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HILBERTIAN MATRIX CROSS NORMED SPACES ARISING FROM NORMED IDEALS

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ABSTRACT. Generalizing Pisier's idea, we introduce a Hilbertian matrix cross normed space associated with a pair of symmetric normed ideals. When the two ideals coincide, we show that our construction gives an operator space if and only if the ideal is the Schatten class. In general, a pair of symmetric normed ideals that are not necessarily the Schatten class may give rise to an operator space. We study the space of completely bounded mappings between the matrix cross normed spaces obtained in this way and show that the multiplicator norm naturally appears as the completely bounded norm.

1. Introduction

An operator space is a subspace of the set of bounded operators on a Hilbert space, which is abstractly characterized as a Banach space equipped with matrix norms satisfying certain properties. An operator space whose base space is a Hilbert space is said to be a Hilbertian operator space. The theory of homogeneous Hilbertian operator space is one of the central topics in operator space theory and it plays an essential role in various situations. For example, it is used to analyze the structures of the space of operator spaces with the metric which is analogous to the Banach–Mazur distance (cf. [18]) and to obtain an embedding of operator spaces into noncommutative L_p-spaces (cf. [11] and [16]).

The relationships between homogeneous Hilbertian operator spaces and operator ideals are first studied by Mathes and Paulsen. Mathes and Paulsen considered in [14] a larger category, called matricially cross normed spaces (m.c.n. spaces), than that of operator spaces. They showed that if H_1 and H_2 are homogeneous Hilbertian m.c.n. spaces with the common base space H,

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then the space of completely bounded mappings $CB(H_1, H_2)$ becomes a symmetric normed ideal (s.n. ideal) [14, Proposition 1.2] and showed that every s.n. ideal on $B(H)$ which is not equivalent to the ideal of compact operators or the ideal of trace class operators is isomorphic as a set to the space of completely bounded mappings on some homogeneous Hilbertian m.c.n. spaces [14, Theorem 2.2].

Pisier showed that the norm of the elements in the interpolating spaces between the row Hilbert space and the column Hilbert space is represented by the operator norm on the Schatten ideals [18, Theorem 8.4]. Inspired by this analysis, in our paper we introduce a Hilbertian m.c.n. space $H(\Phi, \Psi)$ for a pair of symmetric norming functions (s.n. functions) Φ, Ψ with $\Phi \geq \Psi$ and investigate the structure of the space. The matrix norm of $H(\Phi, \Psi)$ is defined by

$$\|T\|_{H(\Phi,\Psi)} = \left(\sup_x \frac{\| \sum T_i x T_i^* \|_\Psi}{\|x\|_\Phi} \right)^{1/2},$$

where $T = \sum \xi_i \otimes T_i \in H \otimes M_n$ and (ξ_i) is an orthonormal basis of a separable Hilbert space H. We also focus on the space of completely bounded mappings between two spaces arising in this way. The m.c.n. space $H(\Phi, \Psi)$ is not always an operator space. In Section 3, we show that if the m.c.n. space $H(\Phi, \Psi)$ is an operator space, then for all $x, y, z \in \mathfrak{S}_\Phi$ the following inequality

$$\frac{\|x \otimes y\|_\Psi}{\|x\|_\Psi} \leq \frac{\|z \otimes y\|_\Phi}{\|z\|_\Phi}$$

is satisfied, where $\mathfrak{S}_\Phi$ is the s.n. ideal arising from Φ. In particular, when $\Phi = \Psi$ we show that the m.c.n. space $H(\Phi) = H(\Phi, \Phi)$ is an operator space if and only if Φ is the Schatten norm. However, the situation differs for $\Phi \neq \Psi$. Indeed, when Φ is a Q*-norm and Ψ is a Q-norm, $H(\Phi, \Psi)$ is always an operator space.

We also study the space of completely bounded mappings between m.c.n. spaces we constructed. We determine the completely bounded norm from the row Hilbert space R to $H(\Phi, \Psi)$ as

$$\|x\|_{CB(R,H(\Phi,\Psi))} = \left(\sup_y \frac{\| |x|^2 \otimes y\|_\Psi}{\|y\|_\Phi} \right)^{1/2}.$$

This implies that if $H(\Phi, \Psi)$ is an operator space, then we have the isometric isomorphisms $CB(R, H(\Phi, \Psi)) = \mathfrak{S}_{\tilde{\Psi}}$ and $CB(C, H(\Phi, \Psi)) = \mathfrak{S}_{\tilde{\Phi}^*}$ for the column Hilbert space C (see Section 3 for the definition of $\tilde{\Phi}$).

The above result leads us to consider the condition:

$$\exists c > 0, \quad \|x \otimes y\|_\Psi \leq c\|x\|_\Psi \|y\|_\Psi, \quad \forall x, y \in \mathfrak{S}_\Psi.$$

This condition implies that there exists a constant

$$p = \lim_{n\to\infty} \frac{\log n}{\log \|P_n\|_\Phi} \quad (P_n \text{ is any rank } n \text{ projection})$$

such that $\|x\|_p \leq c\|x\|_\Phi$, where $\|x\|_p$ is the Schatten p-norm. This together with a dual version implies the above mentioned fact that $H(\Phi)$ is an operator space only if Φ is the Schatten norm.

2. Preliminaries

In this section, we collect the basics of the theory of operator spaces and operator ideals, which are often used in the paper. We refer to [9] and [17] for the theory of operator spaces and to [10] for the theory of operator ideals.

An operator space is abstractly characterized as follows. We consider a Banach space E such that for each $n \in \mathbb{N}$ there is a norm $\|\cdot\|_n$ on the matrix space $M_n(E)$ of $n \times n$ matrices with entries in the elements of E and the family $\{M_n(E), \|\cdot\|_n\}$ with $\|\cdot\|_1$ equal to the original norm of E. Then we can consider the two properties:

(M1) $\|\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\|_{m+n} = \mathrm{Max}\{\|x\|_m, \|y\|_n\}$ for any $x \in M_m(E), y \in M_n(E)$, and $m, n \in \mathbb{N}$, and

(M2) $\|axb\|_n \leq \|a\|\|x\|_m\|b\|$ for any $x \in M_m(E)$, $a \in M_{n\times m}$, $b \in M_{m\times n}$, and $m, n \in \mathbb{N}$, where $M_{m\times n} = M_{m\times n}(\mathbb{C})$ and axb means the matrix product.

(M1) may be replaced with

(M1)′ $\|\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\|_{m+n} \leq \mathrm{Max}\{\|x\|_m, \|y\|_n\}$, for any $x \in M_m(E)$, $y \in M_n(E)$, and $m, n \in \mathbb{N}$.

For a Hilbert space H, an operator space $E \subseteq B(H)$ is a Banach space satisfying the properties (M1) and (M2) under the identification of $M_n(E)$ as a subspace of $M_n(B(H)) = B(H^n)$. Conversely, Ruan [15, Theorem 3.1] showed that a Banach space having the matrix norm structure with the properties (M1) and (M2) has an isometric embedding into the space $B(H)$ for some Hilbert space H such that the matrix norms come from $M_n(B(H)) = B(H^n)$. The properties (M1) and (M2) are called Ruan's axioms. In the operator space category, the morphisms are the completely bounded (c.b.) mappings. Let E, F be operator spaces and u be a linear mapping from E to F. We say that u is completely bounded if

$$\|u\|_{cb} = \sup_n \|id_n \otimes u : M_n(E) \to M_n(F)\| < \infty,$$

where $M_n(E)$ is identified with the algebraic tensor product $M_n \otimes E$. The completely bounded norm of u is defined by $\|u\|_{cb}$. An operator space E is said to be homogeneous if for any bounded linear mapping u on E we have $\|u\| = \|u\|_{cb}$. We denote the Banach space of completely bounded mappings from E to F with norm $\|\cdot\|_{cb}$ by $CB(E, F)$.

The category of matrix cross normed spaces is larger than that of operator spaces. Let H be a separable Hilbert space with a sequence of matrix norms $\{\|\cdot\|_n\}_{n=1}^\infty$ on the family $\{M_n(H)\}_{n=1}^\infty$ such that $\|\cdot\|_1$ coincides with the norm of H. We call H a matrix cross normed space (m.c.n. space) if

$$\|x \otimes A\|_n = \|x\|\|A\|_{M_n}$$

for all $x \in H, A \in M_n$, and $n \in \mathbb{N}$.

For a finite-dimensional or separable infinite-dimensional Hilbert space K with dimension n, identifying $B(K)$ with the matrix space M_n we denote the matrix whose (i,j)-entry is 1 and the other entries are 0 by e_{ij}.

Next, we introduce the basic theory of the operator ideals (cf. [10 Chapter III]). Let c_0, $\hat{c}$, and $\hat{k}$ be the spaces of sequences of real numbers defined by

$$\begin{aligned}
c_0 &= \Big\{\xi = \{\xi_i\} : \lim_{i\to\infty} \xi_i = 0\Big\},\\
\hat{c} &= \big\{\xi = \{\xi_i\} \in c_0 : \text{only finitely many } \xi_i\text{'s are nonzero}\big\},\\
\hat{k} &= \big\{\xi = \{\xi_i\} \in \hat{c} : \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \geq \cdots \geq 0\big\},
\end{aligned}$$

respectively. A real valued function Φ on $\hat{c}$ is called a symmetric norming (s.n.) function if it satisfies the followings:

(1) Φ is a norm on $\hat{c}$;
(2) $\Phi(1,0,0,\ldots) = 1$;
(3) $\Phi(\xi_1,\xi_2,\ldots,\xi_n,0,0,\ldots) = \Phi(|\xi_{j_1}|,|\xi_{j_2}|,\ldots,|\xi_{j_n}|,0,0,\ldots)$ for all $\xi \in \hat{c}$, where $\{j_1,j_2,\ldots,j_n\}$ is any permutation of $\{1,2,\ldots,n\}$.

