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CYCLIC MODULES OF FINITE GORENSTEIN INJECTIVE DIMENSION AND GORENSTEIN RINGS

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Dedicated to the achievements of Phil Griffith

ABSTRACT. The main result asserts that a local commutative Noetherian ring is Gorenstein, if it possesses a non-zero cyclic module of finite Gorenstein injective dimension. From this follows a classical result by Peskine and Szpiro stating that the ring is Gorenstein, if it admits a non-zero cyclic module of finite (classical) injective dimension. The main result applies to local homomorphisms of local rings and yields the next: if the source is a homomorphic image of a Gorenstein local ring and the target has finite Gorenstein injective dimension over the source, then the source is a Gorenstein ring. This, in turn, applies to the Frobenius endomorphism when the local ring is of prime equicharacteristic and is a homomorphic image of a Gorenstein local ring.

1. Introduction

Throughout this paper $(R, \mathfrak{m}, k)$ is a commutative Noetherian local ring.

The *Gorenstein injective dimension* in the title was introduced by Enochs, Jenda, and Xu, and it is recalled in 3.1, and for any R–module N this integer is denoted $\operatorname{Gid}_R N$. It is a *refinement* of the classical injective dimension $\operatorname{id}_R N$ in the sense that there is always the inequality $\operatorname{Gid}_R N \leq \operatorname{id}_R N$, and if $\operatorname{id}_R N$ is finite, then $\operatorname{Gid}_R N = \operatorname{id}_R N$. It turns out that the ring R is Gorenstein if and only if every R–module has finite Gorenstein injective dimension; see [6, Chapter 6].

The title refers to the next theorem, which is the main result of the paper.

THEOREM A. *If there exists a non-zero cyclic R–module N with the Gorenstein injective dimension $\operatorname{Gid}_R N$ finite, then R is Gorenstein.*

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The theorem has been proved in special cases by Takahashi; see [24, (3.5)]. As an immediate consequence of our theorem it follows that R is Gorenstein if and only if R has finite Gorenstein injective dimension as an R–module; this is the local commutative version of the main result in Holm [16]. Another immediate consequence is that R is Gorenstein if and only if the residue field k has finite Gorenstein injective dimension as an R–module. Since the Gorenstein injective dimension is a refinement of the injective dimension, the above result implies the next result, which is [22, Théorème (5.5)].

Theorem (Peskine and Szpiro). *If there exists a non-zero cyclic R–module N with the injective dimension $\mathrm{id}_R N$ finite, then R is Gorenstein.*

Theorem A is restated in 4.5, and it is proved there. The proof relies on two auxiliary techniques. The first involves the Bass series $\mathrm{I}_R^M(t)$ and Poincaré series $\mathrm{P}_M^R(t)$ for an R–module M. These series are formal power series, and their coefficients are the Bass numbers and the Betti numbers, respectively. The series are presented in Section 4, where also the next key result is proved.

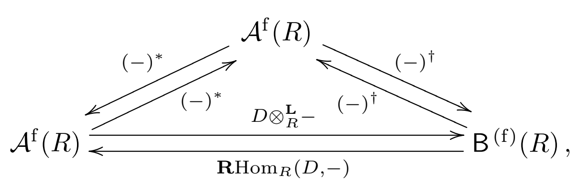
Proposition B. *To any finitely generated R–module N of finite Gorenstein injective dimension there exists then a finitely generated R–module K of finite Gorenstein projective dimension such that there is the next equality:*

$$\mathrm{P}_N^R(t)t^{\operatorname{depth} R} = \mathrm{P}_K^R(t)\,\mathrm{I}_R^R(t)\,.$$

Moreover, if N is of finite (classical) injective dimension, then K has finite (classical) projective dimension.

The second auxiliary technique requires the ring R to admit a dualizing complex D, and it involves two categories $\mathcal{A}^{\mathrm{f}}(R)$ and $\mathcal{B}^{\mathrm{f}}(R)$ consisting of complexes R–modules; see 2.4 and 2.5. Here it is important that a finitely generated R–module is of finite Gorenstein injective dimension if and only if it belongs to $\mathcal{B}^{\mathrm{f}}(R)$; see 3.5, which is main result in [8] by Christensen, Frankild, and Holm. The next result is also used in the proof of Theorem A.

Proposition C. *The two functors $(-)^* = \mathbf{R}\mathrm{Hom}_R(-,R)$ and $(-)^\dagger = \mathbf{R}\mathrm{Hom}_R(-,D)$ fit into the diagram*



in such a way that the inner and outer triangles are both commutative, the left pair of parallel tilted arrows as well as the right pair of parallel tilted arrows provide dualities of categories, and the pair of horizontal arrows provides an equivalence of categories.

This is restated as Proposition 2.6, and it is proved there. The next result is an application of Theorem A.

THEOREM D. *Assume that $\varphi\colon R \longrightarrow S$ is local homomorphism such that the source R is a homomorphic image of a Gorenstein local ring. If the target S has finite Gorenstein injective dimension over the source R, then R is Gorenstein and S has finite Gorenstein flat dimension over R.*

Endomorphisms. Let $\varphi\colon R \longrightarrow R$ be a local endomorphism, let M be an R–module, and let n be a natural number. In this setup, ${}^{\varphi^n}\!M$ denotes M viewed as an R–module via φ^n, that is, the abelian group M equipped with the multiplication $(r,m) \longmapsto \varphi^n(r)m$.

The next result is part of Theorem 5.5.

THEOREM E. *Let $\varphi\colon (R,\mathfrak{m}) \longrightarrow (R,\mathfrak{m})$ be a local endomorphism and assume that R is a homomorphic image of a Gorenstein local ring. The following conditions are then equivalent.*

(i) *R is Gorenstein.*
(ii) $\operatorname{Gid}_R {}^{\varphi^n}\!R$ *is finite for some integer $n \geqslant 1$.*
(iii) $\operatorname{Gid}_R {}^{\varphi^n}\!R$ *is finite for all integers $n \geqslant 1$.*

If one of the above conditions is met, then $\operatorname{Gid}_R {}^{\varphi^n}\!R = \operatorname{depth} R = \dim R.$

When the local ring R is of prime characteristic p, Theorems D and E apply, in particular, to the Frobenius endomorphism $R \to R\ ,\ r \longmapsto r^p$.

