Illinois Journal of Mathematics
Volume 51, Number 1, Spring 2007, Pages 137–150
S 0019-2082

LOCAL DUALITY FOR BIGRADED MODULES

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ABSTRACT. In this paper we study local cohomology of finitely generated bigraded modules over a standard bigraded ring with respect to the irrelevant bigraded ideals and establish a duality theorem. Several applications are considered.

Introduction

Let R be a standard bigraded K-algebra with bigraded irrelevant ideals P generated by all elements of degree $(1,0)$ and Q generated by all elements of degree $(0,1)$. We want to relate the local cohomology functors $H_P^i(-)$ and $H_Q^j(-)$ via duality in the category of bigraded modules. In the ordinary local duality theorem Matlis duality establishes isomorphisms between the local cohomology modules of a module and its Ext-groups.

In our situation we have to consider Matlis duality for bigraded modules. Given a bigraded R-module M we define the bigraded Matlis-dual of M to be $M^\vee$, where the (i,j)th bigraded component of $M^\vee$ is given by $\operatorname{Hom}_K(M_{(-i,-j)},K)$.

As the main result of our paper we have the following duality theorem:

THEOREM. *Let R be a standard bigraded K-algebra with irrelevant bigraded ideals P and Q, and let M be a finitely generated bigraded R-module. Then there exists a convergent spectral sequence*

$$E_{i,j}^2 = H_P^{m-j}(H_{R_+}^i(M)^\vee) \underset{j}{\Longrightarrow} H_Q^{i+j-m}(M)^\vee$$

of bigraded R-modules, where m is the minimal number of homogeneous generators of P and R_+ is the unique graded maximal ideal of R.

Note that the above spectral sequence degenerates when M is Cohen-Macaulay and one obtains for all k the following isomorphims of bigraded

Received May 1, 2006; received in final form July 25, 2006.
2000 *Mathematics Subject Classification.* 13D45, 13C14.

R-modules

$$H_P^k(H_{R_+}^s(M)^\vee) \cong H_Q^{s-k}(M)^\vee, \tag{1}$$

where $s = \dim M$; see Corollary 2.6.

Let R_0 be the K-subalgebra of R which is generated by the elements of bidegree $(1,0)$, and let N be any bigraded R-module. Then for all j, the module $N_j = \bigoplus_i N_{(i,j)}$ is a graded R_0-module with grading $(N_j)_i = N_{(i,j)}$. Moreover, if N is finitely generated, then each N_j is a finitely generated R_0-module. In particular, if M is an s-dimensional Cohen-Macaulay module and if we set $N = H_{R_+}^s(M)^\vee$, then N is again an s-dimensional Cohen-Macaulay module and by (1) we obtain for all j the isomorphisms of graded R_0-modules

$$H_{P_0}^k(N_j) \cong (H_Q^{s-k}(M)_{-j})^\vee, \tag{2}$$

where P_0 is the graded maximal ideal of R_0. Here we used that $H_P^k(N)_j \cong H_{P_0}^k(N_j)$ for all k and j.

Brodmann and Hellus [4] raised the question whether the modules $H_Q^k(M)$ are tame if M is a finitely generated graded R-module, in other words, whether for each k there exists an integer j_0 such that either $H_Q^k(M)_j = 0$ for all $j \leq j_0$, or else $H_Q^k(M)_j \neq 0$ for all $j \leq j_0$. In various cases this problem has been answered in the affirmative; see [3], [4], [16], [10], [12] and [2] for a survey on this problem. In case M is Cohen-Macaulay the tameness problem translates, due to (2), to the following question: Given a finitely generated bigraded R-module N, does there exist an integer j_0 such that $H_{P_0}^k(N_j) = 0$ for all $j \geq j_0$, or else $H_{P_0}^k(N_j) \neq 0$ for all $j \geq j_0$? More generally, one would expect that for a finitely generated graded R_0-module W and a finitely generated bigraded R-module N there exists for all k an integer j_0 such that $\operatorname{Ext}_{R_0}^k(N_j, W) = 0$ for all $j \geq j_0$, or else $\operatorname{Ext}_{R_0}^k(N_j, W) \neq 0$ for all $j \geq j_0$. However, this is not the case as has been recently shown by Cutkosky and the first author; see [7]. Their example also provides a counterexample to the general tameness problem. To show this, Proposition 2.5 of this paper is used. On the other hand, in a recent paper [14] the second author of this paper has shown that tameness holds for all local cohomology modules of a ring with monomial relations and with respect to monomial prime ideals.

In Section 2 we use our duality to give new proofs of known cases of the tameness problem and also to add a few new cases in which tameness holds; see Corollaries 2.4, 2.8 and 2.12. The duality is also used in Corollaries 2.9 and 2.10 to prove some algebraic properties of the modules $H_Q^k(M)_j$ in case M is Cohen-Macaulay.

