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UNUSUAL GEODESICS IN GENERALIZATIONS OF THOMPSON'S GROUP F

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Abstract. We prove that seesaw words exist in Thompson's group $F(N)$ for $N = 2, 3, 4, \ldots$ with respect to the standard finite generating set X. A seesaw word w with swing k has only geodesic representatives ending in g^k or g^{-k} (for given $g \in X$) and at least one geodesic representative of each type. The existence of seesaw words with arbitrarily large swing guarantees that $F(N)$ is neither synchronously combable nor has a regular language of geodesics. Additionally, we prove that dead ends (or k-pockets) exist in $F(N)$ with respect to X and all have depth 2. A dead end w is a word for which no geodesic path in the Cayley graph Γ which passes through w can continue past w, and the depth of w is the minimal $m \in \mathbb{N}$ such that a path of length $m+1$ exists beginning at w and leaving $B_{|w|}$. We represent elements of $F(N)$ by tree-pair diagrams so that we can use Fordham's method for computing word length. This paper generalizes results by Cleary and Taback, who proved the case $N = 2$.

1. Generalizations of Thompson's groups F

1.1. Introduction. Thompson's group $F(N)$ is a generalization of the group F. R. Thompson introduced F in the early 1960s (see [15]) while constructing the groups V and T (also often referred to in the literature as Thompson's groups); V and T were the first known examples of infinite, simple, finitely-presented groups. Here, $F \subseteq T \subseteq V$. Higman in [14] later generalized T into an infinite class of groups, and Brown applied this same generalization to the groups F and V in [3]. This paper only considers generalizations of the group F.

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Definition 1.1 (Thompson's group $F(N)$). Thompson's group $F(N)$, for $N \in \{2, 3, 4, \ldots\}$, is the group of piecewise-linear orientation-preserving homeomorphisms of the closed unit interval with finitely-many breakpoints in the ring $\mathbb{Z}[\frac{1}{N}]$ and slopes in the cyclic multiplicative group $\langle N \rangle$ in each linear piece.

F is then simply the group $F(2)$. Throughout this paper, we use the convention that $N = p + 1$ for $p \in \mathbb{Z}_+$ (we note that p need not be prime, but is rather a positive integer); this is because the numbering of tree-pair diagrams and some algebraic expressions will be simpler with the use of p rather than N.

$F(p+1)$, $p \in \mathbb{N}$, is finitely-presented, infinite-dimensional, torsion-free and of type FP_∞ (see [4]). This paper is specifically interested in the Cayley graph of $F(p+1)$ with respect to the standard finite generating set, about which relatively little is known. One known result is that $F(p+1)$ satisfies no nontrivial convexity condition with respect to the standard finite generating set (see [1], [8], and [16]). More detailed information about Thompson's groups can be found in [5].

1.2. Unusual geodesics.

The first unusual kind of geodesic behavior in the Cayley graph of $F(p+1)$ with respect to the standard finite generating set to that we will explore in this paper is illustrated by the existence of seesaw words.

1.2.1. *Seesaw words.*

Groups with seesaw words with arbitrarily large swing are not synchronously combable by geodesics and do not have a regular language of geodesics. A proof of this fact will follow in Section 2.2. In [10], Cleary and Taback show that Thompson's group $F(2)$ has seesaw words of arbitrarily large swing with respect to the standard finite generating set; we generalize this argument to $F(p+1)$ for $p \geq 2$. Cleary and Taback have shown in [7] that the Lamplighter groups and certain generalized wreath products also have seesaw words of arbitrarily large swing with respect to the natural generating sets.

Definition 1.2 (Seesaw word). A word w with length $|w|$ in the generating set X is a seesaw word with swing $k \in \mathbb{N}$ with respect to $g \in X$ if the following hold:

(1) $|wg^l| = |w| - |l|$ for $0 < |l| \leq k$,
(2) $|wg^l h| \geq |wg^l|$ for all $h \in X \cup X^{-1}$ such that $h \neq g$, when $0 < |l| < k$.

In other words, all geodesic representatives of a seesaw word w end in either g^k or g^{-k}, and there is at least one geodesic representative of each type.

Definition 1.3 (Synchronous k-fellow traveller property). Let λ and η be geodesic paths in the Cayley graph $\Gamma(G, X)$ of the group G from the identity to w and v, respectively. Then λ and η synchronously k-fellow travel if for some constant k:

(1) $d_\Gamma(w,v)=1$ and
(2) For any 2 vertices h on λ and g on η, if $|h|=|g|$, then $d_\Gamma(h,g)\leq k$.

For the remainder of this paper, we will simply refer to this property as the k-fellow traveler property.

Definition 1.4 (Synchronously combable). A group is *synchronously combable* if it can be represented by a language of words satisfying the synchronous k-fellow traveler property.

We will use the term combable in this paper to mean synchronously combable.

1.2.2. *Dead ends.* Dead ends were first defined by Bogopolski in 1997 in [2]. For any group G with Cayley graph $\Gamma(G,X)$, any geodesic representative of a dead end element cannot be extended past that element in the Cayley graph. The depth of a dead end then measures how severe this behavior is: for a dead end element w of length m, a depth of k means that only paths beginning at w of length greater than k can leave the ball B_m.

Definition 1.5 (Dead ends). An element w of a group G is a dead end with respect to the given generating set X if $|wg^{\pm 1}|\leq |w|$ for all $g\in X$.

In this paper, we give a general form for all dead end elements in $F(p+1)$.

Definition 1.6 (Depth of a dead end element). For a dead end element w, let $|w|=n$. The depth of a dead end element w with respect to the generating set X is the smallest number m such that $|wg_1\cdots g_m|\leq n$ for all possible $g_1,\dots,g_m\in X\cup X^{-1}$ and $|wg_1\cdots g_{m+1}|>n$ for some choice of $g_1,\dots,g_{m+1}\in X\cup X^{-1}$. If no such m exists, we say that the dead end has infinite depth.

In other words, the depth of a dead end is the smallest integer m such that all paths of length m or less emanating from w remain in the ball B_n (centered at the identity), but for which there exists a path of length $m+1$ which leaves B_n.

Clearly, all dead ends have depth greater than or equal to 1 (and for groups with all relators of even length this depth is greater than or equal to 2). If a group has a dead end w with depth $k\geq 1$, we can also say that w is a *k-pocket* in the Cayley graph of the group. We will show that while $F(p+1)$ has dead ends, it does not have deep k-pockets, because all dead ends in $F(p+1)$ have depth 2.

The property of having dead ends has been explored for several groups already. Thompson's group $F(2)$ has dead ends with respect to the standard finite generating set, and all of them have depth 2, as Cleary and Taback show in [9]; our results simplify to this case when $p=1$. In contrast, dead ends with arbitrary depth exist in the Lamplighter groups, and in some more general wreath products with respect to the natural generating sets (see [7]).

1.3. Tree-pair diagram representatives. What follows for the remainder of this section is summarized from [16]; greater detail can be found there.

Because elements of $F(p+1)$ are piecewise linear maps which take the ith subinterval of the domain to the ith subinterval of the range, any element of $F(p+1)$ is wholly determined by the subdivisions present in its domain and range. In fact, any element $x \in F(p+1)$ can be entirely determined by an ordered pair of two sets of consecutive subintervals of $[0,1]$:

$$D = \{I_0 = [a_0, a_1], I_1 = [a_1, a_2], \ldots, I_k = [a_k, a_{k+1}]\},$$
$$R = \{J_0 = [b_0, b_1], J_1 = [b_1, b_2], \ldots, J_k = [b_k, b_{k+1}]\},$$

where $a_i < a_{i+1}, b_i < b_{i+1}$ for all $i \in \{0, \ldots, k\}$, and x is the map that takes I_i to J_i for all $i = 0, \ldots, k$. *Tree-pair diagrams*, which we will use to represent elements of $F(p+1)$, are a geometric representation of this idea.

A graph of $p+2$ vertices, one with degree $p+1$ (*parent vertex*) and the rest with degree 1 (*child vertices*), and $p+1$ edges which connect each of the child vertices to the parent vertex is a $(p+1)$-*ary caret.* A diagram which consists of $(p+1)$-ary carets, each with parent vertex oriented upwards and sharing at least one vertex with another caret, is called a $(p+1)$-*ary tree.* The graph consisting of an ordered pair of $(p+1)$-ary trees with the same number of leaves (or equivalently, the same number of carets) is a $(p+1)$-*ary tree-pair diagram.* Figure 1 is an example of a $(p+1)$-ary tree-pair diagram.

Definition 1.7 (Nodes and leaves). Within a $(p+1)$-ary tree, any vertex which is the parent vertex of a caret (i.e., which has degree $p+1$ or $p+2$) is a *node*; any vertex which has degree 1 is a *leaf.* We note that here, the

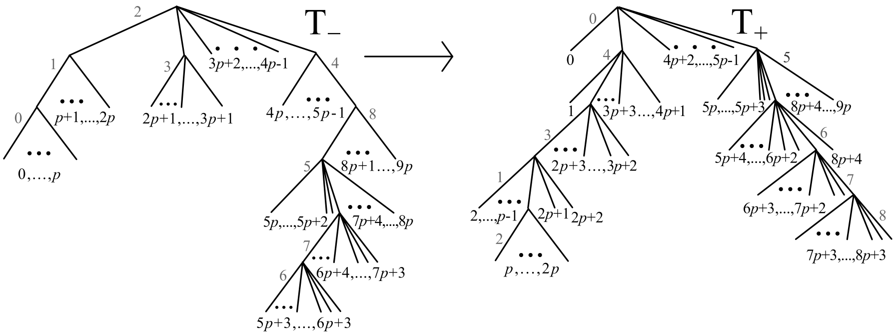

Figure 1. Tree-pair diagram representative of an element of $F(p+1)$ with all carets and leaves numbered.

term node refers only to vertices which are not leaves; it is not a synonym for vertex.

The top node of a $(p+1)$-ary tree is the *root* or *root node*, and the caret which contains it is called the *root caret*. We refer to the leftmost or rightmost directed edge of a tree as the *left* or *right edge* of the tree, respectively.

1.3.1. *Leaf ordering in a tree-pair diagram.* We recall that an arbitrary element x of $F(p+1)$ can be entirely determined by an ordered pair of sets of consecutive subintervals of $[0,1]$: $(D=\{I_0,\dots,I_k\}, R=\{J_0,\dots,J_k\})$. Each leaf in a tree-pair diagram will correspond to one of the intervals $I_0,\dots,I_k,J_0,\dots,J_k$ in the following way: if the parent node of a caret represents an interval $[a,b]$, then the child nodes of that caret represent the subintervals $[a,a+\frac{b-a}{p+1}],[a+\frac{b-a}{p+1},a+\frac{2(b-a)}{p+1}],\dots,[a+\frac{p(b-a)}{p+1},b]$; we let the root node of each tree in a tree-pair diagram represent $[0,1]$, so each leaf in the first or second tree in the the tree-pair diagram now represents one of the subintervals $I_0,\dots,I_k$ or $J_0,\dots,J_k$ respectively. We then number the leaves in the tree by assigning each of them the index number of the interval which they represent. For more details, see [16]. We can see a tree-pair diagram with all its leaves numbered in Figure 1.

1.3.2. *Minimal tree-pair diagrams.* The group $F(p+1)$ induces an equivalence relation on the set of $(p+1)$-ary tree-pair diagrams.

Definition 1.8 (Equivalent tree-pair diagrams). Two $(p+1)$-ary tree-pair diagrams are *equivalent* if they represent the same element of $F(p+1)$.

Definition 1.9 (Minimal tree-pair diagram representative). The tree-pair diagram which has the smallest number of leaves of any diagram in its equivalence class is the *minimal tree-pair diagram representative* of the element of $F(p+1)$ represented by that equivalence class.

Within a $(p+1)$-ary tree-pair diagram, the domain tree is referred to as the *negative* tree and is often denoted by T_-, whereas the range tree is referred to as the *positive* tree and is denoted by T_+. We will denote a tree-pair diagram with negative tree T_- and positive tree T_+, by (T_-,T_+). In general, for any tree-pair diagram (T,S), the first tree listed is the domain tree and the second tree listed is the range tree (whether or not the (T_-,T_+) notation is used).

We describe how we may obtain the equivalent minimal tree-pair diagram representative of an element of $F(p+1)$ from an arbitrary representative. We say that a caret is *exposed* if all of its children are leaves. If there is an exposed caret in both the negative and positive trees, and all the leaves of the exposed caret in each tree have the same index numbers, then we can remove the pair of exposed carets in the tree-pair diagram without changing the element which the tree-pair diagram represents. This is the only way in which a tree-pair diagram can be reduced. So, every element of $F(p+1)$ has a unique representation as a minimal tree-pair diagram. We will write $w = (T_-, T+)$ to denote that (T_-, T_+) is the minimal tree-pair diagram representative of w.

NOTATION 1.1 $(((Tx)_-, (Tx)_+), ((Tx)'_-, (Tx)'_+))$. When $w = (T_-, T_+)$ and $x \in F(p+1)$, we denote the (possibly nonminimal) tree-pair diagram representative of the product wx by $((Tx)_-, (Tx)_+)$. We will denote the minimal tree-pair diagram representative of wx by $((Tx)'_-, (Tx)'_+)$.

1.3.3. *Multiplying tree-pair diagrams.* Multiplication of two elements of $F(p+1)$ is simply function composition. We will use functional notation so that multiplying x by y on the right will be written xy, which denotes $x \circ y$.