The Frobenius endomorphism is a particular instance of a local endomorphism $\varphi\colon (R,\mathfrak{m}) \longrightarrow (R,\mathfrak{m})$ such that $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some integer $i \geqslant 1$; this condition is equivalent to the condition that for every element x from $\mathfrak{m}$ the sequence $(\varphi^i(x))_{i\geqslant 1}$ converges to zero in the $\mathfrak{m}$–adic topology. Such an endomorphism is called a *contraction*. Note that φ in Theorem D and E is *not* supposed to be a contraction.

Theorem E is a Gorenstein version of the next theorem, which is [4, (13.3)].

THEOREM (AVRAMOV–IYENGAR–MILLER). *Let $\varphi\colon (R,\mathfrak{m}) \longrightarrow (R,\mathfrak{m})$ be a contraction. If* $\operatorname{id}_R {}^{\varphi^n}\!R$ *is finite for some integer $n \geqslant 1$, then R is regular.*

The last result should be compared to the classical results by Kunz [20, (2.1)] and Rodicio [23] showing that the ring is regular precisely when the R–module ${}^{\varphi^n}\!R$ has finite flat dimension (or, equivalently, finite projective dimension). Moreover, the result [17, (6.5)] by Iyengar and Sather-Wagstaff implies that the ring is Gorenstein exactly when the Gorenstein flat dimension of R–module ${}^{\varphi^n}\!R$ is finite.

Organization of the paper. The main results, Theorems A, D, and E, belong to *classical homological algebra.* Their proofs, however, take—of necessity—place in the derived category $\mathcal{D}(R)$ of the category of R–modules.

For now there is no suitable description of the applications of this *hyperhomological algebra* to commutative ring theory. Thus, necessary background material is scattered throughout Sections 2–4.

2. Dualities and equivalences

2.1. Derived category. Throughout the paper, $(R, \mathfrak{m}, k)$ is a local commutative Noetherian ring. We will work within the derived category $\mathcal{D}(R)$; see, for example, Gelfand and Manin [15].

The objects in $\mathcal{D}(R)$ are complexes of R–modules. Homological notation is used, so when M is a complex, the differential has degree -1, that is, $\partial_n^M \colon M_n \longrightarrow M_{n-1}$. The symbol $\simeq$ denotes isomorphisms in the derived category. If n is an integer, $\Sigma^n M$ denotes the complex M shifted n degrees to the left, that is, against the direction of the differential.

The full subcategory of $\mathcal{D}(R)$ consisting of complexes with bounded homology is denoted $\mathcal{D}_\square(R)$, while $\mathcal{D}_\square^f(R)$ denotes the full subcategory of $\mathcal{D}_\square(R)$ consisting of complexes with each homology module finitely generated; the objects in $\mathcal{D}_\square^f(R)$ will be referred to as *finite complexes*. Each R–module M is viewed as a complex concentrated in degree zero. Moreover, each complex M of R–modules with homology concentrated in degree zero is isomorphic in $\mathcal{D}(R)$ to the module $\mathrm{H}_0(M)$. Thus we identify R–modules with complexes homologically concentrated in degree zero.

The homological size of a complex M is given by its *homological infimum*, its *homological supremum*, and its *amplitude:* $\inf M = \inf\{\, \ell \mid \mathrm{H}_\ell(M) \neq 0 \,\}$, $\sup M = \sup\{\, \ell \mid \mathrm{H}_\ell(M) \neq 0\}$, and $\operatorname{amp} M = \sup M - \inf M$. We set $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$, so M belongs to $\mathcal{D}_\square(R)$ if and only if $\sup M < \infty$ and $\inf M > -\infty$. Moreover, $\operatorname{amp} M = 0$ if and only if M is isomorphic (in the derived category) to $\Sigma^n K$ for some non-zero R–module K and some integer n; namely $K = \mathrm{H}_n(M)$.

If M is a homologically bounded complex, then M is said to be of *finite projective dimension*, *finite injective dimension*, or *finite flat dimension*, when M is isomorphic (in $\mathcal{D}(R)$) to a bounded complex consisting of, respectively, projective modules, injective modules, or flat modules, in which case we write, respectively, $\operatorname{pd}_R M < \infty$, $\operatorname{id}_R M < \infty$, or $\operatorname{fd}_R M < \infty$. For details consult [1].

The left derived tensor product functor $-\otimes_R^{\mathbf{L}} \sim$ is defined, up to isomorphism in $\mathcal{D}(R)$, by taking an appropriate projective resolution of the first argument or of the second one. The right derived homomorphism functor $\mathbf{R}\mathrm{Hom}_R(-, \sim)$ is obtained by taking an appropriate projective resolution of the first argument or by taking an appropriate injective resolution of the second one. If M and N are R–modules, then there are isomorphisms $\mathrm{H}_\ell(M \otimes_R^{\mathbf{L}} N) \cong \mathrm{Tor}_\ell^R(M, N)$ and $\mathrm{H}_\ell(\mathbf{R}\mathrm{Hom}_R(M, N)) \cong \mathrm{Ext}_R^{-\ell}(M, N)$ for all integers ℓ.

2.2. Functorial isomorphisms. The next standard isomorphisms are used throughout the paper. To facilitate the description here, also the other ring S is supposed to be commutative, and not all the boundedness conditions imposed on the complexes are strictly necessary. Let K, L, and M belong to $\mathcal{D}(R)$, let P belong to $\mathcal{D}(S)$, and let N belong to the derived category $\mathcal{D}(R,S)$ of the category of R–S–bimodules. There are then the next functorial isomorphisms in $\mathcal{D}(R,S)$.

