QUASI-EXTREMALS FOR CONVOLUTION WITH SURFACE MEASURE ON THE SPHERE

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ABSTRACT. If T is the operator given by convolution with surface measure on the sphere, (E, F) is a quasi-extremal pair of sets for T if $\langle T\chi_E, \chi_F \rangle \gtrsim |E|^{d/(d+1)}|F|^{d/(d+1)}$. In this article, we explicitly define a family $\mathcal{F}$ of quasi-extremal pairs of sets for T. We prove that $\mathcal{F}$ is fundamental in the sense that every quasi-extremal pair (E, F) is comparable (in a rather strong sense) to a pair from $\mathcal{F}$. This extends work carried out by Christ for convolution with surface measure on the paraboloid.

1. Introduction

Let T be the linear operator which acts on the continuous functions on $\mathbb{R}^d$ by convolution with surface measure on the unit sphere (which we will denote by S^{d-1}). That is for a continuous function f on $\mathbb{R}^d$, Tf is defined by

$$Tf(x) = \int_{S^{d-1}} f(x - \omega) \, d\sigma(\omega).$$

Then it is well known that T extends to a continuous operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if and only if (p^{-1}, q^{-1}) lies in the closed triangle with vertices $(0, 0)$, $(1, 1)$, $(\frac{d}{d+1}, \frac{1}{d+1})$. Our goal in this article will be to study the behavior of this operator at the endpoint $(p, q) = (\frac{d+1}{d}, d+1)$ in more detail. In particular, we continue work begun by Christ in [2] by partially characterizing "quasi-extremal" and "ε-quasi-extremal" pairs for T.

Let f and g be measurable functions, not identically zero. Then by the boundedness of T from $L^{(d+1)/d}$ to L^{d+1} and duality, we have that

$$|\langle Tf, g \rangle| \lesssim \|f\|_{L^{(d+1)/d}} \|g\|_{L^{(d+1)/d}}.$$

We say that (f, g) is an ε-*quasi-extremal* pair if we have, in addition, the lower bound

$$|\langle Tf, g \rangle| \geq \varepsilon \|f\|_{L^{(d+1)/d}} \|g\|_{L^{(d+1)/d}}.$$

We say that (f, g) is simply *quasi-extremal* if (f, g) is ε-quasi-extremal for some $\varepsilon \gtrsim 1$.

If E and F are Borel sets having positive Lebesgue measures, then (E, F) is an ε-quasi-extremal or a quasi-extremal pair if (χ_E, χ_F) is.

NOTATION. As indicated above, we will write

$$S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}.$$

We will employ the symbols $\lesssim$, $\ll$, and their related symbols $\gtrsim$, $\sim$, and $\gg$ as follows: Let A and B be positive numbers and P be some statement. We will say that P implies that $A \lesssim B$ when there exists a (large) universal constant C such that P implies that $A \leq CB$. We will say that $A \ll B$ implies P when there exists a (small) universal constant c such that $A \leq cB$ implies that P holds. This use is fairly standard in the harmonic analysis literature. We will also use the somewhat less standard notation $\mathcal{T}(E, F) := \langle T\chi_E, \chi_F \rangle$, for measurable sets E and F. Here, $\langle g, f \rangle$ is the L^2 inner product, $\int_{\mathbb{R}^d} f\bar{g}\, dx$, where dx denotes Lebesgue measure. Finally, we will write $|\cdot|$ to indicate Lebesgue measure of a subset E of $\mathbb{R}^n$, where n will be clear from the context.

2. Statement of results

Before stating our results, we give an example of a quasi-extremal pair of sets. Let $\rho \leq 1$ be a positive number, and let $\mathbf{r} = (r_1, \ldots, r_{d-1})$ be a $(d-1)$-tuple of positive numbers satisfying

$$(2.1) \qquad\qquad \rho \leq r_i \leq 1, \quad 1 \leq i \leq d-1,$$

$$(2.2) \qquad\qquad r_i \leq r_{i+1}, \quad 1 \leq i \leq d-2,$$

$$(2.3) \qquad\qquad r_i \geq \rho^{1/2} r_j, \quad 1 \leq i, j \leq d-1.$$

Note that (2.3) is equivalent to $r_1 \geq \rho^{1/2} r_{d-1}$ on account of (2.2). We use the redundant formulation to avoid confusion later on. Then we define $E(\mathbf{r}; \rho)$ to be the set of $x \in \mathbb{R}^d$ such that

$$|x_1| < r_1, \qquad \ldots, \qquad |x_{d-1}| < r_{d-1}, \qquad \text{dist}(x + e_d, S^{d-1}) < \rho, \qquad x_d > -1,$$

and $F(\mathbf{r}; \rho)$ to be the set of $y \in \mathbb{R}^d$ such that

$$|y_1| < \frac{\rho}{r_1}, \qquad \ldots, \qquad |y_{d-1}| < \frac{\rho}{r_{d-1}}, \qquad \text{dist}(y, S^{d-1}) < \rho, \qquad y_d < 0.$$

Here, $e_d = (0, \ldots, 0, 1)$.

These pairs of sets are essentially thin neighborhoods of "dual" ellipsoids lying on the sphere. In particular, the ellipsoids share an orientation, the product of corresponding radii is constant, and the ellipsoids lie at opposite poles (though of different spheres). The pair above is quasi-extremal, as will

be shown in Section 4. Our main theorem states that every ε quasi-extremal pair (E, F) (after a rotation and translation) is comparable to one of the pairs given above. This comparability is rather strong and its extent is quantitative in ε.

Certainly if $R \in O(d)$ is a rotation and $x_0 \in \mathbb{R}^d$, then the sets

$$(2.4) \qquad RE(\mathbf{r}; \rho) + \{x_0\}, \qquad RF(\mathbf{r}; \rho) + \{x_0\}$$

are also quasi-extremal. Throughout, we will refer to such translated and rotated versions of our original pairs, assuming (2.1)–(2.3), as the *basic quasi-extremal pairs*.

The following theorems will involve constants C and A which depend only on the dimension d, and in particular, not on the radii $\mathbf{r}, \rho$ or the sets E, F.

THEOREM 2.1. *For every $\varepsilon > 0$ and every ε-quasi-extremal pair (E, F), there exists a basic quasi-extremal pair $(\tilde{E}, \tilde{F})$ such that*

$$\mathcal{T}(\tilde{E} \cap E, \tilde{F} \cap F) \geq C^{-1} \varepsilon^{(d+1)/(d-1)} \mathcal{T}(E, F)$$

and

$$|\tilde{E}| \leq C\varepsilon^{-A}|E|, \qquad |\tilde{F}| \leq C\varepsilon^{-A}|F|.$$

Hence, a quasi-extremal pair (E, F) may be compared to a basic quasi-extremal pair, whose elements are not too much bigger than E and F. We will also show that Theorem 2.1 implies that a quasi-extremal pair (E, F) may be compared to a basic quasi-extremal pair whose elements are smaller than E, F.

THEOREM 2.2. *For every $\varepsilon > 0$ and every ε-quasi-extremal pair (E, F), there exists a basic quasi-extremal pair $(\tilde{E}, \tilde{F})$ such that*

$$\mathcal{T}(\tilde{E} \cap E, \tilde{F} \cap F) \geq C^{-1} \varepsilon^{A} \mathcal{T}(E, F)$$

and

$$|\tilde{E}| \leq |E|, \qquad |\tilde{F}| \leq |F|.$$

Theorem 2.2 is perhaps more aesthetically pleasing than Theorem 2.1 because only one comparison of (E, F) with the basic quasi-extremal pair involves the loss of a power of ε, and Theorem 2.1 is actually the stronger of the two.

Finally, Theorem 2.2 is the analogue of the main theorem of [2], and by the arguments in that work, implies the following theorem on ε quasi-extremal pairs of functions.

THEOREM 2.3. *For every $\varepsilon > 0$ and every ε quasi-extremal pair of non-negative functions (f, g), there exist sets E, F and real numbers $s, t > 0$, such that*

$$s\chi_E \leq f, \qquad t\chi_F \leq g;$$

furthermore, there exists a basic quasi-extremal pair $(\tilde{E}, \tilde{F})$ *with* $|\tilde{E}| \leq |E|$, $|\tilde{F}| \leq |F|$ *such that*

$$s \cdot t \cdot \mathcal{T}(E \cap \tilde{E}, F \cap \tilde{F}) \geq C^{-1} \varepsilon^A \langle Tf, g \rangle.$$

Except for the verification of condition (2.3), our proof is a more or less straightforward adaptation of [2]; where possible, we will refer to that article for details. In particular, for the proof of Theorem 2.3, we refer the reader to [2].