To compute the product xy of $x = (T_-, T_+)$ and $y = (S_-, S_+)$ using the tree-pair diagram representatives, we first make S_+ identical to T_-. This is possible because we can add a caret to any leaf in S_+ as long as we add a caret to the leaf with the same index number in S_-, because this is just the reverse of the process removing exposed caret pairs. In the same way, we can add a caret to any leaf in T_-. We continue adding carets to the tree-pair diagrams in this way until T_- and S_+ are identical. If we let (T^*_-, T^*_+) and (S^*_-, S^*_+) denote the tree-pair diagrams for x and y, respectively once carets have been added as needed so that $S^*_+ = T^*_-$, then (S^*_-, T^*_+) is the (possibly nonminimal) tree-pair diagram representative of xy. To see an example of multiplication of tree-pair diagrams, see Figure 2.

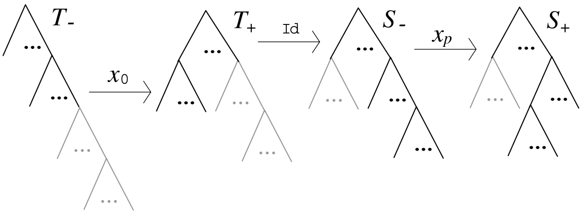

FIGURE 2. Multiplication of tree-pair diagrams representing the product $x_p x_0$ in $F(p+1)$ (each caret has $p+1$ edges) where $x_0 = (T_-, T_+), x_p = (S_-, S_+)$. Here, T_-, T_+, S_-, S_+ are the trees represented by only black carets. The grey carets are the carets that must be added in order for multiplication to take place, so $T^*_-, T^*_+, S^*_-, S^*_+$ are then the trees represented by the union of black and grey carets, and $x_p x_0 = (T^*_-, S^*_+)$. We note that because the operation here is function composition, the order in which x_p and x_0 appear in the product $x_p x_0$ and the order in which the tree-pair diagram representatives for x_p and x_0 appear above is reversed.

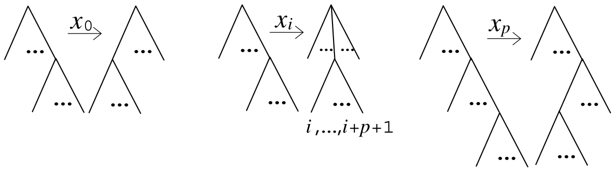

FIGURE 3. The standard finite generators of $F(p+1)$, where $i \in \{1,\dots,p-1\}$ (each caret has $p+1$ edges, which we denote by dots.

1.4. Caret types. In order to understand the metric on $F(p+1)$ developed by Fordham in [11], which we will need to prove the results of this paper, we must first categorize the carets in a tree into the following types:

(1) $\mathcal{L}$. This is a left caret; a left caret is any caret that has one edge on the left side of the tree. The root caret is defined to be of this type.
(2) $\mathcal{R}$. This is a right caret; a right caret is any caret (except the root caret) that has one edge on the right side of the tree.
(3) $\mathcal{M}$. This is a middle caret; all carets which are neither left nor right carets are middle carets.

1.5. Group presentations. $F(p+1)$ has a standard infinite presentation and a standard finite presentation; the infinite presentation can be obtained from the finite presentation by induction.

The standard infinite presentation is [3]:

$$F(p+1) = \{x_0, x_1, x_2, \dots \mid x_i x_j = x_{j+p} x_i \text{ for } i<j\}.$$

The standard finite presentation is [3] (see Figure 3):

$$\left\{ x_0, x_1, \dots, x_p \;\middle|\; \begin{array}{l} [x_0 x_i^{-1}, x_j] \text{ when } i<j, [x_0^2 x_i^{-1} x_0^{-1}, x_j] \text{ when } i \geq j-1, \\ [x_0^3 x_p^{-1} x_0^{-2}, x_1]. \text{ Here } i,j = 0,\dots,p. \end{array} \right\}$$

From now on, we will use the notation X to represent the generating set $\{x_0, \dots, x_p\}$.

In [11], Fordham developed a metric to calculate geodesic lengths in the Cayley graph of $F(p+1)$ generated by X (this is a generalization of his work in [12] and [13]). The material in this section is primarily paraphrased from [11]. This metric depends upon the exact types of carets within a $(p+1)$-ary tree, so before we proceed to present the metric, we further classify caret types.

1.6. Further classification of carets of type $\mathcal{M}$. We further subcategorize the middle carets into p subtypes: $\mathcal{M}^i$ for $i = 1, 2, \dots, p$. The value of i depends upon the type of the middle caret's parent caret and its relative location with respect to its parent caret. Figure 4 shows the subtype of each child caret

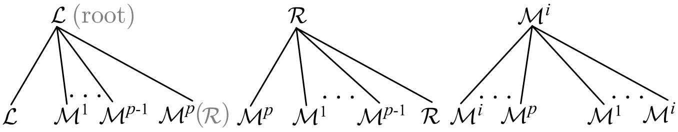

FIGURE 4. For each of the parent caret types given above: $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{M}^i$ for $i = 1,\ldots,p$, the caret type listed below each child is the type of the child caret in that position, if one exists. The root caret will be of type $\mathcal{L}$, but note that the type of any caret hanging off of the rightmost child vertex will by type $\mathcal{R}$ rather than $\mathcal{M}^p$ (as in the case of all other type $\mathcal{L}$ parent carets). Additionally, note that every caret of type $\mathcal{M}^i$ will have two children of type $\mathcal{M}^i$: the children hanging off the leftmost and the rightmost child vertices.

for a given parent caret type. For example, in Figure 1, $\wedge_3, \wedge_5, \wedge_6, \wedge_7 \in T_-$ have types $\mathcal{M}^1, \mathcal{M}^p, \mathcal{M}^3, \mathcal{M}^3$, respectively, and $\wedge_1, \wedge_2, \wedge_3, \wedge_4, \wedge_6, \wedge_7, \wedge_8 \in T_+$ have types $\mathcal{M}^2, \mathcal{M}^p, \mathcal{M}^2, \mathcal{M}^1, \mathcal{M}^4, \mathcal{M}^3, \mathcal{M}^3$, respectively.

1.7. Caret/node order. The metric is based on numbering all the carets in each tree of a tree-pair diagram and pairing up each caret in the negative tree with the caret in the positive tree with the same index number. The type of each caret in the pair then determines the contribution of that pair of carets to the length of the element which the tree-pair diagram represents.

DEFINITION 1.10 (Ancestor, descendant). For any two vertices a and b on an n-ary tree, vertex a is an *ancestor* of vertex b if it is on the directed path from the root node to vertex b. Similarly, vertex b is a *descendent* of vertex a if vertex a is an ancestor of vertex b.

To order the carets in a $(p+1)$-ary tree, we first order the nodes of the tree. Once we have ordered the nodes within a tree, we can simply number them, beginning with 0 and assigning numbers so that the numbering reflects the placement of the nodes in the order. And once we have numbered the nodes of a tree, we can number the carets in the tree simply by assigning to each caret the index number of its parent node.

To order all the nodes within a tree, we begin by ordering all the nodes within a single caret. Since every caret in a tree has at least one node which is common to another caret in the tree, any absolute order for the nodes within an arbitrary caret induces an absolute order on all the nodes in a tree (i.e. for any 3 nodes within a single caret a, b, c such that $a < b < c$ in the order, for an arbitrary descendant node b' of b, we must also have $a < b' < c$).

Now we describe this absolute order of nodes within a caret. The type of a given caret determines which child nodes will come before the parent node in

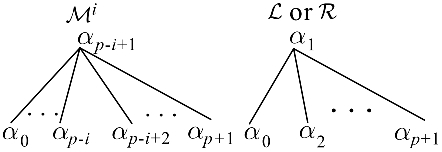

FIGURE 5. For each of the caret types given above: $\mathcal{M}^i$ for $i = 1, \dots, p$ and $\mathcal{L}$ or $\mathcal{R}$, the order of the nodes of the caret is defined so that for arbitrary nodes a and b with vertex index numbers α_j and α_k, $a < b$ if and only if $j < k$. Note that the index numbers here index the nodes of the caret (including the root node) and not simply the child nodes.

the order and which will come after it (see Figure 5). For an arbitrary caret, we assign index numbers $\alpha_0, \dots, \alpha_{p+1}$ to every vertex within the caret; how these index numbers will be assigned depends upon the caret type: For left and right carets, the leftmost child vertex of the caret will have index number α_0, the root vertex will have index number α_1, and the remaining child vertices will have index numbers $\alpha_2, \dots, \alpha_{p+1}$. For carets of type $\mathcal{M}^i$ where $i \in \{1, \dots, p\}$, the $p - i + 1$ leftmost child vertices will have index numbers $\alpha_0, \dots, \alpha_{p-i}$, the parent vertex will have index number α_{p-i+1}, and the remaining child vertices will have index numbers $\alpha_{p-i+2}, \dots, \alpha_{p+1}$. For a visual summary of these details, see Figure 5. Then these vertex index numbers induce an ordering of the nodes of the caret as follows: for arbitrary nodes a and b in the caret with vertex index numbers α_j and α_k, $a < b$ if and only if $j < k$.

Within a tree-pair diagram, the carets in the negative and positive trees with the same index number are paired together and referred to as a caret pair. The caret pair with index number i is called the *ith caret pair*, and is denoted by $\wedge_i$.

NOTATION 1.2 ($\wedge_i$). We use the notation $\wedge_i$ to represent both a single caret with index number i and to represent the ith caret pair in a tree-pair diagram; when we use this notation, which of these is meant should be clear from the context.

1.8. Final classification of caret types. The following definitions will further refine our categories of caret types so that we can finally proceed to the metric.

DEFINITION 1.11 (Successor, predecessor). For two carets $\wedge_i$ and $\wedge_j$ in a tree, we say that $\wedge_i$ is a *successor* of $\wedge_j$ whenever $i > j$, and we say that $\wedge_i$ is a *predecessor* of $\wedge_j$ whenever $i < j$.

REMARK 1.1 (Ancestor/descendant vs. successor/predecessor). We must not confuse successors with children (or descendants) and predecessors with parents (or ancestors). $\wedge_B$ is a child of $\wedge_A$ if and only if the parent vertex of $\wedge_B$ is a child vertex of $\wedge_A$, but $\wedge_B$ is a successor of $\wedge_A$ if and only if $B > A$. The properties of being a child or successor of some fixed caret are wholly independent. For example, in Figure 1, in T_+ $\wedge_1$ is a child but not a successor of $\wedge_3$, and in T_-, $\wedge_8$ is a successor but not a child of $\wedge_6$; in contrast, in T_-, $\wedge_7$ is both a child and a successor of $\wedge_5$.

Definition 1.12 (Leftmost caret). When we refer to a caret as the leftmost caret with some property X, we mean precisely the caret with property X whose index number is smallest. So, for example, the leftmost child of $\wedge_i$ would be the child of $\wedge_i$ with the smallest index number and the leftmost child successor would be the caret with the smallest index number which is both a child and a successor of $\wedge_i$.

And now we enumerate the final set of categories of caret type:

(1) $\mathcal{L}_\emptyset$. This is the first and leftmost caret of the tree. There is one and only one caret of this type in any nonempty tree.
(2) $\mathcal{L}_\mathrm{L}$. Any left caret not of type $\mathcal{L}_\emptyset$ is of this type.
(3) $\mathcal{R}_\emptyset$. This is any right caret for which all successor carets are right carets. For example, in Figure 1, $\wedge_8 \in T_-$ is the only caret of type $\mathcal{R}_\emptyset$.
(4) $\mathcal{R}_\mathrm{R}$. This is a right caret whose immediate successor is a right caret, but which has at least one successor which is not a right caret. For example, in Figure 8, $\wedge_{m+2} \in S_+$ is of type $\mathcal{R}_\mathrm{R}$ because its immediate successor is $\wedge_{m+3}$, which is type $\mathcal{R}$, but its successor $\wedge_{m+np+n}$ is not type $\mathcal{R}$.
(5) $\mathcal{R}_j$. This is a right caret whose immediate successor is not a right caret and whose leftmost child successor is type $\mathcal{M}^j$ when $j < p$, or $\mathcal{R}$ when $j = p$. For example, in T_+ in Figure 1, the leftmost child successor of $\wedge_5$ is $\wedge_6$; since $\wedge_6$ is type $\mathcal{M}^4$, $\wedge_5$ is type $\mathcal{R}_4$. A caret of type $\mathcal{R}_p$ can be seen in T_-: $\wedge_4$ has as its immediate successor $\wedge_5$, which is not a right caret, and the leftmost child successor of $\wedge_4$ is $\wedge_8$, which is type $\mathcal{R}$, so $\wedge_4$ is type $\mathcal{R}_p$.
(6) $\mathcal{M}^i_\emptyset$. This is a middle caret of type $\mathcal{M}^i$ that has no child successor carets. For example, in Figure 1, the only carets of type $\mathcal{M}^i_\emptyset$ for some $i \in \{1, \dots, p\}$ are: $\wedge_3 \in T_-$ is type $\mathcal{M}^1_\emptyset$, $\wedge_6, \wedge_7 \in T_-$ are type $\mathcal{M}^3_\emptyset$, $\wedge_2 \in T_+$ is type $\mathcal{M}^p_\emptyset$, $\wedge_1, \wedge_3 \in T_+$ are type $\mathcal{M}^2_\emptyset$, $\wedge_4 \in T_+$ is type $\mathcal{M}^1_\emptyset$, $\wedge_8 \in T_+$ is type $\mathcal{M}^3_\emptyset$. Note that a caret may have child carets but no child successor carets (i.e., if the child carets are predecessors rather than successors).
(7) $\mathcal{M}^i_j$. This is a middle caret of type $\mathcal{M}^i$ with leftmost child successor of type $\mathcal{M}^j$. We note that we will always have $j \leq i$ because all child successor carets of a type $\mathcal{M}^i$ caret will have type $\mathcal{M}^j$ such that $j \leq i$ (see Figure 4). For example, in Figure 1, $\wedge_5 \in T_-$ is type $\mathcal{M}^p_3$, $\wedge_6 \in T_+$ is type $\mathcal{M}^4_3$, and $\wedge_7 \in T_+$ is type $\mathcal{M}^3_3$.