1. Proof of the duality theorem

Let $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be the standard bigraded polynomial ring over the field K. We set $K[x] = K[x_1, \ldots, x_m]$ and $K[y] = K[y_1, \ldots, y_n]$ and consider both as standard graded polynomial rings.

If A is a standard (bi)graded K-algebra and M a (bi)graded A-module, we set $M^\vee = \operatorname{Hom}_K(M, K)$ and view $M^\vee$ as (bi)graded A-module with the (bi)grading

$$(M^\vee)_a = \operatorname{Hom}_K(M_{-a}, K)$$

for $a \in \mathbb{Z}$ (respectively $a \in \mathbb{Z}^2$ in the bigraded case).

The following simple fact is needed for the proof of the next lemma.

Lemma 1.1. *Let M be a graded $K[x]$-module and N be a graded $K[y]$-module. Then there exists a natural bigraded isomorphism of bigraded S-modules*

$$(M \otimes_K N)^\vee \cong M^\vee \otimes_K N^\vee.$$

Proof. Let $S = K[x] \otimes_K K[y] = K[x, y]$. Note that $M \otimes_K N$ is a bigraded free S-module with the natural bigrading

$$(M \otimes_K N)_{(i,j)} = M_i \otimes_K N_j.$$

Thus we see that

$$((M \otimes_K N)^\vee)_{(i,j)} = \operatorname{Hom}_K((M \otimes_K N)_{(-i,-j)}, K) = \operatorname{Hom}_K(M_{-i} \otimes_K N_{-j}, K).$$

By using the universal property of the tensor product one has the following natural isomorphism of K-vector spaces

$$\operatorname{Hom}_K(M_{-i} \otimes_K N_{-j}, K) \cong \operatorname{Hom}_K(M_{-i}, K) \otimes_K \operatorname{Hom}_K(N_{-j}, K).$$

Thus we have

$$\begin{aligned} ((M \otimes_K N)^\vee)_{(i,j)} &\cong \operatorname{Hom}_K(M_{-i}, K) \otimes_K \operatorname{Hom}_K(N_{-j}, K) \\ &= (M^\vee)_i \otimes_K (N^\vee)_j \\ &= (M^\vee \otimes_K N^\vee)_{(i,j)}. \end{aligned}$$

So the desired isomorphism follows. □

Lemma 1.2. *Let $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be the standard bigraded polynomial ring over the field K with the irrelevant bigraded ideals $P = (x_1, \ldots, x_m)$ and $Q = (y_1, \ldots, y_n)$. Then we have the following isomorphism of bigraded S-modules*

$$H^m_P(\omega_S) \cong H^n_Q(S)^\vee,$$

where ω_S is the bigraded canonical module of S.

Proof. We denote by P_0 the graded maximal ideal of $K[x]$ and by Q_0 the graded maximal ideal of $K[y]$. First we notice that there is a natural isomorphism of bigraded S-modules

$$H_P^m(S) \cong H_{P_0}^m(K[x]) \otimes_K K[y].$$

By the graded version of the local duality theorem (see [5, Example 13.4.6]) we have

$$H_{P_0}^m(K[x])^\vee \cong K[x](-m).$$

Thus we see that

$$\begin{aligned} H_P^m(\omega_S) = H_P^m(S(-m,-n)) &= H_P^m(S)(-m,-n) \\ &\cong (K[x](-m)^\vee \otimes_K K[y])(-m,-n) \\ &= K[x]^\vee \otimes_K K[y](-n). \end{aligned}$$

On the other hand, using again the local duality theorem, Lemma 1.1 yields

$$\begin{aligned} H_Q^n(S)^\vee \cong (K[x] \otimes_K H_{Q_0}^n(K[y]))^\vee &\cong (K[x] \otimes_K K[y](-n)^\vee)^\vee \\ &\cong K[x]^\vee \otimes_K K[y](-n)^{\vee\vee} \\ &\cong K[x]^\vee \otimes_K K[y](-n), \end{aligned}$$

as desired. $\square$

Corollary 1.3. *Let F be a finitely generated bigraded free S-module, and set $F^* = \operatorname{Hom}_S(F, \omega_S)$. Then there exists a natural isomorphism of bigraded S-modules*

$$H_P^m(F^*) \cong H_Q^n(F)^\vee.$$

Proof. Let $F = \bigoplus_{k=1}^t S(-a_k,-b_k)$. Thus $F^* = \bigoplus_{k=1}^t (\omega_S)(a_k,b_k)$ and hence by Lemma 1.2 we have

$$\begin{aligned} H_P^m(F^*) \cong \bigoplus_{k=1}^t H_P^m(\omega_S)(a_k,b_k) &\cong \bigoplus_{k=1}^t H_Q^n(S)^\vee(a_k,b_k) \\ &\cong H_Q^n(\bigoplus_{k=1}^t S(-a_k,-b_k))^\vee \\ &\cong H_Q^n(F)^\vee. \end{aligned}$$ $\square$

The previous result can easily be extended as follows.

