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THE BEHAVIOUR IN SHORT INTERVALS OF EXPONENTIAL SUMS OVER SIFTED INTEGERS

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ABSTRACT. We consider the Hardy–Littlewood approach to the Twin prime problem, which uses a certain exponential sum over prime numbers. We propose a conjecture on the behaviour of the exponential sum in short intervals of the argument. We first show that this conjecture implies the Twin prime conjecture. We then prove that an analogous conjecture is true for exponential sums over integers without small prime factors.

1. Introduction

Some of the famous unsolved problems in Analytic Number Theory are the Goldbach Problem and the Twin prime problem. The ternary Goldbach problem (see [3]), namely the representation of an odd integer N as a sum of three primes:

$$N = p_1 + p_2 + p_3 \tag{1.1}$$

has been treated successfully by Vinogradov. The method of approach (see [1]) is based on the exponential sum $S(\alpha) = \sum_{p \le X} e(p\alpha)$ (where in this case choose $X = N$). The number $r(N)$ of representations of N in the form (1.1) is given by

$$r(N) = \int_0^1 S^3(\alpha) e(-N\alpha)\, d\alpha. \tag{1.2}$$

The evaluation of the integral (1.2) is done by evaluating the contribution from the "major arcs,"

$$\mathfrak{M} = \bigcup_{q \le (\log X)^C} \bigcup_{a;(a,q)=1} \left(\frac{a}{q} - \delta_0, \frac{a}{q} + \delta_0 \right) \tag{1.3}$$

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(with $\delta_0 = X^{-1}(\log X)^A$) asymptotically, whereas for the reminder of the unit interval, one uses upper bounds for the exponential sum.

This approach has not been successful, so far, for the binary Goldbach problem and the Twin prime problem. The number

$$\pi_2(X) := \#\{p \le X-2 : p \text{ and } p+2 \text{ both are primes}\}$$

is given by the integral:

$$\pi_2(X) = \int_0^1 |S(\alpha)|^2 e(-2\alpha)\, d\alpha.$$

From (1.3), we arrive at

$$S\left(\frac{a}{q}+\eta\right) \asymp \frac{X}{\log X}\frac{\mu(q)}{\phi(q)}\sum_{n\le X} e(n\eta)$$

and

$$\int_{\mathfrak{M}} |S(\alpha)|^2 e(-2\alpha)\, d\alpha = \frac{X}{(\log X)^2}\rho\big(1+o(1)\big), \tag{1.4}$$
$$\text{where } \rho = 2\prod_{p>2}\big(1-(p-1)^{-2}\big) \sim 1.320\cdots,$$

with the “singular series”

$$\rho = \sum_{q=1}^{\infty}\frac{\mu^2(q)}{\phi^2(q)} c_q(-2),$$

where c_q being the *Ramanujan sum*

$$c_q(m) = \sum_{\substack{a \bmod q,\\ (a,q)=1}} e\left(\frac{am}{q}\right).$$

If one had more detailed information on the behaviour of $S(\alpha)$ outside the major arcs, this information together with (1.4) might imply the Twin prime conjecture.

We expect the following statement to hold.

Short-interval Conjecture. *There exist positive constants A, B, C, and D such that*:

$$\int_{I\cap\mathfrak{M}^c} |S(\alpha)|^2\, d\alpha = C|I|\frac{X}{\log X}\big(1+O((\log X)^{-A})\big) \tag{1.5}$$

for each subinterval $I \subset (0,1)$ of length $|I| \ge (\log X)^{-B}$ if we choose $\delta_0 = X^{-1}(\log X)^D$ in (1.3).

REMARK 1. We first claim that (1.4) and (1.5) imply the Twin prime conjecture. This can be proved as follows. From (1.4), we get

$$\begin{aligned}\pi_2(X) &= \int_{\mathfrak{M}} |S(\alpha)|^2 e(-2\alpha)\,d\alpha + \int_{\mathfrak{M}^c} |S(\alpha)|^2 e(-2\alpha)\,d\alpha \\ &= \frac{X}{(\log X)^2}\rho\big(1+o(1)\big) + \int_{\mathfrak{M}^c} |S(\alpha)|^2 e(-2\alpha)\,d\alpha.\end{aligned} \tag{1.6}$$

We partition the interval $[0,1]$ into $[\frac{1}{2}(\log X)^B]$ abutting subintervals $I_l = [a_l, b_l]$ of equal length. We then have:

$$\begin{aligned}&\int_{I_l\cap\mathfrak{M}^c} |S(\alpha)|^2 e(-2\alpha)\,d\alpha \\ &\quad = e(-2a_l)\int_{I_l\cap\mathfrak{M}^c} |S(\alpha)|^2\,d\alpha \\ &\qquad + O\bigg(\int_{I_l\cap\mathfrak{M}^c} |S(\alpha)|^2 |e(-2\alpha)-e(-2a_l)|\,d\alpha\bigg).\end{aligned} \tag{1.7}$$

For $\alpha \in I_l$, we have:

$$|e(-2\alpha)-e(-2a_l)| \ll (\log X)^{-B}$$

and thus from the conjectural estimate (1.5)

$$\int_{I_l\cap\mathfrak{M}^c} |S(\alpha)|^2 |e(-2\alpha)-e(-2a_l)|\,d\alpha \ll |I_l| X(\log X)^{-(B+1)}.$$

From (1.7), summing over l and using the conjectural estimate (1.5), we obtain

$$\begin{aligned}\int_{\mathfrak{M}^c} |S(\alpha)|^2 e(-2\alpha)\,d\alpha = |I_l|\bigg(\sum_l e(-2a_l)\bigg) C\frac{X}{\log X} \\ + O(X(\log X)^{-A-1}).\end{aligned} \tag{1.8}$$

We note that all the $|I_l|$ are equal, and hence independent of l. If we let $L = [\frac{(\log X)^B}{2}]$, then $a_l = \frac{l-1}{L}$ and the sum $\sum_l e(-2a_l)$ is a complete geometric sum. Therefore, we have

$$\sum_l e(-2a_l) = \sum_{l \bmod L} e\bigg(\frac{2-2l}{L}\bigg) = 0 \tag{1.9}$$

provided $L \geq 3$.

From (1.6) to (1.9), we observe that

$$\begin{aligned}\pi_2(X) &= \frac{X}{(\log X)^2}\rho\big(1+o(1)\big) + \int_{\mathfrak{M}^c} |S(\alpha)|^2 e(-2\alpha)\,d\alpha \\ &= \frac{X}{(\log X)^2}\rho\big(1+o(1)\big) + O(X(\log X)^{-A-1}).\end{aligned} \tag{1.10}$$

For example with $A \geq 2, B \geq 2$, the claim now follows.

The purpose of this paper is to prove an analogue to (1.5) of the conjecture which is obtained by replacing the prime numbers in the exponential sum by integers without small prime factors. The precise result is stated in the next section.

2. Notation, preliminaries and results

1. We write $e(x)$ for $e^{2\pi i x}$.
2. ε, η and δ will denote arbitrarily small positive constants.
3. Let $\|x\|$ denote the distance of x from the nearest integer, i.e., $\|x\| =: \min_{n\in\mathbb{Z}} |x-n|$.
4. Vinogradov's notation $f \ll g$ means that $|f| < C|g|$ where C is a positive constant depending not on X and Y but at most on the functions $w_1(X), w_2(X)$ and the constants in our theorem.
5. The positive constants $A, B, C, \ldots$ need not be the same at each occurrence.

