GENERALIZED DIVISORS AND BILIAISON

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ABSTRACT. We extend the theory of generalized divisors so as to work on any scheme X satisfying the condition S_2 of Serre. We define a generalized notion of Gorenstein biliaison for schemes in projective space. With this we give a new proof in a stronger form of the theorem of Gaeta, that standard determinantal schemes are in the Gorenstein biliaison class of a complete intersection.

We also show, for schemes of codimension three in $\mathbb{P}^n$, that the relation of Gorenstein biliaison is equivalent to the relation of even strict Gorenstein liaison.

0. Introduction

In this paper we generalize further the theory of generalized divisors introduced in [6] by partially removing the Gorenstein hypotheses. This, we feel, puts the theory in its natural state of generality. The main difference is that instead of requiring the sheaves of ideals defining a generalized divisor to be reflexive, we require only the condition S_2 of Serre. If a scheme X satisfies G_1 and S_1, then a coherent sheaf is reflexive if and only if it satisfies S_2 [6, 1.9]. Here we show that if X satisfies S_1 only, then a coherent sheaf satisfies S_2 if and only if it is ω-reflexive: this means that the natural map $\mathcal{F} \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \omega), \omega)$ is an isomorphism, where ω is the canonical sheaf. With this weaker condition we are able to establish a theory of generalized divisors on schemes X satisfying only the condition S_2.

We apply this theory to define a notion of generalized biliaison for schemes in projective space. Let D be a (generalized) divisor on an ACM scheme X in $\mathbb{P}^n_k$. If $D' \sim D + mH$, meaning D' is linearly equivalent to D plus m times the hyperplane section H of X, we say D' is obtained by an *elementary biliaison* from D. We call *biliaison* the equivalence relation generated by the elementary biliaisons.

If we do biliaisons using only complete intersection schemes X in $\mathbb{P}^n$, the resulting notion of biliaisons is equivalent to even complete intersection liaison (CI-liaison) [6, 4.4]. If we do biliaisons using an ACM scheme X satisfying

G_0, we will show (3.5) that any such biliaison (called G-biliaison) is an even
Gorenstein liaison. We do not know if the converse is true. If we use arbitrary
ACM schemes X, we obtain a notion of biliaison that is possibly more general
than G-biliaison. Note that even this more general type of biliaison preserves
the Rao modules up to shift (3.2).

As an application we give a new proof (4.1) of the theorem of KMMNP–
Gaeta [11, Sec. 3] using biliaisons. I would like to thank Marta Casanellas
for explaining the old proof and helping to discover the new proof. Since this
paper was written in 2003, the theorem of Gaeta and the proof given here
have been further generalized by E. Gorla [3], to include all determinantal
schemes.

Examining the proof of (3.5) we see that the Gorenstein linkages there are
all of a special kind: they use only arithmetically Gorenstein schemes of the
form $M + mH$ on some ACM scheme satisfying G_0 (cf. 3.3). These we call
strict Gorenstein linkages, and so (3.5) actually tells us that every G-biliaison
is an even strict Gorenstein liaison. In Section 5 we prove a partial converse:
Every even strict Gorenstein liaison of codimension 3 subschemes of $\mathbb{P}^n$ is a
Gorenstein biliaison (5.1).

1. ω-reflexive modules

We will need some well-known results about the canonical module, or du-
alizing module as it is sometimes called, of a ring or scheme. We restrict our
attention to equidimensional *embeddable* noetherian rings and schemes. For
a ring A, this means that it is a quotient of a regular ring. For a scheme X,
it means that it can be embedded as a closed subscheme of a regular scheme.
This includes all quasiprojective schemes over a field, which will be our most
common application.

An equidimensional embeddable ring or scheme always has a canonical
module or sheaf unique up to isomorphism. It is finitely generated (resp.
coherent). Its formation commutes with localization, and with completion
of a local ring. If the ring A is a quotient of a regular ring P, and r is the
difference of dimensions, then the canonical module ω of A can be obtained as
$\omega = \mathrm{Ext}_P^r(A, P)$, and similarly for a closed subscheme X of a regular scheme
P. If A is a Cohen–Macaulay ring, then ω is a Cohen–Macaulay module
of the same dimension as A, and for any maximal Cohen–Macaulay module
M, the natural map $M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, \omega), \omega)$ is an isomorphism. For
references see [10] for the case of Cohen–Macaulay rings; [4, II.7] for the case
of projective schemes; and see also [1].

We will expand these results somewhat by weakening their hypotheses to
suit our situation. We define a module M over a ring A (as above) to be ω-
reflexive if the natural map $M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, \omega), \omega)$ is an isomorphism.
Sometimes we will denote by M^ω the module $\mathrm{Hom}_A(M, \omega)$, and call it the

ω-*dual* of M. We say an A-module M satisfies the condition S_r of Serre if depth $M_{\mathfrak{p}} \geq \min(r, \dim A_{\mathfrak{p}})$ for all prime ideals $\mathfrak{p} \subseteq A$.

LEMMA 1.1.   *If A is a local ring of dimension 0, every finitely generated module M is ω-reflexive.*

Proof. Since A is Cohen–Macaulay, this follows from [10, 6.1]. It also follows from the local duality theorem, which says in this case that M^{ω} is the dual of $H^0_{\mathfrak{m}}(M) = M$, so that $M^{\omega\omega}$ is the double dual, which is isomorphic to M. Here $\mathfrak{m}$ denotes the maximal ideal of A. $\square$

LEMMA 1.2.   *For any local ring A, the canonical module ω_A satisfies the condition S_1 of Serre.*

Proof. Write A as a quotient of a regular local ring P of codimension r. Then $\omega_A = \mathrm{Ext}^r_P(A, P)$. By reason of dimension and local duality on P, the functor $\mathrm{Ext}^r_P(\cdot, P)$ is contravariant and left-exact on A modules. If $\dim A = 0$, there is nothing to prove. If $\dim A \geq 1$, let $x \in \mathfrak{m}_A$ be an element such that $\dim A/xA < \dim A$. Then from the sequence

$$A \xrightarrow{x} A \to A/xA \to 0$$

we obtain

$$0 = \mathrm{Ext}^r_P(A/xA, P) \to \omega_A \xrightarrow{x} \omega_A.$$

Thus ω_A has depth at least 1. Since formation of ω commutes with localization, we conclude that ω_A satisfies S_1. $\square$

