Illinois Journal of Mathematics
Volume 53, Number 1, Spring 2009, Pages 135–161
S 0019-2082

NEW CONSTANT MEAN CURVATURE SURFACES IN THE HYPERBOLIC SPACE

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ABSTRACT. Applying Ribaucour transformations, we construct two new 3-parameter families of complete surfaces, immersed in H^3, with constant mean curvature 1 and infinitely many embedded horosphere type ends. Each surface of the first family is locally associated to an Enneper cousin. It has one irregular end and infinitely many regular ends asymptotic to horospheres. The surfaces of the second family are locally associated to a catenoid cousin. Each surface of this family has infinitely many embedded horosphere type ends and one regular end with infinite total curvature.

0. Introduction

In the last two decades, the construction of new complete cmc surfaces (surfaces of constant mean curvature) in space forms has been a very active topic of research. The following methods were used in the construction of cmc surfaces: the method of perturbation [K1], [K2]; integrable systems [PS], [Bo]; conjugate prime surfaces [La], [Ka]; Weierstrass type representation [DH] [KMS]. The geometry of surfaces of constant mean curvature one (cmc1) in hyperbolic space H^3, in terms of Weierstrass representation, was introduced by Bryant [Br], relating such surfaces to minimal surfaces in the Euclidean space $\mathbb{R}^3$. Later, Umehara–Yamada [UY1], following Bryant's ideas, made an important contribution that provides a method to obtain cmc1 surfaces in H^3 from minimal surfaces in $\mathbb{R}^3$.

Received June 22, 2008; received in final form August 13, 2008.
This work was partially supported by CNPq.
2000 *Mathematics Subject Classification.* 53C20.

More recently (see [CFT2]), complete cmc surfaces were constructed by applying another method, based on Ribaucour transformations. These transformations for hypersurfaces in space forms were studied by Bianchi in 1918–1919 [Bi]. The classical theory shows that Ribaucour transformations may be used to construct surfaces of constant Gaussian curvature and constant mean curvature surfaces from a given such surface [Bi2]. However, the first application of this method to minimal and cmc surfaces in $\mathbb{R}^3$ was obtained in recent years by Corro, Ferreira, and Tenenblat in [CFT1], providing new families of complete minimal surfaces associated to the Enneper surface and to the catenoid. Moreover, 3-parameter families of complete cmc surfaces were obtained, in [CFT2], by applying the theory to the cylinder and to the Delaunay surfaces. These families contain the cmc n-bubble surfaces described by Sievert [S], [GB], and [SW]. The results on minimal surfaces obtained in [CFT1] were exended in [LT] to a class of minimal surfaces given in terms of Weierstrass data. In [CFT2], the classical theory of Ribaucour transformations was extended to linear Weingarten surfaces in $\mathbb{R}^3$, providing a unified version of the classical results. As an application, the existence of complete hyperbolic linear Weingarten surfaces immersed in $\mathbb{R}^3$ was shown, in contrast to Hilbert's theorem that asserts the nonexistence of complete surfaces of constant negative curvature immersed in $\mathbb{R}^3$.

In [TW], we considered the theory of Ribaucour transformations for surfaces M^2 in a space form $\overline{M}^3(\overline{k}), \overline{k} = 0, 1, -1$, and extended the results of [CFT2], providing a method of constructing linear Weingarten surfaces in $\overline{M}^3(\overline{k})$ from a given such surface. This theory was applied to construct a 1-parameter family of complete cmc surfaces in the unit sphere S^3, associated to the flat torus. For special values of the parameter, we obtained a family of complete cmc cylinders immersed in S^3. In particular, we exhibited a family of complete minimal surfaces and minimal cylinders immersed in S^3, associated to the Clifford torus by a Ribaucour transformation.

In this paper, we apply the theory to construct two new 3-parameter families of complete cmc1 surfaces in H^3, with infinitely many embedded regular ends with finite total curvature, that are asymptotic to horospheres. An end of a cmc1 surface, in H^3, is called regular if the hyperbolic Gauss map of the surface has no essential singularity at the end. Otherwise, the end is called irregular. Results of Earp–Toubiana [ET] and Rossman–Levi [RL], prove that a regular end with finite total curvature is proper and it is asymptotic to a catenoid cousin or a horosphere. We will show that in our cmc1 surfaces all such ends are asymptotic to horospheres. The cmc1 surfaces we obtain in the first family are locally associated, by a Ribaucour transformation, to an Enneper cousin. Each surface of this 3-parameter family has infinitely many regular ends and one irregular end. The regular ends have finite total curvature, they are proper and asymptotic to horospheres, i.e., they are *horosphere* *type ends.* The surfaces in the second family are locally associated, by a Ribaucour transformation, to a catenoid cousin. Each surface of this family has one regular end with infinite total curvature and infinitely many embedded horosphere type ends.

One should mention that Rossman–Umehara–Yamada [RUY] have found cmc1 surfaces in H^3, with infinitely many ends, all of them regular. On the other hand, Karcher [Ka2] obtained cmc1 surfaces in H^3, with infinite topology and one end.

1. Ribaucour transformations for linear Weingarten surfaces in space forms

Ribaucour transformations provide an integrable system of differential equations whose solutions enable us to obtain linear Weingarten surfaces in a space form $\overline{M}^3$ from a given such surface and in particular cmc surfaces from a given such surface. The precise results will be given in this section. For more details and proofs see [TW] and its references.

Let $M^3(\overline{k}), \overline{k} = \pm 1, 0$, be the simply connected space form of sectional curvature $\overline{k}$ and let L^4 be the set of $x = (x_0, x_1, \ldots, x_3) \in \mathbb{R}^4$ endowed with the pseudo-Riemannian inner product given by $\langle x, y\rangle = -x_0y_0 + \sum_{i=1}^{3} x_iy_i$. A model for the hyperbolic three space is the submanifold $H^3 = \{x \in L^4 | \langle x, x\rangle = -1\}$.

Consider an orientable surface M in $\overline{M}^3(\overline{k})$, where $\overline{M}^3(k) = S^3 \subset \mathbb{R}^4$ when $\overline{k} = 1$ and $\overline{M}^3(\overline{k}) = H^3 \subset L^4$ when $\overline{k} = -1$. Let $e_i, i = 1, \ldots, 2$, be an orthonormal frame tangent to M and N a unit normal vector field defined on M. Denote by ω_i the 1-forms dual to e_i and $\omega_{ij}, 1 \leq i, j \leq 2$, the connection forms which are defined by $d\omega_i = \sum_{j\neq i} \omega_{ij} \wedge w_j, \omega_{ij} + \omega_{ji} = 0$. The Gauss equation is given by $d\omega_{12} = \omega_{13} \wedge \omega_{32} - \overline{k}\omega_1 \wedge \omega_2$, where $\omega_{i3} = -\omega_{3i} = \langle de_i, N\rangle$, and the Codazzi equations are $d\omega_{i3} = \sum_{j=1}^{2} \omega_{ij} \wedge \omega_{j3}$. Let the vector fields e_i be principal directions corresponding to the principal curvatures $-\lambda_i, 1 \leq i \leq n$, then

$$\omega_{i3} = -\lambda^i\omega_i, \qquad dN(e_i) = \lambda^i e_i.$$

An orientable surface $\tilde{M} \subset \overline{M}^3$ is *associated to M by a Ribaucour transformation* if there exist a differentiable function $h : M \to \mathbb{R}$, a diffeomorphism $\psi : M \to \tilde{M}$ and unit normal vector fields N and $\tilde{N}$ of M and $\tilde{M}$, respectively, such that:

(a) $\exp_p h(p)N(p) = \exp_{\psi(p)} h(p)\tilde{N}(\psi(p)), \forall p \in M$, where exp is the exponential map of $\overline{M}$;

(b) the subset $S = \{\exp_p h(p)N(p) | p \in M\}$ is an n-dimensional submanifold of $\overline{M}$;

(c) $d\psi(e_i), i = 1, \ldots, n$, are orthogonal principal directions of $\tilde{M}$.

By considering $\overline{M}$ as $\mathbb{R}^3, S^3 \subset \mathbb{R}^4$ or $H^3 \subset L^4$, we can rewrite condition (a) above as

$$p + h(p)N(p) = \psi(p) + h(p)\tilde{N}(\psi(p)), \quad p \in M, \text{ if } \overline{k} = 0,$$

and

$$h(p) = \begin{cases} \tan(\phi(p)), & \phi : M \to \left(0, \frac{\pi}{2}\right), \quad \text{if } \overline{k} = 1, \\ \tanh(\phi(p)), & \phi : M \to \mathbb{R}, \qquad\quad \text{if } \overline{k} = -1. \end{cases}$$

One can also formulate a definition for surfaces locally associated by Ribaucour transformations.

A characterization of a Ribaucour transformation in a space form is given in terms of a system of nonlinear partial differential equations for h. However, the problem of obtaining h can be linearized, by considering $h = \Omega/W$ and $\Omega_i = d\Omega(e_i)$. The following theorem provides a sufficient condition for a Ribaucour transformation to transform a linear Weingarten surface in a space form into another surface of the same type. We refer to [CFT2] and [TW] for the proofs and details.