For an s.n. function Φ, we set

$$c_\Phi = \Big\{\xi = \{\xi_i\} \in c_0 : \sup_n \Phi\big(\xi^{(n)}\big) < \infty\Big\},$$

where $\xi^{(n)} = (\xi_1,\ldots,\xi_n,0,0,\ldots)$. We extend the domain of Φ by

$$\Phi(\xi) = \lim_{n\to\infty} \Phi\big(\xi^{(n)}\big), \quad \xi \in c_\Phi.$$

For $1 \leq p \leq \infty$, we denote by Φ_p the ℓ_p-norm.

Throughout the paper, H denotes a separable infinite-dimensional Hilbert space with an orthonormal basis $\{\xi_i\}_{i=1}^\infty$ and $\mathfrak{S}_\infty$ denotes the subspace of $B(H)$ consisting of all compact operators on H. For $x \in \mathfrak{S}_\infty$, we denote by $\{s_j(x)\}_{j=1}^\infty$ the singular numbers (s-numbers) of x, i.e., the nonincreasing rearrangement of eigenvalues of $|x|$.

Let $\mathfrak{S}$ be a two-sided ideal of $B(H)$. A functional $\|\cdot\|_s$ on $\mathfrak{S}$ is said to be a symmetric norm if it satisfies the followings:

(1) $\|\cdot\|_s$ is a norm on $\mathfrak{S}$;
(2) for any rank one operator $x, \|x\|_s = \|x\|$;
(3) $\|axb\|_s \leq \|a\|\|x\|_s\|b\|$ $(\forall a,b \in B(H),\ \forall x \in \mathfrak{S})$.

We call $(\mathfrak{S}, \|\cdot\|_s)$ a symmetrically normed ideal if $\|\cdot\|_s$ is a symmetric norm on $\mathfrak{S}$ and makes $\mathfrak{S}$ a Banach space.

For an s.n. function Φ, we denote by $\mathfrak{S}_\Phi$ the set of operators $x \in \mathfrak{S}_\infty$ with $s(x) = \{s_j(x)\} \in c_\Phi$, and put

$$\|x\|_\Phi = \Phi(s(x)).$$

Then $\mathfrak{S}_\Phi$ is an s.n. ideal with the norm $\|\cdot\|_\Phi$. In this paper, we often use the property

$$xx^* \in \mathfrak{S}_\Phi \quad \Leftrightarrow \quad x^*x \in \mathfrak{S}_\Phi \quad \text{and} \quad \|xx^*\|_\Phi = \|x^*x\|_\Phi.$$

Let Φ be an s.n. function. The function

$$\Phi^*(\eta) = \max_{\xi \in \hat{k}} \left\{ \frac{1}{\Phi(\xi)} \sum_i \eta_i^* \xi_i \right\}.$$

makes sense for any $\eta \in \hat{c}$ and Φ^* is an s.n. function. We call Φ^* the adjoint of Φ. Note that for any s.n. function Φ, we have $(\Phi^*)^* = \Phi$ and the following duality

$$\|x\|_\Phi = \sup_{\|y\|_{\Phi^*} \le 1} |\operatorname{Tr}(yx)|.$$

We introduce a few classes of normed ideals used in this paper. We denote by $\mathfrak{S}_p = \mathfrak{S}_{\Phi_p}$ the Schatten ideal for $1 \le p \le \infty$. For $1 \le q \le p < \infty$, the Lorentz ideal $S_{p,q}$ is an s.n. ideal whose norm is given by

$$\|x\|_{p,q} = \left(\sum_{j=1}^\infty \frac{s_j(x)^q}{j^{1-q/p}} \right)^{1/q}.$$

Let $1 = \pi_1 \ge \pi_2 \ge \cdots \ge 0$ be a sequence of nonincreasing positive numbers such that $\lim_{n\to\infty} \pi_n = 0$ and $\sum_{n=1}^\infty \pi_n = \infty$. We say that such a sequence is binormalizing. The s.n. function Φ_π is defined by

$$\Phi_\pi(a) = \sum_{n=1}^\infty \pi_n a_n^*, \quad a = (a_n),$$

where (a_n^*) is the nonincreasing rearrangement of (a_n). Note that if $q = 1$, then the Lorentz ideal $S_{p,1}$ is equal to the ideal $\mathfrak{S}_{\Phi_\pi}$ defined by the binormalizing sequence $\pi_j = j^{1/p-1}$.

Finally, we introduce an important class of operator spaces. If E_0, E_1 are compatible Banach spaces, then we denote by $(E_0, E_1)_\theta$ for $0 < \theta < 1$ the complex interpolation space of them (see [5, Chapter 4]). If E_0, E_1 are operator spaces whose base spaces are compatible, we construct an operator space complex interpolation by identifying $M_n((E_0, E_1)_\theta)$ with $(M_n(E_0), M_n(E_1))_\theta$ for each $n \in \mathbb{N}$. We denote by R and C the row and column operator space respectively [9, Section 3.4]. These spaces are homogeneous Hilbertian operator spaces whose matrix norms are given by

$$\left\| \sum_{i=1}^n \xi_i \otimes T_i \right\|_R = \left\| \sum_{i=1}^n T_i T_i^* \right\|^{1/2}, \qquad \left\| \sum_{i=1}^n \xi_i \otimes T_i \right\|_C = \left\| \sum_{i=1}^n T_i^* T_i \right\|^{1/2},$$

for a finite sequence of matrices $\{T_i\}_{i=1}^n$. Note that $R^* = C$ and $C^* = R$ in the operator space category. We denote by $R(\theta)$ the operator space complex interpolation $(R, C)_\theta$ for $0 < \theta < 1$, which is a homogeneous Hilbertian operator space. We set $R(0)$ to be the row Hilbert space R and $R(1)$ to be the column Hilbert space C. When $\theta = 1/2$, we write $OH = R(1/2)$. Pisier [18, Theorem 1.1] introduced these spaces and showed that for any finite sequence $\{T_i\}$ it holds that

$$\left\|\sum_i \xi_i \otimes T_i\right\|_{OH} = \left\|\sum_i T_i \otimes \bar{T}_i\right\|^{1/2},$$

where $\bar{T}_i$ means the complex conjugate of T_i. Another important property of OH is the self-duality. For an operator space E, the operator space $\bar{E}$ means its complex conjugate. The matrix norms of the elements of $\bar{E}$ are defined by

$$\|(\overline{x_{ij}})\|_{M_n(\bar{E})} = \|(x_{ij})\|_{M_n(E)}.$$

Pisier showed in [18, Theorem 1.1] the completely isometric identification

$$OH = \overline{OH^*}.$$

Another important example of a homogeneous Hilbertian operator space is the minimal operator space $H_{\min}$. Let E be a Banach space. We can embed E into a commutative C^*-algebra (for example the space of all continuous functions on the unit ball of E^* equipped with the weak topology). We denote by $\min(E)$ the operator space whose matrix norms arise form this embedding. The minimal operator space norm is the minimal norm among all operator space norms. When E is a Hilbert space H, we denote the minimal operator space by $H_{\min}$. The matrix norm on $H_{\min}$ satisfies

$$\left\|\sum_{i=1}^m \xi_i \otimes T_i\right\|_{\min} = \sup\left\|\sum_{i=1}^m v_i T_i\right\|,$$

where the supremum is taken over all unit vectors $\{v_i\}$ of ℓ_2^m.

3. Basic properties of the m.c.n. space $H(\Phi, \Psi)$

Let K be a separable Hilbert space which is identified with a subspace of separable infinite-dimensional Hilbert space. For $n \in \mathbb{N} \cup \{\infty\}$, we denote by I_n the identity operator on the Hilbert space of dimension n. Let T be a finite sum $T = \sum_i \xi_i \otimes T_i$ in the algebraic tensor product $H \otimes B(K)$ and we set $T^* = \sum_i \xi_i \otimes T_i^*$. Pisier showed the identification of matrix norms of $R(\theta)$ $(0 \le \theta \le 1)$ in [18, Theorem 8.4] as follows:

$$\left\|\sum_i \xi_i \otimes T_i\right\|_{R(\theta)\otimes_{\min} B(K)} = \sup\left\{\left\|\sum_i T_i x T_i^*\right\|_p^{1/2} : x \in \mathfrak{S}_{p,+}, \|x\|_p \le 1\right\},$$

where $p = \theta^{-1}$. We define the operators ρ_T and ρ_{T^*} on $B(K)$ by

$$\rho_T(x) = \sum T_i x T_i^*, \quad x \in B(K),$$
$$\rho_{T^*}(x) = \sum T_i^* x T_i, \quad x \in B(K).$$

Neither ρ_T nor ρ_{T^*} depends on the choice of the basis $\{\xi_i\}_{i=1}^{\infty}$. If $\mathfrak{S}$ is a two-sided ideal in $B(K)$, we have $\rho_T(\mathfrak{S}) \subseteq \mathfrak{S}$ and $\rho_{T^*}(\mathfrak{S}) \subseteq \mathfrak{S}$. For fixed s.n. functions Φ and Ψ with $\Psi \leq \Phi$, we define a norm $\|\cdot\|_{\Phi,\Psi}$ on the space of finite sums $T \in H \otimes B(K)$ by

$$\|T\|_{\Phi,\Psi} = \|\rho_T : \mathfrak{S}_\Phi \to \mathfrak{S}_\Psi\|^{1/2}.$$

Now, we introduce an m.c.n. space $H(\Phi, \Psi)$ whose matrix norm structure is given by identifying $M_n(H(\Phi, \Psi))$ with $(H \otimes M_n, \|\cdot\|_{\Phi,\Psi})$. We write $H(\Phi) = H(\Phi, \Phi)$ for simplicity. Before proving that $H(\Phi, \Psi)$ is a homogeneous m.c.n. space, we prove a useful formula. We denote by $F(K)$ and $U(K)$ the subsets of $B(K)$ consisting of all finite-rank operators and all unitary operators, respectively. If S is a subset of $B(K)$, we denote by S_+ the subset of S consisting of positive elements in $B(K)$.