Lemma 1.4. *Let $\mathbb{F}$ be a bounded complex of bigraded free S-modules. We set $\mathbb{F}^* = \operatorname{Hom}_S(\mathbb{F}, \omega_S)$. Then we have a functorial isomorphism*

$$H_P^m(\mathbb{F}^*) \cong H_Q^n(\mathbb{F})^\vee$$

of complexes of bigraded modules.

Proof. In order to prove that the complexes of $H_P^m(\mathbb{F}^*)$ and $H_Q^n(\mathbb{F})^\vee$ are isomorphic, we observe that for any bihomogeneous linear map $\varphi : G \to F$ between finitely generated free bigraded S-modules we obtain the following commutative diagram

$$\begin{array}{ccc} H_Q^n(F)^\vee & \xrightarrow{\psi_1^\vee} & H_Q^n(G)^\vee \\ \downarrow & & \downarrow \\ H_P^m(F^*) & \xrightarrow{\psi_2} & H_P^m(G^*), \end{array}$$

where $\psi_1 = H_P^n(\varphi)$ and $\psi_2 = H_Q^m(\varphi^*)$ and where the vertical maps are the isomorphisms given in Corollary 1.3. The commutativity of the diagram results from the fact that all maps in the diagram are functorial. $\square$

PROPOSITION 1.5. *Let M be a finitely generated bigraded S-module, P and Q be the irrelevant bigraded ideals of S. Then we have the following convergent spectral sequence*

$$E_{i,j}^2 = H_P^{m-j}(\operatorname{Ext}_S^{n+m-i}(M, \omega_S)) \underset{j}{\Longrightarrow} H_Q^{i+j-m}(M)^\vee.$$

Proof. Let $(\mathbb{F}, d)$ be a bigraded free resolution of M of length $n+m$, and let $\mathbb{G}$ be the complex of bigraded S-modules with $G_i = \operatorname{Hom}_S(F_{m+n-i}, \omega_S)$ and differential $\partial_i = \operatorname{Hom}_S(d_{m+n-i}, \omega_S)$. Next we choose a bigraded free resolution $\mathbb{C}$ of the complex $\mathbb{G}$. In other words, $\mathbb{C}$ is a double complex C_{ij} of finitely generated bigraded free S-modules with $i, j \geq 0$ such that:

(i) The ith column of $\mathbb{C}$ is a free resolution of G_i for all i, i.e.,

$$H_j(C_{i\bullet}) = \begin{cases} G_i & \text{for } j = 0, \\ 0 & \text{for } j > 0. \end{cases}$$

(ii) For each row the image of $C_{i-1,j} \longleftarrow C_{i,j}$ is a bigraded free direct summand of the kernel of $C_{i-2,j} \longleftarrow C_{i-1,j}$. In particular, the homology of

$$C_{i-2,j} \longleftarrow C_{i-1,j} \longleftarrow C_{i,j}$$

is a bigraded free S-module for all i and j.

(iii) For each i the complex

$$0 \longleftarrow H_i(C_{\bullet,0}) \longleftarrow H_i(C_{\bullet,1}) \longleftarrow H_i(C_{\bullet,2}) \longleftarrow \cdots$$

is a bigraded free resolution of $H_i(\mathbb{G})$.

Now we compute the total homology of the double complex $H_P^m(\mathbb{C})$: Since all G_i are free S-modules, it follows that the complexes

$$0 \longleftarrow G_i \longleftarrow C_{i,0} \longleftarrow C_{i,1} \longleftarrow \cdots$$

are all split exact. Hence the complexes

$$0 \longleftarrow H_P^m(G_i) \longleftarrow H_P^m(C_{i,0}) \longleftarrow H_P^m(C_{i,1}) \longleftarrow \cdots$$

are again exact.

This implies that the E^1-terms of the double complex $H_P^m(\mathbb{C})$ with respect to the column filtration are

$$E^1_{i,j} = \begin{cases} H_P^m(G_i) & \text{for } j = 0, \\ 0 & \text{for } j > 0. \end{cases}$$

As a consequence, for the E^2-terms of $H_P^m(\mathbb{C})$ we have that $E^2_{i,j} = 0$ for $j > 0$, and that $E^2_{i,0}$ is the ith homology of the complex $H_P^m(\mathbb{G})$. Now we use Lemma 1.4 as well as [13, Theorem 1.1] and obtain

$$E^2_{i,j} = \begin{cases} H_Q^{i-m}(M)^\vee & \text{for } j = 0, \\ 0 & \text{for } j > 0, \end{cases}$$

since $H_i(H_P^m(\mathbb{G})) = (H_{n+m-i}(H_Q^n(\mathbb{F})))^\vee$. From this it follows that the $(i+j)$th total homology of $H_P^m(\mathbb{C})$ is equal to $H_Q^{i+j-m}(M)^\vee$.