Theorem 1.1 ([CFT2], [TW]). *Let M be a linear Weingarten surface in $\overline{M}^3(\overline{k})$, which admits orthonormal principal vector fields. Suppose that the Gaussian and mean curvatures of M satisfy $\alpha + \beta H + \gamma(K - \overline{k}) = 0$. Then for any constant $c \neq 0$, the system of the equations*

$$\begin{aligned} d\Omega &= \sum_{i=1}^{2} \Omega_i \omega_i, \\ dW &= \sum_{i=1}^{2} \Omega_i \omega_{i3}, \\ d\Omega_i &= \Omega_j \omega_{ij} + \{(2c\alpha - \overline{k})\Omega - \beta c W\}\omega_i \\ &\quad + \{c\beta\Omega + (2c\gamma - 1)W\}\omega_{i3}, \quad i \neq j. \end{aligned} \tag{1.1}$$

is integrable. Any solution of (1.1), on a simply connected domain U, whose initial condition satisfies

$$\Omega_1^2 + \Omega_2^2 + W^2 + \overline{k}\Omega^2 = 2c(\alpha\Omega^2 + \beta\Omega W + \gamma W^2), \tag{1.2}$$

will satisfy (1.2) identically. If M is locally parametrized by $X : U \subset R^2 \to M \subset \overline{M}^3(\overline{k})$, and Ω, W is a solution of (1.1) satisfying (1.2), then

$$\tilde{X} = \left(1 - \frac{2\overline{k}\Omega^2}{S}\right)X - \frac{2\Omega}{S}\left(\sum_{i=1}^{2} \Omega_i e_i - WN\right),$$

where $S = \Omega_1^2 + \Omega_2^2 + W^2 + \overline{k}\Omega^2$, is a linear Weingarten surface satisfying $\alpha + \beta\tilde{H} + \gamma(\tilde{K} - \overline{k}) = 0$. Moreover, $\tilde{X}$ is locally associated to X by a Ribaucour transformation and it is defined on

$$\tilde{U} = \left\{(u_1, u_2) \in U : S\left(T^2 + 2TQH + Q^2(K - \overline{k})\right) \neq 0\right\},$$

where $T=\alpha\Omega^2-\gamma W^2$ *and* $Q=2\gamma\Omega W+\beta\Omega^2$.

One can show that the first fundamental form of $\tilde{X}$ is given by $\tilde{I}=\tilde{\omega}_1^2+\tilde{\omega}_2^2$, where

$$\tilde{\omega}_i=\frac{1}{P}\big(\gamma W^2-\alpha\Omega^2+(\beta\Omega+2\gamma W)\Omega\lambda^i\big)\omega_i,\quad i=1,2,$$
$$P=\alpha\Omega^2+\beta\Omega W+\gamma W^2$$

and the principal curvatures $-\tilde{\lambda}^i$ are defined by

$$\tilde{\lambda}^i=\frac{2\alpha\Omega W+\beta W^2+\lambda^i(\alpha\Omega^2-\gamma W^2)}{(2\gamma\Omega W+\beta\Omega^2)\lambda^i-(\alpha\Omega^2-\gamma W^2)},\quad i=1,2.$$

Considering H constant, $\alpha=-H,\beta=1$ and $\gamma=0$ in Theorem 1.1, we have the cmc H surfaces ($H=0$ is the case of minimal surfaces).

THEOREM 1.2. *Let* M *be a cmc* H *surface in* $\overline{M}^3(\overline{k})$ *with no umbilic points and let* e_1,e_2 *be principal directions. For any constant* $c\neq 0$, *the system of equations*

$$\begin{aligned} d\Omega&=\sum_{i=1}^{2}\Omega_i\omega_i,\\ dW&=\sum_{i=1}^{2}\Omega_i\omega_{i3},\\ d\Omega_i&=\Omega_j\omega_{ij}-[(2cH+\overline{k})\Omega-cW]\omega_i+(c\Omega-W)\omega_{i3},\quad i\neq j,\end{aligned}\tag{1.3}$$

is integrable. Assume that

$$-\overline{k}+c^2-2Hc>0,\tag{1.4}$$

then any solution of (1.3) *on a simply connected domain* U, *whose initial condition satisfies*

$$\Omega_1^2+\Omega_2^2+W^2+\overline{k}\Omega^2=2c\Omega(-H\Omega+W),\tag{1.5}$$

will satisfy (1.5) *identically. If* M *is locally parametrized by* $X:U\subset R^2\to\overline{M}^3(\overline{k})$, *and* Ω,W *is a solution of* (1.3) *satisfying* (1.4) *and* (1.5), *then*

$$\tilde{X}=\left(1-\frac{\overline{k}\Omega}{c(W-H\Omega)}\right)X-\frac{1}{c(W-H\Omega)}\left(\sum_{i=1}^{2}\Omega_ie_i-WN\right),\tag{1.6}$$

is a cmc H *surface locally associated to* X *by a Ribaucour transformation defined on the subset of* U *where* $\Omega(W-H\Omega)\neq 0$.

We observe that (1.4) is a necessary condition for a nontrivial solution to satisfy (1.5). Moreover, the first fundamental form of $\tilde{X}$ is given by $\tilde{I}=\tilde{\omega}_1^2+\tilde{\omega}_2^2$, where

$$\tilde{\omega}_i=\frac{\Omega(H+\lambda^i)}{W-H\Omega}\omega_i,\quad i=1,2\tag{1.7}$$

and the principal curvatures $-\tilde{\lambda}^i$ are defined by

$$\tilde{\lambda}^i = \frac{(W - H\Omega)^2}{\Omega^2(H + \lambda^i)} - H, \quad i = 1, 2. \tag{1.8}$$

One observes that given a cmc H surface, a Ribaucour transformation, generically, provides (locally) a 3-parameter family of cmc H surfaces (same constant mean curvature) associated to the given surface. In this case, it follows from (1.7) that the associated surfaces are conformal and hence the transformation is a Darboux transformation. In general, a Ribaucour transformation between linear Weingarten surfaces, given by Theorem 1.1, is not a Darboux transformation (see for example [CFT2], p. 278). Due to the fact that Ribaucour transformations, generically, are not conformal, they can be applied to a broader class of associated hypersurfaces than the Darboux transformations.

2. Families of complete cmc1 surfaces in H^3

In this section, applying Theorem 1.2, we construct two new families of complete surfaces, immersed in H^3, with constant mean curvature 1 and infinitely many embedded horosphere type ends. The cmc1 surfaces we obtain in the first family are locally associated, by a Ribaucour transformation, to an Enneper cousin. Each surface of this 3-parameter family has infinitely many proper regular ends and one irregular end. The regular ends are asymptotic to horospheres. The surfaces in the second family are locally associated, by a Ribaucour transformation, to a catenoid cousin. Each surface of this family has one regular end, with infinite total curvature and infinitely many horosphere type ends.

THEOREM 2.1. *Consider the Enneper cousin M in H^3 parametrized by*

$$\begin{aligned} X(u_1, u_2) = \frac{\sqrt{2}}{2}\Bigg(& \frac{|u|^2 + 6}{4}\cosh u_1 - u_1 \sinh u_1, \\ & \frac{|u|^2 + 6}{4}\sinh u_1 - u_1 \cosh u_1, \\ & \frac{2 - |u|^2}{4}\sin u_2 - u_2 \cos u_2, \frac{|u|^2 - 2}{4}\cos u_2 - u_2 \sin u_2 \Bigg), \\ & u = (u_1, u_2) \in R^2. \end{aligned} \tag{2.1}$$

A parametrized surface $\tilde{X}_c$ of constant mean curvature 1 in H^3, not congruent to M, is locally associated to X by a Ribaucour transformation as in Theorem 1.2 if, and only if,

$$\tilde{X}_c = \left(1 - \frac{1}{c}\right)X + \frac{1}{c(W - \Omega)}\left(WY - \sum_{i=1}^{2}\Omega_i e_i\right), \tag{2.2}$$

where $e_i = \frac{X_{u_i}}{|X_{u_i}|}, c \in \mathbb{R}^n \setminus \{0,1\}$, Ω, W *and* Ω_i *are given in terms of two real functions* $R(u_1)$ *and* $G(u_2)$ *by*

$$\Omega = \frac{\sqrt{2}}{4}\left[\frac{\phi}{4}(R+G) + \frac{1}{c-1}(G - R + u_1 R' - u_2 G')\right], \tag{2.3}$$

$$\Omega_1 = \frac{u_1(G-R)}{\phi} + \frac{R'}{2}, \qquad \Omega_2 = -\frac{u_2(G-R)}{\phi} + \frac{G'}{2}, \tag{2.4}$$

$$W = \Omega + \frac{\sqrt{2}(G-R)}{\phi}, \tag{2.5}$$

$$Y = \frac{2\sqrt{2}}{|u|^2+2}(\cosh u_1, \sinh u_1, \sin u_2, -\cos u_2), \tag{2.6}$$

$$\phi = 2 + |u|^2, \tag{2.7}$$

and

$$R(u_1) = \sin\big(\sqrt{c-1}u_1 + A_1\big), \qquad G(u_2) = \delta\cosh\big(\sqrt{c-1}u_2 + A_2\big), \quad \text{when } c > 1, \tag{2.8}$$

$$R(u_1) = \delta\cosh(\sqrt{1-c}u_1 + B_1), \qquad G(u_2) = \sin(\sqrt{1-c}u_2 + B_2), \quad \text{when } c < 1, c \neq 0, \tag{2.9}$$

$\delta = \pm 1, A_1, A_2, B_1$ *and* B_2 *are constants. Moreover,* $\tilde{X}_c$ *is a regular surface defined on*

$$\tilde{U} = \{(u_1,u_2) \in R^2 | [\Omega(\Omega - W)](u_1,u_2) \neq 0\}.$$

Remark 2.2. The surface M parametrized by (2.1) is an Enneper cousin with Weierstrass data $h = 1$, $g = z$ (Cf. [Br], [UY2]).