LEMMA 1.3.   *If a local ring A satisfies S_1, then ω_A satisfies S_2.*

Proof. Write A as a quotient of a regular local ring P of codimension r, as before. Let $x \in \mathfrak{m}_A$ be a non-zero-divisor so that $B = A/xA$ has dimension one less. Then from the exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

and (1.2) we obtain

$$0 \to \omega_A \xrightarrow{x} \omega_A \to \mathrm{Ext}^{r+1}_p(B, P) = \omega_B.$$

Since ω_B satisfies S_1 by (1.2), we see that if $\dim A \geq 2$, then ω_A has depth ≥ 2. Hence ω_A satisfies S_2. $\square$

LEMMA 1.4.   *Let A be a one-dimensional local Cohen–Macaulay ring. Then a finitely generated module M is ω-reflexive if and only if it has depth 1.*

Proof. Since ω has depth 1 (1.2) so does the ω-dual of any module. If M is reflexive, it is the ω-dual of M^{ω} and so has depth 1. The converse is [10, 6.1]. $\square$

PROPOSITION 1.5.   *Let A be a local ring satisfying S_1. A finitely generated module M is ω-reflexive if and only if it satisfies S_2.*

Proof. First we show that the ω-dual of any module N will satisfy S_2. Write N as a cokernel of a map of free modules

$$L_1 \to L_0 \to N \to 0.$$

Taking ω-duals and the image of the second map, we obtain

$$0 \to N^\omega \to L_0^\omega \to K \to 0$$

where K is a submodule of L_1^ω. Now L_0^ω and L_1^ω are direct sums of copies of ω, so satisfy S_2 by (1.3). Hence K satisfies S_1, and then from the exact sequence it follows that N^ω satisfies S_2. In particular, any ω-reflexive module satisfies S_2.

Conversely, suppose M satisfies S_2. The map $\alpha : M \to M^{\omega\omega}$ is an isomorphism in codimension 0, by (1.1), so the kernel of α must have support of codimension ≥ 1. Since M satisfies S_1, there is no kernel. Thus we can write

$$0 \to M \xrightarrow{\alpha} M^{\omega\omega} \to R \to 0.$$

Now, since α is an isomorphism in codimension 1, by (1.4), the module R must have support of codimension ≥ 2. Since both M and $M^{\omega\omega}$ satisfy S_2, this is impossible (cf. proof of [6, 1.9]), so $R = 0$ and α is an isomorphism. $\square$

COROLLARY 1.6.   *Let the local ring A satisfy S_1. The ω-dual of any module is ω-reflexive.*

Proof. This follows from the first step of the proof of (1.5). $\square$

COROLLARY 1.7.   *If the local ring A itself satisfies S_2, then the natural map $A \to \operatorname{Hom}_A(\omega, \omega)$ is an isomorphism.*

REMARK 1.8.   Let X be a scheme satisfying S_1. Using the same arguments as in [6, 1.11,1.12] we see that if $\mathcal{F}$ is a coherent sheaf satisfying S_2 then $\mathcal{F}$ is *normal* in the sense of Barth [5, 1.6], namely for any open set U and any closed subset $Y \subseteq U$ of codimension ≥ 2, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U - Y)$ is bijective. In fact this condition characterizes S_2, if we assume S_1.

If $\mathcal{F}$ is a coherent sheaf satisfying S_1 only, then it is easy to see that the set Y of points of X where it does not satisfy S_2 is a closed subset of codimension ≥ 2, and that the double ω-dual $\mathcal{F}^{\omega\omega}$ can be identified with $j_*(\mathcal{F}|_{X-Y})$ where $j : X - Y \to X$ is the inclusion. Thus the double ω-dual can be regarded as the S_2-ification of the sheaf.

It also follows naturally that for $Y \subseteq X$ closed of codimension ≥ 2, the category of coherent sheaves satisfying S_2 on X is equivalent by restriction to the analogous category on $X - Y$.

REMARK 1.9. To see the connection between the properties reflexive and ω-reflexive, note that the proof of [6, 1.9] shows that a reflexive module over a ring A satisfying S_2 also satisfies S_2. So we see that if A satisfies S_2, then a reflexive module is also ω-reflexive. The converse is not true without the G_1 hypothesis. For example, if X is the union of the three coordinate axes in $\mathbb{A}^3$, a scheme that satisfies G_0 but not G_1, the canonical sheaf ω is ω-reflexive by (1.4), but is easily seen not to be reflexive. On the other hand, the proof of [6, 1.9] does show that if X satisfies S_2, and $\mathcal{F}$ satisfies S_2 and is reflexive in codimension ≤ 1, then $\mathcal{F}$ is reflexive.

Note also that if A satisfies S_2, then the proofs of [6, 1.8,1.9] show that the dual of any module M satisfies S_2, and hence is ω-reflexive.

2. Generalized divisors

Let X be a noetherian, equidimensional, embeddable scheme satisfying the condition S_2 of Serre. We develop the theory of generalized divisors as in [6, §2], noting the differences in our more general setting.

Let $\mathcal{K}_X$ be the sheaf of total quotient rings on X [6, 2.1]. A *fractional ideal* is a subsheaf $\mathcal{I} \subseteq \mathcal{K}_X$ that is a coherent sheaf of $\mathcal{O}_X$-modules. It is *nondegenerate* if for each generic point $\eta \in X$, $\mathcal{I}_\eta = \mathcal{K}_{X,\eta}$.