(Comm) $$L \otimes^{\mathbf{L}}_R M \xrightarrow{\simeq} M \otimes^{\mathbf{L}}_R L\,.$$

(Assoc) $$(M \otimes^{\mathbf{L}}_R N) \otimes^{\mathbf{L}}_S P \xrightarrow{\simeq} M \otimes^{\mathbf{L}}_R (N \otimes^{\mathbf{L}}_S P)\,.$$

(Adjoint) $$\mathbf{R}\mathrm{Hom}_S(M \otimes^{\mathbf{L}}_R N, P) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(N,P))\,.$$

(Swap) $$\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(P,N)) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_S(P, \mathbf{R}\mathrm{Hom}_R(M,N))\,.$$

Moreover, there are the following evaluation morphisms.

(Tensor–eval)
$$\alpha_{KNP}\colon \mathbf{R}\mathrm{Hom}_R(K,N) \otimes^{\mathbf{L}}_S P \longrightarrow \mathbf{R}\mathrm{Hom}_R(K, N \otimes^{\mathbf{L}}_S P) \quad \text{and}$$

(Hom–eval)
$$\beta_{PNM}\colon P \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(N,M) \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(P,N), M)\,.$$

- The morphism α_{KNP} is an isomorphism, if K is finite, $\mathrm{H}(N)$ is bounded, and either $\mathrm{fd}_S\, P$ or $\mathrm{pd}_R\, K$ is finite.
- The morphism β_{PNM} is an isomorphism, if P is finite, $\mathrm{H}(N)$ is bounded, and either $\mathrm{pd}_S\, P$ or $\mathrm{id}_R\, M$ is finite.

For details consult [6, A.4] and the references therein.

2.3. Dimension and depth. For a finite complex M we define its *depth* and *dimension* as follows:

$$\dim_R M = \sup\{\, \dim_R \mathrm{H}_\ell(M) - \ell \mid \ell \in \mathbb{Z}\,\} \quad \text{and}$$
$$\mathrm{depth}_R M = \inf\{\, \ell \mid \mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(k,M)) \neq 0\,\}\,.$$

Here $\dim_R \mathrm{H}_\ell(M)$ denotes the (Krull) dimension of the module $\mathrm{H}_\ell(M)$. If M is a finitely generated module, then these invariants yield the classical depth and dimension of M. It turns out that the dimension of a complex M equals the supremum of the numbers $\dim R/\mathfrak{p} - \inf M_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Spec}\, R$; see [10, (16.3)].

2.4. Dagger duality. In this paragraph we assume that R admits a *normalized dualizing complex* D, that is, D is a finite R–complex, its injective dimension $\mathrm{id}_R\, D$ is finite, $\sup D = \dim R$, and the canonical morphism $\mu_D\colon R \longrightarrow \mathbf{R}\mathrm{Hom}_R(D,D)$ is an isomorphism. It follows that $k \simeq \mathbf{R}\mathrm{Hom}_R(k,D)$ and $\inf D = \mathrm{depth}\, R$.

When M is a finite complex, we consider the *dagger dual* $M^\dagger = \mathbf{R}\mathrm{Hom}_R(M,D)$, and the following relations hold: $\sup M^\dagger = \dim_R M$ and

$\inf M^{\dagger} = \operatorname{depth}_R M$; for details see [10, (16.20)]. In particular, if $\mathrm{H}(M) \neq 0$ then $\operatorname{depth}_R M \leqslant \dim_R M$, and the Cohen–Macaulay defect of M is defined to be the non-negative integer $\operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M$.

If R is a homomorphic image of a local Gorenstein ring Q, then $\Sigma^{\dim Q}\,\mathbf{R}\mathrm{Hom}_Q(R,Q)$ is a normalized dualizing complex over R. Consequently, every complete local ring admits a dualizing complex. On the other hand, if a local ring admits a dualizing complex, then it is a homomorphic image of a Gorenstein ring by Kawasaki [19].

The contravariant functor $(-)^{\dagger} = \mathbf{R}\mathrm{Hom}_R(-,D)$ carries the category $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ into itself, and for every finite M there is the biduality isomorphism $\delta^M_D \colon M \xrightarrow{\simeq} M^{\dagger\,\dagger}$ from (Hom–eval) in 2.2. This induces the next duality of categories.

$$\mathsf{D}^{\mathrm{f}}_{\square}(R) \underset{(-)^{\dagger}}{\overset{(-)^{\dagger}}{\rightleftarrows}} \mathsf{D}^{\mathrm{f}}_{\square}(R).$$

If R is a Cohen–Macaulay ring of dimension d possessing a normalized dualizing complex D, then $\mathrm{H}_n(D) = 0$ for $n \neq d$, and $\mathrm{H}_d(D)$ is said to be the dualizing (or canonical) module for R; see [5, Chapter 3].

2.5. Dualizing equivalence. Let D be a dualizing complex for R and consider the functors $D \otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(D,-)$.

The *Auslander categories* $\mathcal{A}(R)$ and $\mathcal{B}(R)$ are full subcategories of $\mathcal{D}(R)$ defined as follows.

$$\mathcal{A}(R) = \left\{ M \in \mathcal{D}_{\square}(R) \;\middle|\; \begin{array}{l} \eta_M \colon M \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R M) \text{ is an iso-} \\ \text{morphism in } \mathcal{D}(R), \text{ and } D \otimes^{\mathbf{L}}_R M \text{ is bounded} \end{array} \right\},$$

$$\mathcal{B}(R) = \left\{ N \in \mathcal{D}_{\square}(R) \;\middle|\; \begin{array}{l} \varepsilon_N \colon D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,N) \xrightarrow{\simeq} N \text{ is an isomor-} \\ \text{phism in } \mathcal{D}(R), \text{ and } \mathbf{R}\mathrm{Hom}_R(D,N) \text{ is bounded} \end{array} \right\}.$$

The homomorphisms η_M and ε_N are the evaluation morphisms α_{DDM} and β_{DDN} from 2.2, respectively, composed with the isomorphism $\mathbf{R}\mathrm{Hom}_R(D,D) \simeq R$. This setup was introduced in [2], where it was also pointed out that any complex of finite flat dimension belongs to $\mathcal{A}(R)$, and that any complex of finite injective dimension belongs to $\mathcal{B}(R)$.