Proof of Theorem 2.1. It is easy to verify that a unit normal vector field N of M is given by

$$N = -X + Y, \tag{2.10}$$

where X and Y are given by (2.1) and (2.6), respectively, and the first and second fundamental forms of M are given by

$$ds^2 = \langle dX, dX\rangle = \frac{\phi^2}{32}(du_1^2 + du_2^2),$$

$$-\langle dX, dN\rangle = \left(-\frac{1}{2} + \frac{\phi^2}{32}\right)du_1^2 + \left(\frac{1}{2} + \frac{\phi^2}{32}\right)du_2^2,$$

respectively, where ϕ is given by (2.7). Since the coordinate curves of M are lines of curvature, the vector fields $e_1 = 4\sqrt{2}X_{u_1}/\phi, e_2 = 4\sqrt{2}X_{u_2}/\phi$ are orthonormal principal directions and the 1-forms $\omega_1 = \frac{\phi}{4\sqrt{2}}du_1, \omega_2 = \frac{\phi}{4\sqrt{2}}du_2$ are dual to e_1, e_2. The connection form of M is given by

$$\omega_{12} = \frac{1}{a_1 a_2}\left(-\frac{\partial a_1}{\partial u_2}\omega_1 + \frac{\partial a_2}{\partial u_1}\omega_2\right) = \frac{2}{\phi}(-u_2 du_1 + u_1 du_2),$$

where $a_i = \sqrt{2}\phi/8, i = 1, 2$. Moreover, the principal curvatures of M are given by

$$-\lambda^1 = 1 - \frac{16}{\phi^2}, \qquad -\lambda^2 = 1 + \frac{16}{\phi^2}.$$

In order to obtain the cmc1 surfaces, locally associated to M by a Ribaucour transformation, it follows from Theorem 1.2 that we need to solve the following system of equations derived from (1.3):

$$\frac{\partial \Omega_1}{\partial u_2} = \Omega_2 \frac{\phi_{u_1}}{\phi}, \tag{2.11}$$

$$\frac{\partial \Omega_2}{\partial u_1} = \Omega_1 \frac{\phi_{u_2}}{\phi}, \tag{2.12}$$

$$\frac{\partial \Omega}{\partial u_i} = \frac{\sqrt{2}\phi\Omega_i}{8}, \quad i = 1, 2, \tag{2.13}$$

$$\frac{\partial W}{\partial u_i} = -\frac{\sqrt{2}\lambda^i\phi\Omega_i}{8}, \quad i = 1, 2, \tag{2.14}$$

$$\frac{\partial \Omega_i}{\partial u_i} = -\frac{\phi_{u_j}}{\phi}\Omega_j - \frac{\phi}{4\sqrt{2}}[(2c-1)\Omega - cW + (c\Omega - W)\lambda^i], \quad i \neq j, \tag{2.15}$$

with initial conditions satisfying

$$\Omega_1^2 + \Omega_2^2 + W^2 - 2c\Omega W + (2c-1)\Omega^2 = 0, \tag{2.16}$$

where $c \neq 0$ and $(c-1)^2 > 0$ i.e. $c \in \mathbb{R} \setminus \{0, 1\}$. Then such a solution defined on a simply connected domain will satisfy (2.16) identically.

We will now determine the solutions of this system of equations. Considering the derivative of (2.11) with respect to u_1 and using (2.12), we have

$$\frac{\partial^2 \Omega_1}{\partial u_2\, \partial u_1} + \left(\frac{\phi_{u_1}}{\phi} - \frac{1}{u_1}\right)\frac{\partial \Omega_1}{\partial u_2} - \frac{\phi_{u_1}\phi_{u_2}}{\phi^2}\Omega_1 = 0. \tag{2.17}$$

Equation (2.17) can be solved explicitly, since its Laplace invariant is zero. It follows from [T, p. 181] that

$$\Omega_1 = \frac{u_1}{\phi}\left(G(u_2) + \int \phi F(u_1)\, du_1\right), \tag{2.18}$$

where $F(u_1)$ (resp. $G(u_2)$) is an arbitrary function of u_1 (resp. u_2).

Similarly, we have

$$\Omega_2 = \frac{u_2}{\phi}\left(R(u_1) + \int \phi Q(u_2)\, du_2\right), \tag{2.19}$$

where $R(u_1)$ (resp. $Q(u_2)$) is an arbitrary function of u_1 (resp. u_2).

From (2.13), we get

$$\frac{\partial(\phi\Omega_1)}{\partial u_2} = \frac{\partial(\phi\Omega_2)}{\partial u_1}. \tag{2.20}$$

Substituting (2.18) and (2.19) into (2.20), we have

$$u_1\left(G'(u_2)+2u_2\int F(u_1)\,du_1\right)=u_2\left(R'(u_1)+2u_1\int Q(u_2)\,du_2\right).$$

Thus,

$$\frac{G'(u_2)}{u_2}-2\int Q(u_2)\,du_2=\frac{R'(u_1)}{u_1}-2\int F(u_1)\,du_1\equiv d,$$

where d is a constant. Without loss of generality, we can assume that $d=0$.

Therefore, we have

$$G(u_2)=2\int u_2\left(\int Q(u_2)\,du_2\right)du_2,$$

$$R(u_1)=2\int u_1\left(\int F(u_1)\,du_1\right)du_1.$$

Hence,

$$\int \phi Q(u_2)\,du_2=\frac{\phi G'(u_2)}{2u_2}-G(u_2),$$

$$\int \phi F(u_1)\,du_1=\frac{\phi R'(u_1)}{2u_1}-R(u_1).$$

It follows from (2.18) and (2.19) that

$$\phi\Omega_1=u_1\big(G(u_2)-R(u_1)\big)+\frac{\phi R'(u_1)}{2}, \tag{2.21}$$

$$\phi\Omega_2=u_2\big(R(u_1)-G(u_2)\big)+\frac{\phi G'(u_2)}{2}. \tag{2.22}$$

Since

$$\int \phi\Omega_1\,du_1=\left(\frac{u_1^2}{2}+m\right)G(u_2)+\frac{\phi}{2}R(u_1)-2\int u_1R(u_1)\,du_1,$$

we have

$$\frac{\partial}{\partial u_2}\int \phi\Omega_1\,du_1=\left(\frac{u_1^2}{2}+m\right)G'(u_2)+u_2R(u_1),$$

where m is a constant. Therefore, it follows from (2.22) that

$$\phi\Omega_2-\frac{\partial}{\partial u_2}\int \phi\Omega_1\,du_1=\left(1+\frac{u_2^2}{2}-m\right)G'(u_2)-u_2G(u_2).$$

Thus, we deduce from (2.13) that

$$\begin{aligned}\Omega&=\frac{\sqrt{2}}{8}\int \phi\Omega_1\,du_1+\frac{\sqrt{2}}{8}\int\left(\phi\Omega_2-\frac{\partial}{\partial u_2}\int \phi\Omega_1\,du_1\right)du_2\\&=\frac{\sqrt{2}}{16}\phi\big(R(u_1)+G(u_2)\big)-\frac{\sqrt{2}}{4}\left(\int u_1R(u_1)\,du_1+\int u_2G(u_2)\,du_2\right).\end{aligned} \tag{2.23}$$

Introducing the notation $A=-\lambda_1\Omega_1\phi, B=-\lambda_2\Omega_2\phi$, we have,

$$\int A\,du_1=\left(\frac{u_1^2}{2}+m\right)G(u_2)+\frac{8(G(u_2)-R(u_1))}{\phi}+\frac{\phi R(u_1)}{2}-2\int u_1R(u_1)\,du_1.$$

Therefore, we have

$$\frac{\partial}{\partial u_2}\int A\,du_1=\left(\frac{u_1^2}{2}+m\right)G'(u_2)+u_2R+\frac{8(\phi G'-2u_2(G-R))}{\phi^2},$$

and hence

$$B-\frac{\partial}{\partial u_2}\int A\,du_1=\left(1+\frac{u_2^2}{2}-m\right)G'(u_2)-u_2G(u_2).$$

We conclude from (2.14) that W is given by (2.5) and one can verify that the functions Ω_1,Ω_2,Ω and W given by (2.21), (2.22), (2.23), and (2.5) satisfy (2.11)–(2.14).

We will now determine R and G by using (2.16). From (2.15), we have

$$\frac{1}{\phi}\frac{\partial\Omega_1}{\partial u_1}+\frac{2u_2}{\phi^2}\Omega_2-\frac{\sqrt{2}}{8}(\lambda_1+c)W+\frac{\sqrt{2}}{8}(2c-1+c\lambda_1)\Omega=0,$$
$$\frac{1}{\phi}\frac{\partial\Omega_2}{\partial u_2}+\frac{2u_1}{\phi^2}\Omega_1-\frac{\sqrt{2}}{8}(\lambda_2+c)W+\frac{\sqrt{2}}{8}(2c-1+c\lambda_2)\Omega=0.$$

Taking the derivative of (2.21) (resp. (2.22)) with respect to u_1 (resp. u_2), we get

$$\phi\frac{\partial\Omega_1}{\partial u_1}+2u_1\Omega_1=\big(G(u_2)-R(u_1)\big)+\frac{\phi R''(u_1)}{2},$$
$$\phi\frac{\partial\Omega_2}{\partial u_2}+2u_2\Omega_2=\big(R(u_1)-G(u_2)\big)+\frac{\phi G''(u_2)}{2}.$$

It follows from the last four equalities that

$$\frac{R''+G''}{\phi}=\frac{\sqrt{2}}{2}(c-1)(W-\Omega). \tag{2.24}$$

Substituting (2.5) into (2.24), we find

$$R''+(c-1)R=-G''+(c-1)G=\nu, \tag{2.25}$$

where ν is a constant.