3. Some context

First, we compare the results here to those obtained in [2]. In that work, Christ considered the operator T_P, defined by convolution with a measure on the paraboloid $P = \{x \in \mathbb{R}^d : x_d = |x'|^2\}$;

$$T_P f(x) := \int_{\mathbb{R}^{d-1}} f(x' - t', x_d - |t|^2) \, dt.$$

He proved that the basic quasi-extremal pairs for T_P, also at the $L^p \to L^q$ endpoint $(\frac{d+1}{d}, d+1)$, are obtained by applying a rotation $S \in O(d)$ (fixing the x_d component) to the following pairs: If $x_0, y_0 \in \mathbb{R}^d$ with $x_0 - y_0 \in P$ and $\rho, r_1, \ldots, r_{d-1}$ are *any* positive radii, define $E_P(x_0, y_0, \mathbf{r}, \rho)$ to be the set of $x \in \mathbb{R}^d$ such that

$$|x_i - (x_0)_i| < r_i, \quad 1 \leq i \leq d-1, \qquad \left| x_d - (y_0)_d - |x' - y_0'|^2 \right| < \rho$$

and $F_P(x_0, y_0, \mathbf{r}, \rho)$ to be the set of $y \in \mathbb{R}^d$ such that

$$|y_i - (y_0)_i| < \frac{\rho}{r_i}, \quad 1 \leq i \leq d-1, \qquad \left| y_d - (x_0)_d + |y' - x_0'|^2 \right| < \rho.$$

Thus, the families of basic quasi-extremal pairs for the two operators are similar. The chief difference is in the admissible radii. For convolution with surface measure on the paraboloid, any collection of radii $\rho, r_1, \ldots, r_{d-1} > 0$ give rise to quasi-extremal pairs. For convolution with surface measure on the sphere, compactness forces us to take $0 < \rho < 1$ and $\rho < r_i < 1$. The further condition, (2.3), however, comes from geometric properties other than compactness.

We will write $E_P(\mathbf{r}, \rho) := E_P(0, 0, \mathbf{r}, \rho)$ and $F_P(\mathbf{r}, \rho) := F_P(0, 0, \mathbf{r}, \rho)$.

One explanation for the extra rigidity of the quasi-extremals for the sphere is that the paraboloid posses a product structure, while the sphere does not. One can see a manifestation of this by considering the pair

$$E := E_P((1, \ldots, 1, \rho, \ldots, \rho); \rho), \qquad F := F_P((1, \ldots, 1, \rho, \ldots, \rho); \rho).$$

The radii are admissible for the paraboloid, and one may think of E and F as exhibiting the product structure $P = P \cap (\mathbb{R}^k \times \{0\}^{d-1-k}) + P \cap (\{0\}^k \times \mathbb{R}^{d-1-k})$. In the case of the sphere, however, (2.3) fails and the radii are not admissible.

When one considers either the sphere or the paraboloid, two of the basic quasi-extremal pairs of sets are quite well-known. When the r_i all equal ρ, E is a ball of radius ρ and F is a ρ-neighborhood of a patch of the hypersurface. When the r_i all equal $\rho^{1/2}$, (E, F) is essentially the Knapp example; $E \approx [-\rho^{1/2}, \rho^{-1/2}]^{d-1} \times [-\rho, \rho]$, $F \approx E + \{y_0\}$ ($y_0 = -e_d$ for S^{d-1} and 0 for P). Together, these examples are often used to show that $(p, q) = (\frac{d+1}{d}, d+1)$ is an endpoint for the Lebesgue bounds for T or T_P, as in [9], for instance.

The operators studied here and in [2] are merely examples of a much larger class, called generalized Radon transforms in the recent literature.

See the articles [6] by Phong and Stein and more recently [7] by Schlag for some results and an excellent discussion of a larger class of operator defined by integration on curved hypersurfaces. One motivation for studying these Radon-like transforms comes from partial differential equations. For example, in 3 dimensions the solution at fixed times to the initial value problem for the wave equation is solved by convolution with surface measure on the sphere.

At the other dimensional extreme from hypersurfaces, Tao and Wright in [11] have proved $L^p \to L^q$ bounds near the endpoint for operators defined by integration over one-dimensional curves; Christ has reproved their result in [1] using similar techniques. The article by Tao and Wright, in particular, contains an extensive bibliography which may be of interest to the reader.

Between these extremes, the $L^p \to L^q$ bounds are still largely unknown, but curvature is still important; see [4].

A more general discussion of the role curvature plays in harmonic analysis may be found, for instance, in the two expository articles, [10] and [5].

4. Basic quasi-extremal pairs of sets

In this section, we will prove that the basic quasi-extremal pairs of sets are in fact quasi-extremal.

PROPOSITION 4.1. *Let $1 > \rho > 0$ and suppose that $r_1, \ldots, r_{d-1}$ satisfy inequalities (2.1)–(2.3). Then the pair $(E(\mathbf{r}; \rho), F(\mathbf{r}; \rho))$, which was defined in Section 2, is quasi-extremal.*

Because the operator T commutes with translations and rotations, we obtain the following corollary.

COROLLARY 4.2. *If $R \in O(d)$ is a rotation, $x \in \mathbb{R}^d$, and $\rho > 0$ and $r_1, \ldots, r_{d-1} > 0$ satisfy (2.1)–(2.3), then $(RE(\mathbf{r}; \rho) + \{x\}, RF(\mathbf{r}; \rho) + \{x\})$ is a quasi-extremal pair.*

Proof of Proposition 4.1. It suffices to prove the proposition under the additional assumptions that

$$(4.1) \qquad \rho \ll 1, \qquad \rho \ll r_i \ll 1, \quad 1 \le i \le d-1.$$

Let $E := E(\mathbf{r}; \rho)$ and $F := F(\mathbf{r}; \rho)$. Then it is easy to check that

$$|E| \sim \left(\prod_{i=1}^{d-1} r_i\right)\rho, \qquad |F| \sim \left(\prod_{i=1}^{d-1} \frac{\rho}{r_i}\right)\rho,$$

so what we must show to verify quasi-extremality of (E, F) is that

(4.2) $\mathcal{T}(E, F) \gtrsim \rho^d.$

Let $x \in E(c\mathbf{r}; c\rho)$, where $c > 0$ is sufficiently small (depending only on d) for later purposes. Consider the set

$$\mathcal{F}_0(x) := \left\{s \in \mathbb{R}^{d-1} : |s_i - x_i| < c\frac{\rho}{r_i}, 1 \le i \le d-1\right\}.$$

Then by the smallness of the r_i and $\frac{\rho}{r_i}$, $|\mathcal{F}_0(x)| \sim \prod_{j=1}^{k} \frac{\rho}{r_j}$. We will prove that

(4.3) $s \in \mathcal{F}_0(x) \quad \Longrightarrow \quad x - \left(s, \sqrt{1 - |s|^2}\right) \in F(\mathbf{r}, \rho).$

This will imply that $T\chi_F(x) \gtrsim \prod_{j=1}^{k} \frac{\rho}{r_j}$, from which the estimate (4.2) will follow.

First, suppose that $r_{d-1} \le \rho^{1/2}$. Let $x \in E(c\mathbf{r}; c\rho)$ and $s \in \mathcal{F}_0(x)$. Then

$$x_d = cO(\rho), \qquad x_i - s_i = cO\left(\frac{\rho}{r_i}\right),$$

$$x_d - \sqrt{1 - |s|^2} = cO(\rho) + \sqrt{1 - |x' - s|^2},$$

where the second inequality follows from $r_i \le \rho^{1/2} \le \frac{\rho}{r_i}$. Therefore, (4.3) holds.

Similarly, it is not difficult to show that (4.3) holds when $r_1 \ge \rho^{1/2}$. Hence, we may assume that there exists an index k, $1 \le k \le d-2$ so that

$$r_k \le \rho^{1/2} \le r_{k+1};$$

recall the monotonicity assumption (2.2).