Now we compute the homology of $H_P^m(\mathbb{C})$ using the row filtration. Each row $H_P^m(C_{\bullet j})$ of $H_P^m(\mathbb{C})$ is split exact with homology $H_i(H_P^m(C_{\bullet j})) = H_P^m(H_i(C_{\bullet j}))$. In other words, $E^1_{i,j} = H_P^m(H_i(C_{\bullet j}))$. Hence by property (iii) of the complex $\mathbb{C}$ and by [13, Theorem 1.1] it follows that $E^2_{i,j} = H_P^{m-j}(\mathrm{Ext}_S^{m+n-i}(M, \omega_S))$. This yields the desired conclusion. □

Now our main theorem is an easy consequence of Proposition 1.5:

Proof. As R is a standard bigraded K-algebra, it is the homomorphic image of a standard bigraded polynomial ring $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. We may consider R and S as well as standard graded K-algebras with the unique graded maximal ideal R_+ (resp. S_+), and M as a graded R-module (resp. S-module). Then by the graded local duality theorem we have

$$\mathrm{Ext}_S^{m+n-i}(M, \omega_S) \cong H^i_{S_+}(M)^\vee.$$

Since $H^i_{S_+}(M) \cong H^i_{R_+}(M)$, it follows that

$$H_P^{m-j}(H^i_{R_+}(M)^\vee) = H_P^{m-j}(\mathrm{Ext}_S^{m+n-i}(M, \omega_S)).$$

Let $(x) = (x_1, \ldots, x_m)$ and $(y) = (y_1, \ldots, y_n)$ be the irrelevant ideals of S. We note that $H_P^{m-j}(\mathrm{Ext}_S^{m+n-i}(M, \omega_S)) = H_{(x)}^{m-j}(\mathrm{Ext}_S^{m+n-i}(M, \omega_S))$ and that $H_Q^{i+j-m}(M)^\vee = H_{(y)}^{i+j-m}(M)^\vee$. Therefore, Proposition 1.5 yields the desired convergent spectral sequence. □

Corollary 1.6. *Let R be a standard bigraded d-dimensional Cohen-Macaulay K-algebra with irrelevant bigraded ideals P and Q, and let M be a finitely generated bigraded R-module. Then there exists a convergent spectral sequence*

$$E^2_{i,j} = H_P^{m-j}(\mathrm{Ext}_R^{d-i}(M, \omega_R)) \underset{j}{\Longrightarrow} H_Q^{i+j-m}(M)^\vee$$

of bigraded R-modules, where m is the minimal number of homogeneous generators of P.

Proof. The assertion follows from our main theorem by using the fact that $H^i_{R_+}(M)^\vee = \operatorname{Ext}^{d-i}_R(M, \omega_R)$. □

2. Some applications

In this section, unless otherwise stated, R denotes a standard bigraded K-algebra of dimension d, and M a finitely generated bigraded R-module.

We note that for the E^2-terms in the spectral sequence of our main theorem we have $E^2_{i,j} = H^{m-j}_P(H^i_{R_+}(M)^\vee) = 0$ if $i < \operatorname{depth} M$ or $i > \dim M$ or $j < 0$ or $j > m$. Thus the possible non-zero E^2-terms are in the shadowed region of the following picture.

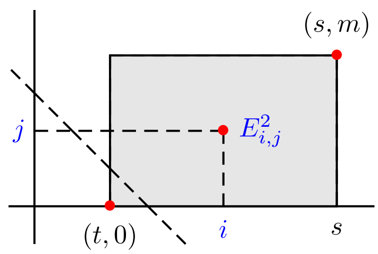

FIGURE 1

Here $t = \operatorname{depth} M$, $s = \dim M$ and $E^2_{i,j} = H^{m-j}_P(H^i_{R_+}(M)^\vee)$.

We first observe that the graded local duality theorem is a special case of our main theorem. In fact, if we assume that $P = (0)$, then $m = 0$, and $\mathfrak{m} = Q$ is the unique graded maximal ideal of R. Moreover, $E^2_{i,j} = E^\infty_{i,j} = 0$ for $j \neq 0$ and all i, since $H^k_{(0)}(-) = 0$ if $k \neq 0$. Therefore we have

$$\operatorname{Ext}^{d-i}_R(M, \omega_R) = H^0_{(0)}(\operatorname{Ext}^{d-i}_R(M, \omega_R)) \cong H^i_{\mathfrak{m}}(M)^\vee.$$

Considering Figure 1 we immediately obtain the following corner isomorphisms.