TABLE 1. Weight of types of caret pairs in a $(p+1)$-ary tree-pair diagram

$(\cdot,\cdot)$	$\mathcal{L}_\emptyset$	$\mathcal{L}_\mathrm{L}$	$\mathcal{R}_\emptyset$	$\mathcal{R}_\mathrm{R}$	$\mathcal{R}_j$	$\mathcal{M}^l_\emptyset$	$\mathcal{M}^t_u$
$\mathcal{L}_\emptyset$	0	–	–	–	–	–	–
$\mathcal{L}_\mathrm{L}$	–	2	1	1	1	2	2
$\mathcal{R}_\emptyset$	–	1	0	2	2	1	3
$\mathcal{R}_\mathrm{R}$	–	1	2	2	2	1	3
$\mathcal{R}_i$	–	1	2	2	2	$\begin{cases}1 \text{ for } i \le l\\ 3 \text{ for } i > l\end{cases}$	3
$\mathcal{M}^k_\emptyset$	–	2	1	1	$\begin{cases}1 \text{ for } j \le k\\ 3 \text{ for } j > k\end{cases}$	2	$\begin{cases}2 \text{ for } k \le u\\ 4 \text{ for } k > u\end{cases}$
$\mathcal{M}^r_s$	–	2	3	3	3	$\begin{cases}2 \text{ for } l \le s\\ 4 \text{ for } l > s\end{cases}$	4

1.9. The metric. We now describe the metric developed by Fordham in [11] for geodesic length in $F(p+1)$ with respect to X. According to this metric, each caret pair in the minimal tree-pair diagram representative of an element of $F(p+1)$ contributes a "weight" which, when summed over all caret pairs in the diagram, yields the length of the element in $F(p+1)$.

NOTATION 1.3 ($|w|$). For given $w \in F(p+1)$, $|w|$ is the length of w w.r.t. X.

The *weight* of a caret pair in a minimal tree-pair diagram representing $w \in F(p+1)$ is the contribution of that caret pair to the length of w (see Table 1). The weight depends upon the type of each caret in the pair and is derived from the cardinality of the set of generators which is required to produce the caret pair.

NOTATION 1.4 ($w_{(T_-,T_+)}(\wedge_i)$, $w_{(T_-,T_+)}(\tau_1,\tau_2)$). If the types of the negative and positive carets in the ith caret pair of (T_-,T_+) are denoted by τ_1 and τ_2 respectively, then we denote the weight of $\wedge_i$ by $w_{(T_-,T_+)}(\wedge_i)$ or $w_{(T_-,T_+)}(\tau_1,\tau_2)$. When the tree-pair diagram itself is obvious from the context, we will often omit the subscript.

REMARK 1.2. Since Table 1 is symmetric, $w(\tau_1,\tau_2) = w(\tau_2,\tau_1)$ for all τ_1,τ_2.

THEOREM 1.1 (Fordham [11], Theorem 2.0.11). *Given an element* $w = (T_-,T_+)$ *in* $F(p+1)$, $|w|$ *is the sum of the weights given in Table* 1 *for each of the pairs of carets in* (T_-,T_+). (*Note that since only* $\wedge_0$ *is of type* $\mathcal{L}_\emptyset$, $(\mathcal{L}_\emptyset,\mathcal{L}_\emptyset)$ *is the only possible pairing in which the caret type* $\mathcal{L}_\emptyset$ *can appear.*)

1.10. How generators act on caret type pairings. Our approach in this paper involves thinking of multiplication on the right by a generator as an

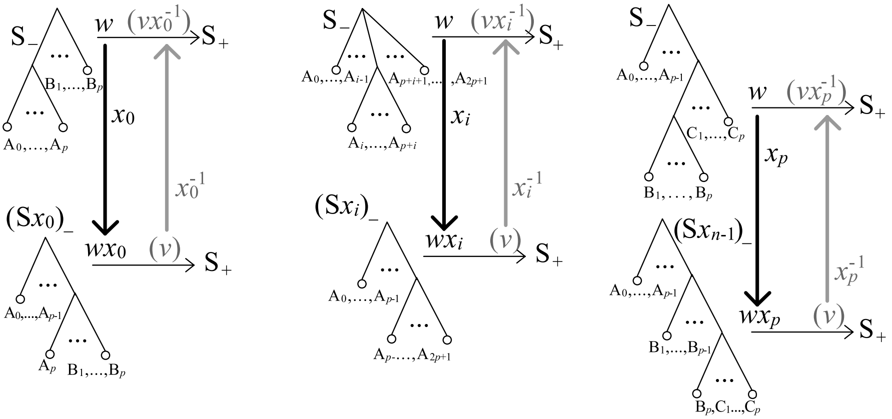

FIGURE 6. The "action" of given $g \in X \cup X^{-1}$ on an arbitrary $(p+1)$-ary tree-pair diagram, where we assume that the tree-pair diagram (S_-, S_+) has already had any carets added which are needed in order to compute the product. Here $i \in \{1, \dots, p-1\}$ and A_i, B_i and C_i all represent (possibly empty) subtrees. Black arrows/labels indicate the "action" of g on the tree-pair diagram representative of an arbitrary word w, and grey arrows/labels indicate the "action" of g^{-1} on the tree-pair diagram representative of an arbitrary word v. Because multiplication on the right has no effect on the positive tree of a tree-pair diagram after all carets have been added for multiplication, the "action" makes no change to the positive trees (see Remark 1.3).

"action" on a tree-pair diagram. When we multiply $x = (T_-, T_+)$ of $F(p+1)$ on the right by y, we view $((Ty)_-, (Ty)_+)$ as the result of this "action" of y on (T_-, T_+). Diagrams depicting this "action" of $g \in X \cup X^{-1}$ on an arbitrary S_- can be seen in Figure 6. Because multiplication by $x_i^{\pm 1}$ is harder to visualize than other product computation (since it does not exist in the case of $F(2)$), we include an example of this computation in Figure 7.

We now define two conditions which will be used in the theorems that follow.

DEFINITION 1.13 (Subtree condition). For fixed $w = (T_-, T_+) \in F(p+1)$, $g \in X \cup X^{-1}$, w and g fulfil the subtree condition when wg can be computed without adding carets.

DEFINITION 1.14 (Minimality condition). For fixed $w = (T_-, T_+) \in F(p+1)$, $g \in X \cup X^{-1}$, w and g fulfil the minimality condition when $((Tg)_-, (Tg)_+)$ is minimal.

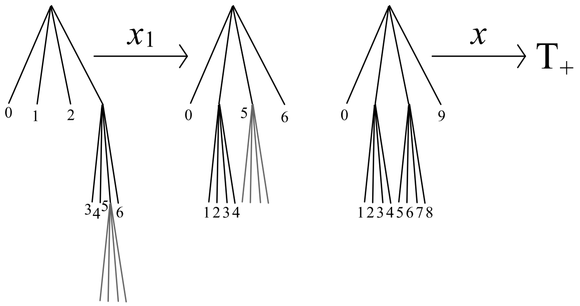

FIGURE 7. The calculation of a product xx_1 in $F(4)$. In this case, we must add a caret to the leaves with index number 5 in x_1 in order for the multiplication to take place; this is indicated by the grey carets in the figure. The tree-pair diagram for xx_1 will then consist of the negative tree of x_1 with the added grey caret as the negative tree and T_+ as the positive tree.

Fordham proves that when these two conditions are met, only one caret pair in the tree-pair diagram changes type as a result of the "action" of g:

THEOREM 1.2 (Fordham [11], Theorem 2.1.1). *If $w = (T_-, T_+) \in F(p+1)$ and $g \in X \cup X^{-1}$ satisfy the subtree and minimality conditions, then there is exactly one caret $\wedge_i$ in the tree-pair diagram that changes type under the multiplication wg; that is, if we let $\tau_{T_-}(\wedge_i)$ denote the caret type of $\wedge_i$ in T_- in the tree-pair diagram (T_-, T_+), then $\exists i$ such that*

$$\tau_{T_-}(\wedge_i) \neq \tau_{(Tg)_-}(\wedge_i) \quad \textit{and} \quad \tau_{T_-}(\wedge_j) = \tau_{(Tg)_-}(\wedge_j) \quad \forall j \neq i.$$

The caret $\wedge_i$ which changes type when the conditions of Theorem 1.2 are met will always be in the negative tree.

REMARK 1.3. When multiplying an element $x = (T_-, T_+)$ in $F(p+1)$ by an element y on the right, if the subtree condition is met, then the type of caret $\wedge_i$ is the same in both T_+ and $(Ty)_+$ for all caret index numbers i, and $T_+ = (Ty)_+$. The type of $\wedge_i$ will be different in $(Ty)'_+$ than in T_+ only if the minimality condition is not met.

When either the subtree or minimality condition fails, we have an alternate theorem which can help us to determine the effect of multiplication on an element's length without computing it directly.

THEOREM 1.3 (Fordham [11], Theorems 2.1.3 and 2.14). *If $g \in X \cup X^{-1}$ and $w = (T_-, T_+) \in F(p+1)$, do not fulfil:*

(1) *the subtree condition when computing wg, then $|wg| = |w| + 1$,*
(2) *the minimality condition when computing wg, then $|wg| = |w| - 1$.*

2. Seesaw words with arbitrary swing exist in $F(p+1)$

2.1. Seesaw words in $F(p+1)$.

THEOREM 2.1. *Any word in $F(p+1)$ with the following normal form, where $m,n \in \mathbb{N}$ is a seesaw word with respect to x_0 in X.*

$$x_0^{m-1} x_p x_{np^2+(m+n)p} \left(\prod_{i=1}^{pn} x^{-1}_{np^2+(m+n-i+1)p-i} \right) x_0^{-m}.$$

The minimal tree-pair diagram representative of an element of this form can be seen in Figure 8. This family of seesaw words will be denoted $\mathcal{S}$.

The proofs that follow will be concerned entirely with showing that all elements with minimal tree-pair diagram representative of the form given in Figure 8 are seesaw words with respect to x_0. The algebraic expression is entirely determined by the minimal tree-pair diagram; to see how this algebraic expression can be obtained from the tree-pair diagram given in Figure 8, see the section on normal forms of $F(p+1)$ in [16]. This family $\mathcal{S}$ is a generalization of the family of seesaw words introduced by Cleary and Taback in [10].

For our proof, we take arbitrary $w = (S_-, S_+) \in \mathcal{S}$. First, we prove that w satisfies part 1 of the definition of seesaw words with respect to $x_0 \in X$.

LEMMA 2.1.

$$|wx_0^{\pm q}| = |w| - q \quad \textit{for all } q \textit{ such that } 0 < q < m-1, n-1,$$

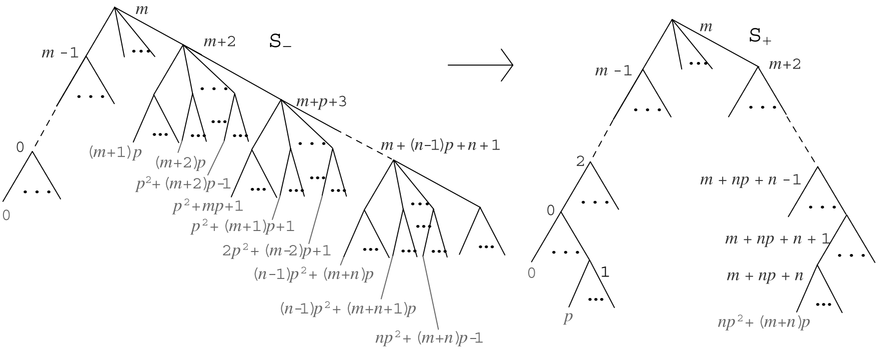

FIGURE 8. Minimal tree-pair diagram representative of an arbitrary seesaw element in the family $\mathcal{S}$. The letter m denotes the number of carets of type $\mathcal{L}_{\rm L}$ in S_- and the letter n denotes the number of carets of type $\mathcal{R}$ on the right side of S_- which are not of type $\mathcal{R}_\emptyset$.

where m denotes the number of carets of type $\mathcal{L}_{\mathrm{L}}$ in S_- and n denotes the number of carets of type $\mathcal{R}$ on the right side of S_- which are not type $\mathcal{R}_\emptyset$.

Proof. We prove this by induction. Throughout this proof, we let (S^q_-, S^q_+) denote $((Sx^{-q})_-, (Sx^{-q})_+)$ and we let (R^q_-, R^q_+) denote $((Sx^q)_-, (Sx^q)_+)$, where $q > 0$ in both cases. Our inductive hypotheses will include the assumption that wx_0^q and wx_0^{-q} have minimal tree-pair diagram representatives of the form given in Figures 9 and 10, respectively.