Definition. For $1 \le Y \le X$, let

$$S(X,Y) := \{n \le X : p \mid n \Rightarrow p > Y\},$$

and

$$S(X,Y,\alpha) := \sum_{n\in S(X,Y)} e(n\alpha).$$

For $w_0 > 0, Q > 0$, we define the "set of Major arcs":

$$\mathfrak{M} = \mathfrak{M}(w_0, Q) = \bigcup_{\substack{q\le Q, a \bmod q,\\ (a,q)=1}} \left(\frac{a}{q} - w_0, \frac{a}{q} + w_0\right).$$

We shall prove the following theorem below.

Theorem. *Let $\varepsilon > 0$ be arbitrarily small, A and C are arbitrarily large positive numbers. Let $w_1, w_2 : [1,\infty) \to \mathbb{R}^+$ be any two functions with $w_1(X)$, $w_2(X) \to 0$ for $X \to \infty$,*

$$(\log\log X)^{\frac{5}{3}+\varepsilon} \le \log Y \le w_1(X)\frac{\log X}{\log\log X}.$$

Let $w_0 = X^{-1}(\log X)^A$. Let V be determined by

$$(\log Y)(\log\log X) = (\log V) \times w_2(X)$$

and assume that

$$Q \ll \min(XV^{-4}, (\log X)^C).$$

Then for any interval $I \subset (0,1)$ with

$$|I| \ge \max\bigl(Q^{-1}(\log X)^{\frac{C}{2}}, (\log X)^{-\frac{A}{2}}\bigr)$$

we have

$$\int_{I\cap\mathfrak{M}^c} |S(X,Y,\alpha)|^2\,d\alpha = C^*|I|\frac{X}{\log X}\bigl(1+O((\log X)^{-B})\bigr)$$

with

$$B=\min\left(\frac{C}{2}-4,\frac{A}{2}-4\right)$$

and $C^*=C^*(Q,X,Y)$ *is a suitable positive quantity* (*independent of the interval* I).

Plan of the proof. In Section 3, using sieving formula, we reduce the problem to the estimation of certain *Basic integrals.* In subsequent sections, we estimate these basic integrals for various cases. In the last section, we establish a good positive lower bound for the quantity C^* and thus complete the proof of the theorem. □

3. The Sieve formula and the function E_d

We set

$$P(Y)=\prod_{p\le Y} p,\qquad E_d=E_d(X,\alpha)=\sum_{\substack{n\le X,\\ n\equiv 0 (\operatorname{mod} d)}} e(n\alpha).$$

By the Sieve formula, we have

$$\begin{aligned} S(X,Y,\alpha) &= \sum_{n\le X} e(n\alpha)\sum_{\substack{d|n,\\ d|P(Y)}} \mu(d)\\ &= \sum_{d|P(Y)}\mu(d)\sum_{\substack{n\le X,\\ n\equiv 0(\operatorname{mod} d)}} e(n\alpha)\\ &= \sum_{d|P(Y)}\mu(d)E_d(X,\alpha). \end{aligned}$$

We thus obtain:

$$\begin{aligned} (3.1)\qquad &\int_{I\cap\mathfrak{M}^c}|S(X,Y,\alpha)|^2\,d\alpha\\ &= \sum_{d_1|P(Y),d_2|P(Y)}\mu(d_1)\mu(d_2)\int_{I\cap\mathfrak{M}^c} E_{d_1}(X,\alpha)\overline{E_{d_2}(X,\alpha)}\,d\alpha. \end{aligned}$$

The expressions $\int_{(d_1,d_2)} := \int_{I\cap\mathfrak{M}^c} E_{d_1}(X,\alpha)\overline{E_{d_2}(X,\alpha)}\,d\alpha$ are called the *basic integrals*. Crucial for their discussion are the sizes of d_1,d_2 as well as the size of $D:=\gcd(d_1,d_2)$.

In the subsequent sections, we treat the following cases:

I. *The small case:* both d_1, d_2 are $\leq V$ and $D \leq Q$.
II. *The intermediate case:* both d_1, d_2 are $\leq V$ and $Q < D$.
III. *The very large case:* $X \geq d_1 > V$ or $X \geq d_2 > V$.

4. The basic integrals—small case

We set $d_1 = Dd_1', d_2 = Dd_2'$ where $D := \gcd(d_1, d_2)$ and thus we have $\gcd(d_1', d_2') = 1$. We partition the interval I into

$$|I|D + O(1) \tag{4.1}$$

subintervals of the form $I_{D,l} = [lD^{-1}, (l+1)D^{-1}]$ and possibly a first and last interval of length $< D^{-1}$ each.

We have for each subinterval $I_{D,l}$:

$$\begin{aligned} &\int_{I_{D,l}\cap \mathfrak{M}^c} |E_{Dd_1'}(X,\alpha)||E_{Dd_2'}(X,\alpha)|\, d\alpha \\ &= \int_{\frac{l}{D}+w_0}^{\frac{l+1}{D}-w_0} |E_{Dd_1'}(X,\alpha)||E_{Dd_2'}(X,\alpha)|\, d\alpha \\ &= \frac{1}{D}\int_{Dw_0}^{1-Dw_0} \left|E_{d_1'}\left(\frac{X}{D},\beta\right)\right|\left|E_{d_2'}\left(\frac{X}{D},\beta\right)\right| d\beta. \end{aligned} \tag{4.2}$$

We note that

$$\left|E_d\left(\frac{X}{D},\beta\right)\right| \ll \min\left(\left[\frac{X}{Dd}\right], \|d\beta\|^{-1}\right). \tag{4.3}$$

We set $\tilde{X} = \frac{X}{D}$ and consider the system of inequalities:

$$\|d_1'\beta\| \leq \left[\frac{\tilde{X}}{d_1'}\right]^{-1} \quad \text{if } \Delta_1 \leq \left[\frac{\tilde{X}}{d_1'}\right]^{-1} \tag{4.4}$$

and

$$\Delta_1 < \|d_1'\beta\| \leq 2\Delta_1, \qquad \Delta_2 < \|d_2'\beta\| \leq 2\Delta_2 \quad \text{if } \Delta_1 > \left[\frac{\tilde{X}}{d_1'}\right]^{-1}. \tag{4.5}$$

We define the set,

$$\rho(\Delta_1, \Delta_2) = \{\beta \in (0,1) : (4.4) \text{ or } (4.5) \text{ hold}\}.$$

In the sequel, we write $(Dw_0, 1 - Dw_0)$ as a union of sets $\rho(\Delta_1, \Delta_2)$ (there could be some overlaps of these sets, however as far as upper bounds are concerned it does not matter). We may also assume that

$$\Delta_1(d_1')^{-1} \le \Delta_2(d_2')^{-1}, \qquad \left(k-\frac{1}{2}\right) \le \beta d_1' \le \left(k+\frac{1}{2}\right), \\ \left(j-\frac{1}{2}\right) \le \beta d_2' \le \left(j+\frac{1}{2}\right). \tag{4.6}$$

If (4.6) is satisfied for the pair $(k,j)=(0,0)$, then $\beta \in J := [0, \frac{1}{2}\min((d_1')^{-1}, (d_2')^{-1})]$. We have by (4.3),

$$\int_{J, \beta \ge Dw_0} |E_{d_1'}(\tilde{X},\beta)||E_{d_2'}(\tilde{X},\beta)|\, d\beta \ll D^{-1}(d_1'd_2')^{-1}w_0^{-1}. \tag{4.7}$$

The same estimate is satisfied in the case of the pair $(k,j)=(d_1',d_2')$ for which we have $\beta \in [1-\frac{1}{2}\min((d_1')^{-1},(d_2')^{-1}),1]$. In the sequel, we estimate

$$\int_{\substack{\beta\in(0,1),\\ \|\beta\|\ge\frac{1}{2}(d_1'd_2')^{-1}}} |E_{d_1'}(\tilde{X},\beta)||E_{d_2'}(\tilde{X},\beta)|\, d\beta$$

and we may assume that $(k,j) \ne (0,0), (d_1',d_2')$.