From the expressions of $\tilde{X}_c$, W, Ω_1 and Ω_2 (see (2.2) and (2.4)), without loss of generality, we can assume that $\nu=0$. Hence, from (2.25) we have that

$$R''+(c-1)R=0,\qquad G''-(c-1)G=0,$$

which implies that

$$\int u_1 R(u_1)\,du_1 + \int u_2 G(u_2)\,du_2 = \frac{1}{c-1}(R - u_1R' - G + u_2G') + m, \tag{2.26}$$

where m is a constant. Thus, one obtains from (2.23) and (2.26) that

$$\Omega = \frac{\sqrt{2}}{16}\phi(R+G) - \frac{\sqrt{2}}{4}\cdot\frac{1}{c-1}(R - u_1R' - G + u_2G') - \frac{\sqrt{2}}{4}m. \tag{2.27}$$

Substituting (2.4), and (2.27) into (2.16), we get

$$\frac{R'^2 + G'^2 + (1-c)(G^2 - R^2)}{4} - (1-c)m\frac{G-R}{\phi} = 0. \tag{2.28}$$

If $c > 1$ then the functions R and G are given by

$$R(u_1) = a_1\cos\big(\sqrt{c-1}u_1\big) + b_1\sin\big(\sqrt{c-1}u_1\big),$$
$$G(u_2) = a_2\cosh\big(\sqrt{c-1}u_2\big) + b_2\sinh\big(\sqrt{c-1}u_2\big),$$

where a_1, a_2, b_1 and b_2 are constants. Substituting these expressions into (2.28) and simplifying, we obtain the identity

$$\frac{(c-1)(a_1^2 + b_1^2 - a_2^2 + b_2^2)}{4} - \frac{(1-c)m(G-R)}{\phi} = 0,$$

which implies

$$a_1^2 + b_1^2 = a_2^2 - b_2^2 \quad \text{and} \quad m = 0. \tag{2.29}$$

Thus, we conclude from (2.27) that Ω is given by (2.3).

Observe that from (1.7) we get the metric of $\tilde{X}_c$ given by

$$d\tilde{s}_c^2 = \tilde{\omega}_1^2 + \tilde{\omega}_2^2 = \frac{\Omega^2}{(W-\Omega)^2}\cdot\frac{8}{\phi^2}(du_1^2 + du_2^2). \tag{2.30}$$

If $(a_1, b_1) = (0,0)$, then R and G reduce to

$$R(u_1) = 0, \quad G(u_2) = a_2\big(\cosh\big(\sqrt{c-1}u_2\big) \pm \sinh\big(\sqrt{c-1}u_2\big)\big).$$

From (2.3) and (2.5), we know that

$$\Omega = \frac{\sqrt{2}}{4}\left(\frac{\phi}{4}G - \frac{1}{c-1}(u_2G' - G)\right) = \frac{\sqrt{2}}{4}\left(\frac{\phi}{4}G - \frac{1}{c-1}\big(-G \pm u_2\sqrt{c-1}G\big)\right),$$

$$W - \Omega = \frac{\sqrt{2}G}{\phi},$$

which implies that

$$d\tilde{s}_c^2 = \frac{l_\mp}{32}(du_1^2 + du_2^2), \quad \text{where } l_\mp = u_1^2 + \left(u_2 \mp \frac{2}{\sqrt{c-1}}\right)^2 + 2.$$

It follows from (1.8) that the principal curvatures of $\tilde{X}_c$ are given by

$$-\tilde{\lambda}_c^1 = 1 - \frac{(W-\Omega)^2}{\Omega^2(\lambda^1+1)} = 1 - \frac{16}{l_\mp^2},$$
$$-\tilde{\lambda}_c^2 = 1 - \frac{(W-\Omega)^2}{\Omega^2(\lambda^2+1)} = 1 + \frac{16}{l_\mp^2},$$

which, combined with the fact that $\tilde{X}_c$ is parametrized by lines of curvature, implies that the second fundamental form of $\tilde{X}_c$ is given by

$$-\langle d\tilde{X}_c, d\tilde{N}_c\rangle = \left(-\frac{1}{2} + \frac{l_\mp^2}{32}\right) du_1^2 + \left(\frac{1}{2} + \frac{l_\mp^2}{32}\right) du_2^2.$$

Observe that $X(u_1,u_2)$ and $\tilde{X}_c(u_1, u_2 \pm \frac{2}{\sqrt{c-1}})$ have the same first and second fundamental forms, hence we conclude that X and $\tilde{X}_c$ are congruent.

By hypothesis we are excluding surfaces congruent to X. Therefore, we can suppose that $(a_1,b_1) \neq (0,0)$, and in this case we have

$$R(u_1) = \sqrt{a_1^2 + b_1^2}\sin\left(\sqrt{c-1}u_1 + A_1\right),$$
$$G(u_2) = \sqrt{a_2^2 - b_2^2}\delta\cosh\left(\sqrt{c-1}u_2 + A_2\right),$$

and $\delta = 1$ or -1 so that $\delta a_2 > 0$. Since (2.29) holds, it follows from (1.6) that we may consider R and G given by (2.8). Similarly, when $c<1$, we show that R and G are given by (2.9).

From (1.6) and (2.10), we conclude that $\tilde{X}_c$ is given by (2.2). From Theorem 1.2, $\tilde{X}_c$ is a regular surface on $\tilde{U} = \{(u_1,u_2) \in R^2 | [\Omega(W-\Omega)](u_1,u_2) \neq 0\}$. This completes the proof of Theorem 2.1. ☐

THEOREM 2.3. *For any $c \in \mathbb{R} \setminus \{1,0\}$, the cmc1 surfaces $\tilde{X}_c$ described in Theorem 2.1 are complete surfaces defined on $R^2 - \{p_k, k \in Z\}$, where*

$$p_k = \begin{cases} \dfrac{1}{\sqrt{c-1}}\left(2k\pi + \delta\frac{\pi}{2} - A_1, -A_2\right), & \text{when } c>1, \\ \dfrac{1}{\sqrt{1-c}}\left(-B_1, 2k\pi + \delta\frac{\pi}{2} - B_2\right), & \text{when } c<1. \end{cases} \tag{2.31}$$

Proof. It is clear from (2.2) and (2.5) that each surface $\tilde{X}_c$ is defined on $R^2 - \{p_k, k \in Z\}$, where p_k is given by (2.31). In order to prove that $\tilde{X}_c$ is complete, we need to show that every divergent curve on $\tilde{X}_c$ has infinite length. We will consider $c>1$, since the proof for the case $c<1$ is similar. A divergent curve is either of the form $\tilde{X}_c(u_1(t),u_2(t)), t \in [0,1)$, where $\lim_{t\to 1}(u_1(t),u_2(t)) = p_k$ for some $k \in Z$, or of the form $\tilde{X}_c(u_1(t),u_2(t)), t \in$

$[0, +\infty)$, where $\lim_{t\to+\infty}(u_1^2(t)+u_2^2(t)) = +\infty$. We may assume that $u_1'(t)^2 + u_2'(t)^2 = 1$.

CLAIM 1. *For any regular curve* $\gamma(t) = (u_1(t), u_2(t)), t \in [0, a)$, *such that* $\lim_{t\to a}(u_1(t), u_2(t)) = p_k$, *for some* $k \in Z$, *the curve* $\tilde{X}_c \circ \gamma$ *has infinite length.*

Proof. By translation, we can assume that $p_k = (0, 0) \equiv 0$. From the expressions of Ω and W, we have

$$(W - \Omega)(0) = 0, \quad \Omega(0) \neq 0,$$

and the derivatives at the origin are given by

$$(W-\Omega)_{u_i}(0) = (W-\Omega)_{u_iu_j}(0) = 0 \quad \text{for } i \neq j,$$
$$(W-\Omega)_{u_iu_i}(0) = \frac{\sqrt{2}\delta}{2}(c-1), \quad \text{for } i = 1, 2.$$

Therefore,

$$(W-\Omega)(u) = \frac{\delta(c-1)\sqrt{2}}{4}|u|^2 + \tilde{R},$$

where $\lim_{|u|\to 0}\frac{\tilde{R}}{|u|^2} = 0$.

Observe that from (2.30), we have the metric of $\tilde{X}_c$ given by $d\tilde{s}_c^2 = \psi^2(du_1^2 + du_2^2)$, where

$$\psi = \frac{2\sqrt{2}|\Omega|}{|(W-\Omega)|\phi}. \tag{2.32}$$

Consequently, we have $\lim_{|u|\to 0}|u|^2\psi = h > 0$.

Let $\gamma : [0, a) \to R^2 \setminus \{p_k, k \in Z\}$ be a curve, $\gamma(t) = (u_1(t), u_2(t))$, such that $\lim_{t\to a}\gamma(t) = p_k = 0$. Then, there exists $t_0 \in (0, a)$ such that $\psi(\gamma(t))|\gamma(t)|^2 \geq h/2$, $\forall t \in [t_0, a)$. Hence, the length of $\tilde{X}_c \circ \gamma$ is

$$\begin{aligned}
l(\tilde{X}_c \circ \gamma) &= \int_0^a \psi(\gamma(t))|\gamma'(t)|\,dt \\
&= \int_0^a \psi(\gamma(t))|\gamma(t)|^2\frac{|\gamma'(t)|}{|\gamma(t)|^2}\,dt \\
&\geq \frac{h}{2}\left|\int_{t_0}^a \frac{|\gamma'(t)|}{|\gamma(t)|^2}\,dt\right| \\
&= +\infty.
\end{aligned}$$

This completes the proof of Claim 1. □

For the remainder of the proof, we introduce the following notation:

$$F = R \times \left[-\frac{A_2}{\sqrt{c-1}} - 1, -\frac{A_2}{\sqrt{c-1}} + 1\right],$$
$$\alpha_k = \frac{1}{\sqrt{c-1}}\left(k\pi + \delta\cdot\frac{\pi}{2} - A_1\right),$$

$$D_k = \left[\alpha_k - \frac{\pi}{8\sqrt{c-1}}, \alpha_k + \frac{\pi}{8\sqrt{c-1}}\right] \times \left[-\frac{A_2}{\sqrt{c-1}} - 1, -\frac{A_2}{\sqrt{c-1}} + 1\right],$$
$$F_k = \left[\alpha_k + \frac{\pi}{4\sqrt{c-1}}, \alpha_{k+1} - \frac{\pi}{4\sqrt{c-1}}\right] \times R, \qquad k \in Z.$$

