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UNIFORM EQUIVALENCE OF SYMBOLIC AND ADIC TOPOLOGIES

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Abstract. Let (R,m) be a local ring. We study the question of when there exists a positive integer h such that for all prime ideals $P \subseteq R$, the symbolic power $P^{(hn)}$ is contained in P^n, for all $n \geq 1$. We show that such an h exists when R is a reduced isolated singularity such that R either contains a field of positive characteristic and R is F-finite or R is essentially of finite type over a field of characteristic zero.

1. Introduction

The purpose of this note is to prove that for a large class of isolated singularities, the symbolic topology defined by a prime ideal P is uniformly linearly equivalent to the P-adic topology. In other words, there exists $h \geq 1$, independent of P, such that for all primes $P \subseteq R$, $P^{(hn)} \subseteq P^n$, for all n. Results of this type were first given by Ein, Lazarsfeld and Smith in [5] and by Hochster and Huneke in [6]. Combining these results, it was shown that if R is a regular local ring containing a field, and d is the Krull dimension of R, then $P^{(dn)} \subseteq P^n$, for all prime ideals $P \subseteq R$ and all $n \geq 1$. More refined results for regular rings along the same lines can be found in [7] or [25].

Prior to the results in [5] and [6], a number of authors had studied equivalences of slightly more general topologies. Let A be a Noetherian ring and $I, J \subseteq A$ ideals. Write $I^n : \langle J \rangle := \bigcup_{m \geq 1}(I^n : J^m)$. In [20], [21], [9] it was studied when the I-adic and $\{I^n : \langle J \rangle\}$ topologies are equivalent, i.e., one requires for each $n \geq 1$ an integer $m \geq 1$ so that $I^m : \langle J \rangle \subseteq I^n$. In [21], [11], [26], [18] it was studied when these topologies are what the authors call *linearly* equivalent. By this, they mean there must exist $t \geq 1$, so that

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$I^{n+t} : \langle J \rangle \subseteq I^n$, for all n. Such "linear" equivalence basically is equivalent to saying that $I^m : \langle J \rangle$ is integral over I^m. When trying to prove something about all primes, this notion of linear equivalence is too weak, and one must allow higher slopes than one. In [23], Swanson proved the following strong result relating the notions of equivalent and linearly equivalent topologies.

THEOREM 1.1 (Swanson). *Let A be a Noetherian ring and I, J ideals. Then the I-adic and $\{I^n : \langle J \rangle\}$ topologies are equivalent if and only if there exits $h \geq 1$, so that for all $n \geq 1$, $I^{hn} : \langle J \rangle \subseteq I^n$.*

Two comments are in order regarding Theorem 1.1. The first is that h depends a priori on I and the second is that the theorem implies that if $P \subseteq A$ is a prime ideal, and the P-symbolic and P-adic topologies are equivalent, then there exists $h \geq 1$ so that $P^{(hn)} \subseteq P^n$, for all n. This set the stage for the results in [5] and [6] mentioned above for regular local rings, namely that $P^{(dn)} \subseteq P^n$, for all P and for all n, with d the dimension of the regular local ring R. The point is that while h in Theorem 1.1 may depend on the ideal I, $h = d$ in [5] and [6] is independent of the ideal. Even very simple examples show that one cannot do better. For example, in the hypersurface $x^2 - yz = 0$, the prime ideal $P = (x, y)$ has $2n$th symbolic power generated by y^n, which is clearly not in P^{2n-t} for any fixed constant t, as n gets large. Moreover, the results in [5] and [6] cannot be improved in an asymptotic sense, as shown by Bocci and Harbourne in [1].

The following theorem is a special case of the main result of this paper. For an arbitrary ideal I, we write $I^{(n)}$ for the nth symbolic power as defined in [6], namely, $I^n R_S \cap R$, where S is the complement of the union of the associated primes of I.

THEOREM 1.2. *Let R be an equicharacteristic, local domain such that R is an isolated singularity. Assume that R is either essentially of finite type over a field of characteristic zero or R has positive characteristic and is F-finite. Then there exists $h \geq 1$ with the following property. For all ideals $I \subseteq R$ such that the symbolic topology of I is equivalent to the I-adic topology, $I^{(hn)} \subseteq I^n$, for all $n \geq 1$.*

We present this theorem in section three. The crucial ingredients in the proof of this theorem are threefold: the relation between the Jacobian ideal and symbolic powers established in [6] (see Theorem 3.2 below), the uniform Artin–Rees theorem [10], and a uniform Chevalley theorem which is proved in Section 2 (see Theorem 2.3). Aside from playing a crucial part in our main result, we feel Theorem 2.3 has independent interest, and moreover, believe it should hold for arbitrary local rings.

We use [4] as a standard reference.

2. A uniform Chevalley theorem

In this section, we will establish a uniform Chevalley theorem for symbolic powers in an analytically unramified local ring. Recall that Chevalley's theorem states that if $(R, \mathfrak{m})$ is a complete local ring and $\{J_n\}_{n\geq 1}$ is a descending collection of ideals with $\bigcap_{n\geq 1} J_n = 0$, then the $\{J_n\}$ topology is finer than the $\mathfrak{m}$-adic topology. In other words, for all $n \geq 1$, there exists $t \geq 1$, such that $J_t \subseteq \mathfrak{m}^n$. Before presenting our theorem, we would first like to make some general comments concerning comparisons of ideal topologies. We begin by stating a theorem of Schenzel (see [21, Theorem 3.2]) which tells when certain ideal topologies are equivalent. For the statement of the theorem, we write $A(I)$ for the union over n of the associated primes of I^n. As is well-known, this is a finite set of prime ideals (see [2] and [17]). We also write $\widehat{T}$ for the completion of the local ring T with respect to its maximal ideal. Here is Schenzel's theorem.