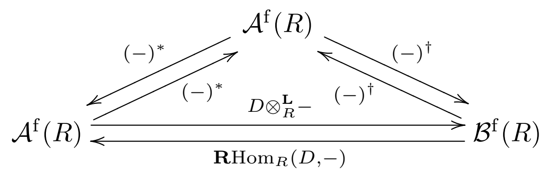
The Auslander categories are defined in such a way that there are the following equivalences of categories; the second equivalence then follows by restriction.

$$\mathcal{A}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathcal{B}(R).$$

$$\mathcal{A}^{\mathrm{f}}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathcal{B}^{\mathrm{f}}(R).$$

In the following result we consider the two functors $(-)^* = \mathbf{R}\mathrm{Hom}_R(-, R)$ and $(-)^\dagger = \mathbf{R}\mathrm{Hom}_R(-, D)$.

2.6. Proposition. *The following statements hold for the next diagram.*



(1) *The inner triangle is commutative (up to canonical isomorphism).*
(2) *The outer triangle is commutative (up to canonical isomorphism).*
(3) *The left pair of parallel tilted arrows provides a duality of categories.*
(4) *The right pair of parallel tilted arrows provides a duality of categories.*
(5) *The pair of horizontal arrows provides an equivalence of categories.*

Proof. The assertion concerning the horizontal arrows follows by dualizing equivalence 2.5. The functor $(-)^*$ provides a duality on $\mathcal{A}^{\mathrm{f}}(R)$; see [2, (4.1.7)] or [7, (4.7)]. The corresponding assertion concerning $(-)^\dagger$ follows by similar arguments; see also the proof of [6, (3.2.9)]. The commutativity of the inner and outer triangles are up to natural isomorphisms composed by the isomorphisms in 2.2 and the isomorphism $R \simeq \mathbf{R}\mathrm{Hom}_R(D, D)$. $\square$

3. Gorenstein homological dimensions

3.1. Gorenstein injective dimension. An R–complex I is said to be a *complete injective resolution*, if it is exact and consists of injective modules, and it is such that $\mathrm{Hom}_R(I', I)$ is exact for all injective R–modules I'. An R–module J is said to be *Gorenstein injective*, if it is a cokernel in a complete injective resolution. Thus, every injective module is Gorenstein injective.

The *Gorenstein injective dimension* $\mathrm{Gid}_R\, M$ of $M \in \mathcal{D}_\square(R)$ is defined to be the infimum of the set of integers n such that there exists a complex I consisting of Gorenstein injective modules satisfying $M \simeq I$ and $I_\ell = 0$ for $-\ell > n$. (Recall that we always use homological notation.) Thus, a complex of R–modules has finite Gorenstein injective dimension if and only if it is isomorphic in $\mathcal{D}(R)$ to a bounded complex of Gorenstein injective modules. Moreover, the Gorenstein injective dimension is a *refinement* of the (classical) injective dimension, that is, $\mathrm{Gid}_R\, M \leqslant \mathrm{id}_R\, M$ with equality if $\mathrm{id}_R\, M$ is finite; see [6, Chapter 6].

3.2. Gorenstein projective dimension. The *Gorenstein projective dimension* $\mathrm{Gpd}_R\, M$ of $M \in \mathcal{D}_\square(R)$ was introduced by Enochs and Jenda; it is defined dually to the injective one above, it is a refinement of the (classical) projective dimension, that is, $\mathrm{Gpd}_R\, M \leqslant \mathrm{pd}_R\, M$ with equality if $\mathrm{pd}_R\, M < \infty$; see [6, Chapter 4].

3.3. Gorenstein flat dimension. The definition of the *Gorenstein flat dimension* $\operatorname{Gfd}_R M$ of $M \in \mathcal{D}_{\square}(R)$ is similar to the Gorenstein injective dimension above. A complex F of R–modules is said to be a *complete flat resolution*, if it is exact and consists of flat modules, and it is such that also $I' \otimes_R F$ is exact for all injective R–modules I'. An R–module G is said to be *Gorenstein flat*, if it is a cokernel in a complete flat resolution. Thus, every flat module is Gorenstein flat.

The *Gorenstein flat dimension* $\operatorname{Gfd}_R M$ of $M \in \mathcal{D}_{\square}(R)$ is defined to be the infimum of the set of integers n such that there exists a complex F consisting of Gorenstein flat modules satisfying $M \simeq F$ and $F_\ell = 0$ for $\ell > n$. Thus, a complex of R–modules has finite Gorenstein flat dimension if and only if it is isomorphic in $\mathcal{D}(R)$ to a bounded complex of Gorenstein flat modules. Moreover, the Gorenstein flat dimension is a refinement of the (classical) flat dimension, that is, $\operatorname{Gfd}_R M \leqslant \operatorname{fd}_R M$ with equality if $\operatorname{fd}_R M < \infty$. For details consult [6, Chapter 5].

3.4. Auslander's G–dimension. If G is a *finitely generated* R–module, then it is Gorenstein projective if and only if it satisfies the next conditions.

- $\operatorname{Ext}^\ell_R(G,R) = 0$ and $\operatorname{Ext}^\ell_R(\operatorname{Hom}_R(G,R),R) = 0$ for $\ell > 0$, and
- the canonical map $G \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(G,R),R)$ is an isomorphism,

see, for example, [6, (4.4.6)]. Auslander's *Gorenstein dimension* $\operatorname{G–dim}_R M$ of a finite R–complex M is defined to be at most n exactly when M is isomorphic in $\mathcal{D}(R)$ to a bounded complex G such that $G_\ell = 0$ for $\ell > n$, and such that each G_ℓ is a finitely generated R–modules satisfying the above two conditions; see [6, (2.3.2)]. By [8, (3.8)] $\operatorname{G–dim}_R M = \operatorname{Gpd}_R M = \operatorname{Gfd}_R M$ for any finite complex M.

Moreover, [6, (3.1.11)] yields $\operatorname{G–dim}_R M < \infty$ if and only if $M \in \mathcal{A}^{\mathrm{f}}(R)$. The last result is extended by the next one, which is the main theorem in Christensen, Frankild and Holm [8].

3.5. Finiteness of Gorenstein dimensions. Let R be a homomorphic image of a Gorenstein ring. If M is an R–complex, then the following are equivalent.