DEFINITION. Let X be a scheme (as above) satisfying S_2. A *generalized divisor* on X is a nondegenerate fractional ideal $\mathcal{I}$ satisfying the condition S_2 as a sheaf of $\mathcal{O}_X$-modules. It is *effective* if $\mathcal{I} \subseteq \mathcal{O}_X$. We say the generalized divisor $\mathcal{I}$ is *principal* if $\mathcal{I} = (f)$ for some global section f of $\mathcal{K}_X$. We say it is *Cartier* if $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module. We say it is *almost Cartier* if there exists a closed subset $Z \subseteq X$ of codimension ≥ 2 so that $\mathcal{I}|_{X-Z}$ is Cartier. We say it is *reflexive* if $\mathcal{I}$ is a reflexive $\mathcal{O}_X$-module.

Note that $\mathcal{I}$ is Cartier if and only if it is locally principal [6, 2.3]. Note that an almost Cartier divisor is reflexive (1.9) and that the sheaf $\mathcal{I}$ being reflexive implies the condition S_2.

PROPOSITION 2.1. *With X satisfying S_2, as above, the effective generalized divisors are in one-to-one correspondence with closed subschemes $Y \subseteq X$ of pure codimension one with no embedded points.*

Proof. Let Y be a closed subscheme of X, defined by a sheaf of ideals $\mathcal{I}$, so that we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

To say that $\mathcal{I}$ is nondegenerate is equivalent to saying Y has codimension ≥ 1. Since X satisfies S_2, to say that $\mathcal{I}$ satisfies S_2 is equivalent to saying that every associated prime of Y has codimension 1 (cf. [6, 1.10]), i.e., that Y is of pure codimension 1 with no embedded points. $\square$

DEFINITION.   Let X satisfy S_2. For any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, let us denote $\mathcal{F}^{\sim}$ its double ω-dual, so that $\mathcal{F}^{\sim}$ satisfies S_2, where ω is the canonical sheaf on X (1.8). If $\mathcal{I} \subseteq \mathcal{K}_X$ is a fractional ideal, then naturally $\mathcal{I}^{\sim}$ is also a fractional ideal, and will satisfy S_2. We may often denote a generalized divisor $\mathcal{I}$ by a letter D, and call $\mathcal{I}$ the ideal of D. Given two (generalized) divisors D_1 and D_2, with corresponding ideal sheaves $\mathcal{I}_1, \mathcal{I}_2$, we define the *sum* $D_1 + D_2$ by the fractional ideal $(\mathcal{I}_1 \cdot \mathcal{I}_2)^{\sim}$. We define the negative $-D$ by $(\mathcal{I}^{-1})^{\sim}$, where $\mathcal{I}^{-1}$ is the sheaf of local sections f of $\mathcal{K}_X$ for which $f \cdot \mathcal{I} \subseteq \mathcal{O}_X$ locally (cf. [6, 2.2]). We denote the divisor with ideal $\mathcal{I} = \mathcal{O}_X$ by 0.

PROPOSITION 2.2.   *Let X satisfy S_2.*

   (a) *Addition of divisors is associative and commutative.*
   (b) $D + 0 = D$ *for all D.*
   (c) $-(-D) = D$ *if and only if D is reflexive.*
   (d) $D + (-D) = 0$ *if and only if D is almost Cartier.*
   (e) *If D is any divisor, and E is almost Cartier, then $-(D+E) = (-D) + (-E)$.*

   Proof. (a) and (b) are obvious. For (c), first note that $\mathcal{I}^{-1} \cong \mathcal{I}^{\vee} = \mathcal{H}om(\mathcal{I}, \mathcal{O}_X)$ by [6, 2.2]. But by (1.9), $\mathcal{I}^{\vee}$ is ω-reflexive, so in fact the defining sheaf of $-D$ is just $\mathcal{I}^{\vee}$. For (d) we follow the proof of [6, 2.5], noting that at a point of codimension 1, every ideal is ω-reflexive (1.4), so that the condition says $\mathcal{I} \cdot \mathcal{I}^{-1} = \mathcal{O}_X$, which implies $\mathcal{I}$ reflexive there [6, 2.3]. For (e) it is the same proof as [6, 2.5].   □

   COROLLARY 2.3.   *The set of almost Cartier divisors forms a group, containing the subgroups of Cartier divisors and of principal divisors. This group acts on the set of all divisors.*

   DEFINITION.   We say two divisors are *linearly equivalent* if one is obtained from the other by adding a principal divisor. We denote the equivalence classes by the group $\operatorname{Pic} X =$ Cartier divisors mod linear equivalence; the group $\operatorname{APic} X =$ almost Cartier divisors mod linear equivalence, and the set $\operatorname{GPic} X =$ generalized divisors mod linear equivalence.

   PROPOSITION 2.4.   *Two divisors D_1 and D_2 are linearly equivalent if and only if their ideal sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ are isomorphic as $\mathcal{O}_X$-modules. Every coherent $\mathcal{O}_X$-module that satisfies S_2 and is locally free of rank 1 at every generic point of X is isomorphic to the ideal of some divisor.*

   Proof. Indeed, an isomorphism $\varphi : \mathcal{I}_1 \to \mathcal{I}_2$ of sheaves of $\mathcal{O}_X$-modules extends to $\mathcal{I}_1 \otimes \mathcal{K}_X \to \mathcal{I}_2 \otimes \mathcal{K}_X$. Each of these is isomorphic to $\mathcal{K}_X$, so the map is given by multiplication by a global section f of $\mathcal{K}_X$. If $\mathcal{F}$ is coherent

satisfying S_2 and locally free of rank 1 at every generic point, then $\mathcal{F} \otimes \mathcal{K}_X \cong \mathcal{K}_X$ and the natural map $\mathcal{F} \to \mathcal{F} \otimes \mathcal{K}_X$ makes $\mathcal{F}$ into a nondegenerate fractional ideal. $\qquad\square$

WARNING 2.5. The usual theory of the sheaf $\mathcal{L}(D) = \mathcal{I}^{-1}$ associated to a divisor D [6, 2.8] does not extend to divisors that may not be reflexive. However, we can get an analogue of [6, 2.10] using the sheaf $\mathcal{M}(D) = \mathcal{H}om(\mathcal{I}, \omega)$. Note that $\mathcal{M}(D)$ is ω-reflexive by (1.6) and therefore satisfies S_2.