Lemma 3.1. *For any operator T, we have the equality*

$$\|T\|_{\Phi,\Psi}^2 = \sup\{\operatorname{Tr}(a\rho_T(b))\} = \|T^*\|_{\Psi^*,\Phi^*}^2,$$

where the supremum is taken over all $a, b \in F(K)_+$ with $\|a\|_{\Psi^*} \leq 1$ and $\|b\|_\Phi \leq 1$.

Proof. Note first that for any $b \in \mathfrak{S}_\Phi$ it holds that

$$\|b\|_\Phi = \sup_{\substack{a \in F(K) \\ \|a\|_{\Phi^*} \leq 1}} |\operatorname{Tr}(ab)|,$$

and if a is positive we can choose b to be also positive [10, proof of Theorem 12.2]. The trace duality implies

$$\|\rho_T : \mathfrak{S}_\Phi \to \mathfrak{S}_\Psi\| = \sup_{\|b\|_\Phi \leq 1} \|\rho_T(b)\|_\Psi = \sup_{\substack{\|b\|_\Phi \leq 1 \\ \|a\|_{\Psi^*} \leq 1}} |\operatorname{Tr}(a\rho_T(b))|.$$

If we let $a = u|a|$ and $b = v|b|$ be the polar decompositions of a and b, respectively, by the Schwarz inequality we have

$$\begin{aligned}
|\operatorname{Tr}(a\rho_T(b))| &\leq \operatorname{Tr}\Big(\sum_i ||a|^{\frac{1}{2}} T_i v |b|^{\frac{1}{2}}|^2\Big)^{1/2} \operatorname{Tr}\Big(\sum_i ||a|^{\frac{1}{2}} u^* T_i |b|^{\frac{1}{2}}|^2\Big)^{1/2} \\
&= \operatorname{Tr}(|a|\rho_T(v|b|v^*))^{1/2} \operatorname{Tr}(u|a|u^*\rho_T(|b|))^{1/2} \\
&\leq \sup_{\substack{x,y \geq 0 \\ \|x\|_{\Psi^*}, \|y\|_\Phi \leq 1}} \operatorname{Tr}(x\rho_T(y)).
\end{aligned}$$

Thus,

$$\begin{aligned}\|\rho_T : \mathfrak{S}_\Phi \to \mathfrak{S}_\Psi\| &= \sup_{\substack{x,y\geq 0\\ \|x\|_{\Psi^*},\|y\|_\Phi\leq 1}} \mathrm{Tr}(x\rho_T(y)) = \sup_{\substack{y\geq 0\\ \|y\|_\Phi\leq 1}} \|\rho_T(y)\|_\Psi \\ &= \sup_{\substack{x\in F(K)_+,y\geq 0\\ \|x\|_{\Psi^*},\|y\|_\Phi\leq 1}} \mathrm{Tr}(x\rho_T(y)) = \sup_{\substack{x\in F(K)_+\\ \|x\|_{\Psi^*}\leq 1}} \|\rho_{T^*}(x)\|_{\Phi^*} \\ &= \sup_{\substack{x,y\in F(K)_+\\ \|x\|_{\Psi^*},\|y\|_\Phi\leq 1}} \mathrm{Tr}(x\rho_T(y)).\end{aligned}$$ □

LEMMA 3.2. *The space $H(\Phi,\Psi)$ is homogeneous.*

Proof. Let $A\in B(H)$. It suffices to show that for any finite sequence

$$T=\sum_{i=1}^m \xi_i\otimes T_i\in H\otimes M_n$$

and $x\in M_{n,+}$, the norm inequality

$$\left\|\rho_{(A\otimes I)T}(x)\right\|_\Psi \leq \|A\|^2\|\rho_T(x)\|_\Psi.$$

holds. Let H_0 be the finite-dimensional subspace of H spanned by $\{A\xi_i\}_{i=1}^m$ and $\{\eta_j\}_{j=1}^k$ be an orthonormal basis of H_0. Then $k\leq m$ and there is an $m\times k$-matrix $B=(b_{ij})$ such that $\|B\|\leq\|A\|$ and $A\xi_i=\sum_{j=1}^k b_{ij}\eta_j$. Note that

$$(A\otimes I_n)T=\sum_i A\xi_i\otimes T_i=\sum_j \eta_j\otimes\left(\sum_i b_{ij}T_i\right).$$

Thus, if we let $S_j=\sum_i b_{ij}T_i$ for $1\leq j\leq k$, then

$$\begin{aligned}&\left\|\rho_{(A\otimes I)T}(x)\right\|_\Psi \\ &=\left\|\sum_j S_jxS_j^*\right\|_\Psi \\ &=\left\|\begin{pmatrix}S_1 & \cdots & S_k\\ & \bigcirc & \end{pmatrix}(I_k\otimes x)\begin{pmatrix}S_1^* & \\ \vdots & \bigcirc\\ S_k^* & \end{pmatrix}\right\|_\Psi \\ &=\left\|(I_k\otimes x^{\frac12})\begin{pmatrix}S_1^* & \\ \vdots & \bigcirc\\ S_k^* & \end{pmatrix}\begin{pmatrix}S_1 & \cdots & S_k\\ & \bigcirc & \end{pmatrix}(I_k\otimes x^{\frac12})\right\|_\Psi \\ &=\left\|(I_k\otimes x^{\frac12})(B^*\otimes I_n)\begin{pmatrix}T_1^* & \\ \vdots & \bigcirc\\ T_m^* & \end{pmatrix}\begin{pmatrix}T_1 & \cdots & T_m\\ & \bigcirc & \end{pmatrix}(B\otimes I_n)(I_k\otimes x^{\frac12})\right\|_\Psi\end{aligned}$$

$$
\begin{aligned}
&\leq \|B\|^2 \left\| (I_m \otimes x^{\frac{1}{2}}) \begin{pmatrix} T_1^* & \\ \vdots & \bigcirc \\ T_m^* & \end{pmatrix} \begin{pmatrix} T_1 & \cdots & T_m \\ & \bigcirc & \end{pmatrix} (I_m \otimes x^{\frac{1}{2}}) \right\|_\Psi \\
&\leq \|A\|^2 \|\rho_T(x)\|_\Psi . \qquad \square
\end{aligned}
$$

Proposition 3.3. *The space $H(\Phi, \Psi)$ is an m.c.n. space and satisfies the Ruan's axiom* (M2).

Proof. Let T and S be finite sums defined by

$$T = \sum_i \xi_i \otimes T_i, \qquad S = \sum_i \xi_i \otimes S_i,$$

and let $a, b \in F(K)_+$. Then

$$
\begin{aligned}
&\operatorname{Tr}(a\rho_{T+S}(b)) \\
&\quad = \sum_i \operatorname{Tr}\big(a(T_i + S_i)b(T_i^* + S_i^*)\big) \\
&\quad = \operatorname{Tr}(a\rho_T(b)) + \operatorname{Tr}(a\rho_S(b)) + \sum_i \big(\operatorname{Tr}(aT_ibS_i^*) + \operatorname{Tr}(aS_ibT_i^*)\big) \\
&\quad \leq \operatorname{Tr}(a\rho_T(b)) + \operatorname{Tr}(a\rho_S(b)) + 2\sqrt{\sum_i \operatorname{Tr}(aT_ibT_i^*)}\sqrt{\sum_i \operatorname{Tr}(aS_ibS_i^*)} \\
&\quad = \operatorname{Tr}(a\rho_T(b)) + \operatorname{Tr}(a\rho_S(b)) + 2\sqrt{\operatorname{Tr}(a\rho_T(b)) \operatorname{Tr}(a\rho_S(b))} \\
&\quad = \big(\operatorname{Tr}(a\rho_T(b))^{1/2} + \operatorname{Tr}(a\rho_S(b))^{1/2}\big)^2 .
\end{aligned}
$$

Thus, $\|T+S\|_{\Phi,\Psi} \leq \|T\|_{\Phi,\Psi} + \|S\|_{\Phi,\Psi}$. If $T = \xi \otimes A$ is a simple tensor product with $\|\xi\| = 1$, then

$$\|\rho_T(x)\|_\Psi = \|AxA^*\|_\Psi \leq \|A\|\|x\|_\Psi\|A\| \leq \|A\|\|x\|_\Phi\|A\|.$$

Conversely,

$$\|T\|^2_{\Phi,\Psi} \geq \sup_p \|ApA^*\|_\Psi = \sup_p \|pA^*Ap\|_\Psi = \|A\|^2,$$

where p runs over all rank one projections. Thus, $\|\xi \otimes A\|_{\Phi,\Psi} = \|\xi\|\|A\|$ and hence, $H(\Phi, \Psi)$ is an m.c.n. space. Finally, if X and Y are scalar matrices, then

$$
\begin{aligned}
\|XTY\|^2_{\Phi,\Psi} &= \sup_{a,b} \frac{|\operatorname{Tr}(\sum_i XT_iYaY^*T_i^*X^*b)|}{\|a\|_\Phi\|b\|_{\Psi^*}} \\
&= \sup_{a,b} \frac{|\operatorname{Tr}(\sum_i XT_iYaY^*T_i^*X^*b)|}{\|YaY^*\|_\Phi\|X^*bX\|_{\Psi^*}} \frac{\|YaY^*\|_\Phi\|X^*bX\|_{\Psi^*}}{\|a\|_\Phi\|b\|_{\Psi^*}} \\
&\leq \|T\|^2_{\Phi,\Psi}\|X\|^2\|Y\|^2.
\end{aligned}
$$

This shows that $H(\Phi, \Psi)$ satisfies Ruan's axiom (M2). $\square$

Let us see some examples. Thanks to [18, Theorem 8.4], we have $H(\Phi_\infty)=R$ and $H(\Phi_1)=C$.