(i) M belongs to $\mathcal{A}(R)$.
(ii) M has finite Gorenstein projective dimension, that is, $\operatorname{Gpd}_R M < \infty$.
(iii) M has finite Gorenstein flat dimension, that is, $\operatorname{Gfd}_R M < \infty$.

Furthermore, if N is an R–complex, then the following are equivalent.

(i) N belongs to $\mathcal{B}(R)$.
(ii) N has finite Gorenstein injective dimension, that is, $\operatorname{Gid}_R N < \infty$.

For details consult [8, (4.1) and (4.4)].

The next result on completion and Gorenstein injective dimension is due to Christensen, Frankild, and Iyengar. We thank Christensen and Iyengar for allowing us to include it here. Note that it does not require R to be a homomorphic image of a Gorenstein ring.

3.6. THEOREM. *Let R be a local ring, and let M be a finite R–complex. If M has finite Gorenstein injective dimension over R, then $M \otimes_R \widehat{R}$ has finite Gorenstein injective dimension over $\widehat{R}$.*

Proof. Let K^R be the Koszul complex on a set of generators for the maximal ideal $\mathfrak{m}$. Because the homology modules of K^R have finite length, there is a isomorphism $K^R \simeq \widehat{R} \otimes_R K^R$ in $\mathcal{D}(R)$. By flatness of the completion map $R \longrightarrow \widehat{R}$, a minimal set of generators for $\mathfrak{m}$ extends to a minimal set of generators of $\widehat{\mathfrak{m}}$; in particular, $K^{\widehat{R}} = \widehat{R} \otimes_R K^R$ is a Koszul complex on a minimal set of generators for $\widehat{\mathfrak{m}}$. Moreover, the isomorphism of R–modules $R/\mathfrak{m} \cong k \cong \widehat{R}/\widehat{\mathfrak{m}}$ together with the fact that $\mathrm{H}_\ell(K^R) \cong \mathrm{H}_\ell(K^{\widehat{R}})$ for all $\ell \in \mathbb{Z}$ shows that $K^R \simeq K^{\widehat{R}}$ (in $\mathcal{D}(R)$); we set $K = K^{\widehat{R}}$.

Under the present assumptions on M, the complex

$$N = M \otimes_R K^R \simeq M \otimes_R K \simeq (M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes_{\widehat{R}} K$$

has finite Gorenstein injective dimension over R; see [9, (5.5)(c')]. Note that the homology modules of N have finite length since M is finite. Hence there is an isomorphism

$$N \xrightarrow{\simeq} \mathrm{Hom}_R(\mathrm{Hom}_R(N, \mathrm{E}_R(k)), \mathrm{E}_R(k)) \,.$$

Here $\mathrm{E}_R(k)$ denotes the injective envelope of the R–module k. In particular,

$$\mathrm{Gid}_R \, \mathrm{Hom}_R(\mathrm{Hom}_R(N, \mathrm{E}_R(k)), \mathrm{E}_R(k)) < \infty$$

and, therefore, $\mathrm{Gfd}_R \, \mathrm{Hom}_R(N, \mathrm{E}_R(k))$ is finite by [6, (6.4.2)]. As the homology modules of $\mathrm{Hom}_R(N, \mathrm{E}_R(k))$ has finite length, the complex

$$(*) \qquad \mathrm{Hom}_R(N, \mathrm{E}_R(k)) \otimes_R \widehat{R} \simeq \mathrm{Hom}_R(N, \mathrm{E}_R(k))$$

has finite Gorenstein flat dimension over $\widehat{R}$. Thus, using $(*)$ and (Adjoint) from 2.2 we conclude

$$N \xrightarrow{\simeq} \mathrm{Hom}_{\widehat{R}}(\mathrm{Hom}_R(N, \mathrm{E}_R(k)), \mathrm{E}_R(k))$$

has finite Gorenstein injective dimension over $\widehat{R}$ by [8, (5.1)]; this uses the fact that the complete ring $\widehat{R}$ admits a dualizing complex D. From [8, (4.4)] it follows that the complex

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, N) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K$$

is homologically bounded. Thus, from [13, 1.3] it follows that the complex $\mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, M \otimes^{\mathbf{L}}_R \widehat{R})$ is homologically bounded as well. Consider the commutative diagram

$$\begin{array}{ccc}
(M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K & \xleftarrow[\simeq]{\varepsilon^{\widehat{R}}_{(M\otimes^{\mathbf{L}}_R \widehat{R})\otimes^{\mathbf{L}}_{\widehat{R}} K}} & D \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, (M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K) \\
\Vert & & \downarrow \simeq \\
(M \otimes^{\mathbf{L}}_R \widehat{R}) \otimes^{\mathbf{L}}_{\widehat{R}} K & \xleftarrow[(\varepsilon^{\widehat{R}}_{M\otimes^{\mathbf{L}}_R \widehat{R}})\otimes^{\mathbf{L}}_{\widehat{R}} K]{} & (D \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\mathrm{Hom}_{\widehat{R}}(D, M \otimes^{\mathbf{L}}_R \widehat{R})) \otimes^{\mathbf{L}}_{\widehat{R}} K
\end{array}$$

where the rightmost vertical isomorphism is by (Tensor-eval) and (Assoc) in 2.2. Again, using [13, 1.3] and a standard mapping cone argument, it follows that $\varepsilon^{\widehat{R}}_{M\otimes^{\mathbf{L}}_R \widehat{R}}$ is a isomorphism. Whence, $M \otimes_R \widehat{R} \simeq M \otimes^{\mathbf{L}}_R \widehat{R}$ has finite Gorenstein injective dimension over $\widehat{R}$ by [8, (4.4)]. □

4. Bass and Poincaré series

4.1. Bass and Poincaré series. For a finite R–complex M and an integer ℓ, the ℓ th *Bass number* and the ℓ th *Betti numbers* are, respectively, the next vector space dimensions over the residue field k:

$$\mu^{\ell}_R(M) = \mathrm{rank}_k \, \mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(k, M)) \quad \text{and} \quad \beta^R_{\ell}(M) = \mathrm{rank}_k \, \mathrm{H}_{\ell}(k \otimes^{\mathbf{L}}_R M)\,.$$