PROPOSITION 2.6. *Let X be a Cohen–Macaulay scheme with canonical sheaf ω, and for any divisor D, corresponding to an ideal sheaf $\mathcal{I}$, let $\mathcal{M}(D) = \mathcal{H}om(\mathcal{I}, \omega)$. If D is an effective divisor, denoting also by D the associated closed subscheme, there are two natural exact sequences*

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

and

$$0 \to \omega_X \to \mathcal{M}(D) \to \omega_D \to 0.$$

Proof. The first is the defining sequence of D. The second is obtained by applying $\mathcal{H}om(\cdot, \omega_X)$ to the first and noting (since X is Cohen–Macaulay) that $\omega_D \cong \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_D, \omega_X)$. $\qquad\square$

DEFINITION–REMARK 2.7. Even though X has a canonical sheaf ω_X, it may not have a canonical divisor. By *canonical divisor* we mean a generalized divisor K whose ideal satisfies $\mathcal{I}_K^{-1} \cong \omega_X$. Since the ideal of any divisor is locally free at the generic points, the existence of a canonical divisor implies that X satisfies G_0. In this case we see also that ω_X must be reflexive, by [6, 1.8]. Since there are schemes satisfying G_0 and S_2 on which ω is not reflexive (1.9), we conclude that G_0 and S_2 are not sufficient conditions for the existence of a canonical divisor.

However, if X satisfies G_0 and S_2, then ω_X is locally free of rank 1 at every generic point, so is isomorphic to a fractional ideal. We choose and fix an embedding $\omega_X \subseteq \mathcal{K}_X$, and call the corresponding divisor M_X the *anticanonical* divisor. As a divisor it depends on the choice of embedding $\omega_X \subseteq \mathcal{K}_X$, but is unique up to linear equivalence. If X satisfies in addition G_1, then ω is invertible in codimension 1, so we can define a *canonical* divisor $K = -M$, which will be an almost Cartier divisor.

DEFINITION. For any two divisors D_1, D_2, we define $D_1(-D_2)$ to be the divisor whose sheaf of ideals is $\mathcal{H}om(\mathcal{I}_2, \mathcal{I}_1)^\sim$. In general, this operation may not be well-behaved, but we do have the following.

PROPOSITION 2.8. *The operation $D_1(-D_2)$ has the following properties.*
(a) $0(-D_2) = -D_2$ *and* $D_1(-0) = D_1$.

(b) *If E is almost Cartier, $(D_1 + E)(-D_2) = D_1(-(D + (-E))) = D_1(-D_2) + E$.*

(c) *In particular, if either D_1 or D_2 is almost Cartier, then $D_1(-D_2) = D_1 + (-D_2)$.*

(d) *If X satisfies G_0, and $D_1 \sim M + E$, where M is the anticanonical divisor and E is almost Cartier, then $D_1(-D_1(-D_2)) = D_2$ for any D_2.*

Proof. (a), (b), (c) are immediate, since an almost Cartier divisor is invertible in codimension 1, and equality of divisors can be tested in codimension 1. (d) corresponds to the fact that any divisor has an ideal sheaf satisfying S_2, and hence is ω-reflexive (1.5). □

REMARK 2.9. We take this opportunity to point out an error in [6, 2.9]. Assuming that X satisfies G_1 and S_2 as in that paper, it is true that every nondegenerate section $s \in \Gamma(X, \mathcal{L}(D))$ gives rise to an effective divisor D' in the complete linear system $|D|$, and all D' arise in this way. Two sections s_1 and s_2 give rise to the same divisor D' if and only if they differ by an isomorphism of $\mathcal{L}(D)$. If D is almost Cartier, the isomorphisms of $\mathcal{L}(D)$ are given by sections of $\Gamma(X, \mathcal{O}_X^*)$ as stated there. So in the familiar case of X integral projective, $\Gamma(X, \mathcal{O}_X^*) = k^*$ and $|D|$ is simply the projective space associated to the vector space $\Gamma(X, \mathcal{L}(D))$.

Suppose, however, that D is not almost Cartier. Then there may be more isomorphisms of $\mathcal{L}(D)$ and the statement of [6, 2.9] is not correct. For example, let $X = L_1 \cup L_2$ be the union of two lines in $\mathbb{P}^2$ meeting at a point P, and let D be the divisor P. Then one can verify that $\dim \Gamma(X, \mathcal{L}(D)) = 2$, and $\mathrm{Isom}(\mathcal{L}, \mathcal{L}) = k^* \oplus k^*$, so that the complete linear system $|D|$ consists just of the single divisor D, as we expect. (Cf. [6, 3.3] for a relevant calculation.)

How does this discussion extend to the case of the present paper, where X is only assumed to satisfy S_2? We cannot use the sheaf $\mathcal{L}(D)$. Instead, for each effective divisor $D' \sim D$, we take ω-duals of $\mathcal{I}_{D'} \subseteq \mathcal{O}_X$ to get $\omega_X \subseteq \mathcal{H}om(\mathcal{I}_{D'}, \omega_X) \cong \mathcal{M}(D)$, and this gives a section s of the sheaf $\mathcal{N}(D) = \mathcal{H}om(\omega_X, \mathcal{M}(D))$. Conversely, nondegenerate sections of $\mathcal{N}(D)$ give effective divisors $D' \sim D$ by reversing the process. The ambiguity of s is again in $\mathrm{Isom}(\mathcal{N}(D), \mathcal{N}(D)) \cong \mathrm{Isom}(\mathcal{I}_D, \mathcal{I}_D)$.

