{β} is a left Δ^{β}-lattice, so $\Delta^{\beta}M^{\beta}y_1^{-1} \times k = M^{\beta}y_1^{-1}k = M^*$. Define $\theta : M \longrightarrow M^*$ by $\theta(m) = m^{\beta}y_1^{-1}k$. It is a bijection. By Theorem 1, the result follows. □

4.2. An HME between maximal orders. A sufficient condition for Λ to be HME to Δ under Assumption 4 follows.

Theorem 3. *Let $a^{\alpha} = y_1a^{\beta}y_1^{-1}$. Let $V = KM$. Assume that $\dim V_D \geq 2$. If y_1 can be chosen to be a unit of Λ, then*

(a) *$(\Lambda, \alpha, \varepsilon)$ is HME to (Δ, β, δ) when $\delta^{-1}y_1^{\beta}y_1^{-1}\varepsilon = 1$ and*

(b) *$(\Lambda, \alpha, \varepsilon)$ is HME to $(\Delta, \beta, -\delta)$ when $\delta^{-1}y_1^{\beta}y_1^{-1}\varepsilon = -1$.*

Proof. $\lambda^{\alpha} = y_1\lambda^{\beta}y_1^{-1}$ for $\lambda \in \Lambda$ and a fixed $y_1 \in \Lambda$. $\Lambda^{\beta} = y_1^{-1}\Lambda^{\alpha}y_1 = y_1^{-1}\Lambda y_1 = \Lambda$. So by the structure of Λ, since $\dim V_D \geq 2$, $\Delta^{\beta} = \Delta$, $J^{-1\beta} = J$ and $M^{*\beta} = M$. Since by Proposition 4, $y_1M^{*\beta}P_1 = M$ for a $(\Delta - \Delta)$-bimodule P_1, so $y_IMP_1 = M$. But since y_1 is a unit of Λ, $y_1M = M$. Hence, $MP_1 = M$,

$M^*MP_1 = M^*M$, $\Delta P_1 = \Delta$, and hence $P_1 = \Delta$. By Theorem 2, the result follows. □

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