The ring of *formal Laurent series with integer coefficients* is denoted $\mathbb{Z}(|t|)$; its elements have the form $\sum_{\ell\in\mathbb{Z}} a_\ell t^\ell$ with $a_\ell \in \mathbb{Z}$ and $a_\ell = 0$ for $\ell \ll 0$. For a finite R–complex M the *Bass series* $\mathrm{I}^M_R(t)$ and the *Poincaré series* $\mathrm{P}^R_M(t)$ are elements in $\mathbb{Z}(|t|)$ defined as follows.

$$\mathrm{I}^M_R(t) = \sum_{\ell\in\mathbb{Z}} \mu^{\ell}_R(M) t^{\ell} \quad \text{and} \quad \mathrm{P}^R_M(t) = \sum_{\ell\in\mathbb{Z}} \beta^R_{\ell}(M) t^{\ell}\,.$$

The Bass series for the ring is denoted $\mathrm{I}_R(t)$, and the ring R is Gorenstein precisely when $\mathrm{I}_R(t) = t^s$ for some integer s. If D is a finite R–complex, then D is a normalized dualizing complex if and only if $\mathrm{I}^D_R(t) = 1$. The next equations for Bass and Poincaré series will be important later. The first two are proved in [2, (1.5.3)], while the last two are proved in [12, (4.3)].

4.2. Bass–Poincaré equalities. For finite complexes M and N there are the next equalities of formal Laurent series, that is, equalities in $\mathbb{Z}(|t|)$.

$$\text{(PP)} \qquad \mathrm{P}^R_{M\otimes^{\mathbf{L}}_R N}(t) = \mathrm{P}^R_M(t)\,\mathrm{P}^R_N(t)\,.$$

$$\text{(PI)} \qquad \mathrm{I}_R^{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) = \mathrm{P}^R_M(t)\,\mathrm{I}^N_R(t)\,.$$

$$\text{(IP)} \qquad \mathrm{I}_R^{M\otimes^{\mathbf{L}}_R N}(t) = \mathrm{I}^M_R(t)\,\mathrm{P}^N_R(t^{-1}) \quad \text{provided} \quad \mathrm{pd}_R N < \infty\,.$$

$$\text{(II)} \qquad \mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) = \mathrm{I}^M_R(t)\,\mathrm{I}^R_N(t^{-1}) \quad \text{provided} \quad \mathrm{id}_R N < \infty\,.$$

4.3. Proposition. *Let R be a homomorphic image of a Gorenstein ring, and let N be a finite R–complex of finite Gorenstein injective dimension. There exists then a finite R–complex K of finite Gorenstein projective dimension with* $\inf K = \inf N$ *and* $\operatorname{amp} K \leqslant \operatorname{amp} N$ *such that there is the next equality of formal Laurent series.*

$$\mathrm{P}^R_N(t)t^{\operatorname{depth} R} = \mathrm{P}^R_K(t)\,\mathrm{I}_R(t)\,.$$

If N is of finite (classical) injective dimension, then K has finite (classical) projective dimension.

Proof. Throughout the proof, we let D be a normalized dualizing complex, and set $M = N^{\dagger}$, $L = \mathbf{R}\mathrm{Hom}_R(D,N)$, and $s = \operatorname{depth} R$. Note that $\operatorname{depth}_R M = \inf N$ by 2.4. As N belongs to $\mathcal{B}^{\mathrm{f}}(R)$ by 3.5, Lemma 2.6 yields that M and L belong to $\mathcal{A}^{\mathrm{f}}(R)$, and that $L^* \simeq M$ and $M^* \simeq L$. As M belongs to $\mathcal{A}^{\mathrm{f}}(R)$, the G–dimension of M is finite by 3.5. Whence, by [6, (1.2.7) and (1.4.8)] we obtain

$$\inf L = \inf M^* = -\operatorname{G–dim}_R M = \operatorname{depth}_R M - \operatorname{depth} R = \inf N - \operatorname{depth} R.$$

Next, [6, (A.4.6)] yields the inequality

$$\sup L = \sup \mathbf{R}\mathrm{Hom}_R(D,N) \leqslant -\inf D + \sup N = \sup N - \operatorname{depth} R.$$

The last equality is by 2.4. Hence, $\operatorname{amp} L \leqslant \operatorname{amp} N$. Set $K = \Sigma^s L$, which has finite Gorenstein projective dimension as it belongs to $\mathcal{A}^{\mathrm{f}}(R)$. It still remains to prove the equation for the Laurent series. Recall that $s = \operatorname{depth} R$. The computation

$$K^{*\dagger} \simeq ((\Sigma^s L)^*)^{\dagger} = (\Sigma^{-s} L^*)^{\dagger} \simeq (\Sigma^{-s} M)^{\dagger} = \Sigma^s M^{\dagger} \simeq \Sigma^s N$$

yields the next isomorphism.

$$\text{(4.3.1)} \qquad N \simeq \Sigma^{-s} K^{*\dagger}.$$

Using that $\mathrm{I}^D_R(t) = 1$ by 2.4 and that $\mathrm{P}_R(t) = 1$, the formulae (II) and (PI) in 4.2 yields the equalities

$$\mathrm{P}^R_N(t) = \mathrm{P}^R_{\Sigma^{-s}K^{*\dagger}}(t) = t^{-s}\,\mathrm{P}^R_{K^{*\dagger}}(t) = t^{-s}\,\mathrm{I}^{K^*}_R(t) = t^{-s}\,\mathrm{P}^R_K(t)\,\mathrm{I}_R(t).$$

Finally, if $\mathrm{id}_R N$ is finite, it follows that $\mathrm{pd}_R N^\dagger$ is finite as well, and this implies that $\mathrm{pd}_R N^{\dagger *} < \infty$, that is, $\mathrm{pd}_R K < \infty$. □

4.4. Remark. When the finite complex N from Proposition 4.3 is a module, the finite complex K is also a module, and it has finite Gorenstein projective dimension. If the ring R is complete, then it is possible to use results in [14, Section 2] to prove that $K \cong \mathrm{Ext}_R^s(\mathrm{E}_R(k), N)$, where $s = \operatorname{depth} R$. The latter module was used by Peskine and Szpiro in the proof of their theorem mentioned in the introduction.

