Illinois Journal of Mathematics
Volume 51, Number 1, Spring 2007, Pages 41–56
S 0019-2082

FACTORIZATIONS OF BIRATIONAL EXTENSIONS OF LOCAL RINGS

STEVEN DALE CUTKOSKY AND HEMA SRINIVASAN

In honor of Phillip Griffith

Abstract. We give a proof of local strong factorization of a birational, monomial extension of regular local rings along a valuation of rank 1 and maximal rational rank. Our proof uses methods from linear algebra, and is in the spirit of Christensen's proof of this result in dimension 3. This has also been proven by Karu using toric geometry.

1. Introduction

Suppose that R and S are regular local rings such that S dominates R ($R \subset S$ and the maximal ideal m_S of S contracts to the maximal ideal m_R of R).

$R \to S$ is monomial if there exist regular parameters $x_1, \ldots, x_m$ in R, $y_1, \ldots, y_n$ in S, an $m \times n$ matrix $A = (a_{ij})$ of rank m whose entries are nonnegative integers and units $\delta_i \in S$ such that

$$x_i = \prod_{j=1}^{n} y_j^{a_{ij}} \delta_i$$

for $1 \leq i \leq m$.

Suppose that $P \subset R$ is a regular prime (R/P is a regular local ring) and $0 \neq f \in P$. The regular local ring $R_1 = R[\frac{P}{f}]_m$, where m is a maximal ideal of $R[\frac{P}{f}]$ containing m_R, is called a monoidal transform of R.

Suppose that V is a valuation ring of the quotient field of S which dominates S (and thus dominates R). Then given a regular prime P of R (or of S) there exists a unique monoidal transform R_1 of R (or S_1 of S) obtained from P such that V dominates R_1 (or V dominates S_1).

Received January 19, 2006; received in final form November 14, 2006.
2000 *Mathematics Subject Classification.* 13B10, 13A18, 14E05.
Research of Steven Dale Cutkosky partially supported by NSF.

The local monomialization theorem of [C2] and [C4] shows that given an extension $R \to S \subset V$ as above such that R, S are essentially of finite type over a field k of characteristic zero, there exists a commutative diagram

$$\begin{array}{lll} R_1 & \to & S_1 \quad \subset V \\ \uparrow & & \uparrow \\ R & \to & S \end{array}$$

such that the vertical arrows are products of monoidal transforms and $R_1 \to S_1$ is monomial.

Suppose that we further have that $R \to S$ is birational (the induced homomorphism of quotient fields is an isomorphism). If $R \to S$ is monomial and birational, then we can find regular parameters $\overline{y}_1, \ldots, \overline{y}_n$ in S such that

$$x_i = \prod_{j=1}^{n} \overline{y}_j^{a_{ij}}$$

for $1 \le i \le n$ (since $B = A^{-1}$ has integral coefficients).

We may now state Abhyankar's local factorization conjecture (page 237 of [Ab]). Suppose that $R \to S$ is a birational extension of regular local rings of dimension $n \ge 3$ and V is a valuation ring of the quotient field of S such that V dominates R. The conjecture is that there exists a commutative diagram

$$\begin{array}{ccccc} & & T & \subset & V \\ & \nearrow & & \nwarrow & \\ R & & \to & & S \end{array} \tag{1.1}$$

where the northeast and northwest arrows are products of monoidal transforms.

It is proven in [Z] and [Ab1] that there is a direct factorization of $R \to S$ by monoidal transforms if $n = 2$. However, examples of the failure of a direct factorization of $R \to S$ by monoidal transforms are given in [Sa] and [Sh] when $n \ge 3$.

The local factorization theorem is proven when $n = 3$ (and R is essentially of finite type over a field of characteristic 0) in [C1, Theorem A].

In [C2, Theorem 1.9] it is proven that the local monomialization theorem ([C2, Theorem 1.1]) and "strong factorization" of birational toric morphisms of nonsingular toric varieties implies the local factorization theorem in all dimensions (in characteristic zero).

There are two published proofs of "strong factorization" of birational toric morphisms, [Mo] and [AMR]. They have both been found to have errors (as explained in the correction [AMR1] to [AMR]).

Suppose that R is essentially of finite type over a field. In [C2], a strong version of local monomialization is used to reduce the proof of local factorization to the following problem, which is essentially in linear algebra.