Let $x \in E(c\mathbf{r}, c\rho)$ and $s \in \mathcal{F}_0(x)$, and define

$$t = (s_1, \ldots, s_k) \quad \text{and} \quad y = (x_{k+1}, \ldots, x_{d-1}).$$

To verify (4.3), it suffices to show that

(4.4) $\left|x_d - \sqrt{1 - |s|^2} + \sqrt{1 - |x' - s|^2}\right| \le \rho.$

We note that $x_d = \sqrt{1 - |x'|^2} - 1 + cO(\rho)$ and that by our choice of k,

$$|x'|^2 = |y|^2 + cO(\rho), \qquad |x' - s|^2 = |t|^2 + cO(\rho), \qquad |s|^2 = |t|^2 + |y|^2 + cO(\rho).$$

Therefore, (4.4) will follow from

(4.5) $\left|\sqrt{1 - |y|^2} + \sqrt{1 - |t|^2} - \sqrt{1 - |y|^2 - |t|^2} - 1\right| \le cO(\rho).$

By condition (2.3) on the r_i, either $r_1 \geq \rho^{3/4}$ or $r_{d-1} \leq \rho^{1/4}$. Assuming the latter, we use a Taylor series expansion to obtain

$$\sqrt{1 - |y|^2} = 1 - \frac{|y|^2}{2} + cO(\rho)$$

and

$$\sqrt{1 - |t|^2 - |y|^2} = \sqrt{1 - |t|^2} - \frac{|y|^2}{2\sqrt{1 - |t|^2}} + cO(\rho).$$

Therefore,

$$\left| \sqrt{1 - |y|^2} + \sqrt{1 - |t|^2} - \sqrt{1 - |y|^2 - |t|^2} - 1 \right|$$
$$= \frac{|y|^2}{2} \left(\frac{1}{\sqrt{1 - |t|^2}} - 1 \right) + cO(\rho).$$

By smallness of the r_i and the $\frac{\rho}{r_i}$, $\sqrt{1 - |t|^2} \sim 1$, so the term on the is bounded by $c(r_{d-1}^2 \cdot (\frac{\rho}{r_1})^2 + O(\rho))$. Inequality (4.5) follows from another application of condition (2.3).

In the other case, $r_1 \geq \rho^{3/4}$, we have that $\frac{\rho}{r_1} \leq \rho^{1/4}$, and the verification of (4.5) is the same as in the previous case, with the roles of y and t reversed. □

5. Setup for the proof of the main theorem

Let (E, F) be an ε-quasi-extremal pair of sets. Using a partition of unity, we may write $T = \sum_{j=1}^{M} T_j$, where T_j is equal to convolution with $a_j d\sigma$ and the T_j and M depend on the dimension alone. Here $d\sigma$ is surface measure on S^{d-1} and a_j is a smooth function supported on a set $U_j \subset S^{d-1}$ having diameter $\ll 1$. By the triangle inequality, (E, F) is $(M^{-1}\varepsilon)$-quasi-extremal for at least one of the T_j. By means of a rotation, we may assume that U_j is contained in a small ball centered at $(0, \ldots, 0, 1)$. Henceforth, we will write $T = T_j$, $U = U_j$, and $M^{-1}\varepsilon = \varepsilon$. Of course, this means, for instance, that T is no longer self-adjoint.

We define

$$\alpha := \frac{\mathcal{T}(E, F)}{|E|} \quad \text{and} \quad \beta := \frac{\mathcal{T}(E, F)}{|F|}.$$

Then α represents the average size of $T\chi_F$ for points in E and β represents the average size of $T^*\chi_E$ for points in F. See for instance [3] for a proof of the following lemma.

LEMMA 5.1. *For each integer $N \geq 1$, there exists $x_0 \in E$ and measurable sets $\Omega_i \subset \mathbb{R}^{i(d-1)}$, $1 \leq i \leq N$, such that $|\Omega_1| \gtrsim \alpha$, $\Omega_i \subset \Omega_{i-1} \times \mathbb{R}^{d-1}$ for $2 \leq i \leq N$ and such that whenever $t = (t_1, \ldots, t_i) \in \Omega_i$, $1 \leq i \leq N$,*

$$|\{s \in \mathbb{R}^{d-1} : (t, s) \in \Omega_{i+1}\}| \gtrsim \begin{cases} \beta, & \text{if } i < N \text{ is odd,} \\ \alpha, & \text{if } i < N \text{ is even,} \end{cases}$$

$$x + \sum_{j=1}^{i} (-1)^j \left(t_j, \sqrt{1 - |t|^2}\right) \in \begin{cases} F, & \text{if } i \text{ is odd}, \\ E, & \text{if } i \text{ is even}. \end{cases}$$

Here, the implicit constants depend on N and d.

Henceforth, we will assume that N is fixed and sufficiently large (say 5 or so) and that x_0 and $\Omega_1, \Omega_2, \ldots$ satisfy the conclusions of the lemma.

6. The shape of Ω_1

Techniques used in this section have previously appeared in the work [2] by Christ and are similar to arguments used by Schlag in [7].

6.1. Inflation bound. Let $s \in \mathbb{R}^{d-1}$ and $(t_1, \ldots, t_{d-1}) = \vec{t} \in \mathbb{R}^{(d-1)(d-1)}$ with $|s|, |t_i| \ll 1$, $1 \le i \le d-1$. We define $\Psi^\natural : \mathbb{R}^{(d-1)d} \to \mathbb{R}^{d(d-1)}$ by

$$[\Psi^\natural(s, \vec{t})]_j := \left(t_j - s, \sqrt{1 - |t_j|^2} - \sqrt{1 - |s|^2}\right),$$

where we are writing $\mathbf{x} \in \mathbb{R}^{d(d-1)}$ as $\mathbf{x} = ([x]_1, \ldots, [x]_{d-1})$ with $[x]_j \in \mathbb{R}^d$. We may compute

$$\det D\Psi^\natural = \det(F_s(t_1), \ldots, F_s(t_{d-1})),$$

where for $s, t \in \mathbb{R}^{d-1}$ with $|s|, |t| < 1$,

$$F_s(t) := \frac{t}{\sqrt{1 - |t|^2}} - \frac{s}{\sqrt{1 - |s|^2}}.$$

See the end of this section for a few remarks concerning this function.

We would like to obtain a lower bound for $|E|$. For $s \in \Omega_1$, $|s| \ll 1$, let

$$\mathcal{F}(s) := \{t \in \mathbb{R}^{d-1} : |t| \ll 1 \text{ and } (s, t) \in \Omega_2\}.$$

If we define

$$\Omega^\natural := \left\{(s, \vec{t}) \in \mathbb{R}^{d-1} \times \mathbb{R}^{(d-1)(d-1)} : s \in \Omega_1, \ \vec{t} \in (\mathcal{F}(s))^{d-1}\right\},$$

then $\Psi^\natural(\Omega^\natural) \subset E^{d-1}$.

Write $\omega_0 := (s, \sqrt{1 - |s|^2})$ and $\omega_i := (t_i, \sqrt{1 - |t_i|^2})$, $1 \le i \le d-1$. Then for each $\mathbf{x} = (x_1, \ldots, x_{d-1}) \in (\mathbb{R}^d)^{d-1}$, the pre-image of $\mathbf{x}$ under $\Psi^\natural$ has cardinality

$$\#\{(s, \vec{t}) : \Psi^\natural(s, \vec{t}) = \mathbf{x}\}$$
$$\le \#\{(\omega_0, \omega_1, \ldots, \omega_{d-1}) \in (S^{d-1})^d : \omega_0 = x_i + \omega_i, 1 \le i \le d-1\}$$
$$\le \prod_{j=0}^{d-1} \# \bigcap_{i=0}^{d-1} (S^{d-1} + \{x_i - x_j\}) = \left[\# \bigcap_{i=0}^{d-1} (S^{d-1} + \{x_i\})\right]^{d-1},$$

where $x_0 := 0$.

By Bezout's Theorem (see [8], for instance), for $\mathbf{x}$ lying off of a measure-zero (indeed, algebraic) subset of $\mathbb{R}^{d(d-1)}$, the side is bounded by a constant depending only on d. An elementary proof of this fact is also possible; one

can take advantage of the fact that the intersection of two spheres, neither a subset of the other, is either empty or a lower dimensional sphere.