PROPOSITION 2.1. *Let* $\dim M = s$ *and* $\operatorname{depth} M = t$. *Then there are natural isomorphisms*

$$H^m_P(H^t_{R_+}(M)^\vee) \cong H^{t-m}_Q(M)^\vee \quad \textit{and} \quad H^0_P(H^s_{R_+}(M)^\vee) \cong H^s_Q(M)^\vee.$$

Moreover, for $i < t - m$ *we have* $H^i_Q(M) = 0$.

Definition 2.2. Let R_0 be a commutative Noetherian ring, R a graded R_0-algebra and N a graded R-module. The R-module N is called *tame*, if there exists an integer j_0 such that

$$N_j = 0 \text{ for all } j \leq j_0, \quad \text{or } N_j \neq 0 \text{ for all } j \leq j_0.$$

In case of a standard bigraded K-algebra R we let R_0 be the K-subalgebra of R generated by all elements of degree $(1,0)$. Then R is a graded R_0-algebra with components $R_j = R_{(*,j)} = \bigoplus_i R_{(i,j)}$. Let N be a bigraded R-module. We may view N as a graded R-module with graded components $N_j = N_{(*,j)} = \bigoplus_i N_{(i,j)}$. Each of the modules N_j is a graded R_0-module, and if N is a finitely generated R-module then each N_j is a finitely generated R_0-module.

Now let M be a finitely generated bigraded R-module. Then $H^i_P(M)_j = H^i_{P_0}(M_j)$, where P_0 is the graded maximal ideal of R_0. Since M_j is a finitely generated R_0-module it follows that $H^i_{P_0}(M_j)$ is a graded Artinian R_0-module. Hence we see that $(H^i_P(M)^\vee)_j = (H^i_P(M)_{-j})^\vee = H^i_{P_0}(M_{-j})^\vee$ is a finitely generated graded R_0-module for all j. Of course this does not imply that $H^i_P(M)^\vee$ is a finitely generated R-module.

We denote by $\operatorname{cd}(M)$ the cohomological dimension of M with respect to Q, i.e., the number

$$\operatorname{cd}(M) = \sup\{i \in \mathbb{N}_0 : H^i_Q(M) \neq 0\}.$$

Corollary 2.3. *Let $N = H^s_{R_+}(M)^\vee$. Then the following statements hold:*

(a) $\operatorname{cd}(M) < \dim(M)$ *if and only if* $\operatorname{depth}_{R_0} N_j > 0$ *for all j.*
(b) *If* $\operatorname{cd}(M) < \dim(M) - 1$, *then* $\operatorname{depth}_{R_0} N_j > 1$ *for all j.*

Proof. We note that $\operatorname{cd}(M) < \dim(M)$, if and only if $H^s_Q(M) = 0$. Hence Proposition 2.1 yields part (a) of the corollary.

For the proof of (b) we notice that $H^i_P(N) = E^\infty_{s,m-i}$ for $i = 0,1$, and that $E^\infty_{s,m-i}$ is a submodule of $H^{s-i}_Q(M)^\vee$ for all i. Thus our assumption implies that $H^i_{P_0}(N_j) = H^i_P(N)_j = 0$ for $i = 0,1$ and all j. This yields the desired conclusion. □

The second statement of the next corollary is well-known (see [2, Theorem 4.8 (e)]).

Corollary 2.4. *Let M be a finitely generated bigraded R-module of dimension s and depth t. Then $H^{t-m}_Q(M)$ and $H^s_Q(M)$ are tame.*

Proof. We first prove $H^{t-m}_Q(M)$ is tame. By [1, Proposition 2.5] the dimension of N_j as an R_0-module is constant for $j \gg 0$. We set $N = H^t_{R_+}(M)^\vee$ and $s_0 = \dim_{R_0} N_j$ for $j \gg 0$. Note that $s_0 \leq \dim R_0 \leq m$. Thus we have

$H^m_P(N)_j = H^m_{P_0}(N_j) = 0$ for $j \gg 0$ if $s_0 < m$ and $H^m_P(N)_j = H^m_{P_0}(N_j) \neq 0$ for $j \gg 0$ if $s_0 = m$. Therefore by Proposition 2.1 there exists an integer j_0 such that

$$H^{t-m}_Q(M)_j = 0 \text{ for all } j \leq j_0, \quad \text{or } H^{t-m}_Q(M)_j \neq 0 \text{ for all } j \leq j_0,$$

as desired. In order to prove that $H^s_Q(M)$ is tame, we set $N = H^s_{R_+}(M)^\vee$. Since $H^s_{R_+}(M)$ is a graded Artinian R-module, N is a finitely generated graded R-module. Thus N_j is a finitely generated R_0-module. By [1, Proposition 2.5] the set of associated prime ideals of $\operatorname{Ass}_{R_0}(N_j)$ is constant for large j. If $P_0 \in \operatorname{Ass}_{R_0}(N_j)$, it follows that $H^0_P(N)_j = H^0_{P_0}(N_j) \neq 0$ for large j, and if $P_0 \notin \operatorname{Ass}_{R_0}(N_j)$, then $H^0_P(N)_j = H^0_{P_0}(N_j) = 0$ for large j. Thus in view of Proposition 2.1, $H^s_Q(M)$ is also tame. $\square$