3. Biliaison

In this section we generalize the notion of biliaison introduced in [6, §4] and [12, §5.4]. Note that the word biliaison is not a synonym for even liaison. We also generalize the results of [11, §5] so as to remove the G_1 hypotheses. In fact, it was the attempt to put those results in a more natural context that led to this paper. A scheme X in $\mathbb{P}^n$ is *arithmetically Cohen–Macaulay*

(ACM) if its homogeneous coordinate ring S_X is a Cohen–Macaulay ring. It is *arithmetically Gorenstein* (AG) if S_X is a Gorenstein ring.

DEFINITION. Let V_1 and V_2 be equidimensional closed subschemes of dimension r of $\mathbb{P}_k^n$. We say that V_2 is obtained by an *elementary biliaison* of height h from V_1 if there exists an ACM scheme X in $\mathbb{P}^n$, of dimension $r + 1$, containing V_1 and V_2, and so that $V_2 \sim V_1 + hH$ as generalized divisors on X, where H denotes the hyperplane class. The biliaison is *ascending* if $h \geq 0$; *descending* if $h \leq 0$. The equivalence relation generated by elementary biliaisons will be called *biliaison*.

If we restrict the schemes X in the definition all to be complete intersection schemes, we will speak of CI-*biliaison*. If we restrict the schemes X to be ACM schemes satisfying G_0, will speak of *Gorenstein biliaison* or G-*biliaison*.

REMARK 3.1. As was shown in [6, 4.4], the relation of CI-biliaison is equivalent to even CI-liaison in the usual sense.

PROPOSITION 3.2. *Suppose V_2 is obtained from V_1 by an elementary biliaison of height h on X, with $\dim V_1 = \dim V_2 = r$.*
  (a) *Then reciprocally, V_1 is obtained from V_2 by an elementary biliaison of height $-h$.*
  (b) *The higher Rao modules $M_V^i = H_*^i(\mathcal{I}_{V,\mathbb{P}^n})$ are related as follows:*
  $$M_{V_2}^i \cong M_{V_1}^i(-h) \text{ for } 1 \leq i \leq r.$$
  (c) *The Hilbert polynomials are related by*
  $$\chi(\mathcal{O}_{V_2}(m)) = \chi(\mathcal{O}_X(m)) - \chi(\mathcal{O}_X(m-h)) + \chi(\mathcal{O}_{V_1}(m-h)).$$

Proof. (a) If $V_2 \sim V_1 + hH$ then $V_1 \sim V_2 - hH$.

(b) and (c) have the same proof as [6, 4.5] since only the ACM property of X was used there. $\square$

LEMMA 3.3. *Let X be an ACM scheme satisfying G_0 in $\mathbb{P}^n$. Let $Y \subseteq X$ be an effective divisor, $Y \sim M + mH$, where M is the anticanonical divisor and H is the hyperplane divisor. Then Y is an arithmetically Gorenstein (AG) scheme in $\mathbb{P}^n$.*

Proof. Let X be of dimension $r+1$ so that Y is of dimension r. If $\dim Y \geq 1$, to show that Y is AG is equivalent to showing that Y is ACM and $\omega_Y \cong \mathcal{O}_Y(\ell)$ for some $\ell \in \mathbb{Z}$.

First to show Y is ACM, we must show $H_*^i(\mathcal{I}_{Y,\mathbb{P}^n}) = 0$ for $1 \leq i \leq r$. From the exact sequence
$$0 \to \mathcal{I}_{X,\mathbb{P}^n} \to \mathcal{I}_{Y,\mathbb{P}^n} \to \mathcal{I}_{Y,X} \to 0$$
and the fact that X is ACM, so that $H_*^i(\mathcal{I}_{X,\mathbb{P}^n}) = 0$ for $1 \leq i \leq r + 1$, it is equivalent to show $H_*^i(\mathcal{I}_{Y,X}) = 0$ for $1 \leq i \leq r$. Now $Y \sim M + mH$ by

hypothesis, so $\mathcal{I}_{Y,X} \cong \omega_X(-m)$. By Serre duality on X, $H_*^i(\omega_X(-m))$ is dual to $H_*^{r+1-i}(\mathcal{O}_X(m))$. These latter are 0 for $1 \leq i \leq r$ since X is ACM. Hence Y is ACM.

To study the canonical sheaf ω_Y, we use the second exact sequence of (2.6), namely

$$0 \to \omega_X \to \mathcal{M}(Y) \to \omega_Y \to 0.$$

Now since $\mathcal{I}_{Y,X} \cong \omega_X(-m)$, we have $\omega_X \cong \mathcal{I}_{Y,X}(m)$ and $\mathcal{M}(Y) = \mathcal{H}om(\mathcal{I}_{Y,X}, \omega_X) \cong \mathcal{O}_X(m)$. Thus $\omega_Y \cong \mathcal{O}_Y(m)$ and Y is arithmetically Gorenstein.

If $\dim Y = 0$, it is automatically ACM. To show it is AG, we must show that its graded canonical module Ω_Y is isomorphic to $S_Y(m)$. This follows from the graded module analogue of the argument just given using (2.6). $\square$

REMARK 3.4.   An algebraic version of this result was given in [11, 5.2], and a geometric version with the added hypothesis G_1 in [11, 5.4].

DEFINITION.   Two subschemes V_1 and V_2 of $\mathbb{P}^n$, equidimensional of the same dimension and without embedded components are *linked* by a scheme Y if Y contains V_1 and V_2 and $\mathcal{I}_{V_i,Y} \cong \mathcal{H}om(\mathcal{O}_{V_j}, \mathcal{O}_Y)$ for $i,j = 1,2, i \neq j$. If Y is a complete intersection, it is called a CI-*linkage*; if Y is arithmetically Gorenstein, it is a *Gorenstein linkage*. If Y is an arithmetically Gorenstein scheme of the form $M + mH$ on some ACM scheme X satisfying G_0 (as in (3.3) above), then we will say it is a *strict Gorenstein linkage*. (This is a slight generalization of the terminology of [7, §1], where we required that X should satisfy G_1.)