THEOREM 2.1 (Schenzel). *Let A be a Noetherian ring and $I, J \subseteq A$ two ideals. Then the following are equivalent.*

(1) *The $\{I^n : \langle J\rangle\}$ topology is equivalent to the I-adic topology.*
(2) $\dim(\widehat{R_P}/(I\widehat{R_P} + z)) > 0$, *for all prime ideals $P \in A(I) \cap V(J)$, and prime ideals $z \in \operatorname{Ass}(\widehat{R_P})$.*

To prove this result, Schenzel gave a modified version of Chevalley's theorem for the I-adic topology in a complete local ring. But in [15], McAdam showed that it is just enough to invoke Chevalley's theorem directly. Namely, he noted that the $\{I^n : \langle J\rangle\}$ topology and the I-adic topology are equivalent if and only if for all $P \in A(I)$, the $\{I_P^n : \langle J_P\rangle\}$ topology is finer than the P_P-adic topology. This was accomplished by passing to each $\widehat{R_P}$ and invoking Chevalley's theorem. Thus, Chevalley's theorem plays a crucial role in any discussion of equivalent ideal topologies.

NOTATION. Let $S \subseteq R$ be a multiplicatively closed set and $L \subseteq R$ an ideal. We will write $L : \langle S\rangle$ for $LR_S \cap R$.

Let I, J be ideals of R and take $s \in J$ with the following property. For all $P \in A(I)$, $s \in P$ if and only if $J \subseteq P$. (Such an s is found by applying the prime avoidance lemma to the primes in $A(I)$ not containing J.) Then for $S := \{s^t\}_{t\geq 1}$, $I^n : \langle S\rangle = I^n : \langle J\rangle$, for all $n \geq 1$. Consequently, any result about the $\{I^n : \langle S\rangle\}$ topology recovers the corresponding result about the $\{I^n : \langle J\rangle\}$ topology. Moreover, if we let S denote the complement of the union of the associated primes of I, then by definition, $I^{(n)} = I^n : \langle S\rangle$, for all n.

The following proposition is implicit in the work of McAdam and Schenzel, but as it is easy to prove and does not seem to be stated in the precise form we use here, we provide the reader with a proof.

PROPOSITION 2.2. *Let $(R, \mathfrak{m})$ be a local ring with completion $\widehat{R}$. Let $I \subseteq R$ be an ideal and $S \subseteq R$ a multiplicatively closed set. Write $z_1, \ldots, z_l$ for the* *associated primes of* $\widehat{R}$. *Then the* $\{I^n : \langle S\rangle\}$ *topology is finer than the* $\mathfrak{m}$*-adic topology if and only if for all* $1 \le i \le l$, $(I\widehat{R} + z_i) \cap S = \emptyset$.

Proof. Since $\widehat{R}$ is faithfully flat over R, it is easy to check that the $\{I^n :_R \langle S\rangle\}$ topology is finer than the $\mathfrak{m}$-adic topology if and only if the $\{I^n\widehat{R} :_{\widehat{R}} \langle S\rangle\}$ topology is finer than the $\mathfrak{m}\widehat{R}$-adic topology. Thus, we may assume that R is a complete local ring with associated primes $z_1, \dots, z_l$. Now, by Chevalley's theorem, the $\{I^n : \langle S\rangle\}$ is *not* finer than the $\mathfrak{m}$-adic topology if and only if

$$\bigcap_{n\ge 1} (I^n : \langle S\rangle) \ne 0.$$

Suppose $0 \ne x \in \bigcap_{n\ge 1}(I^n : \langle S\rangle)$. Then for each $n \ge 1$, there exists $s \in S$ such that $s \in (I^n : x)$. By applying the Artin–Rees lemma to $I^n \cap (x)$, we see that for n large, $s \in (0 : x) + I^{n-k}$, for some k. Taking z_j so that $(0 : x) \subseteq z_j$, we have that $(I + z_j) \cap S \ne \emptyset$.

On the other hand, suppose that $(I + z_j) \cap S \ne \emptyset$, for some j. Then for all $n \ge 1$, $(I^n + z_j) \cap S \ne \emptyset$. Let $z_j = (0 : x)$. Then for each n, there exists $s \in S$ such that $s \in I^n + (0 : x)$, i.e., $sx \in I^n$. Thus, $0 \ne x \in \bigcap_{n\ge 1}(I^n : \langle S\rangle)$. □

We are now ready for our first theorem.

Theorem 2.3. *Let* R *be an analytically unramified local ring. Then there exists* $h \ge 1$ *with the following property. For all ideals* $I \subseteq R$ *and all multiplicatively closed sets* S *in* R *such that the* $\{I^n : \langle S\rangle\}$ *topology is finer than the* $\mathfrak{m}$*-adic topology,* $I^{hn} : \langle S\rangle \subseteq \mathfrak{m}^n$ *for all* $n \ge 1$.

Proof. We begin by reducing to the case that R is a complete local domain. We first reduce to the case that R is a complete local ring. Let $\widehat{R}$ denote the completion of R and suppose that h has the required property for all appropriate ideals and multiplicatively closed subsets in $\widehat{R}$. If $I \subseteq R$ is an ideal and $S \subseteq R$ is a multiplicatively closed subset so that the $\{I^n : \langle S\rangle\}$ topology is finer than the $\mathfrak{m}$-adic topology, then the same holds when we extend the corresponding filtrations to $\widehat{R}$. But then $\{I^n : \langle S\rangle\}\widehat{R} = \{I^n\widehat{R} : \langle S\rangle\}$, so that if $I^{hn}\widehat{R} : \langle S\rangle \subseteq \mathfrak{m}^n\widehat{R}$, contracting back to R gives $I^{hn} : \langle S\rangle \subseteq \mathfrak{m}^n$, for all n.