Then the distance between any two distinct of the six points $(k-1)(d_1')^{-1}$, $k(d_1')^{-1}, (k+1)(d_1')^{-1}, (j-1)(d_2')^{-1}, j(d_2')^{-1}, (j+1)(d_2')^{-1}$ is at least

$$|k(d_1')^{-1} - j(d_2')^{-1}| = \frac{|kd_2' - jd_1'|}{d_1'd_2'} \ge \frac{1}{d_1'd_2'}. \tag{4.8}$$

Thus, it follows that

$$\Delta_2(d_2')^{-1} \ge \frac{1}{2}(d_1'd_2')^{-1}. \tag{4.9}$$

Thus, $\{\beta \in (0,1) : \|\beta\| \ge \frac{1}{2}\min((d_1')^{-1},(d_2')^{-1})\}$ is a union of the sets $\rho(\Delta_1, \Delta_2)$, where

$$\Delta_1 = \left[\frac{\tilde{X}}{d_1'}\right]^{-1}, \quad \text{or} \quad \Delta_1 = 2^{r_1-1}\left[\frac{\tilde{X}}{d_1'}\right]^{-1}, \\ \Delta_2 = 2^{r_2-2}(d_1'd_2')^{-1}; \quad r_1, r_2 \in \mathbb{N}. \tag{4.10}$$

For given s, the equation

$$k(d_1')^{-1} - j(d_2')^{-1} = s(d_1'd_2')^{-1}$$

has at most one solution (k,j). Thus, for given r_2, there are $\ll 2^{r_2}$ pairs (k,j) such that for $\Delta_2 = 2^{r_2}(d_1'd_2')^{-1}$, we have:

$$\Delta_2 \le |k(d_1')^{-1} - j(d_2')^{-1}| \le 2\Delta_2. \tag{4.11}$$

For $\beta \in \rho(\Delta_1,\Delta_2)$ we must have:

$$|\beta - k(d_1')^{-1}| \le 2\Delta_1 \tag{4.12}$$

for a value k belonging to some of these pairs. Thus, we have

$$\operatorname{meas}\rho(\Delta_1,\Delta_2) \ll 2^{r_2}\Delta_1. \tag{4.13}$$

Since for $\beta\in\rho(\Delta_1,\Delta_2)$, we have

$$|E_{d_1'}(\tilde{X},\beta)| \le \Delta_1^{-1}, \qquad |E_{d_2'}(\tilde{X},\beta)| \le \Delta_2^{-1} \tag{4.14}$$

we now obtain (with $\Delta_2 = 2^{r_2}(d_1'd_2')^{-1}$)

$$\int_{\rho(\Delta_1,\Delta_2)} |E_{d_1'}(\tilde{X},\beta)E_{d_2'}(\tilde{X},\beta)|\,d\beta \ll (\Delta_1^{-1}\Delta_2^{-1}) \times (2^{r_2}\Delta_1) \ll d_1'd_2'. \tag{4.15}$$

Since there are $\ll (\log X)^2$ sets $\rho(\Delta_1,\Delta_2)$, we get

$$\int_{\substack{\beta\in(0,1),\\ \|\beta\|\ge\frac{1}{2}(d_1'd_2')^{-1}}} |E_{d_1'}(\tilde{X},\beta)||E_{d_2'}(\tilde{X},\beta)|\,d\beta \ll d_1'\,d_2'(\log X)^2. \tag{4.16}$$

From (4.2), (4.7), and (4.16), we get

$$\begin{aligned}\int_{I_{D,l}\cap\mathfrak{M}^c} &|E_{Dd_1'}(X,\alpha)E_{Dd_2'}(X,\alpha)|\,d\alpha\\ &\ll \frac{d_1'd_2'}{D}(\log X)^2 + D^{-2}(d_1'd_2')^{-1}w_0^{-1}.\end{aligned}$$

This estimate also holds for the first and last subinterval of I. From (4.1), we get

$$\begin{aligned}\int_{(d_1,d_2)} &\ll d_1d_2(\log X)^2 + X(\log X)^{-A}(d_1'd_2')^{-1}\\ &\quad + |I|X(\log X)^{-A}D^{-1}(d_1'd_2')^{-1}.\end{aligned} \tag{4.17}$$

We observe that (4.17) contributes to (3.1) an error which is in absolute value

$$\begin{aligned}&\ll (\log X)^2\biggl(\sum_{d\le V} d\biggr)^2 + X(\log X)^{-A}\biggl(\sum_{d\le V}\frac{1}{d}\biggr)^2\\ &\quad + |I|X(\log X)^{-A}\biggl(\sum_{d\le V}\frac{1}{d}\biggr)^3\\ &\ll V^4(\log X)^2 + X(\log X)^{2-A} + |I|X(\log X)^{3-A}\\ &\ll XQ^{-1}(\log X)^2 + X(\log X)^{2-A} + |I|X(\log X)^{3-A},\end{aligned} \tag{4.18}$$

since $Q \ll \frac{X}{V^4}$.

5. The basic integrals—the intermediate case

As in Section 4, we set $d_1 = Dd_1', d_2 = Dd_2'$ where $D := \gcd(d_1, d_2)$, and thus we have $\gcd(d_1', d_2') = 1$. We partition the interval I into

$$|I|D + O(1) \tag{5.1}$$

subintervals of the form $I_{D,l} = [lD^{-1}, (l+1)D^{-1}]$ and possibly a first and last interval of length $< D^{-1}$ each.

We have

$$\begin{aligned}(5.2)\qquad &\int_{I_{D,l}} E_{Dd_1'}(X,\alpha)\overline{E}_{Dd_2'}(X,\alpha)\,d\alpha \\ &= \frac{1}{D}\int_0^1 E_{d_1'}\left(\frac{X}{D},\beta\right)\overline{E}_{d_2'}\left(\frac{X}{D},\beta\right)d\beta \\ &= \frac{1}{D}\left|\left\{n \le \frac{X}{D} : n \equiv 0(\operatorname{mod} d_1'), n \equiv 0(\operatorname{mod} d_2')\right\}\right| \\ &= \frac{1}{D}\left[\frac{X}{Dd_1'd_2'}\right].\end{aligned}$$

Let $I_D^{(1)}$ be the first subinterval (if it exists). With $\tilde{X} = \frac{X}{D}$ we have:

$$\int_{I_D^{(1)}} |E_{Dd_1'}(X,\alpha)E_{Dd_2'}(X,\alpha)|\,d\alpha \ll \frac{1}{D}\int_0^{\frac{1}{2}} |E_{d_1'}(\tilde{X},\beta)E_{d_2'}(\tilde{X},\beta)|\,d\beta.$$

Again applying (4.3), we have:

$$\int_0^{\tilde{X}^{-1}} |E_{d_1'}(\tilde{X},\beta)E_{d_2'}(\tilde{X},\beta)|\,d\beta \ll \frac{\tilde{X}}{d_1'd_2'},$$

and

$$\int_{\tilde{X}^{-1}}^{\frac{1}{2}(d_1'd_2')^{-1}} |E_{d_1'}(\tilde{X},\beta)E_{d_2'}(\tilde{X},\beta)|\,d\beta \ll (d_1'd_2')^{-1}\int_{\tilde{X}^{-1}}^{\infty} \beta^{-2}\,d\beta \ll \frac{\tilde{X}}{d_1'd_2'}.$$