We assume that $R \to S$ is monomial, with respect to regular parameters $x_1, \ldots, x_n$ in R and $y_1, \ldots, y_n$ in S, the value group of V is contained in $\mathbf{R}$, and if ν is a valuation of the quotient field of S whose valuation ring is V, then

$$\tau_1 = \nu(y_1), \ldots, \tau_n = \nu(y_n)$$

are rationally independent real numbers.

In this special case, we can assume that $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ and $S = k[y_1, \ldots, y_n]_{(y_1,\ldots,y_n)}$, where k is a field. We have expressions $x_i = \prod_{j=1}^n y_j^{a_{ij}}$ for $1 \le i \le n$. If

$$f = \sum \alpha_{i_1,\ldots,i_n} y_1^{i_1} \ldots y_n^{i_n} \in k[y_1, \ldots, y_n],$$

we have $\nu(f) = \min\{i_1\tau_1 + \cdots + i_n\tau_n \mid \alpha_{i_1,\ldots,i_n} \neq 0\}$. We will call the local factorization conjecture in this special case the "monomial problem".

When $n = 3$, the monomial problem is solved by Christensen [Ch]. In [C2, Theorem 1.6], the first author extends this to prove a weaker form of the monomial problem for all n. By combining this with the local monomialization theorem of [C2], it was proved in [C2] that a birational extension $R \to S$ can be factored by $n-2$ triangles of monoidal transforms.

Recently, there has been a proof by Karu [K] of this monomial problem, using toric geometry.

In this paper, we give a self-contained proof of the monomial problem. We solve the problem in the spirit of Christensen's original theorem in dimension 3. In particular, the problem can be stated completely in the language of linear algebra, and we prove it using linear algebra. As a result, we give an explicit algorithm for the solution of the monomial problem. This theorem (Theorem 2.1) is proven in Section 2 of this paper. The solution to the monomial problem is given in Theorem 3.1.

We show in Theorem 3.3 of Section 3 of this paper how the local monomialization theorem, [C2, Theorem 1.1] and Theorem 2.1 of this paper prove the local factorization conjecture. This provides a complete proof to Theorem 1.9 of [C2].

A monoidal transform affects the coefficient matrix A as a column addition. The valuation can be understood as a column vector $\vec{v}$ of positive rational numbers. To preserve the property that the valuation ring dominates the monodial transform of the local ring, we allow only those column operations on A that keep both A and $A^{-1}\vec{v}$ positive. We construct an algorithm here for finding a sequence of permissible column additions and interchanges to be followed by a sequence of permissible subtractions that results in the identity matrix.

2. Matrix factorization

Suppose that $A = (a_{ij})$ is an $n \times n$ matrix with coefficients which are nonnegative integers and $\mathrm{Det}(A) = \pm 1$. Further suppose that $\vec{v} = (v_1, \ldots, v_n)^t$ is a $n \times 1$ column vector with coefficients which are positive rationally independent real numbers, and $\vec{w} = (w_1, \ldots, w_n)^t = A^{-1}\vec{v}$ is a vector with positive coefficients (which are necessarily rationally independent). $(A, \vec{v}, \vec{w})$ satisfying these conditions will be called an n-dimensional triple.

The column addition C_{ij} of A which adds the j-th column of A to the i-th column is called *permissible* for $(A, \vec{v}, \vec{w})$ if $w_j - w_i > 0$. The triple $(A, \vec{v}, \vec{w})$ is transformed under the permissible column addition C_{ij} to the triple $(A(1), \vec{v}(1), \vec{w}(1))$, where $A(1) = (a(1)_{ij})$ is obtained from A by adding the j-th column of A to the i-th column, $\vec{v}(1) = \vec{v}$ and $\vec{w}(1) = (w(1)_1, \ldots, w(1)_n)^t = A(1)^{-1}\vec{v}(1)$. $\vec{w}(1)$ is obtained from $\vec{w}$ by subtracting the i-th coefficient w_i from the j-th coefficient w_j of $\vec{w}$.

The row subtraction R_{ji} of A which subtracts the i-th row of A from the j-th row is called *permissible* for $(A, \vec{v}, \vec{w})$ if $a_{jk} \geq a_{ik}$ for $1 \leq k \leq n$. The triple $(A, \vec{v}, \vec{w})$ is transformed under the permissible row subtraction R_{ji} to the triple $(A(1), \vec{v}(1), \vec{w}(1))$, where $A(1) = (a(1)_{ij})$ is obtained from A by subtracting the i-th row of A from the j-th row, $\vec{v}(1)$ is obtained from $\vec{v}$ by subtracting the i-th coefficient v_i from the j-th coefficient v_j and $\vec{w}(1) = (w(1)_1, \ldots, w(1)_n)^t = A(1)^{-1}\vec{v}(1)$. We have that $\vec{w}(1) = \vec{w}$.