We say that M is a generalized Cohen-Macaulay R-module if $H^i_{R_+}(M)$ has finite length for all $i \neq \dim M$.

Proposition 2.5. *Let M be a generalized Cohen-Macaulay R-module of dimension s. Then we have the following long exact sequence of bigraded R-modules*

$$\begin{aligned} 0 \to H^1_P(H^s_{R_+}(M)^\vee) \to H^{s-1}_Q(M)^\vee \to H^{s-1}_{R_+}(M)^\vee \to \\ H^2_P(H^s_{R_+}(M)^\vee) \to H^{s-2}_Q(M)^\vee \to H^{s-2}_{R_+}(M)^\vee \to \\ \cdots \to H^{s-m}_Q(M)^\vee \to H^{s-m}_{R_+}(M)^\vee \to 0. \end{aligned}$$

Moreover, we have the following isomorphisms

$$H^i_{R_+}(M) \cong H^i_Q(M) \quad \textit{for all } i < s-m.$$

Proof. Since M is a generalized Cohen-Macaulay module, we have that $H^i_{R_+}(M)^\vee$ is of finite length for $i \neq s$. Thus by Grothendieck's vanishing theorem [5, Theorem 6.1.2] we see that $E^2_{i,j} = E^\infty_{i,j} = 0$ for $j = 0, \ldots, m-1$ and $i \neq s$. The following picture will make this clear.
Therefore for all k with $s \leq k < s+m$ we get the following exact sequences

$$0 \to E^\infty_{s,r} \to H^l_Q(M)^\vee \to E^\infty_{l,m} \to 0,$$

$$0 \to E^\infty_{l,m} \to E^2_{l,m} \to E^2_{s,r-1} \to E^\infty_{s,r-1} \to 0,$$

where l and r are defined by the equations $s + r = l + m = k$.

Composing these two exact sequences we get the long exact sequence

$$\begin{aligned} \cdots \to E^2_{s,r} \to H^l_Q(M)^\vee \to E^2_{l,m} \to E^2_{s,r-1} \to \\ H^{l-1}_Q(M)^\vee \to E^2_{l-1,m} \to E^2_{s,r-2} \to \ldots, \end{aligned}$$

which yields the desired exact sequence, observing that

$$H^0_P(H^i_{R_+}(M)^\vee) = H^i_{R_+}(M)^\vee$$

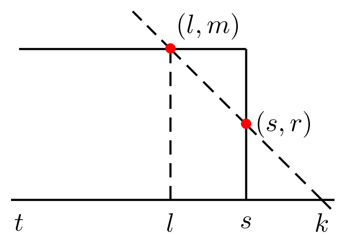

FIGURE 2

for $i \neq s$, since for such i the modules $H^i_{R_+}(M)^\vee$ have finite length. The last statement of the proposition follows similarly. □

COROLLARY 2.6. *Suppose M is a generalized Cohen-Macaulay module of dimension s. Then the following conditions are equivalent:*

(a) *M is Cohen-Macaulay.*
(b) *$H^k_P(H^s_{R_+}(M)^\vee) \cong H^{s-k}_Q(M)^\vee$ for all k.*

Proof. (a) $\Longrightarrow$ (b): Since M is Cohen-Macaulay we have $H^i_{R_+}(M)^\vee = 0$ for all $i \neq s$. Therefore it follows from the long exact sequence in Proposition 2.5 that $H^k_P(H^s_{R_+}(M)^\vee) \cong H^{s-k}_Q(M)^\vee$ for $k = 1, \ldots, m$. The assertion for $k = 0$ follows from Proposition 2.1. The assertion is also clear when $k < 0$. Now assume that $k > m$. Then $s-k < s-m$, and hence by Proposition 2.5 it follows that $H^{s-k}_Q(M)^\vee = H^{s-k}_{R_+}(M)^\vee = 0$. On the other hand, we also have $H^k_P(H^s_{R_+}(M)^\vee) = 0$ because $k > m$.