So, we begin again assuming that R is complete and that $z_1, \dots, z_l$ are the associated primes of R. Set $T_i := R/z_i$, for all i. By Proposition 2.2, it follows that for all $1 \le i \le l$, the $\{I^nT_i : \langle S\rangle\}$ topology is finer than the $\mathfrak{m}T_i$-adic topology. Now suppose that h has been chosen so that the conclusion of the theorem holds for h modulo each minimal (= associated) prime of R. Thus, $I^{hn}T_i : \langle S\rangle \subseteq \mathfrak{m}^nT_i$, for all n and all i. Since $(I^{hn} : \langle S\rangle)T_i \subseteq I^{hn}T_i : \langle S\rangle$, it follows that for each i, $(I^{hn} : \langle S\rangle)T_i \subseteq \mathfrak{m}^nT_i$. It follows from [24, Proposition 1.1.4] that $I^{hn} : \langle S\rangle \subseteq \overline{\mathfrak{m}^n}$, where $\overline{\mathfrak{m}^n}$ denotes the integral closure of $\mathfrak{m}^n$. Since R is analytically unramified, there exists $r \ge 1$ such that $\overline{\mathfrak{m}^n} \subseteq \mathfrak{m}^{n-r}$, for all $n \geq r$ (see [24, Theorem 9.1.2]). Adjusting exponents then gives the required result.

We now assume that R is a complete local domain and show that there exists an $h \geq 1$ so that for all ideals I and all multiplicatively closed subsets S disjoint from I, $I^{hn} : \langle S \rangle \subseteq \mathfrak{m}^n$, for all n. Suppose h is such that for all prime ideals $P \subseteq R$, $P^{(hn)} \subseteq \mathfrak{m}^n$, for all n. Then for I and S as before, take a prime ideal $P \supset I$ such that $P \cap S = \emptyset$. Then for all positive integers $q \geq 1$, $I^q : \langle S \rangle \subseteq P^{(q)}$. Thus, it suffices to show that there exists an $h \geq 1$, so that $P^{(hn)} \subseteq \mathfrak{m}^n$, for all prime ideals P and all $n \geq 1$.

Let $\{v_1, \ldots, v_t\}$ be the Rees valuations of $\mathfrak{m}$. By Izumi's Theorem, there exist positive constants $c_1, \ldots, c_t$ with the property that for any $1 \leq i \leq t$, $v_i(x) \leq c_i \cdot v_j(x)$, for all nonzero $x \in R$ and all $1 \leq j \leq t$. (See [8, Theorem 1.3].)

On the other hand, by a theorem of Rees (see [19, Theorem 2.3]), there exist positive integers $d_1, \ldots, d_t$ with the following property: $e(R/(x)) = \sum_{i=1}^{t} d_i \cdot v_i(x)$, for all nonzero $x \in R$. Here, $e(A)$ denotes the multiplicity of a local ring A.

Now, for each $1 \leq i \leq t$, set

$$r_i := d_1c_1 + \cdots + d_{i-1}c_{i-1} + d_i + d_{i+1}c_{i+1} + \cdots + d_tc_t$$

and take any integer h' such that $h' \geq r_i \cdot v_i(\mathfrak{m})$, for all $1 \leq i \leq t$. We claim that $P^{(h'n)} \subseteq \overline{\mathfrak{m}^n}$, for all prime ideals P and $n \geq 1$. Suppose the claim holds. Then, since R is a complete local domain, there exists $e \geq 1$ such that $\overline{\mathfrak{m}^{n+e}} \subseteq \mathfrak{m}^n$, for all n (see [24, Theorem 9.1.2]). Thus, for all prime ideals P and all $n \geq 1$, we have $P^{(h'n+h'e)} \subseteq \mathfrak{m}^n$. It follows easily from this that there exists $h \geq 1$ required by the theorem.

So it remains to prove the claim. Let $P \subseteq R$ be a (nonzero) prime ideal and take $x \in P^{(h'n)}$. Set $A := R_P$. Then since $x \in P^{h'n}A$,

$$h'n \leq h'n \cdot e(A) \leq e\big(A/(x)\big).$$

On the other hand, since R is a complete local domain, $R/(x)$ and $P/(x)$ meet the requirements of [16, Theorem (40.1)], so $e(A/(x)) \leq e(R/(x))$. Thus,

$$h'n \leq h'n \cdot e(A) \leq e\big(A/(x)\big) \leq e\big(R/(x)\big) = \sum_{i=1}^{t} d_i \cdot v_i(x).$$

Now fix $1 \leq j \leq t$. Then, by the definition of $c_1, \ldots, c_t$, we have

$$\sum_{i=1}^{t} d_i \cdot v_i(x) \leq (d_1c_1 + \cdots + d_{j-1}c_{j-1} + d_j + d_{i+1}c_{j+1} + \cdots + d_tc_t) \cdot v_j(x).$$

Thus,

$$h'n \leq (d_1c_1 + \cdots + d_{j-1}c_{j-1} + d_j + d_{j+1}c_{j+1} + \cdots + d_tc_t) \cdot v_j(x)$$

and therefore, $nr_jv_j(\mathfrak{m}) \leq h'n \leq r_j \cdot v_j(x)$. Since this holds for all j, $x \in \overline{\mathfrak{m}^n}$. Thus, $P^{(h'n)} \subseteq \overline{\mathfrak{m}^n}$ for all n. This completes the proof of the claim and the theorem. □

As a corollary, we can globalize the statement in Theorem 2.3.