CLAIM 2. *There exist $r > 0$ and $L > 0$, such that if $u_1^2 + u_2^2 \geq L$ and $(u_1, u_2) \notin D = \bigcup_{k \in Z} D_k$, then $\psi(u_1, u_2) \geq r$.*

Proof. Since (2.3) and (2.5) hold, we get the following inequality, from (2.32),

$$\psi \geq \frac{\sqrt{2}}{2}\left[\frac{\phi}{4} \cdot \frac{|G+R|}{|G-R|} - \frac{1}{c-1}\left(1 + \left|\frac{u_1 R' + u_2 G'}{G-R}\right|\right)\right].$$

If $(u_1, u_2) \notin D$, then either $(u_1, u_2) \notin F$ or $(u_1, u_2) \in F \setminus D$. Observe that there exists a constant $a > 1$ such that if $(u_1, u_2) \notin F$, then $|G(u_2)| \geq a$. Thus, since $|R| \leq 1$, we conclude that if $(u_1, u_2) \notin F$, then

$$\begin{aligned}\psi &\geq \frac{\sqrt{2}}{2}\left[\frac{\phi}{4} \cdot \frac{|G| - |R|}{|G| + |R|} - \frac{1}{c-1} \cdot \left(1 + \frac{\sqrt{u_1^2 + u_2^2}\sqrt{R'^2 + G'^2}}{|G| - |R|}\right)\right] \\ &\geq \frac{\sqrt{2}}{2}\left[\frac{\phi}{4} \cdot \frac{|G| - 1}{|G| + 1} - \frac{1}{c-1}\left(1 + \sqrt{(c-1)\phi} \cdot \frac{|G| + 1}{|G| - 1}\right)\right].\end{aligned}$$

Since $|G| \geq a$, we have

$$\frac{|G| - 1}{|G| + 1} \geq \frac{a-1}{a+1}, \qquad \frac{|G| + 1}{|G| - 1} \leq \frac{a+1}{a-1}.$$

Therefore,

$$\psi \geq \frac{\sqrt{2}}{2}\left[\frac{\phi}{4} \cdot \frac{a-1}{a+1} - \frac{1}{c-1}\left(1 + \sqrt{(c-1)\phi} \cdot \frac{a+1}{a-1}\right)\right].$$

On the other hand, it is easy to verify that there exists a positive constant $b < 1$ such that if $(u_1, u_2) \in F \setminus D$, then $|R(u_1)| \leq b$. Therefore, we know from $|G| \geq 1$ that when $(u_1, u_2) \in F \setminus D$,

$$\begin{aligned}\psi &\geq \frac{\sqrt{2}}{2}\left[\frac{\phi}{4} \cdot \frac{|G| - b}{|G| + b} - \frac{1}{c-1}\left(1 + \sqrt{(c-1)\phi} \cdot \frac{|G| + 1}{|G| - b}\right)\right] \\ &\geq \frac{\sqrt{2}}{2}\left[\frac{\phi}{4} \cdot \frac{1-b}{1+b} - \frac{1}{c-1}\left(1 + \sqrt{(c-1)\phi} \cdot \frac{2}{1-b}\right)\right].\end{aligned}$$

We conclude that there exist $L > 0$ and $r > 0$, such that if $u_1^2 + u_2^2 \geq L$ and $(u_1, u_2) \notin D$, then $\psi \geq r$. This completes the proof of *Claim* 2. □

CLAIM 3. *Let $\gamma(t) = (u_1(t), u_2(t)), t \in [0, \infty)$ be a regular curve such that $u_1(t)^2 + u_2(t)^2 \to +\infty$, $t \to +\infty$. Then the curve $\tilde{X}_c \circ \gamma$ has infinite length.*

Proof. Consider $r>0$, $L>0$ as in *Claim* 2 and t_1 such that $\forall t \geq t_1, u_1^2(t)+u_2^2(t)\geq L$. If there exists $t_2\geq t_1$ such that $\forall t>t_2, \gamma(t)\notin D$, then the length of $\tilde{X}_c\circ\gamma$ is

$$l(\tilde{X}_c\circ\gamma)\geq \int_{t_2}^{+\infty} r\,dt=+\infty.$$

On the other hand, if $\forall t_2\geq t_1$, there exists $t\geq t_2$, such that $\gamma(t)\in D$, then γ intersects transversally an infinite number of subsets F_k. Therefore, it follows from *Claim* 2 that $\psi\geq r$ on F_k. Since the width of each F_k is $\pi/(2\sqrt{c-1})$, we conclude that $\tilde{X}_c\circ\gamma$ has infinite length, which completes the proof of *Claim* 3. □

The proof of Theorem 2.3 follows from *Claims* 1 and 3. □

THEOREM 2.4. *Any cmc*1 *surface $\tilde{X}_c$, locally associated to the Enneper cousin by a Ribaucour transformation as in Theorem* 2.1, *has the following properties*:

(a) *It is a complete surface corresponding to an immersion of a sphere punctured at an infinite number of points contained on a circle.*

(b) *It has infinite total curvature.*

(c) *It has one irregular end and infinitely many proper regular ends. Each regular end has finite total curvature and it is asymptotic to a horosphere.*

Proof. Any surface described by $\tilde{X}_c$ was shown to be complete and defined on $R^2\setminus\bigcup_{k\in Z}\{p_k\}$ in Theorem 2.3. Hence, any such surface corresponds to a conformal immersion of a sphere punctured at a pole and at an infinite number of points corresponding to p_k, which are contained on a circle. Therefore, the surface has infinite total curvature (cf. [H]).

A unit normal vector field of $\tilde{X}_c$ in H^3 is given by (see [TW])

$$\tilde{N}_c=N+\frac{2W}{S}\left(\sum_{i=1}^{2}\Omega_i e_i - WN-\Omega X\right),$$

where $W,S,\Omega_1,\Omega_2,\Omega$ and N are given in Theorem 2.1. Since $S=2c\Omega(W-\Omega)$, we conclude that

$$\tilde{X}_c+\tilde{N}_c=\left(1-\frac{1}{c}\right)X+\frac{1}{c\Omega}\left(-WN+\sum_{i=1}^{2}\Omega_i e_i\right). \tag{2.33}$$

Observe that the functions $w,\Omega,\Omega_i, i=1,2$ can be defined on $p_k,\forall k$, $\Omega(p_k)\neq 0$ and $\Omega_i(p_k)=0$. Therefore, $\tilde{X}_c+\tilde{N}_c$ can be extended to be a differential map from R^2 to $H^3\subset L^4$. Moreover, since

$$\langle \tilde{X}_c+\tilde{N}_c, \tilde{X}_c+\tilde{N}_c\rangle = 0 \tag{2.34}$$

holds on $R^2 \setminus \bigcup_{k\in Z}\{p_k\}$, it follows by continuity that (2.34) holds on R^2. Denote by $X_c^j + N_c^j$, $1 \le j \le 4$, the jth coordinate of $\tilde{X}_c + \tilde{N}_c$ and denote by $\tilde{n}_c$ the hyperbolic Gauss map of $\tilde{X}_c$, which is defined as

$$\tilde{n}_c = \frac{1}{\tilde{X}_c^1 + \tilde{N}_c^1}(\tilde{X}_c + \tilde{N}_c).$$

Since X, N, e_1, e_2 are linearly independent, it follows from (2.33), that $\tilde{X}_c^1 + \tilde{N}_c^1$ does not vanish and hence $\tilde{n}_c$ can be extended to R^2. Consequently, for any $k \in \mathbf{Z}$, p_k corresponds to a regular end.

Observe that the Gaussian curvature of $\tilde{X}_c$ is given by

$$\tilde{K} = \frac{(W-\Omega)^4\phi^4}{16^2\Omega^4}$$

and the metric by $ds_c^2 = \psi^2(du_1^2 + du_2^2)$, where

$$\psi = \frac{2\sqrt{2}\Omega}{(W-\Omega)\phi}.$$

Therefore, it follows that each end p_k has finite total curvature.