Let H_1 be a homogeneous Hilbertian m.c.n. space and Φ be an s.n. function. Mathes and Paulsen [14, p. 1764] define a new m.c.n. space $H_{1,\Phi}$ whose matrix norm is defined by

$$\|T\|_{H_1,\Phi} = \sup_{x\in\mathfrak{S}_\Phi,\|x\|_\Phi\le 1} \|(x\otimes I)T\|_{H_1}, \quad T\in H\otimes B(K).$$

It is easy to see that $H_{1,\Phi}$ is an m.c.n. space. For example, $H_{\Phi_\infty}=H$ and $H_{\Phi_1}=H_{\min}$ (see [14, Proposition 1.3]). If we are given an s.n. function Φ, let $\tilde{\Phi}$ be the 2-convexification of Φ defined by

$$\tilde{\Phi}(a_1,\dots,a_n,\dots)=\Phi(a_1^2,\dots,a_n^2,\dots)^{1/2}, \quad a\in\hat{k}.$$

LEMMA 3.4. *For any s.n. functions Φ and Ψ with $\Phi\ge\Psi$, we have the completely isometric identifications*

- $H(\Phi_1,\Phi)=C_{\widetilde{\Phi^*}}$,
- $H(\Phi,\Phi_\infty)=R_{\tilde{\Phi}}$,
- $H(\Phi,\Psi)_{\Phi_2}=H_{\min}$.

In particular, $H(\Phi_1,\Phi_\infty)=H_{\min}$.

Proof. We first prove the second equation. Let T be a finite sum defined by

$$T=\sum_i \xi_i\otimes T_i\in H\otimes B(K).$$

Then

$$\|T\|^2_{\Phi,\Phi_\infty} = \sup_{\substack{a,b\in F(K)_+\\ \|a\|_\Phi,\|b\|_{\Phi_1}\le 1}} \operatorname{Tr}(b\rho_T(a)).$$

If we write the spectral decomposition of b by $b=\sum_i\lambda_i p_i$ with rank one projections $\{p_i\}$, then

$$\operatorname{Tr}(b\rho_T(a))=\sum_i\lambda_i\operatorname{Tr}(p_i\rho_T(a))\le\|b\|_1\operatorname*{Max}_i\{\operatorname{Tr}(p_i\rho_T(a))\}.$$

This shows that b can be replaced by rank one projections. Thus, we have

$$\begin{aligned}
\|T\|^2_{\Phi,\Phi_\infty} &= \sup_a \sup_{\substack{p:\text{rank one}\\ \text{projection}}} \operatorname{Tr}(p\rho_T(a))\\
&= \sup_p \|\rho_{T^*}(p)\|_{\Phi^*}\\
&= \sup_p \left\| \begin{pmatrix} T_1^*p & \cdots & T_n^*p \\ & \bigcirc & \end{pmatrix} \begin{pmatrix} pT_1 & \\ \vdots & \bigcirc \\ pT_n & \end{pmatrix} \right\|_{\Phi^*}\\
&= \sup_p \|(pT_iT_j^*p)_{ij}\|_{\Phi^*}.
\end{aligned}$$

We write p as $p\zeta = \langle \zeta, \xi\rangle \xi$ with a unit vector $\xi \in K$. Then for $\eta = (\eta_i)_{i=1}^n \in K^n$ we obtain

$$\begin{aligned}(pT_iT_j^*p)_{ij}\eta &= \left(\sum_j pT_iT_j^*p\eta_j\right)_i \\ &= \left(\sum_j \langle \eta_j, \xi\rangle \langle T_iT_j^*\xi, \xi\rangle \xi\right)_i \\ &= \left(\sum_j \langle T_iT_j^*\xi, \xi\rangle p\eta_j\right)_i = \big((\langle T_iT_j^*\xi, \xi\rangle)_{ij} \otimes p\big)\eta.\end{aligned}$$

So, it holds that

$$\|T\|_{\Phi,\Phi_\infty} = \sup_\xi \|(\langle T_iT_j^*\xi, \xi\rangle)_{ij}\|_{\Phi^*}.$$

We express any positive operator $a \in \mathfrak{S}_\Phi$ with $\|a\|_\Phi \le 1$ in the form

$$a = v^* \operatorname{diag}(a_1, \dots, a_n)v,$$

where v is a unitary matrix and $a_1 \ge \cdots \ge a_n$ are eigenvalues of a. In the following we denote by a the diagonal matrices $\operatorname{diag}(a_1, \dots, a_n)$. We write $v = (v(i)_j)_{ij}$. Then $\{v(k)\}_{k=1}^n$ is an orthonormal basis of $\mathbb{C}^n$. Thus, the above supremum is equal to

$$\begin{aligned}&\sup_\xi \sup_{a\ge 0, \Phi(a)\le 1} \sup_v |\operatorname{Tr}(v^*\operatorname{diag}(a_1, \dots, a_n)v(\langle T_iT_j^*\xi, \xi\rangle)_{ij})| \\ &= \sup_\xi \sup_{a\ge 0, \Phi(a)\le 1} \sup_{\{v(k)\}_{k=1}^n} \left|\sum_{k,i,j} a_k v(k)_i v(k)_j^* \langle T_iT_j^*\xi, \xi\rangle\right| \\ &= \sup_\xi \sup_{a\ge 0, \Phi(a)\le 1} \sup_{\{v(k)\}_{k=1}^n} \left|\left\langle \sum_k a_k T(v(k))T(v(k))^*\xi, \xi\right\rangle\right|,\end{aligned}$$

where $T(v(k))$ is defined by $T(v(k)) = \sum_{i=1}^n v(k)_i T_i$. Hence,

$$\|T\|_{\Phi,\Phi_\infty}^2 = \sup_{a,\{v(k)\}} \left\|\sum_k a_k T(v(k))T(v(k))^*\right\| = \|T\|_{R_{\tilde\Phi}}^2.$$

The second equality follows from

$$\|T\|_{R_{\tilde\Phi}} = \|T^*\|_{C_{\widetilde{\Phi^*}}}.$$

The third equality holds since

$$\begin{aligned}\|T\|_{H(\Phi,\Psi)_{\Phi_2}} &= \sup_{\substack{a_1\ge\cdots\ge a_n\ge 0, \sum_i a_i \le 1 \\ \{v(k)\}_{k=1}^n \\ \|x\|_\Phi \le 1, \|y\|_{\Psi^*}\le 1}} \left|\operatorname{Tr}\left(\sum_k a_k yT(v(k))xT(v(k))^*\right)\right|^{1/2} \\ &\le \sup_k \sup_{\substack{\{v(k)\}_{k=1}^n \\ \|x\|_\Phi \le 1, \|y\|_{\Psi^*}\le 1}} |\operatorname{Tr}(yT(v(k))xT(v(k))^*)|^{1/2}\end{aligned}$$

$$= \sup_{v\in\ell_2^n,\|v\|\le 1}\left\|\xi\otimes\left(\sum_i v_iT_i\right)\right\|_{H(\Phi,\Psi)}$$
$$= \sup_{v\in\ell_2^n,\|v\|\le 1}\left\|\sum_i v_iT_i\right\| = \|T\|_{\min}.$$

Finally, these equalities imply that $H(\Phi_1,\Phi_\infty) = C_{\Phi_2} = H_{\min}$. □

To check whether $H(\Phi,\Psi)$ is an operator space, it suffices to check whether $H(\Phi,\Psi)$ satisfies Ruan's axiom (M1)$'$. The three m.c.n. spaces in Lemma 3.4 are clearly operator spaces. But, not every $H(\Phi,\Psi)$ is an operator space. We give a necessary condition for $H(\Phi,\Psi)$ to be an operator space.

THEOREM 3.5. *Let Φ and Ψ be s.n. functions with $\Phi\ge\Psi$. If the m.c.n. space $H(\Phi,\Psi)$ is an operator space, then for any $x,y,z\in\mathfrak{S}_\Phi$ the following inequality*

$$\frac{\|x\otimes y\|_\Psi}{\|x\|_\Psi}\le\frac{\|z\otimes y\|_\Phi}{\|z\|_\Phi}$$

holds. In particular, if $H(\Phi)$ is an operator space, then Φ is a cross norm.

Proof. We may suppose that x, y, and z are positive diagonal matrices in M_n ($n\in\mathbb{N}$) written by $x=\mathrm{diag}(x_i)$, $y=\mathrm{diag}(y_i)$, and $z=\mathrm{diag}(z_i)$. For each positive diagonal matrix $w_i=\mathrm{diag}(w_i)\in M_n$, let $T=\sum_{i,j=1}^n \xi_i\otimes z_i^{1/2}w_j^{1/2}e_{ij}$, Then $\rho_T(x)=\sum_{i,j} z_iw_jx_je_{ii}$, and thus

$$\|\rho_T\| = \sup_x\frac{|\operatorname{Tr}(xw)|\|z\|_\Psi}{\|x\|_\Phi} = \|w\|_{\Phi^*}\|z\|_\Psi.$$

Let S be the n-tuple of T. Since $\|\rho_T\|\ge\|\rho_S(x\otimes y)\|_\Psi/\|x\otimes y\|_\Phi$,

$$\|w\|_{\Phi^*}\|z\|_\Psi\ge\frac{|\operatorname{Tr}(xw)|\|z\otimes y\|_\Psi}{\|x\otimes y\|_\Phi}.$$

Taking the supremum over w, we obtain the required inequality. When $\Phi=\Psi$, if we let x or z be a rank one projection, then we see that Φ must be a cross norm. □

QUESTION 3.1. Is the converse of Theorem 3.5 true? Namely, if two s.n. functions Φ and Ψ satisfy the conclusion of Theorem 3.5, is $H(\Phi,\Psi)$ always an operator space?