COROLLARY 2.4. *Let R be a Noetherian ring and $J \subseteq R$ an ideal. Suppose that R_P is analytically unramified for all $P \in A(J)$. Then there exists a positive integer h with the following property. For all ideals $I \subseteq R$ and multiplicatively closed sets S for which the $\{I^n : \langle S\rangle\}$ topology is finer than the J-adic topology, $I^{hn} : \langle S\rangle \subseteq J^n$, for all n.*

Proof. The point of the proof is that we can combine Theorem 2.3 with the main result in [22]. Recall that from [22], there exists a positive integer c and for each n an irredundant primary composition $Q_{n,1} \cap \cdots \cap Q_{n,s_n}$ of J^n so that for $P_j := \operatorname{rad}(Q_{n,j})$, $P_j^{cn} \subseteq Q_{n,j}$, for all $n \geq 1$ and $1 \leq j \leq s_n$. It follows from this that $P_j^{(cn)} \subseteq Q_{n,j}$, for all n and all j.

On the other hand, if I, S are such that the $\{I^n : \langle S\rangle\}$ topology is finer than the J-adic topology, then the $\{I^n : \langle S\rangle\}$ topology is finer than the P-symbolic topology for any $P \in A(J)$. By our hypothesis on $A(J)$ and Theorem 2.3, there is a positive integer l such that $(I^{ln} : \langle S\rangle)_P \subseteq P_P^n$, for all n and all $P \in A(J)$. Therefore, $I^{ln} : \langle S\rangle \subseteq P^{(n)}$, for all $P \in A(J)$ and $n \geq 1$. Combining this with the previous paragraph and setting $h := cl$, it follows that $I^{hn} : \langle S\rangle \subseteq J^n$, for all I, S and $n \geq 1$. □

REMARK 2.5. Let R be a local ring with completion $\widehat{R}$ and let $z_1, \ldots, z_s$ be the associated primes of $\widehat{R}$. Set $T_i := \widehat{R}/z_i$, for all i. It follows from Proposition 2.2 that for an ideal $I \subseteq R$ and multiplicatively closed set $S \subseteq R$, the $\{I^n : \langle S\rangle\}$ topology is finer than the $\mathfrak{m}$-adic topology if and only if the $\{I^nT_i :_{T_i} \langle S\rangle\}$ topology is finer than the $\mathfrak{m}T_i$-adic topology, for all $1 \leq i \leq s$. Thus, it would seem that the existence of a uniform h with the property that for all i, $I^{hn}T_i :_{T_i} \langle S\rangle \subseteq \mathfrak{m}^nT_i$, for all I, S and n would give rise to a uniform h' over R for which $I^{h'n} : \langle S\rangle \subseteq \mathfrak{m}^n$, for all appropriate I and S. However, we have only been able to prove this under the extra assumption that R is analytically unramified.

3. Isolated singularities

Before looking at the question of uniform linearity of symbolic powers for isolated singularities, we would first like to look more closely in this context at what it means for the symbolic and adic toplogies of an ideal to be equivalent. For this, we consider McAdam's interpretation of Schenzel's theorem. For an ideal I in a Noetherian ring A, we write $Q(I)$ for the set of prime ideals P containing I for which $\dim(\widehat{R_P}/(I\widehat{R_P} + z)) = 0$, for some $z \in \operatorname{Ass}(\widehat{R_P})$. McAdam refers to $Q(I)$ as the *essential prime divisors* of I. It is not hard to show that if $P \in Q(I)$, then $P \in \operatorname{Ass}(R/I^n)$ for all large n (see [14] for details). In [15], McAdam reinterprets Schenzel's result as follows.

THEOREM 3.1 (Schenzel–McAdam). *Let A be a Noetherian ring, $S \subseteq A$ a multiplicatively closed set and $I \subseteq A$ an ideal. The $\{I^n : \langle S\rangle\}$ topology is equivalent to the I-adic topology if and only if $S \subseteq R \backslash \bigcup\{P | P \in Q(I)\}$.*

Now, take $I \subseteq A$ and let S denote the complement of the union of the associated primes of I, so that $I^n : \langle S\rangle = I^{(n)}$, for all n. Then Theorem 3.1 says that the I-symbolic and I-adic topologies are equivalent if and only if every prime in $Q(I)$ is contained in some prime in $\operatorname{Ass}(R/I)$. Suppose that $A = (R, \mathfrak{m})$ is a local isolated singularity, i.e., R_P is a regular local ring for every prime $P \neq \mathfrak{m}$. Since $\widehat{R_P}$ is an integral domain for any $P \neq \mathfrak{m}$, such a prime belongs to $Q(I)$ if and only if it is minimal over I, and is therefore certainly contained in an associated prime of I. Thus, only $\mathfrak{m}$ is a possible obstruction to the equivalence of the I-symbolic and I-adic topologies. In fact, from the definition of symbolic power and Theorem 3.1, it is now clear that if R is an isolated singularity, the I-symbolic and I-adic topologies are equivalent if and only if either $\mathfrak{m}$ is an associated prime of I or $\mathfrak{m} \notin Q(I)$. Furthermore, suppose $\mathfrak{m} \notin \operatorname{Ass}(R/I)$ and R is analytically unramified, e.g., reduced and excellent. Then it follows that the I-symbolic and I-adic topologies are equivalent if and only if $H^d_I(R) = 0$, by the Hartshorne–Lichtenbaum vanishing theorem (see [3, Theorem 8.2.1]).

In [6], the following property of the Jacobian ideal plays a crucial role in the main results concerning uniform linearity of symbolic powers over regular local rings (see [6, Theorem 4.4]).

THEOREM 3.2 (Hochster–Huneke). *Let R be a an equidimensional local ring essentially of finite type over a field K of characteristic zero. Let J denote the square of the Jacobian ideal of R over K. Then there exists $k \geq 1$ such that*

$$J^n I^{(kn+ln)} \subseteq \left(I^{(l+1)}\right)^n$$

for all ideals I with positive grade and all $l \geq 0$, $n \geq 1$.