By (4.16), we have

$$\int_{\frac{1}{2}(d_1'd_2')^{-1}}^{\frac{1}{2}} |E_{d_1'}(\tilde{X},\beta)E_{d_2'}(\tilde{X},\beta)|\,d\beta \ll d_1'd_2'(\log X)^2.$$

Thus we get

$$\int_{I_D^{(1)}} |E_{Dd_1'}(X,\alpha)E_{Dd_2'}(X,\alpha)|\,d\alpha \ll \frac{X}{D^2d_1'd_2'} + \frac{d_1'd_2'}{D}(\log X)^2. \tag{5.3}$$

The same estimate holds for the last subinterval too. From (5.1), (5.2), and (5.3), we obtain

$$
\begin{aligned}
(5.4)\qquad \int_{(d_1,d_2)} &= \int_{I\cap\mathfrak{M}^c} E_{d_1}(X,\alpha)\overline{E_{d_2}(X,\alpha)}\,d\alpha \\
&= \int_{I\cap\mathfrak{M}^c} E_{Dd_1'}(X,\alpha)\overline{E_{Dd_2'}(X,\alpha)}\,d\alpha \\
&= \big(|I|D+O(1)\big) \\
&\quad \times\left(\frac{X}{D^2d_1'd_2'}+O\left(\frac{1}{D}\right)\right) \\
&\quad + O\left(\frac{X}{D^2d_1'd_2'}\right)+O\left(\frac{d_1'd_2'}{D}(\log X)^2\right) \\
&= |I|\frac{X}{Dd_1'd_2'}+O\left(\frac{X}{D^2d_1'd_2'}\right) \\
&\quad + O(|I|)+O\left(\frac{1}{D}\right)+O(d_1d_2(\log X)^2) \\
&= |I|\frac{XD}{d_1d_2}++O\left(\frac{X}{D^2d_1'd_2'}\right)+O(|I|) \\
&\quad + O\left(\frac{1}{D}\right)+O(d_1d_2(\log X)^2) \\
&= |I|\frac{XD}{d_1d_2}++O\left(\frac{X}{D^2d_1'd_2'}\right)+O(d_1d_2(\log X)^2).
\end{aligned}
$$

We note that (since $d_1'\le V, d_2'\le V, D\le V$)

$$
\frac{X}{D^2d_1'd_2'}\ge\frac{X}{V^4}\gg 1;\qquad \frac{X}{D^2d_1'd_2'}\ge\frac{1}{D}\frac{X}{V^3}\gg\frac{1}{D}.
$$

We observe that (5.4) contributes to (3.1), the main term $C^*|I|\frac{X}{\log X}$ with

$$
C^*=(\log X)\sum_{\substack{d_1\le V, d_1|P(Y);\\ d_2\le V, d_2|P(Y);\\ D=\gcd(d_1,d_2)>Q}}\frac{D\mu(d_1)\mu(d_2)}{d_1d_2}
$$

and an error which is in absolute value

$$
\ll XQ^{-1}(\log X)^2+V^4(\log X)^2\ll XQ^{-1}(\log X)^2,
$$

since $Q\ll\min(XV^{-4},(\log X)^C)$.

6. The basic integrals—the very large case

In this connection, we need several definitions and results related to smooth numbers.

Definition. For an integer n, we denote by $P^+(n)$ the largest prime factor of n.

We call n, Y-*smooth* if $P^+(n) \le Y$. Let $\Psi(X,Y)$ be the number of Y-*smooth* integers $\le X$, i.e.,

$$\Psi(X,Y) = \#\{n \le X : P^+(n) \le Y\}.$$

We shall need the following results.

Let $u = \frac{\log X}{\log Y}$. It is known from the result of Hildebrand (see [2]) that

$$\Psi(X,Y) \ll X\exp(-u\log u) \quad \text{for } u \le Y^{1-\varepsilon} \text{ with } u \to \infty, \text{ i.e. } Y \ge (\log X)^{1+\varepsilon}.$$

A result of Xuan (see [4]), which can be found in [5] as a *Corollary A* says that:

Let $\varepsilon > 0$ be fixed, $\exp\{(\log\log X)^{\frac{5}{3}+\varepsilon}\} \le Y \le X$. We write $u = \frac{\log X}{\log Y}$. Then uniformly in X and Y, we have

$$\sum_{\substack{d \le X \\ P^+(d) \le Y}} \tau(d) = 2^{u+O(\frac{u}{\log u})}\Psi(X,Y)\log Y.$$

We first prove two lemmas below (which are necessary to deal this case).

Lemma 6.1. *Let n be a Y-smooth integer, $Y < n^{\frac{1}{8}}$. Then n has a Y-smooth divisor d satisfying*

$$n^{\frac{1}{4}}Y^{-1} < d \le n^{\frac{1}{4}}. \tag{6.1}$$

Proof. Let d_1 be the largest divisor of n with $d_1 \le n^{\frac{1}{4}}$. Let p be a prime divisor of n such that $p|(\frac{n}{d_1})$. By definition, $d_1 p > n^{\frac{1}{4}}$. Since $p \le Y$, we have $d_1 > n^{\frac{1}{4}}Y^{-1}$. This proves the lemma. □

Lemma 6.2. *Let n be a positive integer, $Z \ge 100$, and set $Y := \exp(Z^{-1}\log n)$. Assume that*

$$\tau(n) \ge \exp(Z\log 2). \tag{6.2}$$

Then either: (i) *n has a Y-smooth divisor d with*

$$n^{\frac{1}{32}} \le d \le n^{\frac{1}{4}} \tag{6.3}$$

or (ii) *we have the estimate*

$$\tau(n) \le 2^{Z+2}\Bigg(\sum_{\substack{t|n \\ t \le n^{1/4}}} 1\Bigg). \tag{6.4}$$

Proof. Let $\Omega(n)$ be the total number of prime divisors of n and let $n = p_1 p_2 \cdots p_{\Omega(n)}$ with $p_1 \le p_2 \le \cdots \le p_{\Omega(n)}$. Then, $\tau(n) \le 2^{\Omega(n)}$. From (6.2), it follows that $\Omega(n) \ge Z$. Let $\Omega_1 := \Omega(n) - [Z] - 1$. Then we have

$$p_{\Omega_1}^Z \le p_{\Omega_1+1} \cdots p_{\Omega(n)} \le n$$

and thus

$$\log p_{\Omega_1} \le \frac{\log n}{Z} = \log Y.$$

Therefore, $d^{(n)} := p_1 \cdots p_{\Omega_1}$ is Y-smooth. If $d^{(n)} \ge n^{\frac{1}{4}}$, then by Lemma 6.1, $d^{(n)}$ has a Y-smooth divisor d with

$$\left(d^{(n)}\right)^{\frac{1}{8}} \le d \le \left(d^{(n)}\right)^{\frac{1}{4}}$$

and hence we have (i) $n^{\frac{1}{32}} \le d \le n^{\frac{1}{4}}$.