The row interchange T_{ij} of A interchanges the i-th and j-th rows of A. T_{ij} transforms the triple $(A, \vec{v}, \vec{w})$ into the triple $(A(1), \vec{v}(1), \vec{w}(1))$, where $A(1)$ is obtained from A by interchanging the i-th and j-th row, $\vec{w}(1) = \vec{w}$ and $\vec{v}(1)$ is obtained from $\vec{v}$ by interchanging the i-th and j-th row of $\vec{v}$.

In this section, we prove the following theorem:

THEOREM 2.1. *Suppose that $A = (a_{ij})$ is an $n \times n$ matrix with coefficents which are nonnegative integers and* $\mathrm{Det}(A) = \pm 1$. *Further suppose that $\vec{v} = (v_1, \ldots, v_n)^t$ is a $n \times 1$ vector with coefficients which are positive rationally independent real numbers. Then there exists a sequence of permissible column additions and row interchanges*

$$(A, \vec{v}, \vec{w}) \rightarrow (A(1), \vec{v}, \vec{w}(1)) \rightarrow \cdots \rightarrow (A(s), \vec{v}, \vec{w}(s))$$

followed by a sequence of permissible row subtractions

$$(A(s), \vec{v}, \vec{w}(s)) \rightarrow (A(s+1), \vec{v}(s+1), \vec{w}(s)) \rightarrow \cdots \rightarrow (A(t), \vec{v}(t), \vec{w}(t))$$

such that $A(t)$ is the $n \times n$ identity matrix.

We will denote the inverse of a matrix A by $B = (b_{ij}) = A^{-1}$. If a permissible column addition C_{ij} is performed by adding the j-th column of A to the i-th column, with a resulting transformation of triples $(A, \vec{v}, \vec{w}) \rightarrow (A(1), \vec{v}, \vec{w}(1))$, then $B(1) = (b(1)_{ij}) = A(1)^{-1}$ is obtained from $B = A^{-1}$ by subtracting the i-th row of B from the j-th row, since $C_{ij}^{-1} = R_{ji}$ and

$$B(1) = A(1)^{-1} = (AC_{ij})^{-1} = C_{ij}^{-1}A^{-1} = R_{ji}A^{-1}.$$

Similarly, if a permissible row subtraction R_{ji} is performed by subtracting the i-th row of A from the j-th row, with a resulting transformation of triples $(A, \vec{v}, \vec{w}) \to (A(1), \vec{v}(1), \vec{w})$, then $B(1) = (b(1)_{ij}) = A(1)^{-1}$ is obtained from $B = A^{-1}$ by adding the j-th column of B to the i-th column.

If a permissible row interchange T_{ij} is performed, then $B(1) = A(1)^{-1}$ is obtained from B by interchanging the i-th and j-th column.

Given a triple $(A, \vec{v}, \vec{w})$, we define $\beta = \max_k\{|b_{k1}|\}$. We will write $A = (C_1, \ldots, C_n)$.

To simplify notation, we will denote the inverse of a matrix $A(t)$ by $B(t) = (b_{ij}(t))$, and define $\beta(t) = \max_k\{|b_{k1}(t)|\}$. We will denote $A(t) = (C_1(t), \ldots, C_n(t))$.

Remark 2.2. Fix i and j. Either C_{ji} is permissible or C_{ij} is permissible (but not both). If C_{ij} is permissible, then after performing C_{ij} a finite number of times, C_{ji} becomes permissible. This is because C_{ij} decreases w_j by a positive integral multiple of w_i.

Definition 2.3. A permissible C_{ij} is allowable for the triple $(A, \vec{v}, \vec{w})$ if b_{i1} and b_{j1} are both non-zero and have the same sign.

Definition 2.4. A permissible C_{ij} is *-allowable for the triple $(A, \vec{v}, \vec{w})$ if either $b_{i1}b_{j1} = 0$, or C_{ij} is allowable.