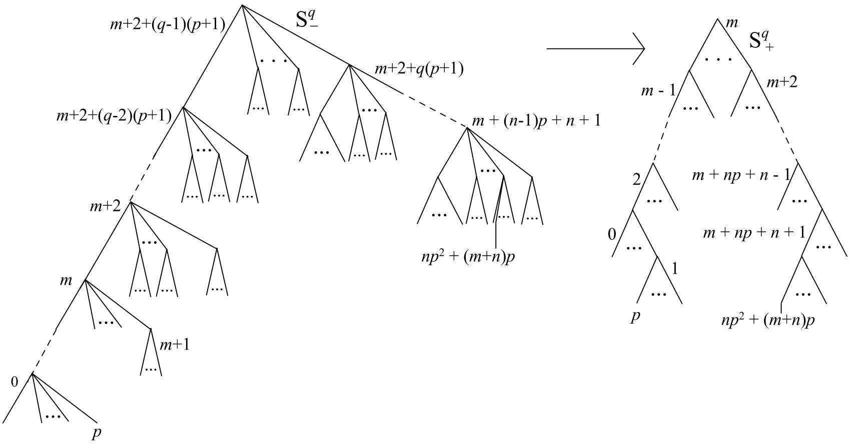

FIGURE 9. Minimal tree-pair diagram representative of wx_0^{-q} (when $0 < q < n-1$) for $w \in \mathcal{S}$.

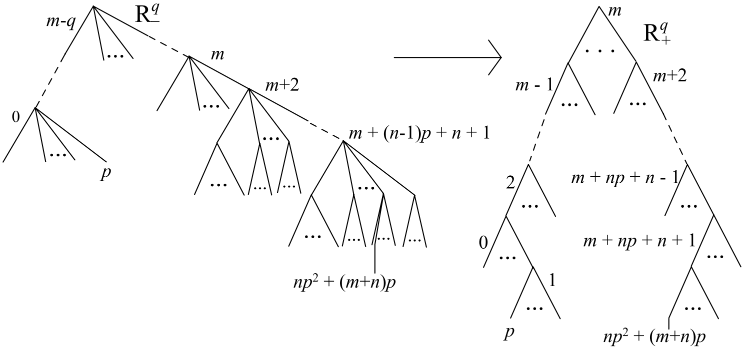

FIGURE 10. Minimal tree-pair diagram representative of wx_0^q (when $0 \le q < m-1$) for $w \in \mathcal{S}$.

(1) $|wx_0^{-q}|$: We begin by considering the case when $q=1$. Performing the multiplication wx_0^{-1} using the minimal tree-pair diagram representatives of w and x_0^{-1} in Figures 8 and 3, respectively, we obtain Figure 9 (when $q=1$); (S_-^1, S_+^1) is minimal because there are only two exposed carets in S_+^1: the carets with leftmost leaf index numbers p and $np^2+(m+n)p$, but neither of the leaves with these index numbers in S_-^1 is the leftmost leaf of an exposed caret.

Our inductive hypothesis will be that $|wx_0^{-q}| = |w| - q$ for some q such that $0 < q < m-1, n-1$ and that wx_0^{-q} has minimal tree-pair diagram representative (S_-^q, S_+^q) given in Figure 9. Now we assume our hypotheses hold for some $q = j-1$ such that $0 < j < n-2$ and we consider what happens when we multiply $wx_0^{-(j-1)}$ by x_0^{-1} on the right. By our inductive hypothesis, the tree-pair diagram in Figure 9 is the minimal representative of $wx_0^{-(j-1)}$ when $q=j-1$. Because $wx_0^{-(j-1)}$ and x_0^{-1} satisfy the subtree condition, the positive tree S_+^{j-1} remains unchanged after multiplication by x_0^{-1} (see Remark 1.3). So we consider which changes x_0^{-1} makes to the negative tree.

By looking at Figures 6 and 9 which represent $wx_0^{-(j-1)}$ and x_0^{-1} respectively, we can see that multiplying $wx_0^{-(j-1)}$ by x_0^{-1} changes $\wedge_{m+2+(j-1)(p+1)}$ (the rightmost child of the root) in S_-^{j-1} from type $\mathcal{R}_1$ to type $\mathcal{L}_{\mathrm{L}}$. This is the only change in the negative tree. So we can see that the resulting tree-pair diagram representative for wx_0^{-j} will have $\wedge_{m+2+(j-1)(p+1)}$ as the root caret and $\wedge_{m+2+(j-2)(p+1)}$ as the leftmost child of the root. The relative location of all other carets in the tree will be identical to their placement in the minimal tree-pair diagram representative for $wx_0^{-(j-1)}$. So it is clear that Figure 9 (when $q=j$) is a tree-pair diagram representative for wx_0^{-j}. Now we need only show that it is minimal; we note that any carets which are exposed in (S_-^j, S_+^j) would also have been exposed in (S_-^{j-1}, S_+^{j-1}), so minimality of (S_-^{j-1}, S_+^{j-1}) implies minimality of (S_-^j, S_+^j).

Now we consider the effect of multiplication of $wx_0^{-(j-1)}$ by x_0^{-1} on the length of $wx_0^{-(j-1)}$. The caret $\wedge_{m+2+(j-1)(p+1)}$ will always be a successor of the caret $\wedge_m$ in both S_+^{j-1} and S_+^j, and the only successors of $\wedge_m$ in S_+^{j-1} and S_+^j which are not of type $\mathcal{R}_{\mathrm{R}}$ are $\wedge_{m+np+n}$ and $\wedge_{m+np+n+1}$. Since $j<n$, it is clear that $m+2+(j-1)(p+1) < m+np+n$ and therefore $\wedge_{m+2+(j-1)(p+1)}$ is of type $\mathcal{R}_{\mathrm{R}}$ in S_+^{j-1} and S_+^j. Therefore, this change in the caret $\wedge_{m+2+(j-1)(p+1)}$ in the negative tree from type $\mathcal{R}_1$ to type $\mathcal{L}_{\mathrm{L}}$ changes the pairing from $(\mathcal{R}_1, \mathcal{R}_{\mathrm{R}})$, which has weight 2, to $(\mathcal{L}_{\mathrm{L}}, \mathcal{R}_{\mathrm{R}})$, which has weight 1 (see Table 1). So $|wx_0^{-j}| = |wx_0^{-(j-1)}| - 1$. And since by our inductive hypotheses $|wx_0^{-(j-1)}| = |w| - (j-1)$,

$$|wx_0^{-j}| = |w| - (j-1) - 1 = |w| - j \quad \text{for all } j \text{ s.t. } 0 < j < n-1.$$

(2) $|wx_0^q|$: The proof that $|wx_0^q| = |w| - q$ is similar to the proof that $|wx_0^{-q}| = |w| - q$. The primary difference is that the caret in (R_-^{j-1}, R_+^{j-1}) in Figure 10 whose type is changed by multiplication by x_0 is $\wedge_{m-(j-1)}$ (the root caret) in R_-^{j-1}, which is changed from type $\mathcal{L}_{\mathrm{L}}$ to type $\mathcal{R}_{\mathrm{R}}$ (or $\mathcal{R}_p$ in the case $j = 1$). In the same way as for the x_0^{-1} case, this leads to the conclusion that Figure 10 is a minimal tree-pair diagram representative of wx_0^j when $q = j$. Then to compute the effect of multiplication by x_0 on length, we note that the caret $\wedge_{m-(j-1)}$ in R_+^{j-1} or R_+^j will always be of type $\mathcal{L}_{\mathrm{L}}$ for any given $j = 1, \dots, m-2$ because $\wedge_{m-(j-1)}$ is a predecessor of the root $\wedge_m$ in R_+^{j-1} and R_+^j since $m - (j-1) < m$ and, and the only predecessors of the root in R_+^{j-1} or R_+^j which are not of type $\mathcal{L}_{\mathrm{L}}$ are $\wedge_1$ and $\wedge_0$. Since $j \leq m-2$ guarantees that $m - (j-1) > 1$ for all possible j, $\wedge_{m-(j-1)} \neq \wedge_1$ or $\wedge_0$. Therefore, this change in the caret $\wedge_{m-(j-1)}$ from type $\mathcal{L}_{\mathrm{L}}$ to type $\mathcal{R}_{\mathrm{R}}$ (or $\mathcal{R}_p$ when $j = 1$) changes the pairing from $(\mathcal{L}_{\mathrm{L}}, \mathcal{L}_{\mathrm{L}})$, which has weight 2, to $(\mathcal{R}_{\mathrm{R}}, \mathcal{L}_{\mathrm{L}})$ (or $(\mathcal{R}_p, \mathcal{L}_{\mathrm{L}})$ when $j = 1$), which has weight 1 (see Table 1). Then similarly to the x_0^{-1} case, we can use induction to conclude that $|wx_0^q| = |w| - q$ and that Figure 10 is a minimal tree-pair diagram representative of wx_0^q for all q such that $0 \leq q < m-1$. □

Now we show that all $w \in \mathcal{S}$ satisfy part 2 of the definition of a seesaw word by considering the "action" of each $g \in X \cup X^{-1}$ on $wx_0^{\pm q}$ for arbitrary q such that $0 \leq q < m-1, n-1$, and showing that this "action" always results in increased length.

LEMMA 2.2. *For $w \in \mathcal{S}$, $\varepsilon \in \{-1, 1\}$, and arbitrary q s.t. $0 < |q| < m-1, n-1$,*

$$|wx_0^{\varepsilon q} g| \geq |wx_0^{\varepsilon q}|$$

for all $g \in X \cup X^{-1}$.

Proof. We consider each possible combination of values of ε and g:

(1) $|wx_0^{-q} x_i^{\pm 1}|$, $i \in \{1, 2, \dots, p\}$: First, we note that wx_0^{-q} and $x_i^{\pm 1}$ when $0 \leq q < m-1, n-1$ and $i = 1, 2, \dots, p$ satisfy both the subtree and minimality conditions of Theorem 1.2 except when $q = 0$ and $i = 1, \dots, p-1$. So only one caret will change type in the negative tree and the positive tree will remain unchanged after multiplication in these cases.

We begin with the case $q = 0$.

(a) $|wx_i^{-1}|$: Multiplying w by x_i^{-1} changes $\wedge_{m+2}$ from type $\mathcal{R}_1$ to type $\mathcal{M}_1^i$ and changes no other caret types. Since all the carets in S_+ and S_+^1 which succeed $\wedge_m$ and precede $\wedge_{m+np+n}$ have type $\mathcal{R}_{\mathrm{R}}$ and $m < m+2 < m+np+n$, $\wedge_{m+2}$ is of type $\mathcal{R}_{\mathrm{R}}$ in S_+ and S_+^1. So the change in the type pair of $\wedge_{m+2}$ goes from $(\mathcal{R}_1, \mathcal{R}_{\mathrm{R}})$ which has weight 2 to type $(\mathcal{M}_1^i, \mathcal{R}_{\mathrm{R}})$ which has weight 3, and clearly $|wx_i^{-1}| > |w|$.

(b) $|wx_i|$: Multiplying w by x_i when $i = 1, \dots, p-1$ does not satisfy the subtree condition and therefore by Theorem 1.3, $|wx_i| > |w|$. Multiplying w by x_p changes $\wedge_{m+1}$ from type $\mathcal{M}_\emptyset^p$ to type $\mathcal{R}_\mathrm{R}$ and changes no other caret types. Since $m < m+1 < m+np+n$, $\wedge_{m+1}$ is of type $\mathcal{R}_\mathrm{R}$ in S_+ and R_+^1. So this change in the type pair of $\wedge_{m+1}$ goes from $(\mathcal{M}_\emptyset^p, \mathcal{R}_\mathrm{R})$ which has weight 1 to $(\mathcal{R}_\mathrm{R}, \mathcal{R}_\mathrm{R})$ which has weight 2, so $|wx_p| > |w|$.

Now we consider multiplying wx_0^{-q} for $0 < q < m-1, n-1$ by $x_i^{\pm 1}$ for $i = 1, 2, \dots, p$, when both conditions of Theorem 1.2 are met.

(a) $|wx_0^{-q}x_i^{-1}|$: Multiplying wx_0^{-q} by x_i^{-1} changes $\wedge_{m+2+q(p-1)}$ (the right child of the root) in S_-^q from type $\mathcal{R}_1$ to type $\mathcal{M}_1^i$. In S_+^q and S_+^{q+1}, all carets which succeed $\wedge_m$ and precede $\wedge_{m+np+n}$ have type $\mathcal{R}_\mathrm{R}$, so since $m < m+2+q(p-1) < m+np+n$ (because $q < n-1$), $\wedge_{m+2+q(p-1)}$ is of type $\mathcal{R}_\mathrm{R}$ in S_+^q and S_+^{q+1}. So this multiplication changes the type pair of $\wedge_{m+2+q(p-1)}$ from $(\mathcal{R}_1, \mathcal{R}_\mathrm{R})$, which has weight 2, to $(\mathcal{M}_1^i, \mathcal{R}_\mathrm{R})$, which has weight 3. So $|wx_0^{-q}x_i^{-1}| = |w| - q + 1$.