In fact, [6, Theorem 4.4] shows that if $\mathfrak{J}$ is the Jacobian of R over K, then $\mathfrak{J}^{n+1} I^{(kn+ln)} \subseteq (I^{(l+1)})^n$, for all ideals I with positive grade and all $l \geq 0$, $n \geq 1$. Squaring $\mathfrak{J}$ gives the formulation above.

In our first proposition, we formalize a consequence of the Hochster–Huneke relation in Theorem 3.2.

PROPOSITION 3.3. *Let R be a Noetherian ring in which the uniform Artin–Rees lemma holds and $J \subseteq R$ an ideal with positive grade. Assume that R_P is analytically unramified for all $P \in A(J)$. Let $\mathcal{J}$ denote the collection of ideals $I \subseteq R$ for which the I-symbolic topology is finer than the J-adic topology.*

Suppose there exists $k \geq 1$ with the following property. For all ideals $I \in \mathcal{J}$,

$$J^n I^{(kn+ln)} \subseteq \left(I^{(l+1)}\right)^n$$

for all $l \geq 0$, $n \geq 1$. Then there exists a positive integer h such that for all ideals $I \in \mathcal{J}$, $I^{(hn)} \subseteq I^n$, for all n.

Proof. By Corollary 2.4, we can choose h_0 so that $I^{(h_0 n)} \subseteq J^n$, for all ideals $I \in \mathcal{J}$. Moreover, we may assume that $h_0 \geq k$. Taking $l = 0$ in the Hochster–Huneke relation gives $J^n I^{(kn)} \subseteq I^n$, for all n and all ideals $I \in \mathcal{J}$. Thus, $I^{(h_0 n)} I^{(kn)} \subseteq I^n$ for all n and all ideals $I \in \mathcal{J}$. On the other hand, if we take $n = 2$ and $l = h_0 r$ in the Hochster–Huneke relation we get

$$J^2 I^{(2k+2h_0 r)} \subseteq \left(I^{(h_0 r+1)}\right)^2 \subseteq I^{(h_0 r)} I^{(kr)} \subseteq I^r,$$

which holds for all r. Thus, there exists a positive integer K such that for all r, $J^2 I^{(Kr)} \subseteq I^r$. Choose a non-zerodivisor $c \in J^2$. Then $cI^{(Kn)} \subseteq I^n$, for all n. Thus, $I^{(Kn)} \subseteq (I^n : c)$ for all n and all I. Now apply the uniform Artin–Rees theorem [10, Theorem 4.12] to the pair $(c) \subseteq R$. This theorem gives us $q \geq 1$ with the property that $I^{n+q} \cap (c) \subseteq I^n(c)$ for all n and all ideals I in $\mathcal{J}$. But $I^{n+q} \cap (c) = c(I^{n+q} : c)$, thus, $(I^{n+q} : c) \subseteq I^n$, for all n and all ideals I. Therefore,

$$I^{(Kn+Kq)} \subseteq (I^{n+q} : c) \subseteq I^n$$

for all n and all ideals $I \in \mathcal{J}$. Taking $h := K + Kq$ gives $I^{(hn)} \subseteq I^n$, for all $I \in \mathcal{J}$ and all $n \geq 1$, as required. ☐

The next proposition will enable us to find an ideal J that, in positive characteristic, plays the same role as the square of the Jacobian in Theorem 3.2. Recall that a ring R of positive characteristic p is *F-finite* if R is a finitely generated R^p-module. If R is reduced this is equivalent to saying that $R^{1/p}$ is a finitely generated R-module.

PROPOSITION 3.4. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. Assume that R is F-finite and an isolated singularity. Then there exists an $\mathfrak{m}$-primary ideal J such that for all ideals I, all $x \in R$ and all $q = p^e$*

$$J^{[q]}\left(I^{[q]} : x^q\right) \subseteq (I : x)^{[q]}.$$

Proof. We begin by noting that if R is F-finite, i.e., R is finite over R^p, then R/z is also F-finite for any minimal prime $z \subseteq R$. Likewise, R/z is also an isolated singularity for z a minimal prime, if R is an isolated singularity. We now reduce to the case that R is a local domain.

Assume first the proposition holds for R/N, where N is the nilradical of R. Let $J' \subseteq R$ be such that its image in R/N satisfies the conclusion of the proposition. Since R has isolated singularity R_Q is a domain for all primes

$Q \neq \mathfrak{m}$. Therefore, $N_Q = 0$ for all primes $Q \neq \mathfrak{m}$. Hence, $0 :_R N$ is an $\mathfrak{m}$-primary ideal. Therefore, the ideal $J := J'(0 :_R N)$ is also $\mathfrak{m}$-primary. By assumption, we have

$$J'^{[q]}\big(I^{[q]} : x^q\big) \subseteq \big((I+N) : x\big)^{[q]} + N.$$

Therefore,

$$J^{[q]}\big(I^{[q]} : x^q\big) \subseteq (0 :_R N)^{[q]}\Big(\big((I+N) : x\big)^{[q]} + N\Big) \subseteq (I : x)^{[q]}.$$

To prove the last inclusion, let $y \in ((I+N) : x))^{[q]}$. Then there exists elements y_i in $(I+N) : x$ and elements $r_i \in R$ such that $y = \sum_i r_i y_i^q$. Let $a \in (0 :_R N)$. Then $a^q y = \sum_i r_i (a y_i)^q$, and $(a y_i)x \in a(I+N) \subset I$. Hence, $a^q y \in (I : x)^{[q]}$.