Remark 2.5. (1) If we perform a *-allowable C_{ij} on the triple $(A, \vec{v}, \vec{w})$ to get $(A(1), \vec{v}, \vec{w}(1))$, then $b_{j1}(1) = b_{j1} - b_{i1}$, $b_{k1}(1) = b_{k1}$ if $k \neq j$ and thus

$$\beta(1) = \max_k\{|b_{k1}(1)|\} \leq \max_k\{|b_{k1}|\} = \beta.$$

(2) Suppose that we fix i and j. Then after a finite sequence consisting of allowable C_{ij} and C_{ji}, both C_{ij} and C_{ji} are not allowable. If at least one of b_{i1}, b_{j1} is nonzero, then after a finite sequence consisting of *-allowable C_{ij} and C_{ji}, both C_{ij} and C_{ji} are not *-allowable.

Proof of (2). If b_{i1} and b_{j1} are nonzero of the same sign, and we perform C_{ij} (or C_{ji}) to obtain the new triple $(A(1), \vec{v}, \vec{w}(1))$, and $b_{i1}(1)$, $b_{j1}(1)$ have the same sign, we then obtain that $(|b_{i1}|, |b_{j1}|) > (|b_{i1}(1)|, |b_{j1}(1)|)$ in the Lex order on $\mathbf{Z}^2$.

Suppose that $b_{i1} \neq 0$ and $b_{j1} = 0$. If C_{ij} is *-allowable, then after performing C_{ij}, we obtain that both C_{ij} and C_{ji} are not *-allowable. If C_{ji} is *-allowable, and we perform C_{ji}, then $b_{i1}(1) = b_{i1}$, $b_{j1}(1) = 0$. By Remark 2.2, we can only perform C_{ji} a finite number of consecutive times. $\square$

LEMMA 2.6. *There exists a sequence of allowable column additions*

$$(A, \vec{v}, \vec{w}) \to (A(1), \vec{v}, \vec{w}(1)) \to \cdots \to (A(t), \vec{v}, \vec{w}(t))$$

such that at most two entries of the first column of $B(t)$ are nonzero.

The proof of this lemma is immediate from [C2, Theorem 6.3].

LEMMA 2.7. *There exists a sequence of *-allowable column additions*

$$(A, \vec{v}, \vec{w}) \to (A(1), \vec{v}, \vec{w}(1)) \to \cdots \to (A(s), \vec{v}, \vec{w}(s))$$

such that there are indices i and j with $b_{i1}(s) = 1$, $b_{j1}(s) = -1$ and $b_{l1} = 0$ if $l \neq i$ and $l \neq j$.

Proof. By Lemma 2.6, there exists a sequence of allowable column additions $(A, \vec{v}, \vec{w}) \to (A(t_1), \vec{v}, \vec{w}(t_1))$ such that at most two entries of the first column of $B(t_1)$ are nonzero. Without loss of generality, we may assume that $b_{k1} = 0$ if $k \neq 1$ or 2.

First assume that one of b_{11} or b_{21} is zero. We may suppose that $b_{21} = 0$. Then since $\mathrm{Det}(B) = \pm 1$, we have that $b_{11} = \pm 1$. As in Remark 2.2, we can (if necessary) perform the permissible column addition C_{21} a finite number of times so that the column addition C_{12} is permissible. We can then perform C_{12} to get a matrix which satisfies the conclusions of the lemma in this case.

Now assume that both b_{11} and b_{21} are nonzero. Since $b_{11}C_1 + b_{21}C_2 = e_1$, where $e_1 = (1, 0, \ldots, 0)^t$, it follows that b_{11} and b_{21} have opposite signs. Recall that $\beta = \max\{|b_{11}|, |b_{21}|\}$. If $\beta = 1$, then we have obtained the conclusions of the theorem.

Assume that $\beta > 1$. We will show that we can construct a sequence of column additions in the first 3 columns which are *-allowable

$$(A, \vec{v}, \vec{w}) \to (A(1), \vec{v}, \vec{w}(1)) \to \cdots \to (A(s_1), \vec{v}, \vec{w}(s_1)) \tag{2.1}$$

such that $\beta(s_1) < \beta$.

Once we have established the existence of the sequence (2.1), we can apply Lemma 2.6 to construct a sequence of allowable column additions

$$(A(s_1), \vec{v}, \vec{w}(s_1)) \to (A(s_1+1), \vec{v}, \vec{w}(s_1+1)) \to \cdots \to (A(s_2), \vec{v}, \vec{w}(s_2)) \tag{2.2}$$

such that at most two of the entries in the first column of $B(s_2)$ are nonzero, and $\beta(s_2) \leq \beta(s_1) < \beta$. We can thus alternate sequences (2.1) and (2.2) to eventually obtain the conclusions of the theorem.