Now, we assume that $d^{(n)} < n^{\frac{1}{4}}$. Let Ω_2 be the largest integer such that $p_{\Omega_2} < p_{\Omega_1}$ (possibly $\Omega_2 = 0$) and Ω_3 be the smallest integer such that $p_{\Omega_3} > p_{\Omega_1}$ (Ω_3 does not exist if $p_j = p_{\Omega_1}$ for all $j > \Omega_2$). Let κ be the exact power of p_{Ω_1} such that $p_{\Omega_1}^{\kappa}$ divides n. Let κ_1 be chosen such that

$$p_1 \cdots p_{\Omega_2} p_{\Omega_1}^{\kappa_1} \le n^{\frac{1}{4}}, \quad \text{but} \quad p_1 \cdots p_{\Omega_2} p_{\Omega_1}^{\kappa_1+1} > n^{\frac{1}{4}}.$$

We thus can write,

$$n = \prod_1 \prod_2 \prod_3$$

with

$$\prod_1 = p_1 \cdots p_{\Omega_2} \qquad \left(\prod_1 = 1 \text{ if } \Omega_2 = 0\right); \qquad \prod_2 = p_{\Omega_1}^{\kappa} \quad \text{with } \kappa \ge \kappa_1;$$

$$\prod_3 = p_{\Omega_3} \cdots p_{\Omega(n)}.$$

(We note that $\prod_3 = 1$ if Ω_3 does not exist.) Then we factor each divisor of $t|n$ as $t = t_1 t_2 t_3$ with $t_j | \prod_j$ $(j = 1, 2, 3)$. We have:

$$\sum_{\substack{t|n \\ t \le n^{1/4}}} 1 \ge \tau\left(\prod_1\right)(\kappa_1 + 1) \tag{6.5}$$

and

$$\tau(n) \le \tau\left(\prod_1\right)(\kappa+1)2^{[Z]+1-(\kappa-\kappa_1)}. \tag{6.6}$$

If $\kappa = \kappa_1$, then (ii) follows immediately from (6.5) and (6.6). Now, if $\kappa > \kappa_1$ (> 0), we have $\kappa + 1 = \kappa_1 + 1 + (\kappa - \kappa_1) \le 2(\kappa_1 + 1)(\kappa - \kappa_1) \le 2(\kappa_1 + 1)2^{(\kappa - \kappa_1)}$. Now, again (ii) follows from (6.5) and (6.6). □

We start with the representation of the characteristic function of I by a Fourier-series. Let

$$\chi_I(\alpha) = \begin{cases} 1 & \text{if } \alpha \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

Then we have for all $\alpha \in (0,1)$ (except for the end points of I):

$$\chi_I(\alpha) = \sum_m a_m e(m\alpha) \tag{6.8}$$

with

$$a_m = \int_I e(-m\alpha)\, d\alpha \ll \min(|I|, m^{-1}). \tag{6.9}$$

We replace $\chi_I(\alpha)$ by the approximation

$$\chi_I^{(1)}(\alpha) = \frac{1}{\sqrt{\pi}} X^4 \int_{-\infty}^{\infty} \chi_I(\alpha+u) \exp(-X^8u^2)\, du. \tag{6.10}$$

We have

$$\frac{1}{\sqrt{\pi}} X^4 \int_{-\infty}^{\infty} \exp(-X^8u^2)\, du = 1. \tag{6.11}$$

First, we consider $\alpha \in (0,1)$, whose distance to any of the end points of I is $\geq X^{-3}$. Then $\chi_I(\alpha) = \chi_I(\alpha+u)$ for $|u| < X^{-3}$ and

$$|\chi_I(\alpha+u) - \chi_I(\alpha)| \leq 1 \quad \text{(for all } u\text{)}. \tag{6.12}$$

Therefore, we get

$$\begin{aligned} \left|\chi_I^{(1)}(\alpha) - \chi_I(\alpha)\right| &= \left|\frac{1}{\sqrt{\pi}} X^4 \int_{-\infty}^{\infty} \big(\chi_I(\alpha+u) - \chi_I(\alpha)\big) \exp(-X^8u^2)\, du\right| \\ &\leq \frac{1}{\sqrt{\pi}} X^4 \int_{|u|\geq X^{-3}} \exp(-X^8u^2)\, du \\ &\ll \exp\left(-\frac{1}{2}X^2\right). \end{aligned}$$

For the other α, we have because of (6.11) and (6.12)

$$\left|\chi_I^{(1)}(\alpha) - \chi_I(\alpha)\right| \leq 1.$$

Now, we have

$$\chi_I^{(1)}(\alpha) = \sum_m a_m^{(1)} e(m\alpha)$$

with the Fourier coefficients

$$\begin{aligned} a_m^{(1)} &= \frac{X^4}{\sqrt{\pi}} \int_{-\infty}^{\infty} e(-m\alpha) \left(\sum_m a_m e\big(m(\alpha+u)\big)\right) \exp(-X^8u^2)\, du \\ &= a_m \frac{X^4}{\sqrt{\pi}} \int_{-\infty}^{\infty} e(mu) \exp(-X^8u^2)\, du \\ &= a_m \exp(-\pi^2 m^2 X^{-8}) \quad \text{(by substituting } v = X^4 u\text{)}. \end{aligned}$$

We have

$$a_m^{(1)} \ll \min(|I|, m^{-1} \exp(-\pi^2 m^2 \ X^{-8})). \tag{6.13}$$

We set

$$\chi_I^{(2)}(\alpha) = \sum_{|m| \le X^5} a_m^{(1)} e(m\alpha). \tag{6.14}$$

We have

$$\left|\chi_I(\alpha) - \chi_I^{(2)}(\alpha)\right| \ll X^{-3}$$

except for α from neighbourhoods of length X^{-3} of the two end points of I. For these α, $|\chi_I(\alpha) - \chi_I^{(2)}(\alpha)| \ll 1$. We obtain

$$\begin{aligned}\int_{(d_1,d_2)} = \int_0^1 \chi_I^{(2)}(\alpha) E_{d_1}(X,\alpha)\overline{E_{d_2}(X,\alpha)}\, d\alpha \\ - \int_{I \cap \mathfrak{M}} E_{d_1}(X,\alpha)\overline{E_{d_2}(X,\alpha)}\, d\alpha + O\left(\frac{X^{-1}}{d_1 d_2}\right).\end{aligned} \tag{6.15}$$

We have

$$\sum_{\substack{X \ge d_1 > V, \\ P^+(d_1) \le Y}} \sum_{d_2 \le X} \int_{(d_1,d_2)} \ll |S_1| + |S_2| + O\big(X^{-\frac{1}{2}}\big)$$

where

$$S_1 = \sum_{\substack{X \ge d_1 > V, \\ P^+(d_1) \le Y}} \sum_{d_2 \le X} \int_0^1 \chi_I^{(2)}(\alpha) E_{d_1}(X,\alpha)\overline{E_{d_2}(X,\alpha)}\, d\alpha,$$

$$S_2 = \sum_{\substack{X \ge d_1 > V, \\ P^+(d_1) \le Y}} \sum_{d_2 \le X} \int_{I \cap \mathfrak{M}} E_{d_1}(X,\alpha)\overline{E_{d_2}(X,\alpha)}\, d\alpha.$$

By (6.14) and (6.15), we have

$$\begin{aligned}|S_1| &= \left| \sum_{|m| \le X^5} a_m^{(1)} \sum_{\substack{(d_1,d_2,n_1,n_2) \\ X \ge d_1 > V, P^+(d_1) \le Y \\ d_2 \le X, d_1 n_1 \le X, d_2 n_2 \le X}} \int_0^1 e\big((m + d_1 n_1 - d_2 n_2)\alpha\big)\, d\alpha \right| \\ &= \left| \sum_{|m| \le X^5} a_m^{(1)} \sum_{\substack{X \ge d_1 > V \\ P^+(d_1) \le Y}} \sum_{d_2 \le X} \sum_{\substack{n_1, n_2, \\ d_2 n_2 - d_1 n_1 = m \\ d_1 n_1 \le X, d_2 n_2 \le X}} 1 \right| \\ &\le \sum_{|m| \le X^5} \left|a_m^{(1)}\right| \sum_{\substack{X \ge d_1 > V \\ P^+(d_1) \le Y}} \sum_{n_1 \le [X/d_1]} \tau^*(m + d_1 n_1).\end{aligned} \tag{6.16}$$