4.5. Theorem. *If there exists a non-zero cyclic R–module N with the Gorenstein injective dimension $\mathrm{Gid}_R N$ finite, then R is Gorenstein.*

Proof. Note first that we may assume that R is complete and thus possesses a dualizing complex; see 3.6. As N is cyclic, we have $N \cong R/\operatorname{Ann}_R N$, and hence the constant term in $\mathrm{P}_N^R(t)$ is 1. Thus, the equlity of power series in 4.3 yields that the constant term in $\mathrm{P}_K^R(t)$ is also 1. In particular, the module K occurring in 4.3 is cyclic; whence $K \cong R/\operatorname{Ann}_R K$. The formula (4.3.1) gives that $\operatorname{Ann}_R N \supseteq \operatorname{Ann}_R K$. Applying the functor $(-)^{\dagger *}$ to (4.3.1) we obtain the equation $K \simeq \Sigma^s N^{\dagger *}$. This yields $\operatorname{Ann}_R N \subseteq \operatorname{Ann}_R K$. Thus $\operatorname{Ann}_R N = \operatorname{Ann}_R K$, and it follows that $N \cong K$, so the equation in 4.3 implies that $\mathrm{I}_R(t) = t^s$, that is, R is Gorenstein. □

4.6. Cohen–Macaulay injective dimension. Theorem 4.6 below is an immediate consequence of 4.5, and it characterizes Cohen–Macaulay rings in terms finiteness of the Cohen–Macaulay injective dimension introduced by Holm and Jørgensen [18]. Recall from [7] that a finitely generated R–module C is *semi-dualizing* if the natural homomorphism $R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^i(C, C) = 0$ for all $i > 0$, that is, the homothety morphism $R \longrightarrow \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathcal{D}(R)$.

The *Cohen–Macaulay injective dimension* of an R–module M is defined as

$$\mathrm{CMid}_R M = \inf\{\, \mathrm{Gid}_{R\ltimes C} M \mid C \text{ is a semi-dualizing module over } R\,\}.$$

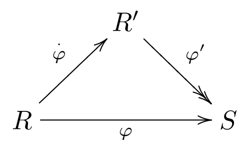
Here $R \ltimes C$ denotes the trivial extension ring; it is the R–module $R \times C$ equipped with the multiplication $(r, c)(r', c') = (rr', rc' + r'c)$. If $(R, \mathfrak{m}, k)$ is local, then so is $(R \ltimes C, \mathfrak{m} \times C, k)$. The ring homomorphism $R \ltimes C \longrightarrow R$ defined by $(r, c) \longmapsto r$ turns every R–module into an $R \ltimes C$– module; if N is cyclic over R, then it is so over $R \ltimes C$. Finally, the module C is a dualizing module precisely when the ring $R \ltimes C$ is Gorenstein; for details see Foxby [11].

4.6. Theorem. *If there exists a non-zero cyclic R–module N with the Cohen–Macaulay injective dimension $\mathrm{CMid}_R N$ finite, then R is Cohen–-Macaulay with dualizing module.* □

5. Local homomorphisms

In this section we apply Theorem 4.5 to a local homomorphism of local rings $\varphi\colon (R,\mathfrak{m},k) \longrightarrow (S,\mathfrak{n},\ell)$.

5.1. Cohen factorizations. In this section we will use the *Cohen factorizations* of local homomorphisms introduced in Avramov, Foxby, and Herzog [3]. A Cohen factorization of φ is a commutative diagram of local homomorphisms



such that φ' is surjective, and $\dot{\varphi}$ is flat with the closed fiber $R'/\mathfrak{m}R'$ a regular ring and with the target R' complete.

Cohen factorizations often exist: the *semi-completion* $\grave{\varphi}\colon R \longrightarrow S \longrightarrow \widehat{S}$ always admits a Cohen factorization; see [3, (1.1)].

5.2. Theorem. *If $\varphi\colon R \longrightarrow S$ is a local homomorphism and R is a homomorphic image of a Gorenstein ring, then $\operatorname{Gid}_R S < \infty$ if and only if R is Gorenstein.*

Proof. If R is Gorenstein, then $\operatorname{Gid}_R S$ is finite. Next, assume that $\operatorname{Gid}_R S$ is finite. Lemma 5.3 below implies that $\operatorname{Gid}_R \widehat{S}$ is finite as well. Choose a Cohen factorization $R \longrightarrow R' \longrightarrow \widehat{S}$ of the semi-completion $\grave{\varphi}$, and note that 5.3 below yields that also the cyclic R'–module $\widehat{S}$ has finite Gorenstein injective dimension. Thus, it follows from the main theorem 4.5 that R' is Gorenstein, and by flat descent [21, (23.4)], so is R. □

5.3. Lemma. *Assume R is a homomorphic image of a Gorenstein ring. Let $\varphi\colon R \longrightarrow S$ be a local homomorphism, let $R \longrightarrow R' \longrightarrow \widehat{S}$ be a Cohen factorization of its semi-completion, let N be a bounded complex of S–modules, and set $\widetilde{N} = N \otimes_R \widehat{S}$. The next numbers are then simultaneously finite.*

$$\operatorname{Gid}_R N, \quad \operatorname{Gid}_R \widetilde{N}, \quad \textit{and} \quad \operatorname{Gid}_{R'} \widetilde{N}$$

Proof. Let D denote the normalized dualizing complex for R. According to [8, thm. 4.4] we are required to show the following two equivalences

$$N \in \mathcal{B}(R) \iff \widetilde{N} \in \mathcal{B}(R) \iff \widetilde{N} \in \mathcal{B}(R')\,.$$

The latter equivalence follows from the second part of [8, (5.3)].