(b) $|wx_0^{-q}x_i|$: Multiplying wx_0^{-q} by x_i changes $\wedge_{m+2+(q-1)(p-1)+i}$ (the *ith* child of the root) in S_-^q from type $\mathcal{M}_\emptyset^i$ to type $\mathcal{R}_{i+1}$ when $i < p$ and to type $\mathcal{R}_\mathrm{R}$ when $i = p$. Again, since $m < m+2+(q-1)(p-1)+i < m+np+n$ (because $q < n-1$), $\wedge_{m+2+(q-1)(p-1)+i}$ is of type $\mathcal{R}_\mathrm{R}$ in R_+^q and R_+^{q+1}. So this multiplication changes the type pair of $\wedge_{m+2+(q-1)(p-1)+i}$ from $(\mathcal{M}_\emptyset^i, \mathcal{R}_\mathrm{R})$, which has weight 1, to $(\mathcal{R}_{i+1}, \mathcal{R}_\mathrm{R})$ when $i < p$ and $(\mathcal{R}_\mathrm{R}, \mathcal{R}_\mathrm{R})$ when $i = p$, both of which have weight 2. So $|wx_0^{-q}x_i| = |w| - q + 1$.

(2) $|wx_0^q x_i^{\pm 1}|$, $i \in \{1, 2, \dots, p\}$: Now we consider multiplying wx_0^q for $0 < q < m-1, n-1$ by $x_i^{\pm 1}$ for $i = 1, 2, \dots, p$. First we note that wx_0^q and x_i^{-1} when $0 \leq q < m-1, n-1$ and $i = 1, 2, \dots, p$ satisfy both the subtree and minimality conditions of Theorem 1.2. So only one caret will change type in the negative tree and the positive tree will remain unchanged after multiplication in this case.

(a) $|wx_0^q x_i^{-1}|$: If we let $i = 1, \dots, p$, multiplying wx_0^q by x_i^{-1} changes the rightmost child of the root, which is $\wedge_{m-q+1}$ when $q > 0$ and $\wedge_{m+2}$ when $q = 0$. When $q = 0$, we can conclude that $\wedge_{m+2}$ is of type $\mathcal{R}_\mathrm{R}$ in both S_+ and R_+^1 (since $m < m+2 < m+np+n$), and $\wedge_{m+2}$ is changed from type $\mathcal{R}_1$ to type $\mathcal{M}_1^i$, changing the type pairing from $(\mathcal{R}_1, \mathcal{R}_\mathrm{R})$ which has weight 2 to $(\mathcal{M}_1^i, \mathcal{R}_\mathrm{R})$ which has weight 3. When $q > 0$, we can conclude that $\wedge_{m-q+1}$ is of type $\mathcal{L}_\mathrm{L}$ in both R_+^q and R_+^{q+1} since all carets which succeed $\wedge_1$ and precede $\wedge_{m+1}$ in R_+^q and R_+^{q+1} are of type $\mathcal{L}_\mathrm{L}$ and clearly $1 < m-q+1 < m+1$ (since $q < m-1$). When $q = 1$, $\wedge_{m-q+1}$ is changed from type $\mathcal{R}_\mathrm{R}$ to type $\mathcal{M}_\emptyset^i$, changing the type pairing from $(\mathcal{R}_p, \mathcal{L}_{\mathrm{L}})$ or $(\mathcal{R}_{\mathrm{R}}, \mathcal{L}_{\mathrm{L}})$, both of which have weight 1, to $(\mathcal{M}^i_\emptyset, \mathcal{L}_{\mathrm{L}})$ which has weight 2. So $|wx_0^q x_i^{-1}| = |w| - q + 1$.

(b) $|wx_0^q x_i|$, $i \in \{1, 2, \ldots, p\}$:

- (i) $|wx_0^q x_i|$, $i \in \{1, 2, \ldots, p\}$, except when $q = 0$ and $i = p$: In this case, multiplying wx_0^q by x_i does not satisfy the required conditions of Theorem 1.2 because we must add a caret before we can complete the multiplication, so we know from Theorem 1.3 that $|wx_0^q x_i| > |wx_0^q|$ in this case.
- (ii) $|wx_0^q x_p|$: When $q = 0$, wx_0^q and x_p satisfy the required subtree and minimality conditions of Theorem 1.2 and therefore only one caret changes type in the negative tree and the positive tree remains unchanged. The caret $\wedge_{m+1}$ is changed from type $\mathcal{M}^p_\emptyset$ to type $\mathcal{R}_{\mathrm{R}}$. Since $m < m+1 < m+np+n$, it is clear that $\wedge_{m+1}$ is of type $\mathcal{R}_{\mathrm{R}}$ in S_+ and R^1_+, and so the change in type pairing goes from $(\mathcal{M}^p_\emptyset, \mathcal{R}_{\mathrm{R}})$ which has weight 1 to $(\mathcal{R}_{\mathrm{R}}, \mathcal{R}_{\mathrm{R}})$ which has weight 2. So we can conclude that that $|wx_0^q x_i| > |wx_0^q|$ in this case. ☐

Proof of Theorem 2.1. This proof follows immediately from Lemma 2.1, Lemma 2.2, and Definition 1.2. So all $w \in \mathcal{S}$ are seesaw words, and we can create such words with any given swing k (where $0 < k < \min\{m-1, n-1\}$) by choosing m and n such that $m, n > k+1$. ☐

COROLLARY 2.1. *Thompson's group $F(p+1)$ contains seesaw words of arbitrarily large swing with respect to $x_0 \in X$.*

2.2. Consequences.

LEMMA 2.3. *Given any constant k, there exists a word $w \in \mathcal{S}$ such that no geodesics paths from the identity to wx_0, w, or wx_0^{-1} satisfy the k-fellow traveler property.*

Proof. This holds for the same reasons that Proposition 4.2 in [10] holds for $p = 1$.

Let γ be a geodesic path from the identity to a seesaw word w with swing n. Then γ passes through the vertex wx_0 or wx_0^{-1}; without loss of generality, we suppose that γ passes through wx_0. Let η be a geodesic path from the identity to wx_0^{-1}. Clearly, $d_\Gamma(w, wx_0^{-1}) = 1$. We can rewrite $\gamma = \gamma' x_0^s$ and $\eta = \eta' x_0^{-(s-1)}$ as long as we choose $s \leq n$. So wx_0^{s-1} is on the path γ and $wx_0^{-(s-1)}$ is on the path η. Because of the properties of seesaw words, $d_\Gamma(wx_0^{s-1}, wx_0^{-(s-1)}) = 2(s-1)$.

Because seesaw words exist with arbitrarily large swing, given any constant k, we can choose a seesaw word w with swing $s > \frac{k}{2} + 1$ so that $2(s-1) > k$. ☐

THEOREM 2.2. *Thompson's group $F(p+1)$ is not combable by geodesics.*

Proof. This holds for the same reasons that Theorem 4.2 in [10] holds for $p=1$. Suppose there does exist a combing of $F(p+1)$ by geodesics. If we let γ represent the geodesic combing path from the identity to w, then we know that the vertex wx_0^{ε} is on the path γ where $\varepsilon \in \{-1,1\}$, but the vertex $wx_0^{-\varepsilon}$ is not. Let η denote the geodesic combing path from the identity to $wx_0^{-\varepsilon}$. Lemma 2.3 shows that γ and η do not have the k-fellow traveler property, and therefore $F(p+1)$ is not combable by geodesics. □

THEOREM 2.3 (Theorem 30 in [6]). *A group G generated by a finite set X with seesaw elements of arbitrary swing w.r.t. X has no regular language of geodesics.*

COROLLARY 2.2. *There does not exist a regular language of geodesics for $F(p+1)$ with respect to X.*

3. Dead ends exist in Thompson's group $F(p+1)$

Cleary and Taback in [9] have shown that $F(2)$ has dead ends with respect to the standard finite generating set, and that all these dead ends have depth 2. In this section, we use a similar approach to extend their results to $F(p+1)$ for all $p \in \mathbb{N}$.

3.1. Dead ends in $F(p+1)$.

The proofs in this section will contain many tree-pair diagrams which use the following notational convention.

NOTATION 3.1 (Subtrees in tree-pair diagrams). When depicting tree-pair diagrams, the symbol ● indicates the presence of a nonempty subtree, and the the symbol ○ indicates the presence of a (possibly empty) subtree. When neither of these symbols are used, it is assumed that there is no subtree present.

Now we proceed to show that elements of $F(p+1)$ are dead ends if and only if they have a minimal tree-pair diagram representative with a specific form.

THEOREM 3.1. *An element in $F(p+1)$ is a dead end with respect to X if and only if it has a minimal tree-pair diagram of the form given in Figure 11.*

We note that in Theorem 3.1 we mean that the minimal form of the dead end tree-pair diagram representative must include all of the carets explicitly given in Figure 11, so, for example, at least one of the subtrees labeled $f_1, \dots, f_p$ in T_- and at least one of the subtrees labeled $f_1', \dots, f_p'$ in T_+ are nonempty because otherwise $\wedge_F$ would cancel. The proof of this theorem is based upon recognizing how the "action" of each $g \in X \cup X^{-1}$ affects an arbitrary tree-pair diagram (T_-, T_+).

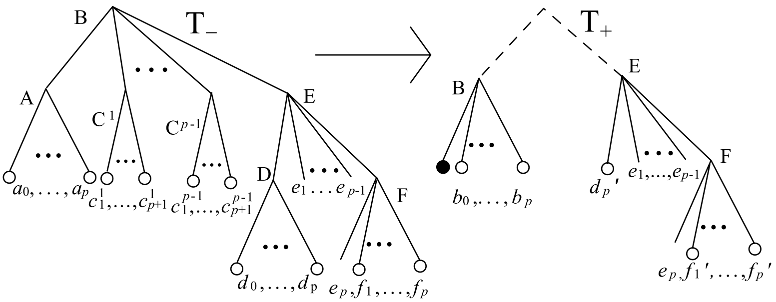

FIGURE 11. *Form of Minimal Tree-pair Diagram for All Dead Ends in* $F(p+1)$ where we impose the additional conditions that whenever $\wedge_{C^i}$, for some $i \in \{1, \dots, p-1\}$, is of type $\mathcal{M}^i_\emptyset$ in T_-, then $\wedge_{C^i}$ must be of type $\mathcal{L}_{\mathrm{L}}$, $\mathcal{R}_k$ $\mathcal{M}^l_\emptyset$ or $\mathcal{M}^l_k$ in T_+, where $k, l \le i$; and similarly, whenever $\wedge_D$ is of type $\mathcal{M}^p_\emptyset$ in T_-, then $\wedge_D$ cannot be of type $\mathcal{R}_{\mathrm{R}}$ or $\mathcal{R}_\emptyset$ in T_+.

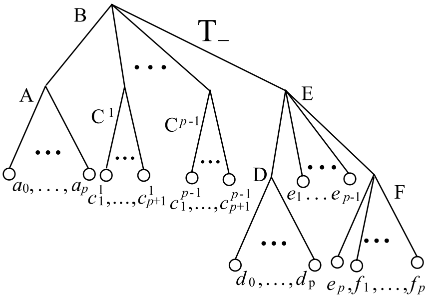

FIGURE 12. A representation of a (possibly non-minimal) negative tree in an arbitrary $(p+1)$-ary tree-pair diagram.

REMARK 3.1. The negative tree of any $(p+1)$-ary tree-pair diagram (whether or not it is a dead end) can be written in the (possibly nonminimal) form given by Figure 12, and for any negative tree in this form, the "action" of any $g \in X \cup X^{-1}$ on T_- will change only one caret type in that tree. (This is because the only other changes in type that can occur when multiplying by a generator are caused by the addition of carets to the tree-pair diagram, but by definition, negative trees in this form will belong to tree-pair diagrams to which all carets needed in order to multiply by a generator or its inverse have already been added—see Theorem 1.2 and Remark 1.3).

The "action" of g on this negative tree will produce the following caret type change (see Figure 6):

(1) x_0 takes the type of $\wedge_B$ from $\mathcal{L}_{\mathrm{L}}$ to $\mathcal{R}$.
(2) x_0^{-1} takes the type of $\wedge_E$ from $\mathcal{R}$ to $\mathcal{L}_{\mathrm{L}}$.
(3) x_i for $i = 1, \ldots, p-1$ takes the type of $\wedge_{C^i}$ from $\mathcal{M}^i$ to $\mathcal{R}$.
(4) x_i^{-1} for $i = 1, \ldots, p-1$ takes the type of $\wedge_E$ from $\mathcal{R}$ to $\mathcal{M}^i$.
(5) x_p takes the type of $\wedge_D$ from $\mathcal{M}^p$ to $\mathcal{R}$.
(6) x_p^{-1} takes the type of $\wedge_E$ from $\mathcal{R}$ to *mpp*.

Because a dead end w by definition must not increase in length when multiplied by $g \in X \cup X^{-1}$ (by Theorem 1.3), w in the product wg must satisfy the subtree condition with respect to any g.

Lemma 3.1. *All dead ends must have a minimal tree-pair diagram with negative tree of the form given by Figure 12, and any dead end w must satisfy the subtree and minimality conditions with respect to all possible $g \in X \cup X^{-1}$.*

Proof. A minimal $(p+1)$-ary tree-pair diagram representing an arbitrary element $x \in F(p+1)$ will have a negative tree of this form if and only if x satisfies the subtree condition with respect to all $g \in X \cup X^{-1}$ (see Remark 3.1). For an arbitrary dead end w, we cannot have $|wg| > |w|$, so by Theorem 1.3, w must satisfy the subtree conditions with respect to all possible g.