It remains to prove that we can construct a sequence (2.1).

Since $\mathrm{Det}(B) = \pm 1$, and $\beta > 1$, we must have that the maximum β is obtained by only one of $|b_{11}|$ and $|b_{21}|$. Without loss of generality, we may assume that

$$\beta = |b_{11}| > |b_{21}|.$$

We now perform a finite sequence of *-allowable column additions C_{32}, followed by a *-allowable column addition C_{23} to obtain a sequence of transformations of triples

$$(A, \vec{v}, \vec{w}) \to \cdots \to (A(t_1), \vec{v}, \vec{w}(t_1)),$$

where the first column of $B(t_1)$ is

$$(b_{11}(t_1), b_{21}(t_2), \ldots, b_{n1}(t_1))^t = (b_{11}, b_{21}, -b_{21}, 0, \ldots, 0)^t,$$

with $\beta(t_1) = |b_{11}(t_1)| = |b_{11}| = \beta$, and either C_{13} or C_{31} is allowable.

If C_{31} is allowable on $(A(t_1), \vec{v}, \vec{w}(t_1))$, we perform it to get

$$|b_{11}(t_1 + 1)| = |b_{11}(t_1) - b_{31}(t_1)| = |b_{11} + b_{21}| < \beta(t_1) = \beta$$

and we stop.

If not, we have that C_{13} is allowable and after that, $b_{11}(t_1 + 1) = b_{11}$ and

$$b_{31}(t_1 + 1) = b_{31}(t_1) - b_{11}(t_1) = -b_{21} - b_{11}$$

have opposite signs. Further, $\beta(t_1+1) = \beta(t_1)$ and $w_3(t_1+1) = w_3(t_1) - w_1 \leq w_3 - w_1$. Now, C_{32} or C_{23} must be allowable.

Now we perform a finite sequence of *-allowable column additions C_{32}, and *-allowable column additions C_{23}, to obtain a sequence of transformations of triples

$$(A(t_1 + 1), \vec{v}, \vec{w}(t_1 + 1)) \to \cdots \to (A(t_2), \vec{v}, \vec{w}(t_2)),$$

where $\beta(t_2) = \beta(t_1 + 1) = \beta$,

$$\max\{| b_{21}(t_2) |, | b_{31}(t_2) |\} <| b_{11}(t_2) |= \beta(t_2) = \beta,$$

and $b_{21}(t_2)$ and $b_{31}(t_2)$ have opposite signs. One of C_{13}, C_{31}, C_{12} or C_{21} must now be allowable.

Performing an allowable C_{31} or C_{21} decreases β and we stop. If not, we perform C_{13} or C_{12} to get $\beta(t_2+1) = \beta(t_2)$ and none of the four C_{13}, C_{31}, C_{12} and C_{21} are allowable.

Further, $w_2(t_2 + 1)$ or $w_3(t_2 + 1)$ is reduced by w_1, so that,

$$w_2(t_2 + 1) + w_3(t_2 + 1) = w_2(t_2) + w_3(t_2) - w_1 \leq w_2 + w_3 - 2w_1.$$

Now either C_{32} or C_{32} becomes allowable and we repeat this process. Since we can perform a C_{13} or a C_{12} at most $[(w_2 + w_3)/w_1]$ times, we must achieve a reduction in β after a finite number of steps. $\square$

LEMMA 2.8. *Let $(A, \vec{v}, \vec{w})$ be a triple such that $A = (C_1, \ldots, C_n)$ satisfies the relation*

$$C_k = C_1 - e_1$$

for some k, where $e_1 = (1, 0, \ldots, 0)^t$. Let A_{11} be the matrix obtained from A be deleting the first row and column. Then

$$\mathrm{Det}(A_{11}) = \mathrm{Det}(A) = \pm 1.$$

Let $\tilde{v} = (v_2, \ldots, v_n)^t$ *and* $\tilde{w} = (\tilde{w}_2, \ldots, \tilde{w}_n) = A_{11}^{-1}\tilde{v}$. *Then*