Replacing R by R/N, we now assume that R is reduced, that the conclusion of the proposition holds modulo each minimal prime, and show that the conclusion holds for R. Let $z_1, \ldots, z_n$ be the minimal primes of R. Let $\mathfrak{a}_i = z_1 \cap \cdots \cap z_{i-1} \cap z_{i+1} \cap \cdots \cap z_n$, so $\mathfrak{a}_i z_i = 0$. Let J_i be the lift of corresponding $\mathfrak{m}$-primary ideal modulo z_i and put

$$J := (J_1 \cap \cdots \cap J_n)(\mathfrak{a}_1 + \cdots + \mathfrak{a}_n).$$

Therefore,

$$J^{[q]} = (J_1 \cap \cdots \cap J_n)^{[q]}(\mathfrak{a}_1 + \cdots + \mathfrak{a}_n)^{[q]} \subseteq \sum_i J_i^{[q]} \mathfrak{a}_i^{[q]}.$$

Note that J is an $\mathfrak{m}$-primary ideal since $\mathfrak{a}_1 + \cdots + \mathfrak{a}_n$ is $\mathfrak{m}$-primary. The latter follows because for any prime ideal $Q \neq \mathfrak{m}$, R_Q is a domain, so Q contains a unique minimal prime z_i and hence $\mathfrak{a}_i$ is not contained in Q.

By assumption for any ideal I, all $x \in R$ and all $q = p^e$, we have

$$J_i^{[q]}\big(I^{[q]} : x^q\big) \subseteq \big((I + z_i) : x\big)^{[q]} + z_i.$$

Therefore,

$$J^{[q]}\big(I^{[q]} : x^q\big) \subseteq \sum_i \mathfrak{a}_i^{[q]}\Big(\big((I + z_i) : x\big)^{[q]} + z_i\Big) \subseteq (I : x)^{[q]},$$

where the last inclusion follows by an argument similar to that used above to go modulo the nilradical.

It remains to prove the proposition for F-finite local domains of characteristic $p > 0$ with an isolated singularity. We first note that there exists $c_1, \ldots, c_l$ which generate an $\mathfrak{m}$-primary ideal such that for each $c = c_i$, there is a free R-module $F \subseteq R^{1/p}$ such that $cR^{1/p} \subseteq F$. Note that F depends on c. In general, if M is a finitely generated torsion-free R-module such that M_Q is free over R_Q for all primes $Q \neq \mathfrak{m}$, then there exists $c_1, \ldots, c_l$ which generate an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ such that for each $c = c_i$, there is a free R-module $F \subseteq M$ such that $cM \subseteq F$. Now take M to be $R^{1/p}$, which is a finitely generated R-module since R is F-finite. Then $M_Q = (R^{1/p})_Q \cong (R_Q)^{1/p}$. By a theorem of Kunz [12], $(R_Q)^{1/p}$ is free over R_Q since R_Q is regular for all primes $Q \neq \mathfrak{m}$.

Now we show that there exists an $\mathfrak{m}$-primary ideal $J \subseteq R$ with a generating set $d_1, \ldots, d_l$ such that for each $d = d_i$, there are free R-modules $F_q \subseteq R^{1/q}$ for every $q = p^e$ such that $dR^{1/q} \subseteq F_q$. We construct the free modules F_q inductively. We start with the $\mathfrak{m}$-primary ideal $\mathfrak{a}$, a fixed generator $c = c_i$ of $\mathfrak{a}$, and free module F from above and set $F_p := F$. Then $c \cdot R^{1/p} \subseteq F_p$. Assume that we have constructed $F_q = \bigoplus_i Ry_i^{1/q} \subseteq R^{1/q}$ such that $c^2 \cdot R^{1/q} \subseteq F_q$. Therefore, we have $F_q^{1/p} = \bigoplus_i R^{1/p} y_i^{1/pq} \subseteq R^{1/pq}$ and $c^{2/p} \cdot R^{1/pq} \subseteq F_q^{1/p}$. Write $F := \bigoplus_j Rv_j^{1/p} \subseteq R^{1/p}$ and consider the free R-module

$$F'_{pq} := \bigoplus_{j,i} Rv_j^{1/p} y_i^{1/pq} \subseteq \bigoplus_i R^{1/p} y_i^{1/pq}.$$

Note that $c \cdot F_q^{1/p} \subseteq F'_{pq}$. Set $F_{pq} := F'_{pq} c^{(p-2)/p}$. Then for any $c \in \mathfrak{a}$,

$$c^2 \cdot R^{1/pq} = c^{(p-2)/p} \cdot c \cdot c^{2/p} \cdot R^{1/pq} \subseteq c^{(p-2)/p} \cdot c \cdot F_q^{1/p} \subseteq c^{(p-2)/p} \cdot F'_{pq} = F_{pq}.$$

Therefore, we obtain $c^2 R^{1/pq} \subseteq F_{pq}$. Thus we may take $d_i = c_i^2$ for $i = 1, \ldots, l$.

Let I be an ideal, $x \in R$ and $q = p^e$. Take $u \in (I^{[q]} : x^q)$. Then $ux^q \in I^{[q]}$. Hence, $u^{1/q} x \in IR^{1/q}$. Fix $d = d_i$, let F_q be as in the above paragraph, and let J be the ideal generated by all the d_i. Then $du^{1/q} x \subseteq IF_q$. Thus,

$$Ju^{1/q} \subseteq (IF_q :_{F_q} x) = (I :_R x)F_q \subseteq (I :_R x)R^{1/q}.$$

Therefore, $J^{[q]}(I^{[q]} : x^q) \subseteq (I : x)^{[q]}$. ☐

We are ready to prove the main result in this paper. For ease of exposition, we use the following notation. $\mathcal{S}_R$ will denote the set of ideals $I \subseteq R$ for which the I-symbolic and I-adic topologies are equivalent.