The equivalence relation generated by CI-linkages is CI-*liaison*, by Gorenstein linkages, *Gorenstein liaison*, and by strict Gorenstein linkages, *strict Gorenstein liaison*. If the liaison can be accomplished by an even number of linkages, then it is an *even* CI-liaison (resp. Gorenstein liaison, resp. strict Gorenstein liaison).

THEOREM 3.5.   *Suppose that V_2 is obtained from V_1 by an elementary biliaison on an* ACM *scheme X satisfying G_0. Then V_2 can be obtained from V_1 by two strict Gorenstein linkages.*

Proof. The proof is almost the same as [6, 4.3], transposed into our context. We assume that $V_2 \sim V_1 + hH$. Thus there is a principal divisor (f) such that $V_2 = V_1 + hH + (f)$. Taking M to be the anticanonical divisor and using (2.8), we can write

$$M(-V_1) = M(-V_2) + hH + (f).$$

Now by [6, 2.11], which still holds in our case, we can find an effective Cartier divisor $E \sim mH$ such that $W = E + M(-V_1) = (M+E)(-V_1)$ is effective. Now let $Y = M + E$. Then Y is an effective divisor that is arithmetically

Gorenstein by (3.4), and I claim that V_1 and W are Gorenstein linked by Y. Indeed, the same argument as in the proof of [6, 4.1] shows that $\mathcal{I}_{W,Y} \cong \mathcal{H}om(\mathcal{O}_{V_1}, \mathcal{O}_Y)$. Since by (2.8) we also have $V_1 = (M + E)(-W)$, we obtain the reverse isomorphism $\mathcal{I}_{V_1,Y} \cong \mathcal{H}om(\mathcal{O}_W, \mathcal{O}_Y)$, so V_1 and W are linked by Y.

We also have $W = E + M(-V_2) + hH + (f)$, so if we let $Y' = E + M + hH + (f)$, then Y' will also be an effective divisor that is an arithmetically Gorenstein scheme, and as above, we see that W and V_2 are linked by Y'. Thus V_2 is obtained from V_1 by two strict Gorenstein linkages. $\qquad\square$

COROLLARY 3.6.   *Every Gorenstein biliaison is an even strict Gorenstein liaison.*

REMARK 3.7.   This theorem was proved for a trivial elementary biliaison $V_2 = V_1 + hH$ (with no linear equivalence) in [11, 5.10], and with the extra hypothesis G_1 in [11, 5.14].

REMARK 3.8.   In Section 5 below we will prove a converse to this theorem in codimension 3.

EXAMPLE 3.9.   Let P be a point in $\mathbb{P}^3$, and let X be the union of three non-coplanar lines through P. Then X satisfies G_0 but not G_1. If H is a hyperplane section of X containing P, then $V = P + H$ is the divisor defined by the square of the ideal of P. Thus P and V are related by one G-biliaison, and hence are evenly G linked. Cf. [11, 4.1], where this was proved by a different method.

4. The theorem of Gaeta

To illustrate the theory of biliaison, we give a new proof the theorem of KMMNP–Gaeta [11, 3.6]. The statement given there is that every standard determinantal scheme is glicci. We prove a slightly stronger result.

THEOREM 4.1.   *Every standard determinantal scheme in $\mathbb{P}^n$ can be obtained from a linear variety by a finite number of ascending Gorenstein biliaisons. In particular, it is glicci by (3.3).*

Proof. We follow the terminology and notation of [11, 3.6]. Let $V \subseteq \mathbb{P}^n$ be a standard determinantal scheme, i.e., a scheme of codimension $c + 1$ whose ideal I_V is generated by the $t \times t$ minors of a $t \times (t + c)$ homogeneous matrix A for some $t > 0$. Let B be the matrix obtained by omitting the last column of A. Then V is contained in the determinantal scheme S defined by the $t \times t$ minors of B. By Step I of the proof of [11, 3.6], S is good determinantal. Hence it is generically a complete intersection [11, 3.2], and so satisfies G_0.

Let A' be the matrix obtained by omitting the last row of B. Then V', defined by the $(t - 1) \times (t - 1)$ minors of A', is also contained in S. We

will show that $V \sim V' + mH$ on S for some $m > 0$, so that V is obtained by an ascending elementary Gorenstein biliaison from V'. Continuing in this manner, after a finite number of G-biliaisons, we reduce to the case $t = 1$, when V is a complete intersection. From these one can perform descending CI-biliaisons to a linear variety.

Let R be the homogeneous coordinate ring of $\mathbb{P}^n$, and let $R_S = R/I_S$ be the homogeneous coordinate ring of S. The ideal of V in S is generated by the images in R_S of the $t \times t$ minors of A that include the last column. The $t \times t$ minors that do not include the last column are just the generators of I_S. On the other hand, the ideal of V' in S is generated by the images of the $(t-1) \times (t-1)$ minors of A'. So there is a one-to-one correspondence between generators N of V in S and generators N' of V' in S, obtained by omitting the last row and column of the corresponding $t \times t$ matrix. We will show that the quotient N/N' of corresponding generators is an element of $H^0(\mathcal{K}_S(m))$, independent of the choice of N, where $\mathcal{K}_S$ is the sheaf of total quotient rings of S, and m is the difference in degrees of N and N'. This will show that $\mathcal{I}_{V,S} \cong \mathcal{I}_{V',S}(-m)$, and so we have the desired biliaison. Note that m is the degree of the element in the lower right-hand corner of the original matrix A.

To show that N/N' is independent of the choice of $N(\mod I_S)$, it will be sufficient to compare two such that differ by one column only. So let M be a $t \times t$ minor of B, let N_1 be obtained by deleting the first column of M and adding the last column of A; let N_2 be obtained by deleting the second column of M and adding the last column of A. Then N_1 and N_2 are two generators of I_V, and the corresponding generators N_1', N_2' of $I_{V'}$ are just M_{t1} and M_{t2}, where M_{ij} denotes the minor of M obtained by deleting the ith row and the jth column. We need to show that $N_1/N_1' = N_2/N_2' \mod I_S$. By making general row and column operations on A at the beginning, we may assume that all the N_i' are non-zero-divisors in R_S. So we must show that $N_1 N_2' - N_2 N_1' \in I_S$.