$$\tilde{w}_j = w_j \textit{ for } j \neq k$$

and

$$\tilde{w}_k = w_1 + w_k.$$

Proof. Set $\lambda = \mathrm{Det}(A) = \pm 1$. Subtracting the k-th column of A from the first column, we see that $\mathrm{Det}(A_{11}) = \lambda$. We thus have that $B = A^{-1} = \lambda\,\mathrm{adj}(A)$, and $A_{11}^{-1} = \lambda\,\mathrm{adj}(A_{11})$. Let

$$A_{11}^{-1} = \lambda\,\mathrm{adj}(A_{11}) = \begin{pmatrix} x_{22} & x_{23} & \cdots & x_{2n} \\ x_{32} & x_{33} & \cdots & x_{3n} \\ & \vdots & & \\ x_{n2} & x_{n3} & \cdots & x_{nn} \end{pmatrix}.$$

Since $C_k = C_1 - e_1$ and $\mathrm{adj}(A) = \lambda A^{-1}$, the first column of $\mathrm{adj}(A)$ is

$$(\lambda, 0, \ldots, 0, -\lambda, 0, \ldots, 0)^t,$$

where $-\lambda$ occurs in the k-th row.

We will compute the entry λb_{ij} in the i-th row and j-th column of $\mathrm{adj}(A)$. Let A_{ji} be the matrix obtained from A by deleting the j-th row and i-th column.

First suppose that $i \neq 1$, $i \neq k$ and $j > 1$. Subtracting the k-th column of A_{ji} (the $(k-1)$-st column of A_{ji} if $i < k$) from the first column, and expanding the determinant along the first column, we get that the (i,j)-th entry of $\mathrm{adj}(A)$ is

$$\begin{aligned}(-1)^{i+j}\,\mathrm{Det}(A_{ji}) &= (-1)^{i+j}\,\mathrm{Det}[(A_{ji})_{11}] \\ &= (-1)^{i+j}\,\mathrm{Det}[(A_{11})_{j-1,i-1}] \\ &= \lambda x_{i,j}.\end{aligned}$$

For $j > 1$, set

$$t_j = \lambda(-1)^{1+j}\,\mathrm{Det}(A_{j1}).$$

We have that the $(1,j)$-th entry of $\mathrm{adj}(A)$ is λt_j. We expand

$$
\begin{aligned}
(-1)^{1+j}\operatorname{Det}(A_{j1}) &= (-1)^{1+j}\operatorname{Det}\begin{pmatrix} a_{12} & \cdots & a_{11}-1 & \cdots \\ & \vdots & & \\ a_{n2} & \cdots & a_{n1} & \cdots \end{pmatrix} \\
&= (-1)^{1+j}\operatorname{Det}\begin{pmatrix} a_{12} & \cdots & a_{11} & \cdots \\ & \vdots & & \\ a_{n2} & \cdots & a_{n1} & \cdots \end{pmatrix} \\
&\qquad + (-1)^{1+j+1+k-2}\operatorname{Det}[(A_{11})_{j-1,k-1}] \\
&= (-1)^{1+j+k-2}\operatorname{Det}(A_{jk}) + (-1)^{j+k}\operatorname{Det}[(A_{11})_{j-1,k-1}] \\
&= -(-1)^{j+k}\operatorname{Det}(A_{jk}) + \lambda x_{kj}.
\end{aligned}
$$

We see that, for $j > 1$, the (k, j)-th entry of $\operatorname{adj}(A)$ is $\lambda(x_{k,j} - t_j)$.

In conclusion,

$$
B = A^{-1} = \lambda \operatorname{adj}(A) = \begin{pmatrix} 1 & t_2 & \cdots & t_n \\ 0 & x_{22} & \cdots & x_{2n} \\ & \vdots & & \\ -1 & x_{k2} - t_2 & \cdots & x_{kn} - t_n \\ & \vdots & & \\ 0 & x_{n2} & \cdots & x_{nn} \end{pmatrix}.
$$

Now we see that

$$
\begin{aligned}
\tilde{w}_i &= (x_{i2}, \ldots, x_{in})(v_2, \ldots, v_n)^t \\
&= (0, x_{i2}, \ldots, x_{in})(v_1, \ldots, v_n)^t \\
&= w_i
\end{aligned}
$$

if $i \neq k$, and

$$
\begin{aligned}
\tilde{w}_k &= \sum_{j=2}^{n} x_{kj} v_j \\
&= \left[-v_1 + \sum_{j=2}^{n} (x_{kj} - t_j) v_j \right] + \left[v_1 + \sum_{j=2}^{n} t_j v_j \right] \\
&= w_k + w_1
\end{aligned}
$$

□

Now we prove Theorem 2.1.