To establish the former one, note that as the completion $\widehat{S}$ is flat over S there is the isomorphism $\mathbf{R}\mathrm{Hom}_R(D,N) \otimes^{\mathbf{L}}_S \widehat{S} \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D,\widetilde{N})$. Thus, the homology of $\mathbf{R}\mathrm{Hom}_R(D, \widetilde{N})$ is bounded if and only if that of $\mathbf{R}\mathrm{Hom}_R(D, N)$ is. Moreover, from the commutative diagram

$$\begin{array}{ccc}
\widetilde{N} & \xleftarrow{\varepsilon^R_{\widetilde{N}}} & D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, \widetilde{N}) \\
\Vert & & \uparrow \simeq \\
N \otimes^{\mathbf{L}}_S \widehat{S} & \xleftarrow[\varepsilon^R_N \otimes^{\mathbf{L}}_R \widehat{S}]{} & D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, N) \otimes^{\mathbf{L}}_S \widehat{S}
\end{array}$$

it follows that $\varepsilon^R_{\widetilde{N}}$ and ε^R_N are simultaneously isomorphisms. □

The following result is [17, (8.14) and (8.15)] and it concerns contractions.

5.4. Theorem (Iyengar–Sather-Wagstaff). *Let $\varphi\colon (R, \mathfrak{m}) \longrightarrow (R, \mathfrak{m})$ be a contraction, that is, $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some integer $i \geqslant 1$. The following conditions are then equivalent.*

(i) *R is Gorenstein.*
(ii) *$\mathrm{Gfd}_R\, {}^{\varphi^n}\! R$ is finite for all integers $n \geqslant 1$.*
(iii) *There exists a finite R–complex P with $\mathrm{H}(P) \neq 0$ and $\mathrm{pd}_R\, P$ finite such that $\mathrm{Gfd}_R\, {}^{\varphi^n}\! P$ is finite for some integer $n \geqslant 1$.*

If one of the above equivalent conditions is satisfied, then $\mathrm{Gfd}_R\, {}^{\varphi^n}\! R = 0$.

What follows may be thought of as a Gorenstein injective version of the above. However, the result is not restricted to contracting endomorphisms.

5.5. Theorem. *Let $\varphi\colon (R, \mathfrak{m}) \longrightarrow (R, \mathfrak{m})$ be a local homomorphism, and assume that R is a homomorphic image of a Gorenstein ring. The following conditions are equivalent*

(i) *R is Gorenstein.*
(ii) *$\mathrm{Gid}_R\, {}^{\varphi^n}\! R$ is finite for all integers $n \geqslant 1$.*
(iii) *There exists a finite R–complex P with $\mathrm{H}(P) \neq 0$ and $\mathrm{pd}_R\, P$ finite such that $\mathrm{Gid}_R\, {}^{\varphi^n}\! P$ is finite for some integer $n \geqslant 1$.*

Proof. The equivalence of (i) and (ii) results from Theorem 5.2. Clearly (iii) is stronger than (ii), and it is trivial that (i) implies (iii). □

If the equivalent conditions of the theorem are satisfied, then it is possible to prove that $\mathrm{Gid}_R\, {}^{\varphi^n}\! R = \operatorname{depth} R = \dim R$. However, because the only proof that the remaining author knows is quite long, it has been left out to keep this article at a reasonable length.

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References

[1] L. L. Avramov and H.-B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), 129–155. MR 1117631 (93g:18017)

[2] ———, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), 241–270. MR 1455856 (98d:13014)

[3] L. L. Avramov, H.-B. Foxby, and B. Herzog, *Structure of local homomorphisms*, J. Algebra **164** (1994), 124–145. MR 1268330 (95f:13029)

[4] L. L. Avramov, S. Iyengar, and C. Miller, *Homology over local homomorphisms*, Amer. J. Math. **128** (2006), 23–90. MR 2197067 (2007e:13027)

[5] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956 (95h:13020)

[6] L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, vol. 1747, Springer-Verlag, Berlin, 2000. MR 1799866 (2002e:13032)

[7] ———, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), 1839–1883. MR 1813596 (2002a:13017)

[8] L. W. Christensen, A. Frankild, and H. Holm, *On Gorenstein projective, injective and flat dimensions—a functorial description with applications*, J. Algebra **302** (2006), 231–279. MR 2236602 (2007h:13022)

[9] L. W. Christensen and H. Holm, *Ascent properties of Auslander categories*, Canad. J. Math, to appear.

[10] H.-B. Foxby, *Hyperhomological algebra & commutative rings*, notes in preparation.

[11] ———, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973). MR 0327752 (48 #6094)

[12] ———, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), 5–19. MR 0447269 (56 #5584)

[13] H.-B. Foxby and S. Iyengar, *Depth and amplitude for unbounded complexes*, Commutative algebra (Grenoble/Lyon, 2001), Contemp. Math., vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 119–137. MR 2013162 (2004k:13028)

[14] A. Frankild, *Vanishing of local homology*, Math. Z. **244** (2003), 615–630. MR 1992028 (2004d:13027)

[15] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR 1950475 (2003m:18001)

[16] H. Holm, *Gorenstein derived functors*, Proc. Amer. Math. Soc. **132** (2004), 1913–1923. MR 2053961 (2004m:16009)

[17] S. Iyengar and S. Sather-Wagstaff, *G-dimension over local homomorphisms. Applications to the Frobenius endomorphism*, Illinois J. Math. **48** (2004), 241–272. MR 2048224 (2005c:13016)

[18] P. Jørgensen and H. Holm, *Cohen–Macaulay homological dimensions*, Rend. Sem. Mat. Univ. Padova, to appear.

[19] T. Kawasaki, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), 123–149. MR 1859029 (2002i:13001)

[20] E. Kunz, *Characterizations of regular local rings for characteristic p*, Amer. J. Math. **91** (1969), 772–784. MR 0252389 (40 #5609)

[21] H. Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461 (90i:13001)

[22] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. (1973), 47–119. MR 0374130 (51 #10330)

[23] A. G. Rodicio, *On a result of Avramov*, Manuscripta Math. **62** (1988), 181–185. MR 963004 (89k:13014)

[24] R. Takahashi, *The existence of finitely generated modules of finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. **134** (2006), 3115–3121. MR 2231892 (2007d:13020)

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