From the cardinality bound established above, we have the estimate

$$|E|^{d-1} \gtrsim \int_{\Omega_1} \int_{\mathcal{F}(s)^{d-1}} |\det(F_s(t_1), \ldots, F_s(t_{d-1}))| \, d\vec{t} \, ds.$$

6.2. Approximation by convex sets. We will use an argument from [2] to obtain a lower bound for this term and to describe a typical set $\mathcal{F}(s)$; by symmetry a similar description will apply to Ω_1.

A set $V \subset \mathbb{R}^{d-1}$ is *balanced* if $-V = V$.

LEMMA 6.1. *Let $\eta > 0$. Then for any Lebesgue measurable set $A \subset \mathbb{R}^{d-1}$ with $0 < |A| < \infty$, there exists a bounded, balanced convex set $V \subset \mathbb{R}^{d-1}$ such that whenever $V' \subset V$ is a balanced convex set with $|V'| \le \frac{1}{2}|V|$, then*

$$|A \cap (V \backslash V')| \gtrsim \left(\frac{|A|}{|V|} \right)^{\eta} |A|.$$

Moreover,

$$\int_{A^{d-1}} |\det(\vec{u})| \, d\vec{u} \gtrsim |V| |A|^{d-1} \left(\frac{|A|}{|V|} \right)^{\eta(d-1)}.$$

Here, the implicit constants depend only on η and d.

Note that as a consequence of the lemma, the convex set V satisfies $|V| \gtrsim |A|$.

The proof is contained in two somewhat more general lemmas in [2]. We only present a sketch of the argument here. The first part is proved via a stopping-time procedure. We start with a large balanced convex set V having size $2^m |A|$ so that $|V \cap A| \ge \frac{3}{4}|A|$; if V satisfies the conclusion of the lemma, we are done, otherwise, there is a bad set $V' \subset V$, with which we replace V. The main trick in this portion of the proof is to use η to show that this procedure terminates before $|V|$ reaches $|A|$. The proof of the second part of the lemma uses the defining property of V and the identity

$$|\det(\vec{u})| = \prod_{i=1}^{d-1} \operatorname{dist}(u_i, V_{i-1}),$$

where $V_0 = \{0\}$ and $V_i = \operatorname{span}(u_1, \ldots, u_i)$ to bound $\vec{u}$ away from the sets where $\det(\vec{u})$ vanishes.

We apply the lemma to the set $F_s(\mathcal{F}(s))$, for the moment leaving η undetermined, to obtain a balanced convex set $V(s) \subset \mathbb{R}^{d-1}$. Because near zero, F_s is a diffeomorphism with bounded differential, we have that

$$|V(s)| \gtrsim |F_s(\mathcal{F}(s))| \sim |\mathcal{F}(s)| \sim \beta.$$

Hence, by making the change of variables $u_i = F_s(t_i)$, $1 \le i \le d-1$, we see that

$$\int_{\Omega^\natural} |\det(F_s(t_1), \dots, F_s(t_{d-1}))| \, d\vec{t} \, ds \gtrsim \int_{\Omega_1} \int_{F_s(\mathcal{F}(s))^{d-1}} |\det(\vec{u})| \, d\vec{u} \, ds$$

$$\gtrsim \int_{\Omega_1} |V(s)|^{1-(d-1)\eta} |\mathcal{F}(s)|^{(d-1)(1+\eta)}.$$

Assuming that $\eta < \frac{1}{d-1}$, this implies that

$$|E|^{d-1} \gtrsim \alpha \beta^d.$$

From the definitions of α and β, this immediately yields the (well-known) bound

$$\langle T\chi_E, \chi_F \rangle \lesssim |E|^{d/(d+1)} |F|^{d/(d+1)}.$$

On the other hand, we are assuming that

$$\langle T\chi_E, \chi_F \rangle \ge \varepsilon |E|^{d/(d+1)} |F|^{d/(d+1)},$$

which implies that

(6.1) $|E|^{d-1} \lesssim \varepsilon^{-(d+1)} \beta^d \alpha.$

Since the series of inequalities above imply that

$$|E|^{d-1} \gtrsim \int_{\Omega_1} \beta^{(d-1)(1+\eta)} |V(s)|^{1-(d-1)\eta} \, ds,$$

the above upper bound on $|E|$ means that for most (in particular, at least one) of the $s \in \Omega_1$, we must have $|V(s)| \lesssim \varepsilon^{-C} \beta$.

The above argument (with minor changes) enables us to assume that for some $\tau \in \mathbb{R}^n$ with $|\tau| \ll 1$, $F_\tau(\Omega_1)$ is contained in a balanced convex set V of size $\lesssim \varepsilon^{-C} \alpha$. By John's theorem, we may assume that V is actually an ellipsoid.

6.3.   Comments on F_s. If we let $g(t) = \sqrt{1 - |t|^2}$ for $|t| < 1$, then the mapping F_s is equal to

$$\nabla g - \nabla g(s).$$

All of the material in Sections 6 and 7 is applicable to any sufficiently nice mapping g with $\nabla g(0) = 0$ and $\det D^2 g(0) \ne 0$. In the case of Section 6, this generality was pointed out to the author by Christ, and in the case of Section 7, it can be obtained from his work in [2] together with some additional details in the following section.

For two choices of g, F_s has a particularly nice form.

When $T = T_P$, the operator mentioned in Section 3, $g(s) = |s|^2$. In this case, of course, $F_s(t) = t - s$ for all s and t. Therefore, $F_\tau(\Omega_1)$ is contained in a balanced convex set if and only if the vertical projection of Ω_1 to $\mathbb{R}^{d-1}$ is contained in a convex set which is balanced with respect to τ.

When $g(s) = \sqrt{1 - |s|^2}$, ∇g may be thought of as the function which maps a point $\omega \in S^{d-1}$ to the point on $\mathbb{R}^{d-1} \times 0$ which is collinear with 0 and ω. This function maps great circles to straight lines, so $F_\tau(\Omega_1)$ is contained in a convex set if and only if Ω_1 is contained in a geodesically convex subset of the upper hemisphere. The point $(\tau, \sqrt{1 - |\tau|^2})$ would be contained in this convex set, but the set would not be balanced about that point in any natural (rotationally invariant) sense.

7. The shape of $\mathcal{F}(s)$

7.1. Slicing bound. Let Φ be defined by

$$\Phi(s, t) = \left(t - s, \sqrt{1 - |t|^2} - \sqrt{1 - |s|^2}\right),$$

for $s, t \in \mathbb{R}^{d-1}$ with $|s|, |t| < 1$; then $\Phi(\Omega_2) \subset E$. Let B be the unit ball in $\mathbb{R}^{d-1}$, and let A be a positive definite symmetric linear transformation having norm $\ll 1$.

In this subsection, we prove the following lemma.

LEMMA 7.1. *If* $F_\tau(\Omega_1) \subset A(B)$, *where* $|\tau| \ll 1$, *then*

$$|E| \gtrsim |\det A|^{-1} \int_\Omega |A(DF_\tau(s))^{-1} F_s(t)| \, dt \, ds.$$

Here, we recall that

$$F_s(t) := \frac{t}{\sqrt{1 - |t|^2}} - \frac{s}{\sqrt{1 - |s|^2}}.$$

The proof of Lemma 7.1 is modeled on a proof of the analogous lemma in [2].

If $\nu \in \mathbb{R}^{d-1}$ is a unit vector, and $a \in \nu^\perp \subset \mathbb{R}^{d-1}$, then we define

$$s^\nu(a, r) := F_\tau^{-1}\left(A(r\nu + a)\right).$$

Thus, when ν is fixed, for each $s \in \Omega_1$, there is a unique choice of a, r with $|a|, |r| \ll 1$ such that $s = s^\nu(a, r)$. For the moment, let ν and a be fixed; we will abuse notation by writing

$$s(r) = s^\nu(a, r).$$

For $|s| \le 1$, let $g(s) := \sqrt{1 - |s|^2}$, and for $r \in \mathbb{R}$, and $u \in \mathbb{R}^{d-1}$ with $|r|, |u| \ll 1$, define

$$\Psi(r, u) := \left(u, g\left(u + s(r)\right) - g(s(r))\right).$$

Note that when $s = s(r)$ and $u = t - s(r)$, $\Psi(r, u) = \Phi(s, t)$.