A quadruple $(A, \vec{v}, \vec{w}, k)$ is a triple $(A, \vec{v}, \vec{w})$ and a number k with $1 < k \le n$ such that if $A = (C_1, \ldots, C_n)$, then $C_k = C_1 - e_1$. To a quadruple $(A, \vec{v}, \vec{w}, k)$ we associate an $(n-1)$-dimensional triple $(\tilde{A} = A_{11}, \tilde{v}, \tilde{w})$ with the notation of Lemma 2.8.

A permissible transformation for the quadruple $(A, \vec{v}, \vec{w}, k)$ is a series of permissible column additions and row interchanges which transform the triple $(A, \vec{v}, \vec{w})$ to a triple $(A(1), \vec{v}, \vec{w}(1))$ and a number j with $1 < j \leq n$, such that $(A(1), \vec{v}, \vec{w}(1), j)$ is a quadruple.

By Lemma 2.7, and since $B = A^{-1}$, there exists a sequence of permissible column additions, possibly followed by some row interchanges T_{ij}, $(A, \vec{v}, \vec{w}) \to (A(1), \vec{v}, \vec{w}(1))$, and a number $k(1)$ such that $(A(1), \vec{v}, \vec{w}(1), k(1))$ is a quadruple. Without loss of generality, we may assume that there exists a number k such that $(A, \vec{v}, \vec{w}, k)$ is a quadruple.

If $n = 3$, then after expanding the determinant of $\tilde{A}$, we see that after possibly performing the row interchange T_{23}, there is a sequence of permissible row subtractions R_{ji} which transform $\tilde{A}$ into the identity matrix. If $n > 3$, we assume by induction that there exists a sequence of permissible column additions and row interchanges

$$(\tilde{A}, \tilde{v}, \tilde{w}) \to (\tilde{A}(1), \tilde{v}, \tilde{w}(1)) \to \cdots \to (\tilde{A}(s), \tilde{v}, \tilde{w}(s)) \tag{2.3}$$

followed by a sequence of permissible row subtractions

$$(\tilde{A}(s), \tilde{v}, \tilde{w}(s)) \to (\tilde{A}(s+1), \tilde{v}(s+1), \tilde{w}(s)) \to \cdots \to (\tilde{A}(l), \tilde{v}(l), \tilde{w}(s)) \tag{2.4}$$

such that $\tilde{A}(l)$ is the $(n-1) \times (n-1)$ identity matrix.

We will first construct a sequence of permissible transformations of quadruples

$$(A, \vec{v}, \vec{w}, k) \to (A(1), \vec{v}, \vec{w}(1), k(1)) \to \cdots \to (A(s), \vec{v}, \vec{w}(s), k(s)) \tag{2.5}$$

such that for $1 \leq t \leq s$, we have $A(t)_{11} = \tilde{A}(t)$ and $\tilde{w}(t) = A(t)_{11}^{-1}(v_2, \ldots, v_n)^t$.

Suppose that we have constructed (2.5) out to $(A(t), \vec{v}, \vec{w}(t), k(t))$, and $t < s$. We will construct $(A(t+1), \vec{v}, \vec{w}(t+1), k(t+1))$.

First suppose that $\tilde{A}(t+1)$ is obtained from $\tilde{A}(t)$ by interchanging the i-th and j-th row. Let the triple $(A(t+1), \vec{v}(t+1), \vec{w}(t+1))$ be obtained from the triple $(A(t), \vec{v}, \vec{w}(t))$ by performing the row interchange T_{ij}. Then the row interchange T_{ij} determines a permissible transformation of $(A(t), \vec{v}, \vec{w}(t), k(t))$ to $(A(t+1), \vec{v}, \vec{w}(t+1), k(t))$, such that $A(t+1)_{11} = \tilde{A}(t+1)$.

Suppose that $\tilde{A}(t+1)$ is obtained from $\tilde{A}(t)$ by adding the j-th column of $\tilde{A}(t)$ to the i-th column. We necessarily have that $\tilde{w}_j(t) > \tilde{w}_i(t)$. Set $k = k(t)$.