We will show that p_k is a horosphere type end. We consider the family of cmc1 surfaces $\tilde{X}_c$ given by (2.2) in the upper half space model, i.e.,

$$Z = (Z^1, Z^2, Z^3) = \left(\frac{\tilde{X}_c^2}{\tilde{X}_c^1 - \tilde{X}_c^4}, \frac{\tilde{X}_c^3}{\tilde{X}_c^1 - \tilde{X}_c^4}, \frac{1}{\tilde{X}_c^1 - \tilde{X}_c^4}\right),$$

where $\tilde{X}_c^j$ is the jth coordinate function of $\tilde{X}_c$. Observe that for $c > 1$ or $c < 1$ and $c \neq 0$, the corresponding functions $R(u_1)$ and $G(u_2)$, given by (2.8) and (2.9), satisfy the following properties: $R(p_k) = G(p_k) = \delta$ and $R'(p_k) = G'(p_k) = 0$. Moreover,

$$(W-\Omega)(p_k) = (W-\Omega)_{u_i}(p_k) = 0, \quad i = 1, 2.$$

Therefore, a straightforward computation of the derivatives at p_k shows that

$$Z^1_{u_1}(p_k) = ca, \qquad Z^1_{u_2}(p_k) = cb, \qquad Z^2_{u_1}(p_k) = -cb, \qquad Z^2_{u_2}(p_k) = ca,$$

where a and b are the constants determined by evaluating the following functions at p_k

$$a = \frac{1+\cos(u_2)\sinh(u_1)}{(\cosh(u_1)+\cos(u_2))^2}(p_k), \qquad b = \frac{\sin(u_2)\sinh(u_1)}{(\cosh(u_1)+\cos(u_2))^2}(p_k),$$

$$Z^3_{u_1}(p_k) = Z^3_{u_2}(p_k) = Z^3_{u_1u_2}(p_k) = 0,$$

$$Z^3_{u_1u_1}(p_k) = Z^3_{u_2u_2}(p_k) = c(c-1)\ell, \quad \text{where } \ell = \frac{2\sqrt{2}}{\phi(\cosh(u_1)+\cos(u_2))}(p_k).$$

By considering the Taylor expansion of $Z(u_1, u_2)$ around the point $p_k = (u_1^k, u_2^k)$, and changing variables

$$x = \frac{a(u_1 - u_1^k) + b(u_2 - u_2^k)}{a^2 + b^2}, \qquad y = \frac{-b(u_1 - u_1^k) + a(u_2 - u_2^k)}{a^2 + b^2},$$

we conclude that

$$Z(x, y) = Z(p_k) + \Big(cx\big(1 + \mathcal{O}(|z|)\big), cy\big(1 + \mathcal{O}(|z|)\big), \frac{\ell}{2}c(c-1)(x^2 + y^2)\big(1 + \mathcal{O}(|z|)\big)\Big)$$

where $z = (x, y)$ and $\mathcal{O}(|z|)$ denotes any real valued function f such that $\limsup_{z \to 0} |f|/|z|$ is finite. Therefore, p_k is an embedded horosphere type end (see [LR]).

We will now prove that each surface of the family $\tilde{X}_c$ has an irregular end, by showing that $\lim_{|u| \to \infty} \tilde{n}_c(u_1, u_2)$ does not exist. For $c > 1$, we consider the third coordinate of $\tilde{n}_c$ on the line $(0, u_2)$, i.e.,

$$\tilde{n}_c^3(0, u_2) = \frac{\tilde{X}_c^3 + \tilde{N}_c^3}{\tilde{X}_c^1 + \tilde{N}_c^1}(0, u_2) = \frac{(c-1)\Omega X^3 - W N^3 + \Omega_2 e_2^3}{(c-1)\Omega X^1 - W N^1 + \Omega_2 e_2^1}(0, u_2).$$

It follows from a straightforward computation that

$$\tilde{n}_c^3(0, u_2) = \frac{-\sin(u_2)(1 + \frac{R}{G}(0, u_2)) + \sum_{j=1}^4 B_j(u_2) \frac{1}{(2+u_2^2)^j}}{(1 + \frac{R}{G}(0, u_2)) + \sum_{j=1}^4 A_j(u_2) \frac{1}{(2+u_2^2)^j}},$$

where R, G are given by (2.8) and A_j, B_j are functions of u_2, satisfying $A_j/(2 + u_2^2)^j \to 0$ and $B_j/(2 + u_2^2)^j \to 0$ when $|u_2| \to \infty$. Thus, taking the limit when $u_2 \to \infty$, we have

$$\lim_{|u_2| \to \infty} \tilde{n}_c^3(0, u_2) = -\lim_{|u_2| \to \infty} \sin(u_2).$$

Therefore, the surface has an irregular end when $|u| \to \infty$. The proof for $c < 1$ is similar. □

Our next result provides the cmc1 surfaces in H^3 associated to a catenoid cousin.

THEOREM 2.5. *Consider the catenoid cousin M^2 in H^3 parametrized by*

$$X(u_1, u_2) = \left(\frac{1}{2}(|u|^2 + 2)\phi - u_1\phi', -\frac{\phi'}{2} + u_1\phi, u_2\phi, \frac{1}{2}(|u|^2\phi - u_1\phi')\right), \tag{2.35}$$

where $\phi = cosh(2u_1)$ and $u = (u_1, u_2) \in R^2$. A parametrized surface $\tilde{X}_c$ of constant mean curvature 1 in H^3, not congruent to M, is locally associated *to X by a Ribaucour transformation as in Theorem 1.2 if, and only if,*

$$\tilde{X}_c = \left(1 - \frac{1}{c}\right)X + \frac{1}{c(W-\Omega)}\left(WY - \sum_{i=1}^{2}\Omega_i e_i\right), \tag{2.36}$$

where $c \in \mathbb{R}\setminus\{0,1\}, e_i = X_{u_i}/\phi, i = 1,2, \phi = \cosh(2u_1)$, and Ω, Ω_i, W are given, in terms of two real functions $f(u_1)$ and $g(u_2)$, by

$$\Omega = \phi(f+g) - \frac{1}{2(c-1)}(\phi' f' - 4\phi f), \tag{2.37}$$

$$\Omega_1 = \frac{\phi'}{\phi}(g-f) + f', \qquad \Omega_2 = g', \tag{2.38}$$

$$W = \Omega - \frac{2}{\phi}(g-f), \qquad Y = \frac{1}{\phi}(|u|^2+1, 2u_1, 2u_2, |u|^2-1), \tag{2.39}$$

$$f = \delta\cosh\bigl(A_1 + 2\sqrt{c}u_1\bigr), \qquad g = \sin\bigl(B_1 + 2\sqrt{c}u_2\bigr), \qquad \textit{if } c>0, \tag{2.40}$$

$$f = \sin\bigl(A_2 + 2\sqrt{-c}u_1\bigr), \qquad g = \delta\cosh\bigl(B_2 + 2\sqrt{-c}u_2\bigr), \qquad \textit{if } c<0, \tag{2.41}$$

$\delta = \pm 1, A_1, B_1, A_2, B_2 \in R$. Moreover, $\tilde{X}_c$ is a regular surface defined on

$$\tilde{U} = \{(u_1,u_2) \in R^2; (\Omega(\Omega - W))(u_1,u_2) \neq 0\}$$

Remark 2.6. The surface $M^2 \subset H^3 \subset L^4$ described by (2.35) is a catenoid cousin (see [MRR]), parametrized by

$$\left(u_1 - \frac{\sinh 2u_1}{\cosh 2u_1}, u_2, \frac{1}{\cosh 2u_1}\right),$$

in the upper half space model of H^3. The associated family of surfaces $\tilde{X}_c$ described by (2.36), in the upper half space model, is given by $Z_c = (Z_c^1, Z_c^2, Z_c^3)$, where

$$Z_c^1 = u_1 - \frac{c(g-f)\phi' + f'\phi}{E}, \qquad Z_c^2 = u_2 + \frac{g'\phi}{E}, \qquad Z_c^3 = \frac{2c(g-f)}{E},$$

f and g are given by (2.40), (2.41) and

$$E = \frac{c}{c-1}[2(c-1)(g-f)\phi - 4f\phi + f'\phi'].$$

Proof of Theorem 2.5. Consider the unit normal vector field N of M given by

$$N(u_1,u_2) = Y - X, \tag{2.42}$$

where X and Y are given by (2.35) and (2.39), respectively.

The first and second fundamental forms of M are given by

$$ds^2 = \langle dX, dX\rangle = \phi^2(du_1^2 + du_2^2),$$
$$-\langle dX, dN\rangle = (2+\phi^2)\,du_1^2 + (-2+\phi^2)\,du_2^2,$$

respectively. Since the coordinate curves of M are lines of curvature, the vector fields $e_1 = X_{u_1}/\phi, e_2 = X_{u_2}/\phi$ are orthonormal principal directions and the 1-forms $\omega_1 = \phi\, du_1, \omega_2 = \phi\, du_2$ are dual to e_1, e_2. The connection form of M is $\omega_{12} = \frac{\phi'}{\phi}\, du_2$ and the principal curvatures of M are given by $-\lambda^1 = 1 + \frac{2}{\phi^2}$ and $-\lambda^2 = 1 - \frac{2}{\phi^2}$.

In order to obtain the cmc1 surfaces locally associated to M by a Ribaucour transformation, it follows from Theorem 1.2 that we need to solve the following system of equations obtained from (1.3):

$$\frac{\partial \Omega_1}{\partial u_2} = \Omega_2 \frac{\phi'}{\phi}, \tag{2.43}$$

$$\frac{\partial \Omega_2}{\partial u_1} = 0, \tag{2.44}$$

$$\frac{\partial \Omega}{\partial u_i} = \phi \Omega_i, \quad i = 1, 2, \tag{2.45}$$

$$\frac{\partial W}{\partial u_i} = -\lambda_i \phi \Omega_i, \quad i = 1, 2, \tag{2.46}$$

$$\frac{\partial \Omega_i}{\partial u_i} = -\frac{\Omega_j \delta_{i2} \phi'}{\phi} - [(2c-1)\Omega - cW + (c\Omega - W)\lambda_i]\phi, \quad i \neq j, \tag{2.47}$$

with initial conditions satisfying

$$\Omega_1^2 + \Omega_2^2 + W^2 - 2c\Omega W + (2c-1)\Omega^2 = 0, \tag{2.48}$$

where $c \in \mathbb{R} \setminus \{0, 1\}$. Then such a solution defined on a simply connected domain will satisfy (2.48) identically.