(b) $\Longrightarrow$ (a): This is proved is the same way. □

As a generalization of Lemma 1.2 we obtain as an immediate consequence of Corollary 2.6 the following

REMARK 2.7. Let R be a bigraded Cohen-Macaulay K-algebra of dimension d. Then

$$H^k_P(\omega_R) \cong H^{d-k}_Q(R)^\vee \quad \text{for all } k.$$

Recall that for a finitely generated graded R-module N one has that $\dim_{R_0} N_j$ as well as $\operatorname{depth}_{R_0} N_j$ is constant for large j; see [1, Proposition 2.5]. In fact, if N is Cohen-Macaulay, then $\lim_{j\to\infty} \operatorname{depth}_{R_0} N_j = \dim N - \dim N/P_0N$ as shown in [9]. We call these constants the limit depth and limit dimension, respectively. Using this fact we have:

COROLLARY 2.8. *Let M be a bigraded Cohen-Macaulay R-module of dimension s. We set $N = H^s_{R_+}(M)^\vee$, and put $t_0 = \lim_{j\to\infty} \operatorname{depth}_{R_0} N_j$ and*

$s_0 = \lim_{j\to\infty} \dim_{R_0} N_j$. *Then the R-modules $H^j_Q(M)$ are tame for all $j \leq s - s_0$ and $j \geq s - t_0$.*

Proof. We see that $H^{s-i}_P(N)_j = H^{s-i}_{P_0}(N_j) \neq 0$ for $j \gg 0$ if $i = s - s_0$ and $i = s - t_0$, and also $H^{s-i}_P(N)_j = H^{s-i}_{P_0}(N_j) = 0$ for $j \gg 0$ if $i < s - s_0$ and $i > s - t_0$. Therefore by Corollary 2.6 we have the desired conclusion. □

Corollary 2.9. *Assume R_0 is Cohen-Macaulay and M is a bigraded Cohen-Macaulay R-module of dimension s. We set $N = H^s_{R_+}(M)^\vee$. Then:*

(a) *For all k and j we have the following isomorphism of graded R_0-modules*
$$\operatorname{Ext}^{d-k}_{R_0}(N_j, \omega_{R_0}) \cong H^{s-k}_Q(M)_{-j},$$
where $d = \dim R_0$.

(b) $\dim H^{s-k}_Q(M)_{-j} \leq k$ *for all k and j.*

Proof. Corollary 2.6 implies that
$$(H^{s-k}_Q(M)_{-j})^\vee \cong (H^{s-k}_Q(M)^\vee)_j = H^k_{P_0}(N_j).$$
Thus the local duality theorem yields
$$H^{s-k}_Q(M)_{-j} \cong H^k_{P_0}(N_j)^\vee \cong \operatorname{Ext}^{d-k}_{R_0}(N_j, \omega_{R_0}),$$
as desired.

Finally by [6, Corollary 3.5.11(c)] one has $\dim_{R_0} \operatorname{Ext}^{d-k}_{R_0}(N_j, \omega_{R_0}) \leq k$. This proves statement (b). □

Let $N \neq 0$ be a graded R_0-module. We set $a(N) = \inf\{i\colon N_i \neq 0\}$ and $b(N) = \sup\{i\colon N_i \neq 0\}$. If $N = 0$, we set $a(N) = \infty$ and $b(N) = -\infty$.

Recall that the *regularity* of N is defined to be
$$\operatorname{reg} N = \max\{b(H^k_{P_0}(N)) + k\colon k = 0, 1, \ldots\}.$$

With the assumptions and notation introduced in Corollary 2.6 we therefore have
$$\operatorname{reg}(N_j) = -\min\{a(H^{s-k}_Q(M)_{-j}) - k\colon k = 0, 1, \ldots\}.$$
In [8] and [11] it is shown that $\operatorname{reg}(N_j)$ is bounded above by a linear function of j. Thus in view of the preceding formula we get:

Corollary 2.10. *Let M be a Cohen-Macaulay R-module. Then there exist integers c and d such that $a(H^k_Q(M)_j) \geq cj + d$ for all k and all j.*

If the dimension and the depth of M differ at most by 1 or $\dim R_0 \leq 1$ one obtains:

Proposition 2.11. *The following statements hold:*

(a) *If* $\dim M = s$ *and* $\operatorname{depth} M = s-1$*, then we obtain the long exact sequence*

$$\cdots \to H_P^{m-j-2}(H_{R_+}^{s-1}(M)^\vee) \to H_P^{m-j}(H_{R_+}^{s}(M)^\vee) \to H_Q^{s-m+j}(M)^\vee \to H_P^{m-j-1}(H_{R_+}^{s-1}(M)^\vee) \to H_P^{m-j+1}(H_{R_+}^{s}(M)^\vee) \to \cdots.$$