Both (r, u) and $\Psi(r, u)$ are elements of $\mathbb{R}^d$, so one can compute the Jacobian $\det D\Psi(r, u)$. The lemma would follow from the estimate

$$(7.1) \qquad |\Psi(\Omega)| \gtrsim \int_\omega |\det D\Psi(r, u)| \, du \, dr,$$

for measurable sets Ω (details are forthcoming). We will establish the validity of (7.1) by showing that Ψ is nearly injective.

Supposing that $\Psi(r, u) = \Psi(r', u')$, one has that $u = u'$ and

$$g(u + s(r)) - g(s(r)) = g(u + s(r')) - g(s(r')) =: f_u(r).$$

We claim that for $u \neq 0$, $r \mapsto f_u(r)$ is $O(1)$-to-1 on $|r| \ll 1$. By rotating coordinates if necessary, we may assume that $A\nu = e_1$ and set $Aa + \frac{\tau}{\sqrt{1-|\tau|^2}} =:$ a_0. Then

$$s(r) = F_0^{-1}(re_1 + a_0) = \frac{re_1 + a_0}{\sqrt{1 + |re_1 + a_0|^2}}.$$

Using this, one can explicitly compute that $f_u'(r)$ vanishes if and only if a certain polynomial $p_{u,a_0}(r)$ of degree $O(1)$ vanishes. Since p_{u,a_0} is the zero polynomial if and only if $u = 0$, the claim is proved, and we may use the estimate (7.1).

With ν and a fixed, we define

$$\omega_a^\nu := \{(r, t) : (s^\nu(a, r), t) \in \Omega_2\},$$
$$\tilde{\omega}_a^\nu := \{(r, u) : (r, u + s^\nu(a, r)) \in \omega_a^\nu\}.$$

So far, we have shown that for each a and ν,

$$|E| \geq |\Psi(\tilde{\omega}_a^\nu)| = \int_{\omega_a^\nu} \left| \langle A[D^2 g(s(r))]^{-1} F_{s(r)}(t), \nu \rangle \right| dt \, dr.$$

Now, for each ν,

$$|E| \gtrsim \int_{a \in \nu^\perp, |a| \ll 1} |\Psi(\tilde{\omega}_a^\nu)| \, da \gtrsim |\det A|^{-1} \int_{\Omega_2} |\langle A[D^2 g(s)]^{-1} F_s(t), \nu \rangle| \, dt \, ds,$$

where the last inequality follows from the change of variables $s = s^\nu(a, r)$ and the fact that $|\det[DF_\tau](s)| \sim 1$. Averaging with respect to unit vectors, ν completes the proof of the lemma.

7.2. Combining the inflation and slicing bounds. The arguments of this subsection are easy modifications of arguments due to Christ in [2].

We have shown the following. There exist $x_0 \in \mathbb{R}^d$, $\tau \in \mathbb{R}^{d-1}$, $|\tau| \ll 1$, a set Ω_1, and a symmetric, positive-definite linear transformation A, $\|A\| \ll 1$ such that

$$|\Omega_1| \gtrsim \alpha, \qquad \det(A) \lesssim \varepsilon^{-b}\alpha, \qquad F_\tau(\Omega_1) \subset A(B),$$
$$s \in \Omega_1 \implies x_0 - \left(s, \sqrt{1 - |s|^2}\right) \in F$$

where B is the unit ball, and $b > 0$. From the main lemma of Section 7.1,

$$|E| \gtrsim (\det A)^{-1} \int_{\Omega_1} \int_{\mathcal{F}(s)} |A(DF_\tau(s))^{-1} F_s(t)| \, dt \, ds$$
$$\sim (\det A)^{-2} \int_{\Omega_1} \int_{\tilde{\mathcal{F}}(s)} |w| \, dw \, ds,$$

after making the change of variables

$$w = A(DF_\tau(s))^{-1}F_s(t), \qquad \tilde{\mathcal{F}}(s) = A(DF_\tau(s))^{-1}F_s(\mathcal{F}(s))$$

and using the fact that for s, τ small,

$$\det\big(DF_{s\,\text{or}\,\tau}(t \text{ or } s)\big) \sim 1.$$

Now for each $s \in \Omega_1$ and each $\rho > 0$, either

$$(7.2) \qquad \int_{\tilde{\mathcal{F}}(s)} |w|\, dt\, ds \gtrsim \rho|\tilde{\mathcal{F}}(s)| \sim \rho(\det A)|\mathcal{F}(s)| \sim \varepsilon^b \rho\alpha\beta.$$

or

$$(7.3) \qquad |\tilde{\mathcal{F}}(s) \cap B(0,\rho)| \gtrsim |\tilde{\mathcal{F}}(s)|.$$

Recalling (6.1), if we set $\rho = C\varepsilon^{-C'}(\alpha\beta)^{1/(d-1)}$ (C, C' depending only on d), then occurrence of (7.2) over a majority of Ω_1 is impossible; it would contradict the upper bound on $|E|$ which results from quasi-extremality. Hence, (7.3) must hold for most $s \in \Omega_1$. Refining Ω_1 to a subset whose size is still $\gtrsim \alpha$, we may assume that (7.3) holds for each $s \in \Omega_1$.

Unwinding the definition of $\tilde{\mathcal{F}}(s)$, we obtain that

$$|F_s(\mathcal{F}(s)) \cap DF_\tau(s)A^{-1}B(0,\rho)| \gtrsim |F_s(\mathcal{F}(s))| \sim \beta.$$

We also refine $\mathcal{F}(s)$ to a subset whose size is still $\gtrsim \beta$ and assume that $F_s(\mathcal{F}(s)) \subset DF_\tau(s)A^{-1}B(0,\rho)$.

8. Proof of the main theorem

By means of a rotation of $\mathbb{R}^{d-1}$, we may assume that A is diagonal with eigenvalues

$$r_1 \le r_2 \le \cdots \le r_{d-1}.$$

By what we have proved so far, one could prove Theorem 2.1 without too much trouble (with ρ and the r_i indicating the same quantities), were it not for the nondegeneracy condition (2.3) on the r_i. In the article [2], the special structure of the paraboloid meant that no such condition was necessary. The primary work of this section, and one of the main new details of this article, will be to establish the necessity of that inequality.

As $|E(\mathbf{r};\rho)|$ and $|F(\mathbf{r};\rho)|$ are allowed to be a factor of $\varepsilon^{-B'}$ larger than $|E|$ and $|F|$, respectively, if we can show that $r_1 \gtrsim \varepsilon^{B'}\rho^{1/2}r_{d-1}$, then by enlarging the various parameters as needed (for instance increasing the size of r_1 and ρ proportionately to one another) we may achieve the nondegeneracy condition (2.3), while maintaining the other conditions. We assume, by way of contradiction, that

$$(8.1) \qquad r_1 \le B^{-1}\varepsilon^{B'}\rho^{1/2}r_{d-1},$$

with B, B' large positive constants, yet to be determined.

In this section, we will need to differentiate between two types of constants, those over which we have control via our assumption (8.1), and those which we cannot substantially influence (e.g. those appearing in the previous two sections). We will denote the former by B, B' and the latter by C, C', while allowing the constants to vary from line to line (as is typical in the harmonic analysis literature). Expressions such as $\lesssim$ will always involve implicit constants of the second type. Though "B" will also be used to denote the unit ball, we have otherwise exhausted letters $A - F$, and our meaning will be clear from the context.

8.1. An alternative description of $\mathcal{F}(s)$. To simplify the exposition, we assume that $\tau = 0$; if this were not the case, it could be achieved by rotating the sphere and enlarging ρ and the r_i by a bounded factor.

We assume that $s \in F_0(\Omega_1) \subset A(B)$, which implies that

$$|s_i| \lesssim r_i, \quad 1 \le i \le d - 1.$$

From this, $[DF_0(s)]_{i,j} = \delta_{i,j} + O(r_i r_j)$, and $DF_0(s)\rho \cdot A^{-1} = \rho A^{-1} + O(\rho)$. Therefore, by the smallness of the r_i, $DF_0(s)A^{-1}(B(0, \rho)) \subset A^{-1}(B(0, C\rho))$, for some constant C independent of ε, ρ, and the r_i. Henceforth, we will ignore this constant.