We will now determine the solutions of this system of equations. From (2.44), we can consider $\Omega_2 = g'(u_2)$, where $g(u_2)$ is a function of u_2, and from (2.43), we can consider

$$\Omega_1 = \frac{\phi'}{\phi}(g - f) + f',$$

where $f = f(u_1)$ is a differentiable function of u_1. Hence, (2.38) holds. It follows from (2.45) that

$$\Omega = \phi(f + g) - 2\int \phi' f(u_1)\, du_1 \tag{2.49}$$

and from (2.46) we have

$$\frac{\partial W}{\partial u_1} = \left(1 + \frac{2}{\phi^2}\right)\big(\phi'(g - f) + \phi f'\big),$$
$$\frac{\partial W}{\partial u_2} = \left(\phi - \frac{2}{\phi}\right) g'.$$

Thus,

$$W = \phi(f + g) - \frac{2}{\phi}(g - f) - 2\int \phi' f\, du_1 = \Omega - \frac{2}{\phi}(g - f),$$

i.e., W is given by (2.39). It follows from (2.47) that

$$\frac{1}{\phi}\frac{\partial\Omega_1}{\partial u_1}-(\lambda^1+c)W+(2c-1+c\lambda^1)\Omega=0,$$
$$\frac{1}{\phi}\frac{\partial\Omega_2}{\partial u_2}+\frac{\phi'}{\phi^2}\Omega_1-(\lambda^2+c)W+(2c-1+c\lambda^2)\Omega=0.$$

Taking the derivative of Ω_i with respect to $u_i, i=1,2$, we have from (2.38) that

$$\phi\frac{\partial\Omega_1}{\partial u_1}+\phi'\Omega_1=\phi''(g-f)+\phi f'', \qquad \frac{\partial\Omega_2}{\partial u_2}=g''.$$

It follows from the last four equalities that

$$\frac{\phi''}{\phi^2}(g-f)+\frac{g''+f''}{\phi}=2(c-1)(W-\Omega).$$

Since $\phi''=4\phi$, using (2.39) we have

$$\frac{4}{\phi}(g-f)+\frac{g''+f''}{\phi}=-\frac{4(c-1)}{\phi}(g-f),$$

which implies that

$$-f''+4cf=g''+4cg=l, \tag{2.50}$$

where l is a constant.

From the expressions of $\tilde{X}_c$, W, Ω_1 and Ω_2 (see (2.36) and (2.38), (2.39)), without loss of generality, we can assume that $l=0$. Since $f=f''/4c$, we have

$$\int\phi' f\,du_1=\frac{1}{4c}\int\phi' f''\,du_1=\frac{1}{4c}\left(\phi' f'-4\int\phi f'\,du_1\right)$$
$$=\frac{\phi' f'}{4c}-\frac{1}{c}\left(\phi f-\int\phi' f\,du_1\right).$$

Thus,

$$\int\phi' f\,du_1=\frac{1}{4(c-1)}(\phi' f'-4\phi f)+\tilde{l}, \tag{2.51}$$

where $\tilde{l}$ is a constant. Therefore, we get from (2.49) and (2.51) that

$$\Omega=\phi(f+g)-\frac{1}{2(c-1)}(\phi' f'-4\phi f)+m, \tag{2.52}$$

where $m=-2\tilde{l}$. Introducing (2.38), (2.39) and (2.52) into the equality (2.48) and simplifying, we get

$$0=f'^2+g'^2+4c(g^2-f^2)+\frac{4(c-1)m}{\phi}(g-f). \tag{2.53}$$

If $c>0$, then

$$f(u_1)=a_1\cosh\bigl(2\sqrt{c}u_1\bigr)+b_1\sinh\bigl(2\sqrt{c}u_1\bigr),\qquad(2.54)$$
$$g(u_2)=a_2\cos\bigl(2\sqrt{c}u_2\bigr)+b_2\sin\bigl(2\sqrt{c}u_2\bigr),$$

where a_1,a_2,b_1,b_2 are constants. Combining (2.53) and (2.54), we get the identity

$$4c(-a_1^2+b_1^2+a_2^2+b_2^2)+\frac{4(c-1)m}{\phi}(g-f)=0.$$

Therefore, we conclude that

$$a_1^2-b_1^2=a_2^2+b_2^2 \quad\text{and}\quad m=0,\qquad(2.55)$$

and hence, using (2.52) we get Ω given by (2.37).

It follows from (1.7), that the metric of $\tilde{X}_c$ is given by

$$d\tilde{s}_c^2=\frac{4\Omega^2}{\phi^2(W-\Omega)^2}(du_1^2+du_2^2).\qquad(2.56)$$

If $(a_2,b_2)=(0,0)$, then f and g are given by

$$f(u_1)=a_1\bigl(\cosh\bigl(2\sqrt{c}u_1\bigr)\pm\sinh\bigl(2\sqrt{c}u_1\bigr)\bigr),\qquad g(u_2)=0.$$

From (2.37) and (2.39), we know that

$$\Omega=\frac{c+1}{c-1}\phi f-\frac{\phi' f'}{2(c-1)},$$
$$W-\Omega=\frac{2f}{\phi}.$$

Therefore, from (2.56), we get

$$\begin{aligned}d\tilde{s}_c^2&=\left(\frac{c+1}{c-1}\cosh(2u_1)\mp\frac{2\sqrt{c}}{c-1}\sinh(2u_1)\right)^2(du_1^2+du_2^2)\\&=\cosh^2\bigl(2(u_1\mp\alpha)\bigr)(du_1^2+du_2^2),\end{aligned}$$

where $\alpha=\frac{1}{2}\cosh^{-1}(\frac{c+1}{c-1})$. The principal curvatures of $\tilde{X}_c$ are

$$-\tilde{\lambda}_c^1=1-\frac{(W-\Omega)^2}{\Omega^2(\lambda^1+1)}=1+\frac{2}{\cosh^2(2(u_1\mp\alpha))},$$
$$-\tilde{\lambda}_c^2=1-\frac{2}{\cosh^2(2(u_1\mp\alpha))}.$$

Thus, the second fundamental form of $\tilde{X}_c$ is

$$-\langle d\tilde{X}_c,d\tilde{N}_c\rangle=\bigl(\cosh^2\bigl(2(u_1\mp\alpha)\bigr)+2\bigr)\,du_1^2+\bigl(\cosh^2\bigl(2(u_1\mp\alpha)\bigr)-2\bigr)\,du_2^2.$$

Observe that $X(u_1,u_2)$ and $\tilde{X}_c(u_1\pm\alpha,u_2)$ have the same first and second fundamental forms, hence we conclude that X and $\tilde{X}_c$ are congruent.

By hypothesis we are excluding surfaces congruent to X. Therefore, we can assume that $(a_2,b_2)\neq(0,0)$ and in this case, we have

$$f(u_1)=\sqrt{a_1^2-b_1^2}\,\delta\cosh\big(2\sqrt{c}u_1+A_1\big),$$
$$g(u_2)=\sqrt{a_2^2+b_2^2}\sin\big(2\sqrt{c}u_2+A_2\big),$$

and $\delta=1$ or -1 so that $\delta a_1>0$. Since (2.55) holds, one can assume from the expression of $\tilde{X}_c$ that f and g are given by (2.40). Similarly, when $c<0$ we can prove that f and g are given by (2.41).

From (1.6) and (2.42), we know that $\tilde{X}_c$ is given by (2.36) and from Theorem 1.2 it is well defined where $\Omega(W-\Omega)\neq 0$. This completes the proof of Theorem 2.5. □

THEOREM 2.7. *For any $c\in\mathbb{R}\setminus\{0,1\}$, the cmc1 surfaces $\tilde{X}_c$ described in Theorem 2.5 are complete surfaces defined on $R^2\setminus\{p_k,k\in Z\}$, where*

$$p_k=\begin{cases}\frac{1}{2\sqrt{c}}(-A_1,\delta\pi/2-B_1+2k\pi), & \text{if } c\geq 0,\\ \frac{1}{2\sqrt{-c}}(\delta\pi/2-A_2+2k\pi,-B_2), & \text{if } c\leq 0.\end{cases} \tag{2.57}$$

Proof. It is clear from (2.36) and (2.39) that each surface $\tilde{X}_c$ is defined on $R^2\setminus\{p_k,k\in Z\}$ where p_k is given by (2.57). In order to prove that $\tilde{X}_c$ is complete, we need to show that every divergent curve on $\tilde{X}_c$ has infinite length. We will consider $c>0$, since the case $c<0$ is similar. A divergent curve is either of the form $\tilde{X}_c(u_1(t),u_2(t)),t\in[0,a)$, where $\lim_{t\to a}(u_1(t),u_2(t))=p_k$ for some $k\in Z$, or of the form $\tilde{X}_c(u_1(t),u_2(t)),t\in[0,+\infty)$, $\lim_{t\to+\infty}(u_1^2(t)+u_2^2(t))=+\infty$. We may assume that $(u_1')^2+(u_2')^2=1$.

CLAIM 1. *For any regular curve $\gamma(t)=(u_1(t),u_2(t)),t\in[0,a)$, such that $\lim_{t\to a}(u_1(t),u_2(t))=p_k$, for some $k\in Z$, the curve $\tilde{X}_c\circ\gamma$ has infinite length.*

Proof. We may assume that $p_k=(0,0)\equiv 0$. From the expressions of W and Ω, we have

$$(W-\Omega)(0)=0,\qquad \Omega(0)\neq 0,$$

and the derivatives are given by

$$(W-\Omega)_{u_i}(0)=(W-\Omega)_{u_iu_j}(0)=0,\quad i\neq j,$$
$$(W-\Omega)_{u_iu_i}(0)=8\delta c,\quad i=1,2.$$

Therefore,

$$(W-\Omega)(u)=4\delta c|u|^2+\tilde{R},$$

where $\lim_{|u|\to 0}\tilde{R}/|u|^2=0$.