THEOREM 3.5. *Let R be an equicharacteristic reduced local ring such that R is an isolated singularity. Assume either that R is equidimensional and essentially of finite type over a field of characteristic zero, or that R has positive characteristic and is F-finite. Then there exists $h \geq 1$ with the following property. For all ideals $I \in \mathcal{S}_R$ with positive grade, $I^{(hn)} \subseteq I^n$, for all $n \geq 1$.*

Proof. Note in both cases the ring R is excellent (see [13] for the F-finite case), so that R is analytically unramified.

Suppose first that R is essentially of finite type over a field of characteristic zero. Let J denote the square of the Jacobian ideal. By Hochster–Huneke [6],

$$J^n I^{(dn+ln)} \subseteq \left(I^{(l+1)}\right)^n$$

for all ideals I with positive grade and all $n \geq 1$ and all $l \geq 0$. Since J is $\mathfrak{m}$-primary and R is analytically unramified, we may invoke Proposition 3.3 with $k = d$ to obtain the result.

Now suppose that R has characteristic $p>0$ and is F-finite. By Proposition 3.4, there exists an $\mathfrak{m}$-primary ideal J such that for all ideals I, all $x \in R$ and all $q=p^e$

$$J^{[q]}\left(I^{[q]}:x^q\right) \subseteq (I:x)^{[q]}.$$

We begin by noting for all ideals $I \subseteq R$ that are contracted with respect to a multiplicative system W and all $q=p^e$, $J^{[q]}(I^{[q]})^W \subseteq I^{[q]}$, where the superscript ^{W} indicates expansion to R_W followed by contraction to R. Let $u \in (I^{[q]})^W$ and choose $x \in W$ such that $ux \in I^{[q]}$. Therefore, $u \in (I^{[q]}:x^q)$. Thus,

$$J^{[q]}u \in J^{[q]}\left(I^{[q]}:x^q\right) \subseteq (I:x)^{[q]} = I^{[q]}.$$

Now we will prove that for an ideal I and every nonnegative integer l and positive integer n we have

$$J^n I^{(dn+ln)} \subseteq \left(\left(I^{(l+1)}\right)^n\right)^*,$$

where d is the dimension of R and the superscript * denotes the tight closure of ideals. The proof is similar to that of [6, Theorem 3.7]. Let S be the complement of the union of the associated primes of I. For any $q=p^e$, we may write $q=an+r$ with a a nonnegative integer and $0 \le r < n$. Let $u \in I^{(dn+ln)}$. By Lemma 2.4(b) in [6], there exists a nonnegative integer s such that $I^{s+(d+l)(n-1)}u^a \subseteq ((I^{(l+1)})^{[q]})^S$. Therefore, multiplying both sides by $J^{[q]}$ we get

$$J^{[q]}I^{s+(d+l)(n-1)}u^a \subseteq J^{[q]}\left(\left(I^{(l+1)}\right)^{[q]}\right)^S \subseteq \left(\left(I^{(l+1)}\right)\right)^{[q]}.$$

Taking the nth power and setting $b=sn+(d+l)(n-1)n$, we obtain

$$I^b(J^n)^{[q]}u^{an} \subseteq \left(\left(I^{(l+1)}\right)^n\right)^{[q]},$$

for all q and since $q \ge an$ this implies $I^b(J^n u)^{[q]} \subseteq ((I^{(l+1)})^n)^{[q]}$ for all q. Note that b is independent of q, thus we may fix a non-zerodivisor z in I^b and obtain

$$z(J^n u)^{[q]} \subseteq \left(\left(I^{(l+1)}\right)^n\right)^{[q]},$$

which means $J^n u \subseteq ((I^{(l+1)})^n)^*$. By [10, Theorem 4.13] we know there exists a non negative integer t such that

$$\left(\left(I^{(l+1)}\right)^n\right)^* \subseteq \overline{\left(I^{(l+1)}\right)^n} \subseteq \left(I^{(l+1)}\right)^{n-t}$$

for all ideals I and for all n and l. Therefore, $J^n I^{(dn+ln)} \subseteq (I^{(l+1)})^{n-t}$, and the result follows by Proposition 3.3. ☐

In the characteristic zero case of Theorem 3.5, we needed to assume that R is equidimensional and considered ideals with positive grade, because we applied the Hochster–Huneke relation Theorem 3.2 directly to R. We can drop these hypotheses by passing to quotients of R modulo associated primes if we make a modest restriction on the class of ideals to which the conclusion applies. To facilitate what follows, we will need some further notation.

Notation. Let $(R,\mathfrak{m})$ be a local ring. We set $\mathcal{M}_R$ to be the set of ideals for which the $\{I^n:\langle\mathfrak{m}\rangle\}$ topology and the I-adic topology are equivalent. Let $\mathcal{A}_R$ denote the set of ideals $I\subseteq R$ such that $\mathfrak{m}\in\operatorname{Ass}(R/I)$. Finally, we let $\mathcal{S}_R(0)$ denote the ideals $I\in\mathcal{S}_R$ such $I\notin\mathcal{A}_{R/z}$, for all $z\in\operatorname{Ass}(R)$. Note that by the discussion in the paragraph preceding Theorem 3.2, it follows that if R is an isolated singularity, then $\mathcal{S}_R=\mathcal{M}_R\cup\mathcal{A}_R$. For ease of reference, we state this more formally.

Observation 3.6. Let $(R,\mathfrak{m})$ be a local isolated singularity. Then $\mathcal{S}_R=\mathcal{M}_R\cup\mathcal{A}_R$.