We suppose that $t \in \mathcal{F}(s)$; then

$$\left| \frac{t_i}{\sqrt{1 - |t|^2}} - \frac{s_i}{\sqrt{1 - |s|^2}} \right| \lesssim \frac{\rho}{r_i}, \quad 1 \le i \le d - 1.$$

By our assumption (8.1), as well as the assumptions $\rho \ll 1$ and $\rho \ll r_1 \le \cdots \le r_{d-1} \ll 1$, there exists an index k, $1 \le k < d - 1$ such that

(8.2) $r_k \le \rho^{1/2} \le r_{k+1}.$

We let $t = (t_I, t_{II}) \in \mathbb{R}^k \times \mathbb{R}^{d-k-1}$. We will show that t can be approximated by

$$\left(t_I, \sqrt{1 - |t_I|^2} \cdot s_{II} \right),$$

in the sense that for $i > k$,

$$t_i = s_i \sqrt{1 - |t_I|^2} + O\left(\frac{\rho}{r_i} \right).$$

If $k + 1 \le i \le d - 1$, then

$$\left| t_i - s_i \sqrt{1 - |t_I|^2} \right| \le \left| t_i - \frac{s_i \sqrt{1 - |t|^2}}{\sqrt{1 - |s|^2}} \right| + |s_i| \cdot \left| \frac{\sqrt{1 - |t|^2}}{\sqrt{1 - |s|^2}} - \sqrt{1 - |t_I|^2} \right|.$$

The first summand on the is $O(\frac{\rho}{r_i})$, which is acceptable. The second summand is

$$\sim r_i \cdot \left| |t_I|^2 + |s|^2 - |t|^2 - |t_I|^2 |s|^2 \right| \lesssim r_i \cdot \left| |s_{II}|^2 (1 - |t_I|^2) - |t_{II}|^2 \right| + O(\rho),$$

because $|s_I|^2 = O(\rho)$. We continue, ignoring the $O(\rho)$ term. The side is

$$\lesssim r_i \sum_{j=k+1}^{d-1} \left|t_j + s_j\sqrt{1-|t_I|^2}\right| \cdot \left|t_j - s_j\sqrt{1-|t_I|^2}\right|$$

$$\leq r_i \sum_{j=k+1}^{d-1} r_j \left|t_j - s_j\sqrt{1-|t_I|^2}\right|,$$

since $\frac{\rho}{r_j} \leq r_j$ when $k+1 \leq j \leq d-1$. We then have that

$$\sum_{i=k+1}^{d-1} \left|t_i - s_i\sqrt{1-|t_I|^2}\right| \lesssim \sum_{i=k+1}^{d-1} \frac{\rho}{r_i} + \sum_{i=k+1}^{d-1} r_i \sum_{j=k+1}^{d-1} r_j \left|t_j - s_j\sqrt{1-|t_I|^2}\right|,$$

which by the monotonicity and smallness of the r_i implies that

$$\left|t_{k+1} - s_{k+1}\sqrt{1-|t_I|^2}\right| \lesssim \sum_{i=k+1}^{d-1} \frac{\rho}{r_i} \leq (d-k-1)\frac{\rho}{r_{k+1}}.$$

The other inequalities can be established by induction and the assumption that $\rho \ll r_i$, $1 \leq i \leq d-1$.

8.2. A few lower bounds. We define

$$\tilde{B} := \{\tilde{s} = (s_2, \ldots, s_{d-1}) \in \mathbb{R}^{d-2} : |s_i| < r_i, 2 \leq i \leq d-1\};$$

then

$$\alpha \lesssim |\Omega_1| = \int_{\tilde{B}} |\{s_1 \in \mathbb{R} : (s_1, \tilde{s}) \in \Omega_1\}|\, d\tilde{s}$$

$$= r_2 \cdots r_{d-1} \operatorname{avg}_{\tilde{s}\in\tilde{B}} |\{s_1 : (s_1, \tilde{s}) \in \Omega_1\}|.$$

Dividing both sides by $r_2 \cdots r_{d-1}$ and using the fact that

$$\alpha \lesssim \varepsilon^{-C} \det A = \varepsilon^{-C} r_1 \cdots r_{d-1},$$

we must then have that

$$\varepsilon^C r_1 \lesssim \operatorname{avg}_{\tilde{s}\in\tilde{B}} |\{s_1 : (s_1, \tilde{s}) \in \Omega_1\}|.$$

Say s_{d-1} is good if there exists $(s_2, \ldots, s_{d-2}) \in \mathbb{R}^{d-3}$ such that $\tilde{s} := (s_2, \ldots, s_{d-1}) \in \tilde{B}$ and $|\{s_1 : (s_1, \tilde{s}) \in \Omega_1\}| > \varepsilon^C r_1$. Then we may choose C and the implicit constant large enough that

$$|\{s_{d-1} : s_{d-1} \text{ is good}\}| \gtrsim \varepsilon^C r_{d-1}.$$

Given $s \in \Omega_1$, we know that $F_s(\mathcal{F}(s))$ is contained in $A^{-1}B(0,\rho)$. By our assumption (8.1), there exists k, $1 \leq k \leq d-2$ so that

$$r_k \leq \rho^{1/2} \leq r_{k+1}.$$

Arguing as above, we may assume that $t \in \mathcal{F}(s)$ satisfies

$$\left| [F_s(t)]_j \right| \gtrsim \varepsilon^C \frac{\rho}{r_j}, \quad 1 \leq j \leq k,$$

where the index j indicates the component, while maintaining the lower bound

$$|\mathcal{F}(s)| \gtrsim \beta.$$

These assumptions on C and $\mathcal{F}(s)$ will be in force for the remainder of this section.

8.3. More slicing. Let $\tilde{B}$ be as defined above, and let $\tilde{s} \in \tilde{B}$. We define

$$\Omega(\tilde{s}) := \{(s_1, t) \in \mathbb{R}^d : s := (s_1, \tilde{s}) \in \Omega_1 \text{ and } t \in \mathcal{F}(s)\}$$

and

$$\tilde{E}(\tilde{s}) := \left\{ \left(t, \sqrt{1 - |t|^2} \right) - \left(s, \sqrt{1 - |s|^2} \right) : (s_1, t) \in \Omega(\tilde{s}) \right\}.$$

We have established (in Section 7.1) that

$$\Psi : (s_1, t) \mapsto \left(t, \sqrt{1 - |t|^2} \right) - \left(s, \sqrt{1 - |s|^2} \right)$$

is sufficiently injective that

$$|\tilde{E}(\tilde{s})| \gtrsim \int_{\Omega(\tilde{s})} |\det D\Psi(s_1, t)| \, dt \, ds_1.$$

Moreover, this is equal to

$$\int_{\Omega(\tilde{s})} |[F_s(t)]_1| \, dt \, ds_1,$$

and is bounded from below by (a constant times)

$$\varepsilon^C r_1 \cdot \frac{\rho}{r_1} \cdot \beta = \varepsilon^C \rho \beta$$

whenever $\tilde{s}$ satisfies

(8.3) $\varepsilon^C r_1 \lesssim |\{s_1 : (s_1, \tilde{s}) \in \Omega_1\}|.$

By the work of the previous subsection, the set of such admissible $\tilde{s}$'s has size $\gtrsim \varepsilon^C |\tilde{B}|$.

We return to Section 7.2 for the definition of ρ, which enables us to conclude that

$$|\tilde{E}(\tilde{s})| \gtrsim \varepsilon^C \beta^{d/(d-1)} \alpha^{1/(d-1)},$$

again, for $\tilde{s}$ satisfying (8.3).

8.4. A disjointness property. We will prove in this section that the sets $\tilde{E}(\tilde{s}^{(1)})$ and $\tilde{E}(\tilde{s}^{(2)})$ are disjoint when $\tilde{s}^{(1)}$ and $\tilde{s}^{(2)}$ are sufficiently far apart. When combined with our lower bound on $|\tilde{E}(\tilde{s})|$, this will give us a stronger lower bound on $|E|$. As we also have the upper bound

$$|E| \lesssim \varepsilon^{-C} \alpha^{1/(d-1)} \beta^{d/(d-1)}$$

from the assumption of quasi-extremality, we will be able to obtain a contradiction and establish the necessity of (2.3).