Observe that from (2.56) we have the metric on $\tilde{X}_c$ given by $d\tilde{s}_c^2=\psi^2(du_1^2+du_2^2)$, where

$$\psi=2\Omega/[(W-\Omega)\phi].$$

Consequently, we get that $\lim_{|u|\to 0}|u|^2\psi = h > 0$. Now, let $\gamma : [0,a) \to R\setminus\{p_k, k\in Z\}$ be a curve, $\gamma(t) = (u_1(t), u_2(t))$, such that $\lim_{t\to a}\gamma(t) = 0$. Then, there exists $t_0 \in (0,a)$ such that $\psi(\gamma(t))|\gamma(t)|^2 \geq h/2$, $\forall t \in [t_0, a)$. Hence, the length of $\tilde{X}_c\circ\gamma$ is

$$\begin{aligned} l(\tilde{X}_c\circ\gamma) &= \int_0^a \psi(\gamma(t))|\gamma'(t)|\,dt \\ &= \int_0^a \psi(\gamma(t))|\gamma(t)|^2\frac{|\gamma'(t)|}{|\gamma(t)|^2}\,dt \\ &\geq \frac{h}{2}\left|\int_{t_0}^1 \frac{|\gamma'(t)|}{|\gamma(t)|^2}\,dt\right| \\ &= +\infty. \end{aligned}$$

This completes the proof of *Claim* 1. □

For the remainder of the proof, we introduce the following notation:

$$\begin{aligned} \beta_k &= \frac{1}{2\sqrt{c}}\left(k\pi + \delta\cdot\frac{\pi}{2} - B_1\right), \\ r &= \cosh^{-1}\left(2\frac{\sqrt{c}+1}{|\sqrt{c}-1|}\right), \\ s &= \sin^{-1}\left(\frac{|\sqrt{c}-1|}{2(\sqrt{c}+1)}\right), \\ D_k &= \left[-\frac{A_1+r}{2\sqrt{c}}, -\frac{A_1-r}{2\sqrt{c}}\right]\times\left[\beta_k - \frac{1}{2\sqrt{c}}\left(\frac{\pi}{2}-s\right), \beta_k + \frac{1}{2\sqrt{c}}\left(\frac{\pi}{2}-s\right)\right], \\ G &= \left[-\frac{A_1+r}{2\sqrt{c}}, -\frac{A_1-r}{2\sqrt{c}}\right]\times R, \\ F_k &= R\times\left[\beta_k + \frac{2}{3\sqrt{c}}\left(\frac{\pi}{2}-s\right), \beta_{k+1} - \frac{2}{3\sqrt{c}}\left(\frac{\pi}{2}-s\right)\right], \quad k\in Z. \end{aligned}$$

CLAIM 2. *There exist $r > 0$ and $L > 0$, such that, if $u_1^2 + u_2^2 \geq L$ and $(u_1,u_2)\notin D = \bigcup_{k\in Z} D_k$, then $\psi(u_1,u_2)\geq r$.*

Proof. Since (2.37) and (2.39) hold, we get

$$\begin{aligned} \psi &= \frac{|\phi(f+g) - \frac{1}{2(c-1)}(\phi' f' - 4\phi f)|}{|f-g|} \\ &\geq \frac{|2(c+1)\phi f + 2(c-1)\phi g - \phi' f'|}{2|c-1|(|f|+1)} \\ &\geq \frac{2(c+1)\phi|f| - 4\sqrt{c}\phi|f| - 2|c-1|\phi|g|}{2(c-1)(|f|+1)} \\ &= \frac{\phi(|\sqrt{c}-1||f| - (\sqrt{c}+1)|g|)}{(\sqrt{c}+1)(|f|+1)}. \end{aligned}$$

If $(u_1,u_2)\notin D$, then either $(u_1,u_2)\notin G$, or $(u_1,u_2)\in G\setminus D$. If $(u_1,u_2)\notin G$, we have $|f(u_1)|\geq \frac{2(\sqrt{c}+1)}{|\sqrt{c}-1|}$, thus

$$\begin{aligned}\psi(u_1,u_2) &\geq \phi(u_1,u_2)\cdot\frac{\frac{|\sqrt{c}-1|}{\sqrt{c}+1}|f(u_1)|-1}{|f(u_1)|+1}\\ &\geq \frac{\frac{|\sqrt{c}-1|}{\sqrt{c}+1}|f(u_1)|-1}{|f(u_1)|+1}\\ &\geq \frac{\frac{|\sqrt{c}-1|}{\sqrt{c}+1}\cdot\frac{2(\sqrt{c}+1)}{|\sqrt{c}-1|}-1}{2\cdot\frac{\sqrt{c}+1}{\sqrt{c}-1}+1}\\ &= \frac{|\sqrt{c}-1|}{2\sqrt{c}+3}.\end{aligned}$$

If $(u_1,u_2)\in G\setminus D$, then there exists $k\in Z$, such that $\beta_k+\frac{1}{2\sqrt{c}}(\frac{\pi}{2}-s)\leq u_2\leq \beta_{k+1}-\frac{1}{2\sqrt{c}}(\frac{\pi}{2}-s)$. Therefore,

$$|g(u_2)|\leq\frac{|\sqrt{c}-1|}{2(\sqrt{c}+1)},$$

and hence

$$\begin{aligned}\psi(u_1,u_2) &\geq \frac{\phi(|\sqrt{c}-1||f(u_1)|-\frac{1}{2}|\sqrt{c}-1|)}{(\sqrt{c}+1)(|f|+1)}\\ &\geq \frac{|\sqrt{c}-1|}{\sqrt{c}+1}\left(\frac{|f(u_1)|-\frac{1}{2}}{|f(u_1)|+1}\right)\\ &\geq \frac{|\sqrt{c}-1|}{4(\sqrt{c}+1)}.\end{aligned}$$

Consequently, there exist $r>0$ and $L>0$, such that if $u_1^2+u_2^2\geq L$ and $(u_1,u_2)\notin D=\bigcup_{k\in Z}D_k$, then $\psi(u_1,u_2)\geq r$. This concludes the proof of *Claim* 2. □

Claim 3. *Let $\gamma(t)=(u_1(t),u_2(t)), t\in[0,\infty)$ be a regular curve such that $u_1(t)^2+u_2(t)^2\to+\infty$, $t\to+\infty$. Then the curve $\tilde{X}_c\circ\gamma$ has infinite length.*

Proof. Consider $r>0$, $L>0$ as in *Claim* 2 and t_1 such that $\forall t\geq t_1, u_1^2(t)+u_2^2(t)\geq L$. If there exists $t_2\geq t_1$ such that $\forall t>t_2,\gamma(t)\notin D$, then the length of $\tilde{X}_c\circ\gamma$ is

$$l(\tilde{X}_c\circ\gamma)\geq\int_{t_2}^{+\infty} r\,dt=+\infty.$$

On the other hand, if $\forall t_2\geq t_1$, there exists $t\geq t_2$ such that $\gamma(t)\in D$, then γ intersects transversely infinitely many of the subsets F_k. Therefore, it follows from *Claim* 2 that $\psi\geq r$ on F_k. Since the width of each F_k is $(\pi+4s)/[3\sqrt{c}]$, we know that $\tilde{X}_c \circ \gamma$ has infinite length, which completes the proof of *Claim* 3. □

The completeness of $\tilde{X}_c$ follows from *Claims* 1 and 3. □

THEOREM 2.8. *Any cmc*1 *surface* $\tilde{X}_c$, *locally associated to a catenoid cousin by a Ribaucour transformation as in Theorem* 2.5, *has the following properties*:

(a) *It is a complete surface corresponding to an immersion of a sphere punctured at an infinite number of points contained on a circle.*

(b) *It has infinite total curvature.*

(c) *It has infinitely many regular ends. One end has infinite total curvature. The other ends have finite total curvature and are asymptotic to horospheres.*

Proof. Any surface described by $\tilde{X}_c$ was shown to be complete and defined on $R^2 \setminus \bigcup_{k\in Z}\{p_k\}$ in Theorem 2.7. Hence, any such surface corresponds to a conformal immersion of a sphere punctured at a pole and at an infinite number of points corresponding to p_k, which are contained on a circle. Therefore, the surface has infinite total curvature (cf. [H]).

The proof of the fact that each p_k provides a regular end with finite total curvature is similar to the proof in Theorem 2.4 and it will be omitted.

We will show that each $p_k = (u_1^k, u_2^k)$ corresponds to a horosphere type end. We consider the family of cmc1 surfaces $\tilde{X}_c$ given by (2.36) in the upper half space model (see Remark 2.6) Observe that for $c \neq 0$ the corresponding functions $f(u_1)$ and $g(u_2)$, given by (2.40) and (2.41), satisfy the following properties: $f(p_k) = g(p_k) = \delta$ and $f'(p_k) = g'(p_k) = 0$. Moreover, a straightforward computation of the derivatives shows that

$$\begin{aligned} &Z^1_{u_1}(p_k) = c, \qquad Z^1_{u_2}(p_k) = 0, \qquad Z^2_{u_1}(p_k) = 0, \qquad Z^2_{u_2}(p_k) = c,\\ &Z^3_{u_1}(p_k) = Z^3_{u_2}(p_k) = Z^3_{u_1u_2}(p_k) = 0,\\ &Z^3_{u_1u_1}(p_k) = Z^3_{u_2u_2}(p_k) = \frac{2c(c-1)}{\cosh(2u_1^k)}. \end{aligned}$$

By considering the Taylor expansion of $Z(u_1, u_2)$ around the point $p_k = (u_1^k, u_2^k)$, we conclude that p_k corresponds to an embedded horosphere type end (see [LR]).

We now observe that the surface has a regular end with infinite total curvature when $|u|$ tends to infinity. In fact, a straightforward computations shows that, for any $c \neq 0$, the limit of the hyperbolic Gauss map of $\tilde{X}_c$ exists, since

$$\lim_{|u|\to\infty} \tilde{n}_c = \lim_{|u|\to\infty} \frac{1}{\tilde{X}^1_c + \tilde{N}^1_c}(\tilde{X}_c + \tilde{N}_c) = (1,0,0,1).$$

Moreover, it is not difficult prove that the total curvature of this end is infinite. □

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