The fact that $w = (T_-, T_+)$ satisfies the minimality condition with respect to all possible g follows directly from the fact that it satisfies the subtree condition. The subtree condition implies that $T_+ = (Tg)_+$, and therefore, the only way in which exposed caret pairs may exist in $((Tg)_-, (Tg)_+)$, is if the "action" of g on (T_-, T_+) causes carets to be exposed in $(Tg)_-$ which were not exposed in T_-. However, if we consider the "action" of each g on the negative tree of w, which must be of the form given in Figure 12, we can see that for all $g \in X \cup X^{-1}$, the only carets which will be exposed in $(Tg)_-$ are those which are also exposed in T_- (see Figure 6, or consider Figures 13–18 which follow). Therefore, $((Tg)_-, (Tg)_+)$ is minimal for all g. ☐

Corollary 3.1. *For all dead ends $w = (T_-, T_+)$ and all $g \in X \cup X^{-1}$, the "action" of g on (T_-, T_+) only changes the type of one caret in T_- and leaves the types of all carets in T_+ unchanged.*

Proof. This follows immediately from Lemma 3.1, Remark 1.3, and Theorem 1.2. ☐

So now we can proceed to prove Theorem 3.1 by observing which caret changes type in the negative tree when each g "acts" on an arbitrary dead end $w = (T_-, T_+)$ and then enumerating those conditions which must be met by (T_-, T_+) in order for this type change to result in a decrease in length (we note that length cannot remain unchanged after multiplication by g because in $F(p+1)$ all relators are of even length). By showing that these conditions will be met if and only if w satisfies those conditions laid out in Theorem 3.1, we will conclude our proof of the theorem. Before continuing with our proof, we first introduce some notation.

Notation 3.2 ($\tau(\wedge_j)$ and $\Delta_g(\wedge_j)$). $\tau_{T_+}(\wedge_j)$ and $\tau_{(T_-,T_+)}(\wedge_j)$ represent the type of the caret $\wedge_j$ in the tree T_+ and the the type pair of the caret pair $\wedge_j$ in the tree-pair diagram (T_-,T_+), respectively.

$\Delta_g(\wedge_j)$ denotes the change in weight of the caret pair $\wedge_j$ during multiplication by some $g \in X \cup X^{-1}$, where the original tree-pair diagram and the resulting tree-pair diagram should be clear from the context.

Proof of Theorem 3.1. We consider multiplying our dead end element $w = (T_-,T_+)$ by each $g \in X \cup X^{-1}$ and enumerate which caret in the negative tree has its type changed by this multiplication and the effect of this change on the length of the element (see Table 1).

For a clearer organizational structure, we organize this process by the caret in T_- which is affected by the multiplication. The labeled carets in T_- are (see Figure 12): $\wedge_A, \wedge_B\ \wedge_{C^i}$ for $i = 1,\dots,p-1, \wedge_D, \wedge_E, \wedge_F$. To see which g affects which caret pair in (T_-,T_+), we consult Remark 3.1.

(1) Conditions on $\wedge_A$ in (T_-,T_+): We know from Remark 3.1 that there is no $g \in X \cup X^{-1}$ which will change the type of $\wedge_A$ in the negative tree, so we have no conditions on the type of this caret unless they are imposed by the required types of other carets within the tree. By definition, $\wedge_A$ is of type $\mathcal{L}$ in T_-. In T_+, the only conditions on $\wedge_A$ will come from the conditions imposed on $\wedge_B$ (see (2)); because $\wedge_B$ in T_+ must be of type $\mathcal{L}$ and since $\wedge_A$ is a predecessor of $\wedge_B$, $\wedge_A$ in T_+ must be of type $\mathcal{L}$ or of type $\mathcal{M}$ with an ancestor of type $\mathcal{L}$.

(2) Conditions on $\wedge_B$ in (T_-,T_+): We know from Remark 3.1 that only x_0 will change the type of $\wedge_B$ in the negative tree, from type $\mathcal{L}_{\mathrm{L}}$ to type $\mathcal{R}$. If we look at (T_-,T_+), we can see that in this case we can compute the types more specifically: x_0 will change the type of $\wedge_B$ in the negative tree from type $\mathcal{L}_{\mathrm{L}}$ to type $\mathcal{R}_1$ because $\wedge_B$'s leftmost child successor is $\wedge_{C^1}$, which is of type $\mathcal{M}^1$ (see Figure 13). Table 2 lists the change in weight (taken from Table 1) of this caret pair for each possible caret type pair of $\wedge_B$. From this table, we conclude that $\wedge_B$ in T_+ must be of type $\mathcal{L}_{\mathrm{L}}$ because this is the only caret pairing in (T_-,T_+) for $\wedge_B$ which will result in $|wx_0| < |w|$.

(3) Conditions on $\wedge_{C^i}$ in (T_-,T_+) for $i = 1,2,\dots,p-1$: We know from Remark 3.1 that only x_i will change the type of $\wedge_{C^i}$ in the negative tree, from type $\mathcal{M}^i$ to type $\mathcal{R}$ (see Figure 14). First, we enumerate the conditions imposed by the specific subtype of $\wedge_{C^i}$ in T_- on the specific subtype of $\wedge_{C^i}$ in $(Tx_i)_-$ (in Figure 14). First, we note that in both T_- and $(Tx_i)_-$, in $\wedge_{C^i}$, the child carets in the subtrees $c^i_1,\dots,c^i_{p-i+1}$ (if they are nonempty) will be predecessors of $\wedge_{C^i}$ and the child carets in the subtrees $c^i_{p-i+2},\dots,c^i_{p+1}$ (if they are nonempty) will be successors of $\wedge_{C^i}$ (see

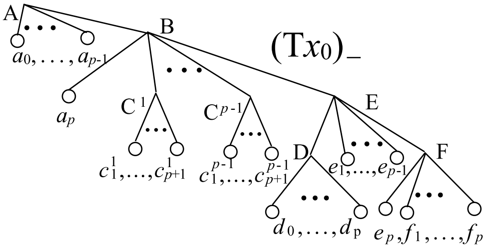

FIGURE 13. $(Tx_0)_-$ (where $(Tx_0)_+ = T_+$).

TABLE 2. How x_0 "acts" on $w(\wedge_B)$ in arbitrary dead end $w = (T_-, T_+)$, listed by possible types of $\wedge_B \in T_+$. Here, $\tau_{T_-}(\wedge_B) = \mathcal{L}_{\mathrm{L}}$

$\tau_{T_+}(\wedge_B)$	$\tau_{(T_-,T_+)}(\wedge_B)$	$\tau_{((Tx_0)_-,(Tx_0)_+)}(\wedge_B)$	$\Delta_{x_0}(\wedge_B)$
$\mathcal{L}_{\mathrm{L}}$	$(\mathcal{L}_{\mathrm{L}}, \mathcal{L}_{\mathrm{L}})$	$(\mathcal{R}_1, \mathcal{L}_{\mathrm{L}})$	-1
$\mathcal{R}_\emptyset$	$(\mathcal{L}_{\mathrm{L}}, \mathcal{R}_\emptyset)$	$(\mathcal{R}_1, \mathcal{R}_\emptyset)$	1
$\mathcal{R}_{\mathrm{R}}$	$(\mathcal{L}_{\mathrm{L}}, \mathcal{R}_{\mathrm{R}})$	$(\mathcal{R}_1, \mathcal{R}_{\mathrm{R}})$	1
$\mathcal{R}_j$	$(\mathcal{L}_{\mathrm{L}}, \mathcal{R}_j)$	$(\mathcal{R}_1, \mathcal{R}_j)$	1
$\mathcal{M}^i_\emptyset$	$(\mathcal{L}_{\mathrm{L}}, \mathcal{M}^i_\emptyset)$	$(\mathcal{R}_1, \mathcal{M}^i_\emptyset)$	1
$\mathcal{M}^i_j$	$(\mathcal{L}_{\mathrm{L}}, \mathcal{M}^i_j)$	$(\mathcal{R}_1, \mathcal{M}^i_j)$	1

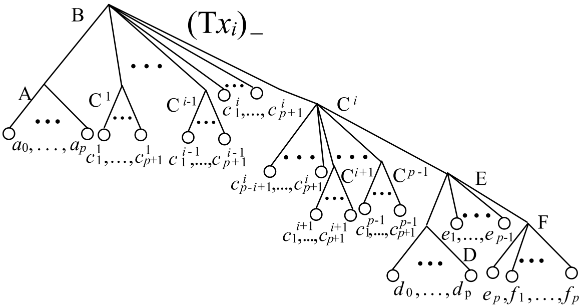

FIGURE 14. $(Tx_i)_-$ when $i = 1,\dots,p-1$ (where $(Tx_i)_+ = T_+$).

Figure 5). Additionally, the root caret of the subtrees $c^i_1,\dots,c^i_{p-i+1}$ (if they exist) will have caret types $\mathcal{M}^i,\dots,\mathcal{M}^p$ respectively, and the root carets of the subtrees $c^i_{p-i+2},\dots,c^i_{p+1}$ (if they exist) will have caret types $\mathcal{M}^1,\dots,\mathcal{M}^i$, respectively (see Figure 4).

TABLE 3. How x_i (for $i = 1, 2, \ldots, p-1$), when $\tau_{T_-}(\wedge_{C^i}) = \mathcal{M}^i_{\emptyset}$, "acts" on $w(\wedge_{C^i})$ in arbitrary dead end $w = (T_-, T_+)$, listed by possible types of $\wedge_{C^i} \in T_+$

$\tau_{T_+}(\wedge_{C^i})$	$\tau_{(T_-,T_+)}(\wedge_{C^i})$	$\tau_{((Tx_i)_-,(Tx_i)_+)}(\wedge_{C^i})$	$\Delta_{x_i}(\wedge_{C^i})$
$\mathcal{L}_{\mathrm{L}}$	$(\mathcal{M}^i_{\emptyset}, \mathcal{L}_{\mathrm{L}})$	$(\mathcal{R}_{i+1}, \mathcal{L}_{\mathrm{L}})$	-1
$\mathcal{R}_{\emptyset}$	$(\mathcal{M}^i_{\emptyset}, \mathcal{R}_{\emptyset})$	$(\mathcal{R}_{i+1}, \mathcal{R}_{\emptyset})$	1
$\mathcal{R}_{\mathrm{R}}$	$(\mathcal{M}^i_{\emptyset}, \mathcal{R}_{\mathrm{R}})$	$(\mathcal{R}_{i+1}, \mathcal{R}_{\mathrm{R}})$	1
$\mathcal{R}_j$	$(\mathcal{M}^i_{\emptyset}, \mathcal{R}_j)$	$(\mathcal{R}_{i+1}, \mathcal{R}_j)$	$\begin{cases} -1 & \text{for } j \leq i \\ 1 & \text{for } j > i \end{cases}$
$\mathcal{M}^k_{\emptyset}$	$(\mathcal{M}^i_{\emptyset}, \mathcal{M}^k_{\emptyset})$	$(\mathcal{R}_{i+1}, \mathcal{M}^k_{\emptyset})$	$\begin{cases} -1 & \text{for } k \leq i \\ 1 & \text{for } k > i \end{cases}$
$\mathcal{M}^l_m$	$(\mathcal{M}^i_{\emptyset}, \mathcal{M}^l_m)$	$(\mathcal{R}_{i+1}, \mathcal{M}^l_m)$	$\begin{cases} -1 & \text{for } l \leq i \\ 1 & \text{for } l > i \end{cases}$

(a) If $\tau_{T_-}(\wedge_{C^i}) = \mathcal{M}^i_{\emptyset}$, then the subtrees $c^i_{p-i+2}, \ldots, c^i_{p+1}$ are all empty, which implies that $\wedge_{C^{i+1}}$ is the leftmost child successor of $\wedge_{C^i}$. Since $\tau(\wedge_{C^{i+1}}) = \mathcal{M}^{i+1}$, $\tau_{(Tx_i)_-}(\wedge_{C^i}) = \mathcal{R}_{i+1}$.

(b) If $\tau_{T_-}(\wedge_{C^i}) = \mathcal{M}^i_j$, then the leftmost child successor of $\wedge_{C^i}$ in T_- is the root caret of the subtree c^i_j, which implies that the subtrees $c^i_{p-i+2}, \ldots, c^i_{j-1}$ are all empty. So the leftmost child successor of $\wedge_{C^i}$ in $(Tx_i)_-$ will also be the root of subtree c^i_j, which is of type $\mathcal{M}^j$, so $\tau_{(Tx_i)_-}(\wedge_{C^i}) = \mathcal{R}_j$.

Table 3 lists the change in weight (taken from Table 1) of this caret pair $\wedge_{C^i}$ when $\tau_{T_-}(\wedge_{C^i}) = \mathcal{M}^i_{\emptyset}$; When $\tau_{T_-}(\wedge_{C^i}) = \mathcal{M}^i_j$, the change in caret type of $\wedge_{C^i}$ from $\mathcal{M}^i_j$ to $\mathcal{R}_j$ results in a decrease in caret weight no matter what the type of $\wedge_{C^i}$ in T_+, so we conclude that if $\tau_{T_-}(\wedge_{C^i}) = \mathcal{M}^i_j$, then $\wedge_{C^i}$ in T_+ may be of any type. If $\tau_{T_-}(\wedge_{C^i}) = \mathcal{M}^i_{\emptyset}$, then we can see from Table 3 that $\wedge_{C^i}$ in T_+ may be of type $\mathcal{L}_{\mathrm{L}}$, $\mathcal{R}_k$ or $\mathcal{M}^k_{\emptyset}$, or $\mathcal{M}^r_s$ for $k, r, s \leq i$.