We know from Section 8.1 that if $s \in \Omega_1$ and $t \in \mathcal{F}(s)$ implies that

$$|t_i| \lesssim \frac{\rho}{r_i}, \quad 1 \le i \le k,$$

$$\left|t_i - s_i\sqrt{1 - |t_I|^2}\right| \lesssim \frac{\rho}{r_i}, \quad k+1 \le i \le d-1.$$

Therefore, $t \in \mathcal{F}(s)$ implies that

$$\left||t_{II}|^2 - |s_{II}|^2(1 - |t_I|^2)\right| \lesssim \sum_{k+1}^{d-1} \frac{\rho}{r_i}\left(r_i + \frac{\rho}{r_i}\right) \lesssim \rho.$$

Let $x \in \tilde{E}(\tilde{s})$; write $x = -(s, \sqrt{1 - |s|^2}) + (t, \sqrt{1 - |t|^2})$, where $t \in \mathcal{F}(s)$. Then

$$|x_i| \lesssim \frac{\rho}{r_i}, \quad 1 \le i \le k,$$

$$\left|x_i - s_i\left(\sqrt{1 - |x_I|^2} - 1\right)\right| \lesssim \frac{\rho}{r_i}, \quad k+1 \le i \le d-1,$$

$$\left|x_d - \sqrt{1 - |s_{II}|^2}\left(\sqrt{1 - |x_I|^2} - 1\right)\right| \lesssim \rho;$$

all of these inequalities are independent of s_1, t, so they describe arbitrary $x \in \tilde{E}(\tilde{s})$. Since we are also assuming that $t \in \mathcal{F}(s)$ implies that $|t_i - s_i| > \varepsilon^C \frac{\rho}{r_i}$, $1 \le i \le k$, $x \in \tilde{E}(\tilde{s})$ also satisfies

$$\varepsilon^C \frac{\rho}{r_i} < |x_i|, \quad 1 \le i \le k.$$

Therefore, if $x \in \tilde{E}(\tilde{s}^{(1)}) \cap \tilde{E}(\tilde{s}^{(2)})$, then

$$(8.4) \qquad \left(1 - \sqrt{1 - |x_I|^2}\right)\left|s_i^{(1)} - s_i^{(2)}\right| \lesssim \frac{\rho}{r_i}, \quad i > k.$$

We are assuming that $|x_I| > \varepsilon^C \frac{\rho}{r_1}$ (recall that the r_j are monotone), $(1 - \sqrt{1 - |x_I|^2}) \gtrsim \varepsilon^C \frac{\rho}{r_1}$, and hence by (8.4),

$$\left|s_i^{(1)} - s_i^{(2)}\right| \lesssim \varepsilon^{-C} \frac{r_1^2}{\rho r_i}, \quad i > k.$$

On the other hand, by the work of Section 8.2, we may choose a sequence $\tilde{s}^{(1)}, \ldots, \tilde{s}^{(N)}$ with $N \gg 1$ so that for each j, $\tilde{s} = \tilde{s}^{(j)}$ satisfies (8.3), and such

that whenever $i \neq j$, $|s^{(i)} - s^{(j)}| \geq \frac{\varepsilon^C}{N} r_{d-1}$. Our assumption (8.1) implies that

$$r_{d-1} \geq B\varepsilon^{-B'} \frac{r_1^2}{\rho r_{d-1}}.$$

We fix N large enough for later purposes and choose B, B' to be sufficiently large that

$$B\frac{\varepsilon^C}{N}\varepsilon^{-B'} \gg \varepsilon^{-C}.$$

Having done this, we ensure that the sets $\tilde{E}(\tilde{s}^{(j)})$ are pairwise disjoint and thus that the size of their union is

$$\left| \bigcup_{j=1}^{N} \tilde{E}(\tilde{s}^{(j)}) \right| = \sum_{j=1}^{N} |\tilde{E}(\tilde{s}^{(j)})| \gtrsim N\varepsilon^C \rho\beta.$$

The above union is contained in E. Therefore, for N sufficiently large, depending only on C and thus ultimately on d, the above implies a contradiction to the upper bound (6.1) on $|E|$.

8.5. Conclusion of proof. Now we complete the proof of the main theorem.

From what we have seen so far, there exist $r_1, \ldots, r_{d-1}, \rho > 0$ satisfying conditions (2.1)–(2.3) such that

$$r_1 \cdots r_{d-1} \lesssim \varepsilon^{-C'} \alpha,$$
$$\frac{\rho}{r_1} \cdots \frac{\rho}{r_{d-1}} \lesssim \varepsilon^{-C'} \beta,$$

where

$$\rho = C\varepsilon^{-C'} \alpha^{1/(d-1)} \beta^{1/(d-1)},$$

for some large (ε-independent) constants C, C'.

From these, one immediately obtains the upper bounds

$$|E(\mathbf{r}; \rho)| \lesssim \varepsilon^{-C} |E| \quad \text{and} \quad |F(\mathbf{r}; \rho)| \lesssim \varepsilon^{-C} |F|.$$

Moreover, by relaxing our assumptions on the size of the Ω_i to

$$|\Omega_1| \gtrsim \alpha,$$

$$s \in \Omega_1 \quad \implies \quad |\mathcal{F}(s)| \gtrsim \beta,$$
$$(s, t) \in \Omega_2 \quad \implies \quad |\mathcal{G}(s, t)| := |\{u \in \mathbb{R}^{d-1} : (s, t, u) \in \Omega_3\}| \gtrsim \alpha,$$

we may assume that

$$\Omega_1 \subset \{s : |[F_0(s)]_i| < r_i, 1 \leq i \leq d-1\},$$
$$\mathcal{F}(s) \subset \left\{t : |[F_s(t)]_i| < \frac{\rho}{r_i}, 1 \leq i \leq d-1\right\},$$
$$\mathcal{G}(s, t) \subset \{u : |[F_t(u)]_i| < r_i, 1 \leq i \leq d-1\}.$$

From this and our definition of the Ω_i, we have that

$$s \in \Omega_1 \quad \Longrightarrow \quad x_0 - \left(s, \sqrt{1-|s|^2}\right) \in F \cap \left(F\left(\frac{\rho}{\mathbf{r}}; \rho\right) + \{x_0\}\right),$$

$$(s,t) \in \Omega_2 \quad \Longrightarrow \quad x_0 - \left(s, \sqrt{1-|s|^2}\right) + \left(t, \sqrt{1-|t|^2}\right)$$
$$\in E \cap \left(E\left(\frac{\rho}{\mathbf{r}}; \rho\right) + \{x_0\}\right),$$

$$(s,t,u) \in \Omega_3 \quad \Longrightarrow \quad x_0 - \left(s, \sqrt{1-|s|^2}\right) + \left(t, \sqrt{1-|t|^2}\right) - \left(u, \sqrt{1-|u|^2}\right)$$
$$\in F \cap \left(F\left(\frac{\rho}{\mathbf{r}}; \rho\right) + \{x_0\}\right).$$

From the material in Section 6.1, one then obtains that

$$|\tilde{E}| = \left|E \cap \left(E(\mathbf{r}, \rho) + \{x_0\}\right)\right| \gtrsim \beta(\alpha\beta)^{1/(d-1)} \gtrsim \varepsilon^{(d+1)/(d-1)}|E|$$

(similarly, $|\tilde{F}| \gtrsim \varepsilon^{(d+1)/(d-1)}|F|$). From this and the lower bound on $|\mathcal{G}(s,t)|$, we finally have the lower bound

$$\mathcal{T}(\tilde{E}, \tilde{F}) \gtrsim \alpha|\tilde{E}| \gtrsim \varepsilon^{(d+1)/(d-1)}\mathcal{T}(E, F),$$

and the theorem is proved.