Theorem 3.5 shows that $H(\Phi)$ is an operator space only if $\|\cdot\|$ is a cross norm. Indeed, we show in Theorem 5.3 that $H(\Phi)$ is an operator space if and only if Φ is the Schatten p-norm for some $p\in[1,\infty]$.

REMARK 3.1. Let C_q ($1\le q\le\infty$) be the operator space defined by $C_q =$

$(C,R)_{1/q}$, and we define the operator space $S_p(C_q) = (\mathfrak{S}_1 \hat{\otimes} C_q, \mathfrak{S}_\infty \otimes_{\min} C_q)_{1/p}$, where $\hat{\otimes}$ means the operator space projective tensor product (cf. [9, Section 7]). Xu showed in [19, Theorem 1] that if we define $2 \leq p \leq \infty$, $0<\theta<1$, $r, r_0(\theta), r_1(\theta)$, and q by

$$\frac{1}{r} = 1 - \frac{2}{p}, \qquad \frac{1}{r_0(\theta)} = \frac{\theta}{2r}, \qquad \frac{1}{r_1(\theta)} = \frac{1-\theta}{2r}, \qquad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{p'},$$

where $1 = 1/p + 1/p'$, then for any $x = (x_1, x_2, \ldots, x_n) \in \mathfrak{S}_p^n$,

$$\|x\|_{S_p(C_q)} = \sup\left\{ \left(\sum_k \|a x_k b\|_2^2 \right)^{1/2} \right\},$$

where the supremum is taken over all $a \in \mathfrak{S}_{r_0(\theta)}$ and $b \in \mathfrak{S}_{r_1(\theta)}$ with norm one. This is an analogue of $H(\Phi_{p_1}, \Phi_{q_1})$, where $1/p_1 = (1-\theta)(1-2/p)$ and $1/q_1 = 1 - \theta(1-2/p)$. In this case, we have $p_1 \geq q_1$.

Remark 3.2. We can introduce another construction of m.c.n. spaces. For any finite sum $T = \sum_i \xi_i \otimes T_i \in H \otimes M_n$, we define

$$\|T\|_{\Phi,\Psi}^\infty = \|\rho_{T \otimes I_\infty} : \mathfrak{S}_\Phi \to \mathfrak{S}_\Psi\|^{1/2},$$

where $T \otimes I_\infty$ acts on $B(K \otimes \ell_2)$ and $K \otimes \ell_2$ is identified with a separable infinite-dimensional Hilbert space. Then we denote by $H^\sharp(\Phi,\Psi)$ the m.c.n. space whose matrix norm structure is given by the family $(H \otimes M_n, \|\cdot\|_{\Phi,\Psi}^\infty)$. There is a case where $H^\sharp(\Phi,\Psi)$ is an operator space though $H(\Phi,\Psi)$ is not an operator space. Let Φ be the KyFan 2-norm, that is $\Phi(a) = a_1^* + a_2^*$. Then $H(\Phi)$ is not an operator space. Indeed, for $x = \mathrm{diag}(1,1) \in M_2$ it holds that $\|x \otimes x\|_\Phi = 2$, but $\|x\|_\Phi^2 = 4$. To determine $H^\sharp(\Phi)$, if we are given Hilbertian operator spaces H_1 and H_2 with the common base space H, we define the matricially normed space $H_1 \vee H_2$ with the base space H by

$$\|x\|_{M_n(H_1 \vee H_2)} = \mathrm{Max}\big\{ \|x\|_{M_n(H_1)}, \|x\|_{M_n(H_2)} \big\}.$$

It is easy to see that $H_1 \vee H_2$ is an operator space.

Proposition 3.6. *Let Φ be an s.n. function defined by*

$$\Phi(a) = a_1^* + \theta a_2^* \quad (0 < \theta \leq 1).$$

Then $H^\sharp(\Phi)$ is an operator space equal to $H^\sharp(\Phi) = H(\Phi_\infty) \vee H(\Phi_1, \Phi)$.

Proof. Let T be a finite sum defined by $T = \sum_i \xi_i \otimes T_i$. For any $x \in F(K)_+$, we write its spectral decomposition as $x = \sum_{j=1}^m s_j(x) p_j$. Then if we let

$$y = s_1(x) p_1 + s_2(x) \sum_{j=2}^m p_j,$$

then y satisfies $\|y\|_\Phi = \|x\|_\Phi$ and $x \le y$. Thus, we have

$$\begin{aligned}\|\rho_{T\otimes I_\infty}\|_\Phi &= \sup_{\frac{1}{1+\theta}\le\alpha\le 1}\ \sup_{p,q}\left\|\rho_{T\otimes I_\infty}\left(\alpha p + \frac{1-\alpha}{\theta}q\right)\right\|_\Phi \\ &= \sup_{p,q}\mathrm{Max}\left\{\|\rho_{T\otimes I_\infty}(p)\|_\Phi, \frac{\|\rho_{T\otimes I_\infty}(p+q)\|_\Phi}{1+\theta}\right\},\end{aligned}$$

where p runs over all rank one projections and q runs over all finite rank projections orthogonal to p. Now for fixed p, it is clear that

$$\|\rho_{T\otimes I_\infty}(p+q)\|_\Phi \le (1+\theta)\left\|\sum_i T_iT_i^*\right\|$$

for any projection q orthogonal to p. To show the converse, represent p as $p\eta = \langle \eta, \xi\rangle\xi$ with a unit vector ξ and write

$$\xi = \sum_{i=1}^n \phi_i \otimes \psi_i, \quad \phi_i \in \ell_2^n, \psi_i \in \ell_2.$$

If we take a projection $r \in B(\ell_2)$ such that the rank of r is not less than 2 and orthogonal to the vectors $\{\psi_i\}$ and let $q = I_n \otimes r$, then we have

$$\|\rho_{T\otimes I_\infty}(p+q)\|_\Phi \ge \left\|\sum_i T_iT_i^* \otimes r\right\|_\Phi = (1+\theta)\left\|\sum_i T_iT_i^*\right\|.$$

Thus,

$$\sup_{p,q}\mathrm{Max}\left\{\|\rho_{T\otimes I_\infty}(p)\|_\Phi, \frac{\|\rho_{T\otimes I_\infty}(p+q)\|_\Phi}{1+\theta}\right\} = \mathrm{Max}\left\{\|T\|^2_{\Phi_1,\Phi}, \left\|\sum_i T_iT_i^*\right\|\right\}.$$

□

QUESTION 3.2. Is $H^\sharp(\Phi,\Psi)$ always an operator space?

As we see below, for many two distinct s.n. functions $\Phi \neq \Psi$, the m.c.n. space $H(\Phi,\Psi)$ is an operator space. Pisier [18, Theorem 8.4] showed the completely isometrically isomorphism $H(\Phi_p,\Phi_p) = R(\theta)$, where $1 \le p \le \infty$ and $\theta = p^{-1}$. We consider whether $H(\Phi_p,\Phi_q)$ is an operator space for general p and q with $1 \le p \le q \le \infty$. In the case of $p = 1$ or $q = \infty$, $H(\Phi_p,\Phi_q)$ is an operator space from Lemma 3.4. To deal with the case $1 \le p \le 2 \le q \le \infty$, we need the following notion.

DEFINITION 3.1. Let Φ be an s.n. function. We call Φ a Q-norm if there is an s.n. function Υ such that $\tilde{\Upsilon} = \Phi$, and Φ is a Q*-norm if Φ is an adjoint of some Q-norm. In other words, an s.n. function Φ is a Q-norm if there is an s.n. function Υ such that for any $A \in \mathfrak{S}_\Phi$, the norm equality

$$\|A\|_\Phi^2 = \|A^*A\|_\Upsilon$$

is satisfied. Note that a Q-norm is smaller than or equal to the Schatten 2-norm and a Q*-norm is greater than or equal to the Schatten 2-norm. For example, the Schatten p-norm Φ_p is a Q-norm when $2 \le p \le \infty$ and is a Q*-norm when $1 \le p \le 2$. The Lorentz ideal $\Phi_{p,q}$ is a Q-norm if $2 \le q \le p$.

We use the following lemma.