Here τ^* denotes the function

$$\tau^*(w) = \begin{cases} \tau(w) \text{ i.e. the number of divisors of } w & \text{if } w > 0, \\ 0 & \text{if } w \leq 0. \end{cases}$$

We now estimate the sum in (6.16). We break it up into $\ll (\log X)^2$ subsums of the form

$$(6.17)\quad \begin{aligned} S_3 &:= \sum_{M<|m|\leq M'} \left|a_m^{(1)}\right| \sum_{\substack{D<d_1<D' \\ P^+(d_1)\leq Y}} \sum_{n_1\leq[X/d_1]} \tau^*(m+d_1n_1) \\ &\leq \Big(\max_{M<m\leq 2M} |a_m|\Big)\Bigg(\sum_{M<|m|\leq 2M} \sum_{\substack{D<d_1<D' \\ P^+(d_1)\leq Y}} \sum_{n_1\leq[X/d_1]} \tau^*(m+d_1n_1)\Bigg) \\ &\leq \Big(\max_{M<m\leq 2M} |a_m|\Big)\Big(\sum(M,D)\Big) \quad \text{(say)}. \end{aligned}$$

Case (i). $M \leq X$.

We have $m + d_1n_1 \leq 2X$. We partition the sum $\sum(M,D)$ into subsums according to the value of $\tau^*(m+d_1n_1)$. For $\mu \in \{0\} \cup \mathbb{N}$, we set:

$$(6.18)\qquad Z_\mu := 2^\mu \frac{\log V}{\log Y}; \quad Y_\mu := \exp((Z_\mu)^{-1}(\log(2X)))$$

and define

$$(6.19)\qquad \sum_0 = \sum_{(n_1,d_1,m)}^{(0)} \tau^*(m+d_1n_1),$$

where we sum over all triplets (n_1, d_1, m) with

$$(6.20)\qquad \begin{aligned} \tau^*(m+d_1n_1) \leq \exp(Z_0), &\qquad M < |m| \leq 2M, \\ D < d_1 \leq 2D, &\qquad P^+(d_1) \leq Y \end{aligned}$$

and for $\mu \geq 1$,

$$(6.21)\qquad \sum_\mu = \sum_{(n_1,d_1,m)}^{(\mu)} \tau^*(m+d_1n_1),$$

where we sum over all triplets (n_1, d_1, m) with

$$(6.22)\qquad \exp(Z_\mu) < \tau^*(m+d_1n_1) \leq \exp(2Z_\mu).$$

For $\mu \geq 1$, we apply Lemma 6.2 with $Z = Z_\mu$. At least one of the following two situations must occur:

(i) $m + d_1n_1$ has a Y_μ-smooth divisor d with $X^{\frac{1}{32}} \leq d \leq 2X^{\frac{1}{4}}$.
(ii) $\tau^*(m+d_1n_1) \leq 2^{Z_\mu+2}(\sum_{t|(m+d_1n_1),t\leq 2X^{\frac{1}{4}}} 1)$.

We have

$$\sum(M,D) \le \sum_0 + \sum_{\mu=1}^{\infty}\sum_{\mu,1} + \sum_{(2)}, \tag{6.23}$$

where

$$\sum_{\mu,1} = \sum_{(n_1,d_1,m)} 1 \tag{6.24}$$

and here we sum over all triplets with

$$M < m \le 2M, \qquad D < d_1 \le 2D, \qquad P^+(d_1) \le Y,$$
$$\exp(Z_\mu) < \tau^*(m + d_1 n_1) \le \exp(Z_{\mu+1}),$$

$m + d_1 n_1$ has a Y_μ-smooth divisor s satisfying

$$X^{\frac{1}{32}} \le s \le 2X^{\frac{1}{4}}.$$

In $\sum_{(2)}$, we sum over all triplets where the condition (ii) is satisfied with $\mu = 0$.

Estimation of $\sum_0$:

To estimate $\sum_0$, we use the result of [2] and partial summation. Thus we get:

$$\begin{aligned}\sum_0 &\ll \exp(Z_0) \sum_{M<|m|\le 2M} \sum_{\substack{D<d_1\le 2D\\ P^+(d_1)\le Y}} \sum_{n_1 \le [X/d_1]} 1 \\ &\ll XM \exp\left(\frac{\log V}{\log Y}\right) \sum_{\substack{D<d_1\le 2D\\ P^+(d_1)\le Y}} \frac{1}{d_1} \\ &\ll XM \exp\left(\frac{\log V}{\log Y} - \frac{\log V}{\log Y}\log\left(\frac{\log V}{\log Y}\right)\right).\end{aligned}$$

Estimation of $\sum_{\mu,1}$:

We have:

$$\begin{aligned}\sum_{\mu,1} &\ll \exp(Z_{\mu+1}) \\ &\times \sum_{M<|m|\le 2M} \sum_{\substack{D<d_1\le 2D,\\ P^+(d_1)\le Y}} \sum_{\substack{X^{1/32}<s\le 2X^{1/4},\\ P^+(s)\le Y_\mu}} \sum_{\substack{n_1 d_1 \le X,\\ n_1 d_1 \equiv -m (\mathrm{mod}\, s)}} 1.\end{aligned} \tag{6.25}$$

Let $g = \gcd(d_1, s)$. Then the congruence $d_1 n_1 \equiv -m (\mathrm{mod}\, s)$ is equivalent to $\frac{d_1}{g} n_1 \equiv -\frac{m}{g} (\mathrm{mod}\, \frac{s}{g})$. For given d_1, g, m, the integer n_1 is uniquely determined mod $\frac{s}{g}$. Thus if we write $s = gs', d_1 = gd_1'$, then the sum in (6.25) is bounded by

$$(6.26)\quad \sum_{\substack{g \le X^{1/4} \\ P^+(g) \le Y_\mu}} \frac{1}{g} \sum_{\substack{M<|m|\le 2M \\ m \equiv 0 (\bmod g)}} \sum_{\substack{X^{1/32} g^{-1} \le s' \le 2X^{1/4} g^{-1} \\ P^+(s') \le Y_\mu}} \sum_{\substack{Dg^{-1} \le d_1' \le 2Dg^{-1} \\ P^+(d_1') \le Y_\mu}} \frac{X}{s'd_1'}$$
$$:= \sum(\mu, M, D) \quad \text{(say)}.$$

We break up $\sum(\mu, M, D)$ into two subsums:

$$(6.27)\qquad \sum(\mu, M, D) = \sum_1(\mu, M, D) + \sum_2(\mu, M, D),$$

where in $\sum_1(\mu, M, D)$, the summation is over all g with $1 \le g \le X^{\frac{1}{64}}$ and in $\sum_2(\mu, M, D)$ over those g with $X^{\frac{1}{64}} \le g \le 2X^{\frac{1}{4}}$. For the estimate of $\sum_1(\mu, M, D)$, we write again $d_1 = gd_1'$ and obtain

$$(6.28)\quad \sum_1(\mu, M, D) \ll X \sum_{M<|m|\le 2M} \sum_{\substack{X^{1/64} \le s' \le X^{1/4} \\ P^+(s') \le Y_\mu}} \frac{1}{s'} \sum_{\substack{D<d_1\le 2D \\ P^+(d_1) \le Y}} \frac{\tau(d_1)}{d_1}.$$

For the estimate of the inner sum in (6.28), we use the result of Xuan mentioned before and obtain:

$$(6.29)\qquad \sum_{\substack{d \le X \\ P^+(d) \le Y}} \tau(d) = 2^{u + O(\frac{u}{\log u})} \Psi(X, Y) \log Y$$
$$\ll X(\log X) \exp\left(-\frac{\log X}{\log Y} \log\left(\frac{\log X}{\log Y}\right)\right).$$