9. Proof of Theorem 2.2

By means of rotations and translations, in proving Theorem 2.2, it suffices to consider the following situation: (E, F) is an ε-quasi-extremal pair, $\mathbf{r}, \rho$ satisfy inequalities (2.1)-(2.3), $\tilde{E} := E \cap E(\mathbf{r}; \rho)$ and $\tilde{F} := F \cap F(\mathbf{r}; \rho)$, and

$$\mathcal{T}(\tilde{E}, \tilde{F}) \gtrsim C^{-1}\varepsilon^{(d+1)/(d-1)}\mathcal{T}(E, F),$$
$$|E(\mathbf{r}; \rho)| \lesssim \varepsilon^{-C}|E| \quad \text{and} \quad |F(\mathbf{r}; \rho)| \lesssim \varepsilon^{-C}|F|.$$

We may further assume that $\rho \ll 1$ and $\rho \ll r_i \ll 1$.

Our strategy will be a typical one in harmonic analysis; we will divide $E(\mathbf{r}; \rho)$ and $F(\mathbf{r}; \rho)$ into $\lesssim \varepsilon^{-C}$ pairs of quasi-extremal sets (E_j, F_j) of the correct size ($|E_j| \leq |E|$ and $|F_j| \leq |F|$), and then use the pigeon-hole principle to pick one pair so that $\mathcal{T}(E_j, F_j) \gtrsim \varepsilon^C \mathcal{T}(E, F)$.

We will begin with an initial decomposition of $E(\mathbf{r}; \rho)$. We choose $\varepsilon^C \leq \lambda < 1$ so that

$$\lambda^d |E(\mathbf{r}; \rho)| \leq |E| \quad \text{and} \quad \lambda|F(\mathbf{r}; \rho)| \leq |F|.$$

Since $\lambda < 1$, inequalities (2.1)–(2.3) still hold. We write

$$E(\mathbf{r}; \rho) = \bigcup_{i=-\lambda^{-1}}^{\lambda^{-1}} E(\mathbf{r}; \lambda\rho) + \lambda\rho \cdot i \cdot e_d.$$

We further decompose $E(\mathbf{r}; \lambda\rho)$ as

$$E(\mathbf{r}; \lambda\rho) = \bigcup_{j=1}^{O(\lambda^{-(d-1)})} B_j E(\lambda\mathbf{r}; \lambda\rho),$$

where B_j is the affine transformation $B_j(x) = R_j(x + e_d) - e_d$, and R_j is the rotation which takes the point $x^j \in S^{d-1} \cap (E(\mathbf{r}; \rho) + e_d)$ to e_d, takes e_d to $(-x_1^j, \ldots, -x_{d-1}^j, x_d^j)$, and fixes all points perpendicular to e_d and x^j.

Thus

$$E(\mathbf{r}; \rho) = \bigcup_{i=-\lambda^{-1}}^{\lambda^{-1}} \bigcup_{j=1}^{O(\lambda^{-(d-1)})} B_j E(\lambda\mathbf{r}; \lambda\rho) + \lambda\rho \cdot i \cdot e_d$$

$$=: \bigcup_{i=-\lambda^{-1}}^{\lambda^{-1}} \bigcup_{j=1}^{O(\lambda^{-(d-1)})} E_{i,j}.$$

Next, for each i and j, we will decompose $F(\mathbf{r}; \rho)$ into a union of sets compatible with $E_{i,j}$; this will be surprisingly easy. We note here that the set $F(\lambda\mathbf{r}; \lambda\rho)$ has the same dimensions as $F(\mathbf{r}; \rho)$ in the directions perpendicular to e_d, but has thickness $\lambda\rho$ rather than ρ in the e_d direction.

First, we wish to know with which portion of $F(\mathbf{r}; \rho)$ an element of $E_{i,j}$ interacts via convolution with the sphere.

Let $B_{i,j}$ be the affine transformation defined by

$$B_{i,j}(x) := B_j(x) + \lambda\rho \cdot i \cdot e_d,$$

so that $E_{i,j} = B_{i,j} E(\lambda\mathbf{r}; \lambda\rho)$. It now suffices to determine which elements of $B_{i,j}^{-1} F(\mathbf{r}; \rho)$ are a distance 1 from elements of $E(\lambda\mathbf{r}; \lambda\rho)$. One can check, for instance by explicitly computing $B_{i,j}^{-1}(y)$ for $y \in F(\mathbf{r}; \rho)$ that $B_{i,j}^{-1} F(\mathbf{r}; \rho) \subset F(\mathbf{r}; C\rho)$ for some constant C depending on the dimension. Now, by increasing ρ as needed, it is sufficient to determine with which elements of $F(\mathbf{r}; \rho)$ lie a distance 1 from a point in $E(\lambda\mathbf{r}; \lambda\rho)$.

Write $x \in E(\lambda\mathbf{r}; \lambda\rho)$ and $y \in F(\mathbf{r}; C\rho)$ as

$$x = \left(x', \sqrt{1 - |x'|^2} - 1 + \delta_1\right), \qquad y = \left(y', -\sqrt{1 - |y'|^2} + \delta_2\right),$$

where $|x_i| < \lambda r_i$, $|y_i| < \frac{\rho}{r_i}$, $|\delta_1| < \lambda\rho$, and $|\delta_2| < \rho$. If we assume that $|x - y| = 1$, then

$$1 = |x'|^2 + |y'|^2 + (x_d - y_d)^2 + O(\lambda\rho)$$
$$= |x'|^2 + |y'|^2 + \left(\sqrt{1 - |x'|^2} - 1 + \sqrt{1 - |y'|^2}\right)^2$$
$$- 2\delta_2\left(\sqrt{1 - |x'|^2} - 1 + \sqrt{1 - |y'|^2}\right) + \delta_2^2 + O(\lambda\rho).$$

Next, one applies the inequality

$$1 - \sqrt{1 - |x'|^2} \lesssim |x'|^2 \lesssim \lambda r_{d-1}^2,$$

so

$$2\big(1 - \sqrt{1 - |x'|^2}\big)\big(1 - \sqrt{1 - |y'|^2}\big) \lesssim \lambda r_{d-1}^2 \frac{\rho^2}{r_1^2} \lesssim \lambda\rho,$$

$$\delta_2\big(1 - \sqrt{1 - |x'|^2}\big) \lesssim \rho\lambda r_{d-1} \leq \lambda\rho.$$

Using these inequalities in the series of equalities above,

$$-2\delta_2\sqrt{1 - |y'|^2} + \delta_2^2 = O(\lambda\rho),$$

which implies that $|\delta_2| \lesssim \lambda\rho$.

The computations in the paragraph above imply that

$$\mathcal{T}(E(\lambda\mathbf{r}; \lambda\rho), F(\mathbf{r}; \rho)) = \mathcal{T}(E(\lambda\mathbf{r}; \lambda\rho), F(\lambda\mathbf{r}; C\lambda\rho)).$$

Therefore,

$$
\begin{aligned}
\mathcal{T}(\tilde{E}, \tilde{F}) &\leq \sum_{i=-\lambda^{-1}}^{\lambda^{-1}} \sum_{j=1}^{O(\lambda^{-(d-1)})} \mathcal{T}\big(E \cap B_{i,j} E(\lambda\mathbf{r}; \lambda\rho), F \cap F(\mathbf{r}; \rho)\big) \\
&= \sum_{i=-\lambda^{-1}}^{\lambda^{-1}} \sum_{j=1}^{O(\lambda^{-(d-1)})} \mathcal{T}\big(E \cap B_{i,j} E(\lambda\mathbf{r}; \lambda\rho), F \cap B_{i,j} F(\lambda\mathbf{r}; C\lambda\rho)\big) \\
&\leq \sum_{i=-\lambda^{-1}}^{\lambda^{-1}} \sum_{j=1}^{O(\lambda^{-(d-1)})} \mathcal{T}\big(E \cap B_{i,j} E(\lambda\mathbf{r}; C\lambda\rho), F \cap B_{i,j} F(\mathbf{r}; C\lambda\rho)\big).
\end{aligned}
$$

By the pigeonhole principle, there exists some choice of i, j so that

$$\mathcal{T}(E, F) \lesssim \mathcal{T}(\tilde{E}, \tilde{F}) \lesssim \varepsilon^{-C}\mathcal{T}\big(E \cap B_{i,j} E(\lambda\mathbf{r}; C\lambda\rho), F \cap B_{i,j} F(\lambda\mathbf{r}; C\lambda\rho)\big),$$

and the Theorem 2.2 is proved.

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