LEMMA 3.7 ([7, Proposition 3]). *Let Φ be a Q^*-norm and $y = \left(\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right)$ with $y_i \in M_n$ $(i = 1, 2, 3, 4$ and $n \in \mathbb{N})$. Then we have the inequality*

$$\sum_{i=1}^{4} \|y_i\|_\Phi^2 \le \|y\|_\Phi^2.$$

THEOREM 3.8. *Let Φ be a Q^*-norm and Ψ be a Q-norm. Then $H(\Phi, \Psi)$ is an operator space.*

Proof. It suffices to check the Ruan's axiom $(\mathrm{M1})'$. Let T and S be finite sums given by

$$T = \sum_{i=1}^{k} \xi_i \otimes T_i \in M_m(H(\Phi, \Psi)) \quad \text{and} \quad S = \sum_{i=1}^{l} \xi_i \otimes S_i \in M_n(H(\Phi, \Psi)).$$

Since for any $t \in \mathbb{N}$, it follows that $\|T \oplus 0_t\|_{H(\Phi,\Psi)} = \|T\|_{H(\Phi,\Psi)}$, we may assume that $m = n$ and clearly that $k = l$. Take matrices y and z given by

$$y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, \qquad z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in M_{2n,+}$$

with $y_j, z_j \in M_n$ $(i = 1, 2, 3, 4)$. Then we have

$$\begin{aligned}
&\left|\mathrm{Tr}\left(\sum_i \begin{pmatrix} T_i & 0 \\ 0 & S_i \end{pmatrix} y \begin{pmatrix} T_i^* & 0 \\ 0 & S_i^* \end{pmatrix} z\right)\right| \\
&= \left|\sum_i \mathrm{Tr}(T_i y_1 T_i^* z_1 + T_i y_2 S_i^* z_3 + S_i y_3 T_i^* z_2 + S_i y_4 S_i^* z_4)\right| \\
&\le \mathrm{Max}\left\{\|T\|_{H(\Phi,\Psi)}^2, \|S\|_{H(\Phi,\Psi)}^2\right\} \sum_{j=1}^{4} \|y_j\|_\Phi \|z_j\|_{\Psi^*} \\
&\le \mathrm{Max}\left\{\|T\|_{H(\Phi,\Psi)}^2, \|S\|_{H(\Phi,\Psi)}^2\right\} \left\{\sum_{j=1}^{4} \|y_j\|_\Phi^2\right\}^{1/2} \left\{\sum_{j=1}^{4} \|z_j\|_{\Psi^*}^2\right\}^{1/2} \\
&\le \mathrm{Max}\left\{\|T\|_{H(\Phi,\Psi)}^2, \|S\|_{H(\Phi,\Psi)}^2\right\} \|y\|_\Phi \|z\|_{\Psi^*}.
\end{aligned}$$

The passage from the second line to the third needs the argument in [18, Remark, p. 85] applying to the second and third terms. In the third line, we use the Schwarz inequality [6, Theorem IX.5.11] and in the last line we do the preceding lemma. This shows that the axiom $(\mathrm{M1})'$ holds. $\square$

4. Completely bounded mappings between $H(\Phi, \Psi)$s

We consider the relationship between the m.c.n. spaces $H(\Phi, \Psi)$ and the space of completely bounded mappings between them. It is possible to describe the space $CB(H(\Phi_\infty), H(\Phi, \Psi))$ in terms of the multiplicator norm, which was discussed by [4] in the case of rearrangement invariant spaces on the interval $[0, 1]$.

THEOREM 4.1. *Let Φ, Ψ be s.n. functions with $\Phi \geq \Psi$ and $x \in B(H)$. Then*

$$\|x\|_{CB(R,H(\Phi,\Psi))} = \left(\sup_{a \in \mathfrak{S}_\Phi} \frac{\| |x|^2 \otimes a\|_\Psi}{\|a\|_\Phi} \right)^{1/2}.$$

In particular, if Φ and Ψ satisfy the condition of Theorem 3.5, then we have the isometric isomorphisms $CB(R, H(\Phi, \Psi)) = \mathfrak{S}_{\tilde{\Psi}}$ and $CB(C, H(\Phi, \Psi)) = \mathfrak{S}_{\tilde{\Phi^*}}$.

Proof. Let $x = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \lambda_1 \geq \cdots \geq \lambda_n \geq 0$ be a positive diagonal matrix. Then from the definition

$$\|x\|_{CB(R,H(\Phi,\Psi))} = \sup_{T \in R, a \in \mathfrak{S}_{\Phi,+}, \|a\|_\Phi \leq 1} \left\{ \left\| \sum_i \lambda_i^2 T_i a T_i^* \right\|_\Psi^{1/2} \right\}.$$

If $\|T\|_R \leq 1$, then $\| \sum_i T_i T_i^* \| \leq 1$ and thus it follows that $(T_i^* T_j)_{ij} \leq I$. Hence, we have

$$\begin{aligned} \left\| \sum_{i=1}^n \lambda_i^2 T_i a T_i^* \right\|_\Psi &= \left\| \begin{pmatrix} T_1 & \cdots & T_n \\ & \bigcirc & \end{pmatrix} \operatorname{diag}(\lambda_1^2 a, \ldots, \lambda_n^2 a) \begin{pmatrix} T_1^* & \\ \vdots & \bigcirc \\ T_1^* & \end{pmatrix} \right\|_\Psi \\ &= \| \operatorname{diag}(\lambda_1 a^{\frac{1}{2}}, \ldots, \lambda_n a^{\frac{1}{2}}) (T_i^* T_j) \operatorname{diag}(\lambda_1 a^{\frac{1}{2}}, \ldots, \lambda_n a^{\frac{1}{2}}) \|_\Psi \\ &\leq \left\| |x|^2 \otimes a \right\|_\Psi. \end{aligned}$$

To show the converse, take a family $\{T_i\}_{i=1}^n$ such that $T_i^* T_j = \delta_{ij} I$, where δ_{ij} is the Kronecker delta.

When Φ and Ψ satisfy the condition of Theorem 3.5, we have

$$\left\| |x|^2 \otimes a \right\|_\Psi \leq \| |x|^2 \|_\Psi \|a\|_\Phi = \|x\|_{\tilde{\Psi}}^2 \|a\|_\Phi$$

and thus, $\|x\|_{CB(H(\Phi_\infty), H(\Phi, \Psi))} \leq \|x\|_{\tilde{\Psi}}$. The converse is verified by putting a to be any rank one projection. The last assertion is obtained from Lemma 3.1. $\square$

Other important Hilbertian operator spaces are $H_{\min}$ and OH. Let us see the space $CB(H_{\min}, H(\Phi_p, \Phi_q))$ next. When $p = q$, this space can be identified with $\mathfrak{S}_2$.

THEOREM 4.2. *For each $\theta \in [0,1]$, the space $CB(H_{\min}, R(\theta))$ coincides with $\mathfrak{S}_2$ up to equivalence of norm.*

Proof. Mathes proved this theorem when $\theta = 0$ or 1 (see [13, Proposition 6]). We use this result and the complex interpolation theory. Since the space of completely bounded mappings between homogeneous m.c.n. spaces is an operator ideal, it suffices to check the cb-norm of the matrices of the diagonal form $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n), \lambda_1 \geq \cdots \geq \lambda_n \geq 0$. We denote by $\|A\|_{cb}$ the c.b. norm of $A : H_{\min} \to R(\theta)$. First we note that

$$\|A\|_{cb} = \sup_T \frac{\|\sum_i \xi_i \otimes \lambda_i T_i\|_{R(\theta)}}{\|T\|_{\min}}.$$

Thus, by the complex interpolation property it follows that

$$\|A\|_{cb} \leq \sup_T \left\{ \frac{\|\sum_i \xi_i \otimes \lambda_i T_i\|_R}{\|T\|_{\min}} \right\}^{1-\theta} \left\{ \frac{\|\sum_i \xi_i \otimes \lambda_i T_i\|_C}{\|T\|_{\min}} \right\}^{\theta} \leq \left(\sum_i \lambda_i^2 \right)^{1/2},$$

where we use the case of $\theta = 0, 1$.

To show the converse inequality let $p = 1/\theta$ and $p' = 1/1-\theta$. Since $\|T\|_{H(\Phi_p)} = \|T^*\|_{H(\Phi_{p'})}$ and $\|T\|_{H_{\min}} = \|T^*\|_{H_{\min}}$ by Lemma 3.1, it follows that $\|A\|_{CB(H_{\min}, R(\theta))} = \|A\|_{CB(H_{\min}, R(1-\theta))}$. Let $B : H_{\min} \to R(1-\theta)$ be the mapping which has the same matrix entries with A. Then the complex interpolation duality yields the isometry $R(\theta)^* = R(1-\theta)$. Thus, we can define the mapping $B^* \circ A : H_{\min} \to H_{\max} = H_{\min}^*$, which satisfies $\|B^* \circ A\|_1 \leq 2\|B^* \circ A\|_{CB(H_{\min}, H_{\max})}$ [14, Proposition 2.1]. Hence, we have

$$\begin{aligned}
\sum_i \lambda_i^2 &= \|B^* \circ A\|_1 \\
&\leq 2\|B^*\|_{CB(R(\theta), H_{\max})} \|A\|_{CB(H_{\min}, R(\theta))} \\
&\leq 2\|B\|_{CB(H_{\min}, R(1-\theta))} \|A\|_{CB(H_{\min}, R(\theta))} \\
&\leq 2\left(\sum_i \lambda_i^2 \right)^{1/2} \|A\|_{CB(H_{\min}, R(\theta))}
\end{aligned}$$

Thus, $\|A\|_2 \leq 2\|A\|_{CB(H_{\min}, R(\theta))}$. □

To deal with the case $p \neq q$, we need the following lemma.

LEMMA 4.3. *Let $1 \leq p \leq q \leq \infty$ and take $\theta, \psi \in [0,1]$ such that*

$$\begin{cases} 1/p = 1 - \psi + \theta\psi, \\ 1/q = \theta\psi. \end{cases}$$

Then for every $T \in H(\Phi_p, \Phi_q)$,

$$\|T\|_{\Phi_p,\Phi_q} \le \|T\|_{(H_{\min}, R(\theta))_\psi}.$$

Proof. For each $t \in [0,1]$, take positive numbers p_t and q_t such that

$$\frac{1}{p_t} = 1 - t + \theta t, \qquad \frac{1}{q_t} = \theta t$$

and let $q_t' = (1 - 1/q_t)^{-1}$. We define a family of bilinear mappings $f_t : \mathfrak{S}_{2q_t'} \times \mathfrak{S}_{2p_t} \to \ell_2(\mathfrak{S}_2)$ by $f_t(a,b) = (aT_ib)_i$ for $0 \le t \le 1$. Then Lemma 3.4 shows that

$$\|f_0\| = \sup_{a,b \in \mathfrak{S}_2} \operatorname{Tr}\left(\sum_i aT_ibb^*T_i^*a^*\right) = \|T\|_{H(\Phi_1,\Phi_\infty)} = \|T\|_{\min}$$

and Pisier [18, Theorem 8.4] showed that $\|f_1\| = \|T\|_{R(\theta)}$. Thus, the multilinear interpolation (see [8, 10.2]) implies that $\|T\|_{\Phi_p,\Phi_q} = \|f_\psi\| \le \|T\|_{(H_{\min},R(\theta))_\psi}$. □

THEOREM 4.4. *Let* $1 \le p \le q \le \infty$. *We have a contractive embedding of* $\mathfrak{S}_r$ *into* $CB(H_{\min}, H(\Phi_p, \Phi_q))$, *where* $r = 2/(1/q - 1/p + 1)$.