Let the last column of A be $u_1, \ldots, u_t$. We will expand N_1 and N_2 along this last column. The coefficient of u_t in $N_1 N_2' - N_2 N_1'$ is just $N_1' N_2' - N_2' N_1' = 0$. For $i \neq t$, the coefficient of u_i is $M_{i1} M_{t2} - M_{i2} M_{t1}$. The proof is then completed by the following identity among determinants, since $M \in I_S$. $\square$

LEMMA 4.2. *Let M be the determinant of a $t \times t$ matrix, let M_{ij} denote the minor obtained by deleting the ith row and the jth column; let $M_{ik,jl}$ denote the minor obtained by deleting the ith and k^{th} rows and the jth and l^{th} columns. Then the determinants satisfy*

$$M_{ij} \cdot M_{kl} - M_{il} \cdot M_{kj} = \pm M_{ij,kl} \cdot M.$$

Proof. [13, p. 132ff]. $\square$

EXAMPLE 4.3.   The 4×4 minors of a general 4×6 matrix of linear forms in $\mathbb{P}^4$ define an irreducible smooth curve C of degree 20 and genus 26 which, according to the theorem, can be obtained by ascending Gorenstein biliaisons from a line. However these curves are not general in the Hilbert scheme, and it is known that a general smooth curve of degree 20 and genus 26 is ACM, but cannot be obtained by ascending biliaisons from a line. It is unknown whether it is glicci [7, 3.9].

5. Strict Gorenstein liaison

The main result of this section is a converse to (3.6) in codimension 3.

THEOREM 5.1.   *For subschemes of codimension 3 in $\mathbb{P}^n$ (equidimensional and without embedded components), any even strict Gorenstein liaison is a Gorenstein biliaison.*

Proof. Suppose V and V' of codimension 3 in $\mathbb{P}^n$ are related by even strict Gorenstein liaison. Then there is a sequence

$$V = V_0, V_1, V_2, \ldots, V_{2k} = V'$$

for some k, where each V_i is related to V_{i+1} by a strict Gorenstein linkage. By composition of biliaisons, it will be sufficient to treat the case $k = 1$, i.e., when there is just one intermediary scheme Z, and V to Z is a strict Gorenstein linkage by Y of the form $M + mH$ on an ACM scheme X satisfying G_0, and Z to V' similarly is linked by a Y' of the form $M' + m'H'$ on an ACM scheme X' satisfying G_0.

Since X and X' are both ACM of codimension 2 in $\mathbb{P}^n$, they are in the same CI-biliaison class, by the classical Gaeta's theorem [12, 6.1.4]. Thus we can apply Lemma 5.2 (below) and find a chain

$$X = X_0, X_1, \ldots, X_r = X'$$

of ACM schemes satisfying G_0 and each containing Z, such that each X_i is directly CI-linked to X_{i+1}, and X_i and X_{i+1} have no common components.

Now for each $i = 1, \ldots, r$, let $D_i = X_{i-1} \cap X_i$. By Lemma 5.3 (below), D_i is an AG scheme of the form $M + mH$ on X_{i-1} and on X_i. Since the X_i all contain Z, so do the D_i. For each $i = 1, \ldots, r$, let W_i be the scheme linked to Z by D_i. We consider the chain of strict Gorenstein linkages

$$V = W_0, Z, W_1, Z, W_2, Z, \ldots, W_r, Z, V' = W_{r+1}.$$

Here, for each $i = 0, \ldots, r$, the two links W_i, Z, W_{i+1} are both strict Gorenstein links on the same ACM scheme X_i. Now, as in the proof of [6, 4.1] we see that W_i being linked to Z by $M + mH$ on X_i is equivalent to saying $Z \sim (M + mH)(-W_i)$ on X_i. Similarly, Z linked to W_{i+1} by $M + m'H$ on X_i says $W_{i+1} \sim (M + m'H)(-Z)$. Substituting the first expression for Z in the

second expression for W_{i+1}, we find using (2.8) that $W_{i+1} \sim W_i + (m' - m)H$ on X_i, which is a single Gorenstein biliaison.

Thus V is joined to V' by the chain of Gorenstein biliaisons

$$V = W_0, W_1, \ldots, W_r, W_{r+1} = V'. \qquad \square$$

LEMMA 5.2. *Suppose given X, X' locally Cohen–Macaulay subschemes of codimension 2 in $\mathbb{P}^n$, both satisfying G_0, and both containing a given closed subscheme Z of codimension at least 3 in $\mathbb{P}^n$, and with X, X' in the same CI-liaison class in $\mathbb{P}^n$. Then there exists a chain*

$$X = X_0, X_1, \ldots, X_r = X'$$

of locally Cohen–Macaulay subschemes of $\mathbb{P}^n$, each containing Z, such that each X_{i+1} is obtained by a single geometric CI-linkage from X_i. In particular, X_i and X_{i+1} will have no common components, so each will be generically locally complete intersection and therefore will satisfy G_0.

Proof. Note first that the hypothesis G_0 implies that X and X' are generically locally complete intersection, since they are in codimension 2. If X to X' is an odd liaison, we can make a single general geometric liaison from X' to a new X'' also containing Z, and thus reduce to the case of an even liaison. Then, since X and X' are in the same even liaison class, by Rao's theorem, they have $\mathcal{N}$-type resolutions with stably equivalent sheaves $\mathcal{N}_i$ up to twist. By adding dissocié sheaves, we can write $\mathcal{N}$-type resolutions

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{L} & \to & \mathcal{N} & \to & \mathcal{I}_X(a) & \to & 0 \\
0 & \to & \mathcal{L}' & \to & \mathcal{N} & \to & \mathcal{I}_{X'}(a') & \to & 0
\end{array}
$$

with the same locally free sheaf $\mathcal{N}$ in the middle, and $\mathcal{L}, \mathcal{L}'$ dissocié.