In order to reduce from the case that R is an isolated singularity to the case that R is a local domain with an isolated singularity, we need a result implicit in the work of Swanson. In [23, Lemmas 2.3 and 2.5] she shows the following. Let A be a Noetherian ring with associated primes $z_1,\dots,z_l$ and set $A_i:=A/z_i$, for all i. For fixed $I,J\subseteq A$, if there exists $h'\geq 1$ such that for all n, $I^{nh'}A_i:\langle J\rangle\subseteq I^nA_i$, for all $1\leq i\leq l$, then there exists $h\geq 1$ such that $I^{nh}:\langle J\rangle\subseteq I^n$, for all n. The proofs use data independent of I together with the Artin–Rees lemma on expressions of the form $I^n\cap C$, where C is one of finitely many ideals that depend only on R. Thus, if A also satisfies the uniform Artin–Rees lemma, h can be chosen independently of I, if h' is also independent of I. In particular, we have the following proposition.

Proposition 3.7 (Swanson). *Let $(R,\mathfrak{m})$ be a local ring with associated primes $z_1,\dots,z_l$ and set $T_i:=R/z_i$. Let $h'\geq 1$ and $\mathcal{C}$ be a class of ideals with the following property. For all $I\in\mathcal{C}$ and all $n\geq 1$, $I^{nh'}T_i:\langle\mathfrak{m}\rangle\subseteq I^nT_i$, for all i. Then there exists $h\geq 1$ such that for all $I\in\mathcal{C}$, $I^{nh}:\langle\mathfrak{m}\rangle\subseteq I^n$, for all n.*

We can now prove a reduction proposition for isolated singularities.

Proposition 3.8. *Let $(R,\mathfrak{m})$ be an isolated singularity containing a field. Let $z_1,\dots,z_l$ be the associated primes of R and set $T_i:=R/z_i$. Suppose there exists $h'\geq 1$ so that for each $1\leq i\leq l$ and each $J\in\mathcal{S}_{T_i}$, $J^{(h'n)}\subseteq J^n$, for all n. Then there exists $h\geq 1$ such that for all $I\in\mathcal{S}_R(0)$, $I^{(hn)}\subseteq I^n$, for all n.*

Proof. Set $d:=\dim(R)$. Let $P\neq\mathfrak{m}$ be a prime ideal. Then for any ideal $I\subseteq R$, it follows from the definition of symbolic power that $(I^{(n)})_P\subseteq(I_P)^{(n)}$ for all n. By the main result of [6], it follows that $(I^{(dn)})_P\subseteq I_P^n$, for all n. Thus, for all n and all I, $I^{(dn)}\subseteq I^n:\langle\mathfrak{m}\rangle$.

Now suppose $I\in\mathcal{S}_R(0)$. If $I\in\mathcal{A}_R$, then $I^{(n)}=I^n$, for all n. Otherwise, $I\in\mathcal{M}_R$, by Observation 3.6, so $IT_i\in\mathcal{M}_{T_i}$, for all $1\leq i\leq l$, by Schenzel's theorem. Now, since $IT_i\notin\mathcal{A}_{T_i}$, $I^nT_i:\langle\mathfrak{m}\rangle\subseteq(IT_i)^{(n)}$ for all i and n. Thus,

$$I^{h'n}:\langle\mathfrak{m}\rangle\subseteq(IT_i)^{(h'n)}\subseteq I^nT_i$$

for all i and n. By Proposition 3.7, there exists $h\geq 1$, so that $I^{hn}:\langle\mathfrak{m}\rangle\subseteq I^n$, for all n. Therefore,

$$I^{(dhn)}\subseteq I^{hn}:\langle\mathfrak{m}\rangle\subseteq I^n$$

for all n, which gives what we want. □

Here is another version of the characteristic zero case of our main theorem.

THEOREM 3.9. *Let* $(R, \mathfrak{m})$ *be an isolated singularity such that* R *is essentially of finite type over a field of characteristic zero. Then there exists* $h \geq 1$ *such that* $I^{(hn)} \subseteq I^n$, *for all* $I \in \mathcal{S}_R(0)$ *and all* $n \geq 1$.

Proof. Note that the hypotheses on R carry over to R/z, for all associated primes z of R. Let $I \in \mathcal{S}_R(0)$. If $\mathfrak{m} \in \operatorname{Ass}(R/I)$, then $I^{(n)} = I^n$ for all n. Otherwise, $\mathfrak{m} \notin Q(I)$, by Observation 3.6. By Schenzel's theorem, for all associated primes z, $\mathfrak{m}/z \notin Q(I \cdot R/z)$, so $I \cdot R/z \in \mathcal{S}_{R/z}$. Thus, the result follows from Theorem 3.5 applied to R/z for each associated prime z together with Proposition 3.8. □

COROLLARY 3.10. *Let* R *be an equicharacteristic, local domain such that* R *is an isolated singularity. Assume that* R *is either essentially of finite type over a field of characteristic zero or* R *has positive characteristic, is* F*-finite and analytically irreducible. Then there exists* $h \geq 1$ *with the following property. For all prime ideals* $P \neq \mathfrak{m}$, $P^{(hn)} \subseteq P^n$, *for all* n.

Proof. By Observation 3.6, a nonmaximal prime ideal belongs to $\mathcal{S}_R$ if and only if it belongs to $\mathcal{M}_R$. On the other hand, since (in each case), $\widehat{R}$ is a domain, every nonmaximal prime belongs to $\mathcal{M}_R$, by Schenzel's theorem. Thus, every nonmaximal prime belongs to $\mathcal{S}_R$. The corollary now follows either from Theorem 3.5 or Theorem 3.9. □

We suspect that uniform linearity of symbolic powers holds much more generally than the cases discussed in this paper. In particular, we offer the following question.

QUESTION. Let $(R, \mathfrak{m})$ be a complete local domain. Does there exist $h \geq 1$ such that for all prime ideals P, $P^{(hn)} \subseteq P^n$, for all $n \geq 1$?

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