Proof. Let $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n), \lambda_1 \ge \cdots \ge \lambda_n \ge 0$. Then

$$\begin{aligned}
\|A\|_{CB(H_{\min},H(\Phi_p,\Phi_q))} &\le \|A\|_{CB(H_{\min},(H_{\min},R(\theta))_\psi)} \\
&\le \|A\|_{(CB(H_{\min},H_{\min}),CB(H_{\min},R(\theta)))_\psi} \\
&\le \|A\|_{(\mathfrak{S}_\infty,\mathfrak{S}_2)_\psi} = \|A\|_r.
\end{aligned}$$

In the first step, we use Lemma 4.3 and in the third, we use Theorem 4.2. □

We observe the c.b. norm of the mappings from OH to $H(\Phi_p, \Phi_q)$.

THEOREM 4.5. *Let* $1 \le p \le q \le \infty$. *Then*

$$CB(OH, H(\Phi_p, \Phi_q)) = \begin{cases} \mathfrak{S}_{4(1-2/p)^{-1}} & (p \ge 2), \\ B(H) & (p \le 2 \le q), \\ \mathfrak{S}_{4(2/q-1)^{-1}} & (q \le 2), \end{cases}$$

with equal norms.

Proof. The second case is obvious and the third one follows from Lemma 3.1 and the first one. We show the first case. Let $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ be a diagonal operator with $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$. Xu showed in [20, Lemma 5.9] that if $1 \le p \ne q \le \infty$, then $CB(H(\Phi_p, \Phi_p), H(\Phi_q, \Phi_q)) = \mathfrak{S}_{2pq/|p-q|}$. From this result, it clearly follows that for any operator A,

$$\|A\|_{CB(OH,H(\Phi_p,\Phi_q))} \le \|A\|_{CB(OH,H(\Phi_p,\Phi_p))} = \|A\|_{4(1-2/p)^{-1}}.$$

To show the converse, for a positive diagonal matrix $B = \mathrm{diag}(b_1, \dots, b_n)$, let $T_{B,i} = b_i e_{1i} \in M_n$ $(i = 1, \dots, n)$. Then

$$\begin{aligned}\left\|\sum_{i=1}^n \xi_i \otimes T_{B,i}\right\|_{OH}^4 &= \left\|\sum_{i=1}^n T_{B,i} \otimes \bar{T}_{B,i}\right\|_{\min}^2 \\ &= \left\|\sum_{i,j=1}^n b_i^2 b_j^2 (e_{1i} \otimes e_{1i})(e_{j1} \otimes e_{j1})\right\|_{M_n \otimes M_n} \\ &= \left\|\sum_{i=1}^n b_i^4 e_{11} \otimes e_{11}\right\|_{M_n \otimes M_n} = \sum_{i=1}^n b_i^4.\end{aligned}$$

However, if we let C be a positive diagonal matrix $\mathrm{diag}(c_1, \dots, c_n)$, then we have

$$\left\|A\left(\sum_{i=1}^n \xi_i \otimes T_{B,i}\right)\right\|_{H(\Phi_p,\Phi_q)} \geq \sup_C \frac{|\sum_{i=1}^n \lambda_i^2 b_i^2 c_i|^{1/2}}{(\sum_{i=1}^n c_i^p)^{1/p}}.$$

Taking the supremum for B in the unit ball of OH, we obtain

$$\|A\|_{CB(OH,H(\Phi_p,\Phi_q))} \geq \sup_C \frac{|\sum_{i=1}^n \lambda_i^4 c_i^2|^{1/4}}{(\sum_{i=1}^n c_i^p)^{1/p}} = \|A\|_{4(1-2/p)^{-1}}. \qquad \square$$

5. Multiplicator in operator ideals

In this section we show that the m.c.n. space $H(\Phi)$ is an operator space if and only if Φ is the Schatten norm.

In view of the result of Theorem 4.1, for an s.n. function Φ we consider the following two conditions:

$(*)$ $\exists c_1 \geq 0, \|x \otimes y\|_\Phi \leq c_1 \|x\|_\Phi \|y\|_\Phi$ for any x and y;
$(**)$ $\exists c_2 \geq 0, \|x \otimes y\|_\Phi \geq c_2 \|x\|_\Phi \|y\|_\Phi$ for any x and y.

Note that if an s.n. function Φ satisfies $(*)$, its adjoint Φ^* satisfies $(**)$ for c_2 with $c_1 c_2 = 1$. The Schatten p-norm is a cross norm and satisfies both $(*)$ and $(**)$ with $c_1 = c_2 = 1$.

Let Φ and Ψ be s.n. functions with $\Phi \geq \Psi$ and $x \in B(\ell_2)$ such that

$$\sup_a \frac{\|x \otimes a\|_\Psi}{\|a\|_\Phi} < \infty.$$

We denote by $M_{\Phi,\Psi}(x)$ the multiplicator from $\mathfrak{S}_\Phi$ to $\mathfrak{S}_\Psi$ defined by

$$M_{\Phi,\Psi}(x)(a) = x \otimes a.$$

For an s.n. function Φ, we denote by $\mathcal{M}(\mathfrak{S}_\Phi)$ the space consisting of $x \in B(\ell_2)$ with $M_{\Phi,\Phi}(x)$ is bounded. We equip $\mathcal{M}(\mathfrak{S}_\Phi)$ with the norm $\|M_{\Phi,\Phi}(x)\|$. It holds that

$$\|x\|_\Psi = \frac{\|x \otimes e_{11}\|_\Psi}{\|e_{11}\|_\Phi} \leq \|M_{\Phi,\Psi}(x)\|.$$

In case of the Schatten norm ($1 \le p \le q \le \infty$), we have

$$\|M_{\Phi_p,\Phi_q}(x)\| = \|x\|_q.$$

If an s.n. function Φ satisfies $(*)$, then

$$\|M_{\Phi,\Phi}(x)\| \le c_1 \|x\|_\Phi,$$

and thus Φ satisfies $(*)$ if and only if $\|x\|_\Phi$ is equivalent to $\|M_{\Phi,\Phi}(x)\|$. Since $M_{\Phi,\Phi}(x)M_{\Phi,\Phi}(y) = M_{\Phi,\Phi}(x \otimes y)$, we have

$$\|M_{\Phi,\Phi}(x \otimes y)\| \le \|M_{\Phi,\Phi}(x)\| \|M_{\Phi,\Phi}(y)\|.$$

The multiplicator is discussed in [4] for the rearrangement invariant space on [0, 1].

The conditions $(*)$ and $(**)$ are closely related to the Schatten norm.

LEMMA 5.1. *If an s.n. ideal $\mathfrak{S}_\Phi$ satisfies $(*)$ or $(**)$, then the limit*

$$p = \lim_{n\to\infty} \frac{\log n}{\log \|P_n\|_\Phi} \in [1, \infty]$$

exists, where P_n stands for any rank n projection.

Proof. We prove the statement in the case that $(*)$ holds. In the case of $(**)$ the proof is similar. By the hypothesis, for fixed $m \in \mathbb{N}$,

$$\|P_{m^k}\|_\Phi \le c_1^{k-1} \|P_m\|_\Phi^k, \quad \forall k \in \mathbb{N}.$$

If $\{t_i\}_{i=1}^\infty$ is a subsequence of $\mathbb{N}$, we can take a nondecreasing sequence $\{k_i\}_{i=1}^\infty$ in $\mathbb{N}$ which tends to infinity such that $m^{k_i} \le t_i < m^{k_i+1}$. Thus, we have

$$\frac{\log t_i}{\log \|P_{t_i}\|_\Phi} \ge \frac{\log m^{k_i}}{\log \|P_{m^{k_i+1}}\|_\Phi} \ge \frac{k_i \log m}{k_i \log c_1 + (k_i+1)\log \|P_m\|_\Phi}.$$

Since $\{t_i\}_{i=1}^\infty$ is arbitrary, it follows that

$$\liminf_{n\to\infty} \frac{\log n}{\log \|P_n\|_\Phi} \ge \frac{\log m}{c_1 + \log \|P_m\|_\Phi}.$$

This implies

$$\liminf_{n\to\infty} \frac{\log n}{\log \|P_n\|_\Phi} \ge \limsup_{m\to\infty} \frac{\log m}{\log \|P_m\|_\Phi}$$

and the limit exists. □

THEOREM 5.2. *Suppose that an s.n. ideal $\mathfrak{S}_\Phi$ satisfies $(*)$ or $(**)$ and let p be as in the preceding lemma. Then the following statements hold.*

(a) *if $\mathfrak{S}_\Phi$ satisfies $(*)$, then*

$$\|x\|_p \le c_1 \|x\|_\Phi, \quad \forall x \in \mathfrak{S}_\Phi.$$

(b) *if $\mathfrak{S}_\Phi$ satisfies $(**)$, then*

$$c_2 \|x\|_\Phi \le \|x\|_p, \quad \forall x \in \mathfrak{S}_\Phi.$$

In particular, if Φ is a cross norm, then $\Phi = \Phi_p$.