Now we will follow the plan of the proof of [2, 3.1] to obtain a chain

$$X = X_0, X_2, X_4, \ldots, X_{2k} = X'$$

of locally Cohen–Macaulay subschemes containing Z, such that for each i, X_{2i} and X_{2i+2} have no common components and are related by a single elementary CI-biliaison on a hypersurface S_i.

Write $\mathcal{L} = \oplus \mathcal{L}_i$ with $\mathcal{L}_i$ invertible, $i = 1, \ldots, t$. Since X is generically locally complete intersection, the rank of the map

$$\mathcal{L}(\xi) \to \mathcal{N}(\xi)$$

is $t - 1$ for each generic point ξ of X. Thus, reordering if necessary, we define $\mathcal{F}$ by

$$0 \to \bigoplus_{i \geq 2} \mathcal{L}_i \to \mathcal{N} \to \mathcal{F} \to 0$$

and $\mathcal{F}$ will be torsion-free of rank 2, and locally free at each generic point ξ of X. Now choose $b \gg 0$ so that $\mathcal{I}_Z \otimes \mathcal{N}(b)$ is generated by global sections and

take $s_1 \in H^0(\mathcal{I}_Z \otimes \mathcal{N}(b))$ a sufficiently general section. Let Y_1 be defined by

$$0 \to \mathcal{O}(-b) \xrightarrow{s_1} \mathcal{F} \to \mathcal{I}_{Y_1}(a_1) \to 0.$$

Then Y_1 contains Z, and Y_1 has no component in common with X, and Y_1 is obtained from X by a single CI-biliaison [2, 3.3]. Furthermore, we can lift s_1 to $\mathcal{N}$ in such a way that $s_1(\xi_1) \neq 0$ in $\mathcal{N}(\xi_1)$ for each generic point of Y_1. In terms of $\mathcal{N}$ we now have

$$0 \to \mathcal{O}(-b) \oplus \bigoplus_{i \geq 2} \mathcal{L}_i \to \mathcal{N} \to \mathcal{I}_{Y_1}(a_1) \to 0.$$

We repeat this process with each $\mathcal{L}_i$ in turn, obtaining a sequence of biliaisons $X, Y_1, Y_2, \ldots, Y_t$, each one containing Z and having no components in common with its neighbors.

We do the same thing with X', obtaining a similar sequence $X', Y_1', \ldots, Y_t'$. Then we observe that one can take the same b in both cases, and since the sections $s_1, \ldots, s_t, s_1', \ldots, s_t'$ are all sufficiently general, we can take $s_i = s_i'$ for each i, and thus $Y_t = Y_t'$. This connects X and X' by biliaisons, all containing Z. Now just relabel Y_i and Y_i' as X_{2j} to get the sequence of biliaison above.

To conclude, let X_2 and X_4 for example be a biliaison on a hypersurface S, where X_2, X_4 both contain Z and have no common component. Then X_2 and X_4 are both generically Cartier divisors on S. When we link them both to a divisor X_3 on S, as in the proof of (3.5), we can take X_3 to have no component in common with X_2 and X_4, and by adding a complete intersection on S containing Z if necessary, we may assume X_3 contains Z. Thus the sequence of biliaisons connecting X and X' can be filled in to a sequence of geometric liaisons as required. $\square$

LEMMA 5.3. *Let X_1, X_2 be* ACM *schemes in $\mathbb{P}^n$ that have no common component and are directly linked by an* AG *scheme S. Then $D = X_1 \cap X_2$ is arithmetically Gorenstein; moreover, it is of the form $M + \ell H$ on each of X_1, X_2, where ℓ is the integer for which $\omega_S \cong \mathcal{O}_S(\ell)$.*

Proof. (cf. [12, 4.2.1]). The fact that D is ACM follows from the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_D \to 0.$$

Since S is AG, its dualizing sheaf ω_S is isomorphic to $\mathcal{O}_S(\ell)$ for some $\ell \in \mathbb{Z}$. Because of the linkage, $\mathcal{I}_{X_1,S} \cong \mathcal{H}om(\mathcal{O}_{X_2}, \mathcal{O}_S)$. Note that $\mathcal{I}_{X_1,S} = \mathcal{I}_{D,X_2}$ by a standard isomorphism theorem for ideals. Since $\omega_{X_2} = \mathcal{H}om(\mathcal{O}_{X_2}, \omega_S)$, we find that $\mathcal{I}_{D,X_2} \cong \omega_{X_2}(-\ell)$. This says $D \sim M + \ell H$ on X_2. The same argument shows that $D \sim M + \ell H$ on X_1 also. $\square$

6. Conclusion

If we reflect on the outstanding problem whether every ACM subscheme of $\mathbb{P}^n$ is glicci, we can appreciate the usefulness of the extended notion of

generalized divisors in this paper. It has allowed us to prove the theorem of KMMNP–Gaeta in a strengthened form, namely that any standard determinantal scheme in $\mathbb{P}^n$ can be obtained by ascending Gorenstein biliaisons from a linear space. This also makes clear the special nature of determinantal schemes, since there are known examples of other ACM schemes that cannot be obtained by ascending Gorenstein biliaisons from a linear space, even though it is still unknown whether they are glicci or not (for curves in $\mathbb{P}^4$, see [7, 3.9], and for points in $\mathbb{P}^3$ see [9, 7.2]).

We also observe that in most known proofs that some class of ACM schemes is glicci (such as the theorem of KMMNP–Gaeta discussed here) the proof could be accomplished using Gorenstein biliaisons, hence using only strict Gorenstein liaisons. Since there are AG schemes in $\mathbb{P}^n$ not of the special form $M + mH$ on some ACM scheme of one dimension higher (for curves in $\mathbb{P}^4$, see [8, 3.6, 3.11] and for points in $\mathbb{P}^3$ see [9, 3.4, 6.8]), this suggests that it would be worthwhile to investigate more deeply what kind of G-liaisons can be accomplished using AG schemes not of this special form.

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