(4) Conditions on $\wedge_D$ in (T_-, T_+): We know from Remark 3.1 that only x_p will change the type of $\wedge_D$ in the negative tree, from type $\mathcal{M}^p$ to type $\mathcal{R}$ (see Figure 15). First, we enumerate the conditions which determine the subtype of $\wedge_D$ in T_- and the conditions imposed by that specific subtype of $\wedge_D$ in T_- on the specific subtype of $\wedge_D$ in $(Tx_p)_-$ in Figure 15. First, we note that in both T_- and $(Tx_p)_-$, in $\wedge_D$, the child carets in the subtree d_0 (if nonempty) will be predecessors of $\wedge_D$ and the child carets in the subtrees $d_1, \ldots, d_p$ (if nonempty) will be successors of $\wedge_D$ (see Figure 5). Additionally, the root caret of the subtrees $d_0, d_1, \ldots, d_p$ (if they exist) will have caret types $\mathcal{M}^p, \mathcal{M}^1, \ldots, \mathcal{M}^p$ respectively (see Figure 4).

(a) If d_j is a leaf for all $j \in \{1, \ldots, p\}$, then $\tau_{T_-}(\wedge_D) = \mathcal{M}^p_{\emptyset}$, because $\wedge_D \in T_-$ will have no child successors (see Figure 11), and $\tau_{(Tx_p)_-}(\wedge_D) =$

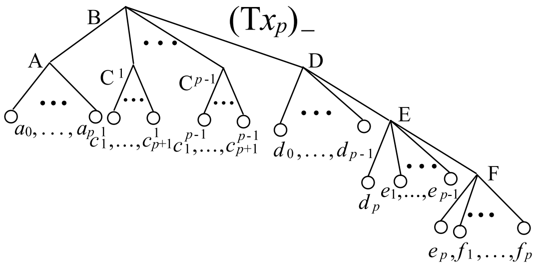

FIGURE 15. $(Tx_p)_-$ (where $(Tx_p)_+ = T_+$).

TABLE 4. How x_p, when $\tau_{T_-}(\wedge_D) = \mathcal{M}_\emptyset^p$, "acts" on $w(\wedge_D)$ in arbitrary dead end $w = (T_-, T_+)$, listed by possible types of $\wedge_D \in T_+$. Case 1 is when $\tau_{(Tx_p)_-}(\wedge_D) = \mathcal{R}_\mathrm{R}$, and case 2 is when $\tau_{(Tx_p)_-}(\wedge_D) = \mathcal{R}_\emptyset$

		$\tau_{((Tx_p)_-,(Tx_p)_+)}(\wedge_D)$		$\Delta_{x_p}(\wedge_D)$	
$\tau_{T_+}(\wedge_D)$	$\tau_{(T_-,T_+)}(\wedge_D)$	case 1	case 2	case 1	case 2
$\mathcal{L}_\mathrm{L}$	$(\mathcal{M}_\emptyset^p, \mathcal{L}_\mathrm{L})$	$(\mathcal{R}_\mathrm{R}, \mathcal{L}_\mathrm{L})$	$(\mathcal{R}_\emptyset, \mathcal{L}_\mathrm{L})$	-1	-1
$\mathcal{R}_\emptyset$	$(\mathcal{M}_\emptyset^p, \mathcal{R}_\emptyset)$	$(\mathcal{R}_\mathrm{R}, \mathcal{R}_\emptyset)$	$(\mathcal{R}_\emptyset, \mathcal{R}_\emptyset)$	1	1
$\mathcal{R}_\mathrm{R}$	$(\mathcal{M}_\emptyset^p, \mathcal{R}_\mathrm{R})$	$(\mathcal{R}_\mathrm{R}, \mathcal{R}_\mathrm{R})$	$(\mathcal{R}_\emptyset, \mathcal{R}_\mathrm{R})$	1	1
$\mathcal{R}_j$	$(\mathcal{M}_\emptyset^p, \mathcal{R}_j)$	$(\mathcal{R}_\mathrm{R}, \mathcal{R}_j)$	$(\mathcal{R}_\emptyset, \mathcal{R}_j)$	-1	-1
$\mathcal{M}_\emptyset^i$	$(\mathcal{M}_\emptyset^p, \mathcal{M}_\emptyset^i)$	$(\mathcal{R}_\mathrm{R}, \mathcal{M}_\emptyset^i)$	$(\mathcal{R}_\emptyset, \mathcal{M}_\emptyset^i)$	-1	-1
$\mathcal{M}_j^i$	$(\mathcal{M}_\emptyset^p, \mathcal{M}_j^i)$	$(\mathcal{R}_\mathrm{R}, \mathcal{M}_j^i)$	$(\mathcal{R}_\emptyset, \mathcal{M}_j^i)$	-1	-1

$\mathcal{R}_\mathrm{R}$ or $\mathcal{R}_\emptyset$ since the leftmost child successor of $\wedge_D \in (Tx_p)_-$ will be $\wedge_E$, which will also be $\wedge_D$'s immediate successor (see Figure 15).

(b) If there is a $j \in \{1, \ldots, p\}$ such that d_j is not a leaf, then $\tau_{T_-}(\wedge_D) = \mathcal{M}_i^p$, where $i = \min\{j | d_j \text{ is not a leaf}\}$, and $\tau_{(Tx_p)_-}(\wedge_D) = \mathcal{R}_i$, because when $j < p$, the root of the subtree d_i will be the leftmost child successor of $\wedge_D$ in both T_- and $(Tx_p)_-$, and will be of type $\mathcal{M}^i$ in both trees, and when $j = p$, the leftmost child successor of $\wedge_D$ will be the root of the subtree d_i (which is type $\mathcal{M}^i$) in T_- and will be $\wedge_E$ (type $\mathcal{R}$) in $(Tx_p)_-$, and the immediate successor of $\wedge_D$ will be in the subtree with root of type $\mathcal{M}^i$ in both trees (see Figures 4 and 5).

Table 4 lists the change in weight (taken from Table 1) of this caret pair $\wedge_D$ when $\tau_{T_-}(\wedge_D) = \mathcal{M}_\emptyset^p$; When $\tau_{T_-}(\wedge_D) = \mathcal{M}_j^p$, the change in caret type of $\wedge_D$ from $\mathcal{M}_j^p$ to $\mathcal{R}_j$ decreases caret weight, no matter what the type of $\wedge_D$ in T_+, so if $\tau_{T_-}(\wedge_D) = \mathcal{M}_j^p$, then $\wedge_D$ in T_+ may be of any

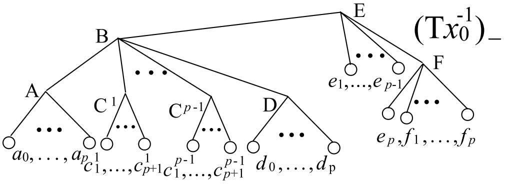

FIGURE 16. $(Tx_0^{-1})_-$ (where $(Tx_0^{-1})_+ = T_+$).

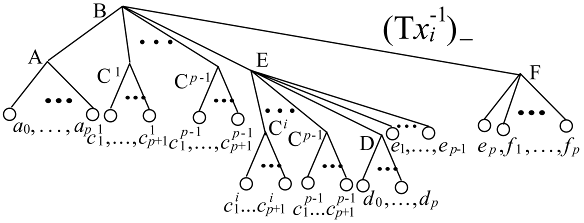

FIGURE 17. $(Tx_i^{-1})_-$ (where $(Tx_i^{-1})_+ = T_+$).

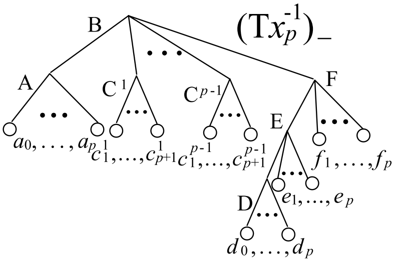

FIGURE 18. $(Tx_p^{-1})_-$ (where $(Tx_p^{-1})_+ = T_+$).

type. If $\tau_{T_-}(\wedge_D) = \mathcal{M}_\emptyset^p$, then we can see from Table 4 that $\wedge_D$ in T_+ must be of type $\mathcal{L}_\mathrm{L}$, $\mathcal{R}_j$, $\mathcal{M}_\emptyset^k$ or $\mathcal{M}_l^k$, where $j,k,l \in \{1,2,\dots,p\}$.

(5) Conditions on $\wedge_E$ in (T_-,T_+): We know from Remark 3.1 that x_i^{-1} for $i = 0,1,2,\dots,p$ will change the type of $\wedge_E$ in the negative tree, from type $\mathcal{R}$ to type $\mathcal{L}_\mathrm{L}$ when $i = 0$ (see Figure 16) and $\mathcal{M}^i$ when $i > 0$ (see Figures 17 and 18). First, we enumerate the conditions that determine the subtype of $\wedge_E$ in T_- (which is $\mathcal{R}$) and in $(Tx_i)_-$, (which is $\mathcal{L}_\mathrm{L}$ when $i = 1$ and $\mathcal{M}^i$ when $i > 0$) by considering Figures 12, 16, 17, and 18. To understand this set of conditions, see Figures 4 and 5.

(a) If e_k is a nonempty subtree in T_- for some $k \in \{1,\dots,p\}$, then:

TABLE 5. How x_0^{-1} "acts" on $w(\wedge_E)$ in arbitrary dead end $w=(T_-,T_+)$, listed by possible types of $\wedge_E \in T_+$. Case 1 is when $\tau_{T_-}(\wedge_E)=\mathcal{R}_\emptyset$, case 2 is when $\tau_{T_-}(\wedge_E)=\mathcal{R}_\mathrm{R}$, and case 3 is when $\tau_{T_-}(\wedge_E)=\mathcal{R}_j$

	$\tau_{(T_-,T_+)}(\wedge_E)$			$\tau(\wedge_E)$	$\Delta_{x_0^{-1}}(\wedge_E)$		
$\tau_{T_+}(\wedge_E)$	case 1	case 2	case 3	in wx_0^{-1}	case 1	case 2	case 3
$\mathcal{L}_\mathrm{L}$	$(\mathcal{R}_\emptyset,\mathcal{L}_\mathrm{L})$	$(\mathcal{R}_\mathrm{R},\mathcal{L}_\mathrm{L})$	$(\mathcal{R}_j,\mathcal{L}_\mathrm{L})$	$(\mathcal{L}_\mathrm{L},\mathcal{L}_\mathrm{L})$	1	1	1
$\mathcal{R}_\emptyset$	$(\mathcal{R}_\emptyset,\mathcal{R}_\emptyset)$	$(\mathcal{R}_\mathrm{R},\mathcal{R}_\emptyset)$	$(\mathcal{R}_j,\mathcal{R}_\emptyset)$	$(\mathcal{L}_\mathrm{L},\mathcal{R}_\emptyset)$	1	−1	−1
$\mathcal{R}_\mathrm{R}$	$(\mathcal{R}_\emptyset,\mathcal{R}_\mathrm{R})$	$(\mathcal{R}_\mathrm{R},\mathcal{R}_\mathrm{R})$	$(\mathcal{R}_j,\mathcal{R}_\mathrm{R})$	$(\mathcal{L}_\mathrm{L},\mathcal{R}_\mathrm{R})$	−1	−1	−1
$\mathcal{R}_k$	$(\mathcal{R}_\emptyset,\mathcal{R}_k)$	$(\mathcal{R}_\mathrm{R},\mathcal{R}_k)$	$(\mathcal{R}_j,\mathcal{R}_k)$	$(\mathcal{L}_\mathrm{L},\mathcal{R}_k)$	−1	−1	−1
$\mathcal{M}^i_\emptyset$	$(\mathcal{R}_\emptyset,\mathcal{M}^i_\emptyset)$	$(\mathcal{R}_\mathrm{R},\mathcal{M}^i_\emptyset)$	$(\mathcal{R}_j,\mathcal{M}^i_\emptyset)$	$(\mathcal{L}_\mathrm{L},\mathcal{M}^i_\emptyset)$	1	1	$\begin{cases}1 & \text{for } i<j\\ -1 & \text{for } i\geq j\end{cases}$
$\mathcal{M}^l_k$	$(\mathcal{R}_\emptyset,\mathcal{M}^l_k)$	$(\mathcal{R}_\mathrm{R},\mathcal{M}^l_k)$	$(\mathcal{R}_j,\mathcal{M}^l_k)$	$(\mathcal{L}_\mathrm{L},\mathcal{M}^l_k)$	−1	−1	−1

   (i) The type of $\wedge_E$ in T_- is $\mathcal{R}_j$ (where $j=\min\{k|e_k$ is nonempty$\}$), because when $j<p$, the root of e_j (which is type $\mathcal{M}^j$) will be the leftmost child successor of $\wedge_E$, and when $j=p$, $\wedge_F$ (which is type $\mathcal{R}$) will be the leftmost child successor of $\wedge_E$ and the immediate successor of $\wedge_E$ will be in e_j (and thus not type $\mathcal{R}$).
   (ii) They type of $\wedge_E$ in $(Tx_i^{-1})_-$ is $\mathcal{L}_\mathrm{L}$ for $i=0$ and $\mathcal{M}^i_j$ for $i>0$, because the leftmost child successor of $\wedge_E$ in $(Tx_i^{-1})_-$ is the root of the subtree e_j, which is of type $\mathcal{M}^j$ (see Figures 4 and 5).