Proof. Let $x = \mathrm{diag}(\lambda_1, \dots, \lambda_m), \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ be a diagonal matrix and let

$$x^{\otimes n} = \sum_{i=1}^{N} t_i e_i$$

be the spectral decomposition of the n-fold tensor product of x. In the above inequality, N is dominated by $\binom{m+n-1}{m-1}$. If we let p_j be the j-th sum of the e_i's given by $p_j = \sum_{i=1}^{j} e_i$, then for all j we have

$$\sum_{i=1}^{n} t_i e_i = \sum_{i=1}^{n} (t_j - t_{j-1}) p_j \geq t_j p_j.$$

Thus, it holds that

$$\operatorname*{Max}_j \{t_j \|p_j\|_\Phi\} \leq \|x^{\otimes n}\|_\Phi \leq N \operatorname*{Max}_j \{t_j \|p_j\|_\Phi\}$$

and hence,

$$\operatorname*{Max}_j \{(t_j \|p_j\|_\Phi)^{1/n}\} \leq \|x^{\otimes n}\|_\Phi^{1/n} \leq N^{1/n} \operatorname*{Max}_j \{(t_j \|p_j\|_\Phi)^{1/n}\}.$$

Note that from the above inequality, if $\Phi = \Phi_p$, then

$$\|x\|_p = \lim_{n\to\infty} \operatorname*{Max}_j \big\{ t_j^{1/n} (\mathrm{rank}\, p_j)^{1/(pn)} \big\}. \tag{1}$$

By the preceding lemma, for any $\varepsilon > 0$, there exists a $D \geq 0$ such that

$$\|p_j\|_\Phi \geq D (\mathrm{rank}\, p_j)^{1/(p+\varepsilon)}, \quad \text{for all } j \in \mathbb{N}.$$

The condition $(*)$ implies that $\|x^{\otimes n}\|_\Phi \leq c_1^{n-1} \|x\|_\Phi^n$, so that

$$\begin{aligned} c_1 \|x\|_\Phi &\geq \|x^{\otimes n}\|_\Phi^{1/n} \\ &\geq \operatorname*{Max}_j \{(t_j \|p_j\|_\Phi)^{1/n}\} \\ &\geq \operatorname*{Max}_j \big\{ (D t_j)^{1/n} (\mathrm{rank}\, p_j)^{1/\{(p+\varepsilon)n\}} \big\}. \end{aligned}$$

The last term converges to $\|x\|_{p+\varepsilon}$ as $n \to \infty$ from the above equality 1, which proves (a). The proof of (b) is similar. □

From Theorem 5.2 and Theorem 3.5, we obtain the following corollary.

COROLLARY 5.3. *Let Φ be an s.n. function. The m.c.n. space $H(\Phi)$ is an operator space if and only if Φ is some Schatten p-norm* $(1 \leq p \leq \infty)$.

REMARK 5.1. Let X be a rearrangement invariant function space X on the interval $[0, 1]$ (cf. [12, Section 2]). For $s > 0$, let σ_s be the dilation operator given by

$$\sigma_s x(t) = x(t/s) 1_{[0, \max\{1, s\}]} \quad (t \in [0, 1], x \in X).$$

This operator is well defined on X and $\|\sigma_s\| \le \max\{1, s\}$. The Boyd indices α_X and β_X of X are defined by

$$\alpha_X = \lim_{s\to 0} \frac{\log \|\sigma_s\|_{X\to X}}{\log s}, \qquad \beta_X = \lim_{s\to\infty} \frac{\log \|\sigma_s\|_{X\to X}}{\log s}.$$

Note that $0 \le \alpha_X \le \beta_X \le 1$. In [3, Theorem 1.5], the embedding $\mathcal{M}(X) \subseteq L_{\alpha_X^{-1}}$ is shown. The Boyd index is discussed in [12] for sequence spaces and in [2] for s.n. ideals. The Boyd index of an s.n. ideal $\mathfrak{S}_\Phi$ is defined by

$$p = \lim_{n\to\infty} \frac{\log n}{\log \|P_n\|_\Phi}$$

when the limit exists (the limit is in $[1, \infty]$). Theorem 5.2 means that if Φ satisfies $(*)$, then $\mathcal{M}(\mathfrak{S}_\Phi) \subset \mathfrak{S}_p$.

In the rest of this paper, we examine the condition $(*)$ for a few classes of s.n. functions.

Theorem 5.4. *Let π be a binormalizing sequence and let S_n be the partial sum defined by $S_n = \sum_{j=1}^n \pi_j$. Then Φ_π satisfies $(*)$ if and only if there is a constant $c > 0$ such that for any $m, n \in \mathbb{N}$, the inequality*

$$\frac{S_{mn}}{S_m S_n} \le c$$

holds.

Proof. Let $x \in F(K)_+$ and we write its spectral decomposition by

$$x = \sum_{j=1}^n s_j(x) p_j.$$

We can represent $\|x\|_\pi$ in the form

$$\begin{aligned}\|x\|_\pi &= \sum_{j=1}^n \pi_j s_j(x) \\ &= \big(s_1(x) - s_2(x)\big)S_1 + \cdots + \big(s_{n-1}(x) - s_n(x)\big)S_{n-1} + s_n(x) S_n,\end{aligned}$$

so that if we let e_j be the partial sum of p_i's given by $e_j = \sum_{i=1}^j p_i$, then

$$x = \big(s_1(x) - s_2(x)\big)e_1 + \cdots + \big(s_{n-1}(x) - s_n(x)\big)e_{n-1} + s_n(x) e_n.$$

Hence, for any $a \in F(K)$,

$$\begin{aligned}\|x \otimes a\|_\pi &\le \left(\sum_{j=1}^n \big(s_j(x) - s_{j+1}(x)\big) S_j \right) \operatorname*{Max}_j \left\{ \frac{\|e_j \otimes a\|_\pi}{S_j} \right\} \\ &\le \|x\|_\pi \operatorname*{Max}_j \left\{ \frac{\|e_j \otimes a\|_\pi}{\|e_j\|_\pi} \right\}.\end{aligned}$$

Similar argument for a yields

$$\sup_{x,a} \frac{\|x \otimes a\|_\pi}{\|x\|_\pi \|a\|_\pi} = \sup_{p,a} \frac{\|p \otimes a\|_\pi}{\|p\|_\pi \|a\|_\pi} = \sup_{p,q} \frac{\|p \otimes q\|_\pi}{\|p\|_\pi \|q\|_\pi},$$

where p and q run over all finite rank projections. If p is a rank n projection, then $\|p\|_\pi = S_n$ and therefore $(*)$ holds if and only if $S_{mn}/S_m S_n \leq c$. $\square$

Remark 5.2. The condition

$$\sup_{m,n} \frac{S_{mn}}{S_m S_n} < \infty$$

appears in [1, Theorem 6], as a necessarily and sufficient condition for the existence of exactly two nonequivalent symmetric basic sequences in Lorentz sequence spaces.

Next, we look out the Lorentz ideals $S_{p,q}$ for $1 \leq q \leq p < \infty$. When $q = 1$, the Lorentz ideal $S_{p,1}$ is equal to the ideal $\mathfrak{S}_{\Phi_\pi}$ with $\pi_j = j^{1/p-1}$, and thus satisfies $(*)$ with $c_1 = 1$ from Theorem 5.4.

Proposition 5.5. *When $1 \leq q \leq p < \infty$ the Lorentz ideal $S_{p,q}$ satisfies $(*)$.*

Proof. Let $x, y \in S_{p,q}$ be positive elements. Note that the spectrum of $x \otimes y$ is equal to $\{s_i(x)s_j(y)\}_{i,j=1}^\infty$ as a set considering multiplicity and each eigenspace is finite-dimensional. We give the product set $\mathbb{N} \times \mathbb{N}$ an order $\prec$ by

$$(m_1, n_1) \prec (m_2, n_2) \quad \Longleftrightarrow \quad \begin{cases} m_1 + n_1 < m_2 + n_2 \\ \text{or} \\ m_1 + n_1 = m_2 + n_2 \quad \text{and} \quad m_1 > m_2. \end{cases}$$

For each eigenvalue α of $x \otimes y$ with index k, let I_α be the finite sequence $\{(m_1, n_1), \ldots, (m_k, n_k)\}$ in $\mathbb{N} \times \mathbb{N}$ such that $s_{m_i}(x)s_{n_i}(y) = \alpha$ and $(m_i, n_i) \prec (m_{i+1}, n_{i+1})$. If $s_{j+1}(x \otimes y) = \cdots = s_{j+k}(x \otimes y) = \alpha$, for all $i = 1, \ldots, k$ we have

$$s_{j+i}(x \otimes y) = s_{m_i}(x)s_{n_i}(y)$$

and $j + i \geq m_i n_i$. Hence,

$$\begin{aligned} \|x \otimes y\|_{p,q} &= \left(\sum_{j=1}^\infty \frac{s_j(x \otimes y)^q}{j^{1-q/p}} \right)^{1/q} \\ &\leq \left(\sum_{i,j=1}^\infty \frac{s_i(x)^q s_j(y)^q}{(ij)^{1-q/p}} \right)^{1/q} \\ &= \left(\sum_{j=1}^\infty \frac{s_j(x)^q}{j^{1-q/p}} \right)^{1/q} \left(\sum_{j=1}^\infty \frac{s_j(y)^q}{j^{1-q/p}} \right)^{1/q} = \|x\|_{p,q} \|y\|_{p,q}. \end{aligned}$$

$\square$

Remark 5.3. In [4, p. 253], it is shown that for the Lorentz function space $L_{p,q}$ $(1 < p < \infty, 1 \leq q \leq \infty)$, we have $\mathcal{M}(L_{p,q}) = L_{p,\min(p,q)}$.

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