From (6.28) and (6.29) we get

$$(6.30)\quad \sum_1(\mu, M, D) \ll XM(\log X)^2 \exp\left(-\frac{2^\mu}{64} \frac{\log X}{\log Z_0} \log\left(\frac{\log X}{\log Z_0}\right)\right)$$
$$\times \exp\left(-\frac{1}{4} \frac{\log V}{\log Y} \log\left(\frac{\log V}{\log Y}\right)\right).$$

The sum $\sum_2(\mu, M, D)$ appears only for $g \ge X^{\frac{1}{64}}$. We obtain:

$$(6.31)\quad \sum_2(\mu, M, D)$$
$$\ll X \sum_{X^{1/64} \le g \le 2X^{1/4}} \sum_{\substack{M<|m|\le 2M \\ m \equiv 0 (\bmod g)}} \sum_{X^{1/64} \le s' \le X^{1/4}} \frac{1}{s'} \sum_{\substack{D<d_1\le 2D \\ P^+(d_1) \le Y}} \frac{\tau(d_1)}{d_1}$$
$$\ll XM(\log X)^3 \exp\left(-\frac{1}{64} \frac{\log X}{\log Y} \log\left(\frac{\log X}{\log Y}\right)\right).$$

Thus, in total we obtain:

$$\sum_{\mu,1} \ll XM(\log X)^3 \exp\left(2^{\mu+2}(\log 2)\frac{\log V}{\log Y}\right) \times \exp\left(-\frac{2^{\mu}}{64}\frac{\log X}{\log Z_0}\log\left(\frac{\log X}{\log Z_0}\right)\right) \times \exp\left(-\frac{1}{4}\frac{\log V}{\log Y}\log\left(\frac{\log V}{\log Y}\right)\right). \tag{6.32}$$

We now estimate $\sum_{(2)}$. We have

$$\sum_{(2)} \ll X2^{Z_0+2} \sum_{M<|m|\le 2M} \sum_{\substack{D<d_1\le 2D\\ P^+(d_1)\le Y}} \sum_{n_1 d_1\le X} \sum_{\substack{t|(m+n_1 d_1)\\ t\le 2X^{1/4}}} 1 \ll XM2^{Z_0+2} \sum_{\substack{D<d_1\le 2D\\ P^+(d_1)\le Y}} \sum_{t\le 2X^{1/4}} \sum_{\substack{n_1 d_1\le X\\ n_1 d_1\equiv -m (\mathrm{mod}\, t)}} 1. \tag{6.33}$$

Writing $t = gt', d_1 = gd_1'$ and reasoning as above, we get:

$$\sum_{(2)} \ll XM2^{Z_0+2} \sum_{\substack{D<d_1\le 2D\\ P^+(d_1)\le Y}} \sum_{1\le g\le 2X^{1/4}} \sum_{1\le t\le 2X^{1/4}} \frac{1}{gd_1 t} \ll XM2^{Z_0+2}(\log X)^3 \exp\left(-\frac{1}{4}\frac{\log V}{\log Y}\log\left(\frac{\log V}{\log Y}\right)\right) \ll XM \exp\left(-\frac{1}{8}\frac{\log V}{\log Y}\log\left(\frac{\log V}{\log Y}\right)\right). \tag{6.34}$$

From (6.23), (6.32), and (6.34), we obtain

$$\sum(M,D) \ll XM \exp\left(-\frac{1}{8}\frac{\log V}{\log Y}\log\left(\frac{\log V}{\log Y}\right)\right).$$

Case (ii). $M > X$.

For the sum $\sum(M,D)$ in (6.17), by Lemma 6.1, we have the bound

$$\sum(M,D) \ll \sum_{t\le 4M} \sum_{\substack{D^{1/4}Y^{-1}<d_1\le 2D^{1/4}\\ P^+(d_1)\le Y}} \sum_{n_1\le X/d_1} \sum_{\substack{M<|m|\le 2M\\ m\equiv -d_1 n_1 (\mathrm{mod}\, t)}} 1 \ll XM(\log X)^2 \sum_{\substack{D^{1/4}Y^{-1}<d_1\le 2D^{1/4}\\ P^+(d_1)\le Y}} \frac{1}{d_1} \ll XM(\log X)^2 \exp\left(-\frac{1}{4}\frac{\log V}{\log Y}\log\left(\frac{\log V}{\log Y}\right)\right).$$

We return to the sum in (6.16). We observe that $|a_m^{(1)}| \ll |I|$ for $|m| \leq |I|^{-1}$ and $|a_m^{(1)}| \ll |m|^{-1}$ for $|m| \geq |I|^{-1}$. From (6.17), we obtain,

$$S_1 \ll X \exp\left(-\frac{1}{16} \frac{\log V}{\log Y} \log\left(\frac{\log V}{\log Y} \right) \right).$$

We now treat S_2. We note that

$$\text{meas}(\mathfrak{M}) \ll Q^2 X^{-1} (\log X)^A$$

and thus we have

$$\begin{aligned} S_2 &\ll \sum_{\substack{X \geq d_1 > V \\ P^+(d_1) \leq Y}} \sum_{d_2 \leq X} \int_{I \cap \mathfrak{M}} |E_{d_1}(X,\alpha) \overline{E_{d_2}(X,\alpha)}| \, d\alpha \\ &\ll X Q^2 (\log X)^A \sum_{\substack{X \geq d_1 > V \\ P^+(d_1) \leq Y}} \sum_{d_2 \leq X} \frac{1}{d_1 d_2} \\ &\ll X \exp\left(-\frac{1}{2} \frac{\log V}{\log Y} \log\left(\frac{\log V}{\log Y} \right) \right), \end{aligned}$$

by the result of [2] mentioned earlier. Thus we finally get

$$\sum_{\substack{X \geq d_1 > V \\ P^+(d_1) \leq Y}} \sum_{d_2 \leq X} \int_{(d_1,d_2)} \ll X \exp\left(-\frac{1}{16} \frac{\log V}{\log Y} \log\left(\frac{\log V}{\log Y} \right) \right). \tag{6.35}$$

The theorem now follows from (3.1), (4.18), (5.4) and (6.35) provided we establish a good positive lower bound for the quantity C^* (this is done in the next section).

7. The size of C^*

We have

$$C^* = (\log X) \sum_{\substack{d_1 \leq V, d_1 | P(Y) \\ d_2 \leq V, d_2 | P(Y) \\ D = \gcd(d_1,d_2) > Q}} \frac{D \mu(d_1) \mu(d_2)}{d_1 d_2}. \tag{7.1}$$

We write $d_1 = Dd_1', d_2 = Dd_2'$ and obtain

$$C^* = (\log X) \sum_{\substack{D | P(Y) \\ Q < D \leq V}} \frac{1}{D} \sum_{\substack{d_1' \leq \frac{V}{D}, d_1' | \frac{P(Y)}{D} \\ d_2' \leq \frac{V}{D}, d_2' | \frac{P(Y)}{D}}} \frac{\mu(d_1') \mu(d_2')}{d_1' d_2'} \sum_{t | \gcd(d_1', d_2')} \mu(t). \tag{7.2}$$

We set $d_1' = d_1'' t, d_2' = d_2'' t$ and obtain

$$C^* = (\log X) \sum_{\substack{D \mid P(Y) \\ Q < D \le V}} \frac{1}{D} \sum_{\substack{t \mid \frac{P(Y)}{D}, \\ t \le V D^{-1}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{\substack{d'' \mid \frac{P(Y)}{(Dt)} \\ d'' \le V (Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg)^2. \tag{7.3}$$