(b) If e_k is a leaf in T_- for all $k\in\{1,\dots,p\}$, then $\wedge_F$ (which is type $\mathcal{R}$) will be the immediate successor of $\wedge_E$ in both T_- and $(Tx_i^{-1})_-$:
   (i) The type of $\wedge_E$ in T_- is $\mathcal{R}_\emptyset$ when $\wedge_F$ in T_- is type $\mathcal{R}_\emptyset$ and $\mathcal{R}_\mathrm{R}$ otherwise. If $\wedge_F$ in T_- is type $\mathcal{R}_\emptyset$, then all of the successors of $\wedge_F$ are type $\mathcal{R}$, and thus all successors of $\wedge_E$ must also be type $\mathcal{R}$. If $\wedge_F$ in T_- is not of type $\mathcal{R}_\emptyset$, then there exists at least one successor of $\wedge_F$, and of $\wedge_E$ by extension, which is not of type $\mathcal{R}$.
   (ii) The type of $\wedge_E$ in $(Tx_i^{-1})_-$ is $\mathcal{L}_\mathrm{L}$ for $i=0$ and $\mathcal{M}^i_\emptyset$ for $i>0$, because $\wedge_E$ will have no nonempty child successor in $(Tx_i^{-1})_-$.

Table 5 lists the change in weight (taken from Table 1) of $\wedge_E$ when $i=0$, and Table 6 lists the change in weight of $\wedge_E$ when $i>0$. So now we proceed to outline the possible caret types of $\wedge_E$ in (T_-,T_+) which result in reduced length after multiplication by x_i^{-1} for $i=0,\dots,p$.

From Tables 5 and 6, we have the following sets of conditions.

(a) $i=0$: The possible caret pairings for $\wedge_E$ in (T_-,T_+), determined because the weight of $\wedge_E$ decreases after multiplication by x_0^{-1} (see Table 5) are:
   (i) $(\mathcal{R},\mathcal{R})$ excluding $(\mathcal{R}_\emptyset,\mathcal{R}_\emptyset)$,

TABLE 6. How x_i^{-1} (for $i=1,2,\dots,p$) "acts" on $w(\wedge_E)$ in arbitrary dead end $w=(T_-,T_+)$, listed by possible types of $\wedge_E \in T_+$. Case 1 is when $\tau_{T_-}(\wedge_E)=\mathcal{R}_\emptyset$, case 2 is when $\tau_{T_-}(\wedge_E)=\mathcal{R}_\mathrm{R}$, and case 3 is when $\tau_{T_-}(\wedge_E)=\mathcal{R}_j$ (where $j>i$)

	$\tau_{(T_-,T_+)}(\wedge_E)$			$\tau(\wedge_E)$	$\Delta_{x_i^{-1}}(\wedge_E)$		
$\tau_{T_+}(\wedge_E)$	case 1	case 2	case 3	in wx_i^{-1}	case 1	case 2	case 3
$\mathcal{L}_\mathrm{L}$	$(\mathcal{R}_\emptyset,\mathcal{L}_\mathrm{L})$	$(\mathcal{R}_\mathrm{R},\mathcal{L}_\mathrm{L})$	$(\mathcal{R}_j,\mathcal{L}_\mathrm{L})$	$(\mathcal{M}_\emptyset^i,\mathcal{L}_\mathrm{L})$	1	1	1
$\mathcal{R}_\emptyset$	$(\mathcal{R}_\emptyset,\mathcal{R}_\emptyset)$	$(\mathcal{R}_\mathrm{R},\mathcal{R}_\emptyset)$	$(\mathcal{R}_j,\mathcal{R}_\emptyset)$	$(\mathcal{M}_\emptyset^i,\mathcal{R}_\emptyset)$	1	-1	-1
$\mathcal{R}_\mathrm{R}$	$(\mathcal{R}_\emptyset,\mathcal{R}_\mathrm{R})$	$(\mathcal{R}_\mathrm{R},\mathcal{R}_\mathrm{R})$	$(\mathcal{R}_j,\mathcal{R}_\mathrm{R})$	$(\mathcal{M}_\emptyset^i,\mathcal{R}_\mathrm{R})$	-1	-1	-1
$\mathcal{R}_k$	$(\mathcal{R}_\emptyset,\mathcal{R}_k)$	$(\mathcal{R}_\mathrm{R},\mathcal{R}_k)$	$(\mathcal{R}_j,\mathcal{R}_k)$	$(\mathcal{M}_\emptyset^i,\mathcal{R}_k)$	1 for $k\le i$, -1 for $k>i$		
$\mathcal{M}_\emptyset^l$	$(\mathcal{R}_\emptyset,\mathcal{M}_\emptyset^l)$	$(\mathcal{R}_\mathrm{R},\mathcal{M}_\emptyset^l)$	$(\mathcal{R}_j,\mathcal{M}_\emptyset^l)$	$(\mathcal{M}_\emptyset^i,\mathcal{M}_\emptyset^l)$	1	1	$\begin{cases}1 & \text{for } l<j\\ -1 & \text{for } l\ge j\end{cases}$
$\mathcal{M}_n^m$	$(\mathcal{R}_\emptyset,\mathcal{M}_n^m)$	$(\mathcal{R}_\mathrm{R},\mathcal{M}_n^m)$	$(\mathcal{R}_j,\mathcal{M}_n^m)$	$(\mathcal{M}_\emptyset^i,\mathcal{M}_n^m)$	1 for $m\le i$, -1 for $m>i$		

  (ii) $(\mathcal{R},\mathcal{M}_u^t)$,
  (iii) $(\mathcal{R}_j,\mathcal{M}_\emptyset^l)$ such that $l\ge j$.

(b) $i>0$: We define the variable $\mathcal{R}'\in\{\mathcal{R}_\emptyset,\mathcal{R}_\mathrm{R},\mathcal{R}_j \mid j>i\}$. The possible caret type pairs for $\wedge_E$ in (T_-,T_+), determined because the weight of $\wedge_E$ decreases after multiplication by x_i^{-1} where $i\in\{1,\dots,p\}$ (see Table 6) are:
  (i) $(\mathcal{R}_\mathrm{R},\mathcal{R}_\emptyset)$,
  (ii) $(\mathcal{R}_j,\mathcal{R}_\emptyset)$ where $j>i$,
  (iii) $(\mathcal{R}',\mathcal{R}_\mathrm{R})$,
  (iv) $(\mathcal{R}',\mathcal{R}_k)$ where $k>i$,
  (v) $(\mathcal{R}_j,\mathcal{M}_\emptyset^l)$ where $j>i$ and $l\ge j$,
  (vi) $(\mathcal{R}',\mathcal{M}_s^r)$ where $s>i$ (and if $\mathcal{R}'=\mathcal{R}_j$, then $r\ge j$).

We note that multiplying by *each* x_i^{-1} for $i=0,\dots,p$ imposes its own set of conditions on the type pair of $\wedge_E$. In order for w to be a dead end, the caret $\wedge_E$ in $w=(T_-,T_+)$ must satisfy *all* $p+1$ sets of conditions, because its length must be reduced whenever we multiply by x_i^{-1} for *any* $i\in\{0,\dots,p\}$. We note that $\bigcap_{i=0}^p\{(\mathcal{R}_j,*)\mid j>i\}=\emptyset$ and $\bigcap_{i=0}^p\{(\mathcal{M}_s^r,*)\mid s>i\}=\emptyset$ for any caret type $*$, so taking the intersection of the set of possible caret type pairs for all $i\in\{0,\dots,p\}$ given in 5a and 5b yields:

$$(\mathcal{R}_\emptyset,\mathcal{R}_\mathrm{R}),(\mathcal{R}_\mathrm{R},\mathcal{R}_\emptyset),(\mathcal{R}_\mathrm{R},\mathcal{R}_\mathrm{R}).$$

These are the only type pairs for $\wedge_E$ which will result in $|wx_i^{-1}|<|w|$ for *all* $i\in\{0,\dots,p\}$, and since $\wedge_E\in T_-$ and $\wedge_E\in T_+$ are both of type $\mathcal{R}_\emptyset$ or $\mathcal{R}_\mathrm{R}$, each $e_1,\dots,e_p$ must be a leaf in both T_- and T_+.

TABLE 7. Possible caret pairings for labeled carets in a dead end $w = (T_-, T_+)$. Here $*$ can be any caret type

$\wedge_A$	$\wedge_B$	$\wedge_{C^i}$, $i = 1, \ldots, p-1$	$\wedge_D$	$\wedge_E$	$\wedge_F$
$(\mathcal{L}, \mathcal{L})$	$(\mathcal{L}_{\mathrm{L}}, \mathcal{L}_{\mathrm{L}})$	$(\mathcal{M}^i, \mathcal{L}_{\mathrm{L}})$	$(\mathcal{M}^p, *)$,	$(\mathcal{R}_\emptyset, \mathcal{R}_{\mathrm{R}})$	$(\mathcal{R}, \mathcal{R})$
$(\mathcal{L}_{\mathrm{L}}, \mathcal{M})$		$(\mathcal{M}^i_\emptyset, \mathcal{R}_k)$ for $k \leq i$	except	$(\mathcal{R}_{\mathrm{R}}, \mathcal{R}_\emptyset)$	
		$(\mathcal{M}^i_\emptyset, \mathcal{M}^l_\emptyset)$ for $l \leq i$	$(\mathcal{M}^p_\emptyset, \mathcal{R}_{\mathrm{R}})$	$(\mathcal{R}_{\mathrm{R}}, \mathcal{R}_{\mathrm{R}})$	
		$(\mathcal{M}^i_\emptyset, \mathcal{M}^r_s)$ for $r, s \leq i$	or		
		$(\mathcal{M}^i_j, *)$	$(\mathcal{M}^p_\emptyset, \mathcal{R}_\emptyset)$		

(6) Conditions on $\wedge_F$ in (T_-, T_+): We know from Remark 3.1 that there is no $g \in X \cup X^{-1}$ which will change the type of $\wedge_F$ in the negative tree, so we have no conditions on the type of this caret unless they are imposed by the required types of other carets within the tree. By definition $\wedge_F$ is of type $\mathcal{R}$ in T_-. Since $e_1, \ldots, e_p$ must all be leaves in T_- and T_+ (see 5), $\wedge_F$ is the immediate successor of $\wedge_E$, so $\wedge_F$ must be type $\mathcal{R}$ in T_+.

We summarize the possible caret pairings outlined above for each of the labeled carets in (T_-, T_+) in Table 7. These are precisely the conditions met by Figure 11. □

3.2. Depth of dead ends.

THEOREM 3.2. *All dead ends in* $F(p+1)$ *have depth* 2 *with respect to* X. *Or, there are no* k*-pockets in* $F(p+1)$ *for* $k \neq 2$.

Proof. We show that for arbitrary dead end w, $|wx_0^{-1}x_ix_j|$ for any $i, j \in \{1, 2, \ldots, p\}$ will have length greater than $|w|$. The word $wx_0^{-1}x_1^2$ which Cleary and Taback use in [9] to prove this theorem for $p = 1$ is a subcase of this construction.

Suppose $|w| = q$; we have seen that $|wg^{\pm 1}| = q - 1$ for $g \in \{x_0, \ldots, x_p\}$. So $|wg_1^{\pm 1}g_2^{\pm 1}| \leq q$ for $g_1, g_2 \in \{x_0, \ldots, x_p\}$, which shows that w cannot have depth 1, and $|wg_1^{\pm 1}g_2^{\pm 1}g_3^{\pm 1}| \leq q + 1$ for $g_1, g_2, g_3 \in \{x_0, \ldots, x_p\}$. So, to show that a dead end w in $F(p+1)$ has depth 2, we need only find $g_1, g_2, g_3 \in \{x_0, \ldots, x_p\}$ such that $|wg_1^{\varepsilon_1}g_2^{\varepsilon_2}g_3^{\varepsilon_3}| \geq q + 1$ where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$.

If we consider the tree-pair diagram for w given in Figure 11, we can see that wx_0^{-1} will have the tree-pair diagram given in Figure 16. We have $|wx_0^{-1}| = q - 1$, and to multiply wx_0^{-1} by x_i for $i = 1, 2, \ldots, p$, we must add a caret to the tree-pair diagram for wx_0^{-1} on the leaf with index number e_i (we note that in a dead end, the subtrees $e_1, \ldots, e_p$ will all be empty—see Figure 11); we call this new caret E^i. So the tree-pair diagram for $wx_0^{-1}x_i$ will have the form given in Figure 19. Since we had to add a caret to the tree-pair diagram for wx_0^{-1} to get $wx_0^{-1}x_i$, by Theorem 1.3, $|wx_0^{-1}x_i| = q$. To multiply $wx_0^{-1}x_i$ by x_j where $j = 1, 2, \ldots, p$, we need to add a caret to the tree-pair diagram for $wx_0^{-1}x_i$ on the leaf with index number e_j, and then

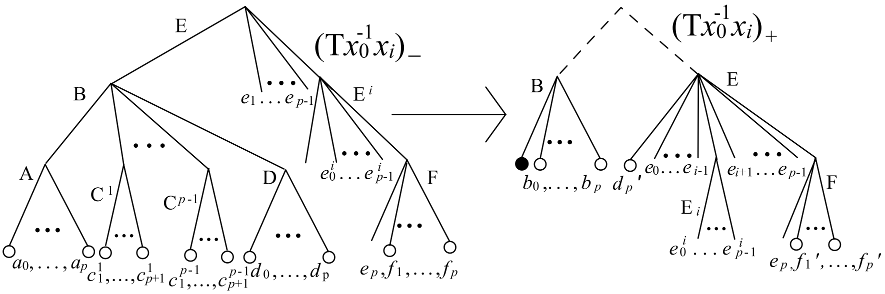

FIGURE 19. Tree-pair diagram representative of $wx_0^{-1}x_i$, for $i = 1, 2, \dots, p$ and w a dead end in $F(p+1)$.

by Theorem 1.3, $|wx_0^{-1}x_ix_j| > q$. Therefore, all dead ends have depth 2 in $F(p+1)$ with respect to X. □

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