(b) *If* $\dim R_0 = 0$*, then* $H_{R_+}^i(M) \cong H_Q^i(M)$ *for all* i*.*

(c) *If* $\dim R_0 = 1$*, then for all* i *we have the short exact sequence*

$$0 \to H_P^1(H_{R_+}^{i+1}(M)^\vee) \to H_Q^i(M)^\vee \to H_P^0(H_{R_+}^{i}(M)^\vee) \to 0.$$

Proof. We first prove (a). Our hypotheses imply the following exact sequences

$$0 \to E_{s,j}^\infty \to H_Q^{s-m+j}(M)^\vee \to E_{s-1,j+1}^\infty \to 0,$$

$$0 \to E_{s-1,j+1}^\infty \to E_{s-1,j+1}^2 \to E_{s,j-1}^2 \to E_{s,j-1}^\infty \to 0.$$

Putting these two exact sequences together we get the long exact sequence

$$\cdots \to E_{s-1,j+2}^2 \to E_{s,j}^2 \to H_Q^{s-m+j}(M)^\vee \to E_{s-1,j+1}^2 \to E_{s,j-1}^2 \to H_Q^{s-m-1+j}(M)^\vee \to E_{s-1,j}^2 \to E_{s,j-2}^2 \to \cdots,$$

which yields the desired exact sequence.

For the proof of (b) we set $N = H_{R_+}^i(M)^\vee$. Since $\dim R_0 = 0$, it follows that N_k is a finitely generated R_0-module of finite length. Thus $H_P^{m-j}(N)_k = H_{P_0}^{m-j}(N_k) = 0$ for all k and $j < m$ and hence $E_{i,j}^2 = 0$ for all i and $j \neq m$. Therefore we have $N = H_P^0(N) \cong H_Q^i(M)^\vee$ for all i, and so $H_{R_+}^i(M) \cong H_Q^i(M)$ for all i. In order to prove (c) we again set $N = H_{R_+}^i(M)^\vee$. Since $\dim R_0 = 1$, it follows that $H_P^{m-j}(N)_k = H_{P_0}^{m-j}(N_k) = 0$ for all k and $j < m-1$ and hence $E_{i,j}^2 = 0$ for all i and $j \neq m, m-1$. Thus for all i we get the exact sequence

$$0 \to E_{i+1,m-1}^\infty \to H_Q^i(M)^\vee \to E_{i,m}^\infty \to 0.$$

Since $E_{i+1,m-1}^\infty = E_{i+1,m-1}^2$ and $E_{i,m}^\infty = E_{i,m}^2$ for all i, the result follows. □

As a simple consequence of Proposition 2.11 (b),(c) we obtain the following tameness result due to [2, Theorem 4.5].

COROLLARY 2.12. *Let* $\dim R_0 \leq 1$*. Then* $H_Q^i(M)$ *is tame for all* i*.*

Proof. First we assume that $\dim R_0 = 0$. Since any Artinian graded R-module is tame, the result follows from Proposition 2.11 (b).

Now we assume that $\dim R_0 = 1$. Let N be a finitely generated bigraded R-module. By Proposition 2.11 (c) it is enough to prove that there exists an integer j_0 such that for $i = 0, 1$ one has:

$$H_{P_0}^i(N_j) = 0 \text{ for all } j \geq j_0, \quad \text{or } H_{P_0}^i(N_j) \neq 0 \text{ for all } j \geq j_0.$$

We set $t_0 = \lim_{j\to\infty} \operatorname{depth}_{R_0} N_j$ and $s_0 = \lim_{j\to\infty} \dim_{R_0} N_j$. Then $H^0_{P_0}(N_j) \neq 0$ for $j \gg 0$ if $t_0 = 0$, and $H^0_{P_0}(N_j) = 0$ for $j \gg 0$ if $t_0 \neq 0$. Similarly, $H^1_{P_0}(N_j) \neq 0$ for $j \gg 0$ if $s_0 = 1$, and $H^1_{P_o}(N_j)$ for $j \gg 0$ if $s_0 = 0$. □

Finally we want to mention two standard 5-term exact sequences arising from our spectral sequence.

Proposition 2.13. *There is a 5-term exact sequence for the corner* $(t, 0)$

$$H_Q^{t+2-m}(M)^\vee \to H_P^{m-2}(H^t_{R_+}(M)^\vee) \to H_P^m(H^{d+1}_{R_+}(M)^\vee) \to H_Q^{t+1-m}(M)^\vee \to H_P^{m-1}(H^t_{R_+}(M)^\vee) \to 0,$$

and a 5-term exact sequence for the corner (s, m)

$$H^s_Q(M)^\vee \to H^0_P(H^{s-1}_{R_+}(M)^\vee) \to H^2_P(H^s_{R_+}(M)^\vee) \to H_Q^{s-1}(M)^\vee \to H^0_P(H^{s-1}_{R_+}(M)^\vee) \to 0.$$

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