We split the D-sum into

$$\begin{aligned} \sum_{\substack{D \mid P(Y) \\ Q < D \le V}} \frac{1}{D} \{\cdots\} &= \sum_{\substack{D \mid P(Y) \\ Q < D \le V^{1/2}}} \frac{1}{D} \{\cdots\} + \sum_{\substack{D \mid P(Y) \\ V^{1/2} < D \le V}} \frac{1}{D} \{\cdots\} \\ &= \sum_{I} + \sum_{II} \quad \text{(say)}. \end{aligned} \tag{7.4}$$

Again, we split $\sum_I$ into

$$\begin{aligned} \sum_I &= \sum_{\substack{D \mid P(Y) \\ Q < D \le V^{1/2}}} \frac{1}{D} \sum_{\substack{t \mid \frac{P(Y)}{D}, \\ t \le V D^{-1}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{\substack{d'' \mid \frac{P(Y)}{(Dt)} \\ d'' \le V (Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg)^2 \\ &= \sum_{\substack{D \mid P(Y) \\ Q < D \le V^{1/2}}} \frac{1}{D} \sum_{\substack{t \mid \frac{P(Y)}{D}, \\ t \le V^{1/4}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{\substack{d'' \mid \frac{P(Y)}{(Dt)} \\ d'' \le V (Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg)^2 \\ &\quad + \sum_{\substack{D \mid P(Y) \\ Q < D \le V^{1/2}}} \frac{1}{D} \sum_{\substack{t \mid \frac{P(Y)}{D}, \\ V^{1/4} < t \le V D^{-1}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{\substack{d'' \mid \frac{P(Y)}{(Dt)} \\ d'' \le V (Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg)^2 \\ &= \sum_{I,1} + \sum_{I,2} \quad \text{(say)}. \end{aligned} \tag{7.5}$$

We have

$$\sum_{I,1} = \sum_{\substack{D \mid P(Y) \\ Q < D \le V^{1/2}}} \frac{1}{D} \sum_{\substack{t \mid \frac{P(Y)}{D}, \\ t \le V^{1/4}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{\substack{d'' \mid \frac{P(Y)}{(Dt)} \\ d'' \le V (Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg)^2. \tag{7.6}$$

From the result of Hildebrand [2] mentioned earlier, we have

$$\Bigg| \sum_{\substack{d'' \mid \frac{P(Y)}{(Dt)}, \\ d'' > V (Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg| \le \sum_{\substack{d'' \mid P(Y), \\ d'' > V^{1/4}}} \frac{1}{d''} \ll \exp\Bigg(-\frac{1}{8} \frac{\log V}{\log Y} \log\Bigg(\frac{\log V}{\log Y} \Bigg) \Bigg). \tag{7.7}$$

Therefore, for the t-sum in (7.6), we have

$$S_t := \sum_{\substack{t \mid \frac{P(Y)}{D},\\ t \le V^{1/4}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{\substack{d'' \mid \frac{P(Y)}{(Dt)}\\ d'' \le V(Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg)^2 \tag{7.8}$$

$$= \sum_{\substack{t \mid \frac{P(Y)}{D},\\ t \le V^{1/4}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{d'' \mid \frac{P(Y)}{(Dt)}} \frac{\mu(d'')}{d''} + O\Bigg(\sum_{\substack{d'' \mid \frac{P(Y)}{(Dt)}\\ d'' > V(Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg) \Bigg)^2$$

$$= \sum_{\substack{t \mid \frac{P(Y)}{D},\\ t \le V^{1/4}}} \frac{\mu(t)}{t^2} \Bigg(\prod_{\substack{p \le Y,\\ p \nmid (Dt)}} \Big(1 - \frac{1}{p}\Big)$$

$$+ O\Bigg(\exp\Bigg(-\frac{1}{8}\frac{\log V}{\log Y}\log\Bigg(\frac{\log V}{\log Y}\Bigg)\Bigg)\Bigg)\Bigg)^2.$$

We thus have

$$\sum_{I,1} = \sum_{\substack{D \mid P(Y)\\ Q < D \le V^{1/2}}} \frac{1}{D} \sum_{\substack{t \mid \frac{P(Y)}{D},\\ t \le V^{1/4}}} \frac{\mu(t)}{t^2} \Bigg(\Bigg(\prod_{\substack{p \le Y,\\ p \nmid (Dt)}} \Big(1 - \frac{1}{p}\Big)\Bigg)^2 \tag{7.9}$$

$$+ O\Bigg(\exp\Bigg(-\frac{1}{8}\frac{\log V}{\log Y}\log\Bigg(\frac{\log V}{\log Y}\Bigg)\Bigg)\Bigg)\Bigg)$$

$$= \sum_{\substack{D \mid P(Y)\\ Q < D \le V^{1/2}}} \frac{1}{D} \sum_{t \mid \frac{P(Y)}{D}} \frac{\mu(t)}{t^2} \Bigg(\prod_{\substack{p \le Y,\\ p \nmid (Dt)}} \Big(1 - \frac{1}{p}\Big)\Bigg)^2$$

$$+ O\Bigg(\exp\Bigg(-\frac{1}{8}\frac{\log V}{\log Y}\log\Bigg(\frac{\log V}{\log Y}\Bigg)\Bigg)\Bigg)\Bigg)$$

$$+ O(V^{-1/4}(\log X))$$

$$= \sum_{\substack{D \mid P(Y)\\ Q < D \le V^{1/2}}} \frac{1}{D} \prod_{\substack{p \le Y,\\ p \nmid D}} \Big(1 - \frac{1}{p^2}\Big) \Bigg(\prod_{\substack{p \le Y,\\ p \nmid (Dt)}} \Big(1 - \frac{1}{p}\Big)\Bigg)^2$$

$$+ O\Bigg(\exp\Bigg(-\frac{1}{8}\frac{\log V}{\log Y}\log\Bigg(\frac{\log V}{\log Y}\Bigg)\Bigg)\Bigg) + O\big(V^{-1/4}(\log X)\big).$$

Therefore, we have the lower bound

$$\sum_{I,1} \gg \frac{\log(\frac{Y}{Q})}{(\log Y)^2}. \tag{7.10}$$

For $\sum_{I,2}$, we get the upper bound:

$$\sum_{I,2} = \sum_{\substack{D|P(Y)\\ Q<D\leq V^{1/2}}} \frac{1}{D} \sum_{\substack{t|P(Y)\\ t>V^{1/4}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{\substack{d''|\frac{P(Y)}{(Dt)}\\ d''\leq V(Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg)^2 \ll V^{-1/4}(\log X)^3. \tag{7.11}$$

For $\sum_{II}$, we get the upper bound:

$$\begin{aligned} \sum_{II} &= \sum_{\substack{D|P(Y)\\ V^{1/2}<D\leq V}} \frac{1}{D} \sum_{\substack{t|\frac{P(Y)}{D}\\ t\leq VD^{-1}}} \frac{\mu(t)}{t^2} \Bigg(\sum_{\substack{d''|\frac{P(Y)}{(Dt)}\\ d''\leq V(Dt)^{-1}}} \frac{\mu(d'')}{d''} \Bigg)^2 \\ &\ll (\log X)^2 \sum_{\substack{D|P(Y)\\ V^{1/2}<D\leq V}} \frac{1}{D} \\ &\ll \exp\Bigg(-\frac{1}{4}\frac{\log V}{\log Y} \log\Bigg(\frac{\log V}{\log Y}\Bigg)\Bigg) \end{aligned} \tag{7.12}$$

by the result of Hildebrand [2]. From (7.10), (7.11), and (7.12), we obtain

$$C^* \gg \frac{\log X}{(\log Y)^2}.$$

This completes the proof of the theorem.

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