If $i \neq k$ and $j \neq k$, then we have (by Lemma 2.8) that $\vec{w}_j(t) > \vec{w}_i(t)$, and thus the column addition C_{ij} determines a permissible transformation of $(A(t), \vec{v}, \vec{w}(t), k(t))$ to $(A(t+1), \vec{v}, \vec{w}(t+1), k(t))$, such that $A(t+1)_{11} = \tilde{A}(t+1)$.

Suppose that $i = k$. Then $\tilde{w}_j(t) > \tilde{w}_k(t)$. Since $\tilde{w}_k(t) = w_1(t) + w_k(t)$ and $\tilde{w}_j(t) = w_j(t)$ (by Lemma 2.8), we can construct a permissible transformation of quadruples $(A(t), \vec{v}, \vec{w}(t), k(t)) \to (A(t+1), \vec{v}, \vec{w}(t+1), k(t))$ by first performing the permissible column addition C_{kj} followed by the permissible column addition C_{1j}. We have that $A(t+1)_{11} = \tilde{A}(t+1)$.

Suppose that $j = k$. Then $\tilde{w}_k(t) > \tilde{w}_i(t)$.

If $w_1(t) > w_i(t)$, then we define a permissible transformation of quadruples

$$(A(t), \vec{v}, \vec{w}(t), k(t)) \to (A(t+1), \vec{v}, \vec{w}(t+1), k(t))$$

by performing the permissible column addition C_{i1}.

Suppose that $w_1(t) < w_i(t)$. If $(\overline{A}, \vec{v}, \overline{w})$ is the triple obtained from $(A(t), \vec{v}, \vec{w}(t))$ by C_{1i}, then we have that the i-th coefficient of $\overline{w}$ is $\overline{w}_i = w_i(t) - w_1(t)$. Since $\tilde{w}_k(t) > \tilde{w}_i(t)$, we must have that $w_1(t) + w_k(t) > w_i(t)$, which implies that $\overline{w}_k(t) > \overline{w}_i(t)$. Thus we can construct a permissible transformation of quadruples

$$(A(t), \vec{v}, \vec{w}(t), k(t)) \to (A(t+1), \vec{v}, \vec{w}(t+1), k(t+1) = i)$$

by first performing the permissible column addition C_{1i} followed by the permissible column addition C_{ik}. We have that $A(t+1)_{11} = \tilde{A}(t+1)$.

We can thus inductively construct the sequence (2.5). Let $k = k(s)$. Since $C_k(s) = C_1(s) - e_1$, where $C_k(s)$, $C_1(s)$ are the k-th and first columns of $A(s)$, the sequence of permissible row subtractions of (2.4) gives a sequence of permissible row subtractions $(A(s), \vec{v}, \vec{w}(s)) \to (A(l), \vec{v}(l), \vec{w}(s))$ such that $A(l)$ is a matrix, where $A(l)_{11}$ is an identity matrix, $a_{1k}(l) = a_{11}(tl) - 1$, $a_{k1}(l) = 1$, and $a_{i1}(l) = 0$ if $i \neq 1$ and $i \neq k$.

Now we perform the successive permissible row subtractions on $A(l)$ of subtracting $a_{11}(l) - 1$ times the k-th row from the first row, subtracting $a_{1i}(l)$ times the i-th row from the first row for all $i \neq k$, and finally subtracting the first row from the k-th row, to transform $A(l)$ into the identity matrix. This completes the proof of Theorem 2.1.

We say that a column subtraction is permissible on $(A, \vec{v}, \vec{w})$ if it leaves the entries of A nonnegative. If we subtract the i-th column from the j-th column, then $\vec{v}$ is unchanged, but the coefficient w_i of $\vec{w}$ is added to w_j. So as a corollary to Theorem 2.1, or by a simple modification of the proof of Theorem 2.1, we obtain:

Theorem 2.9. *Suppose that $(A, \vec{v}, \vec{w})$ is a triple. Then there exists a sequence of permissible column additions and interchanges, followed by a sequence of permissible column subtractions, that transforms A to the identity matrix.*

3. Local factorization of birational extensions

Suppose that R is a regular local ring with quotient field K, and ν is a valuation of K, with valuation ring V, such that V dominates R ($R \subset V$ and the maximal ideal of V intersects R in its maximal ideal). A monoidal transform of R along ν is a regular local ring $R(1)$ such that $R(1) = R[\frac{P}{f}]_m$, where P is a regular prime of R, $f \in P$ is such that $\nu(f) = \min\{\nu(g) \mid g \in P\}$, and $m = \{g \in R[\frac{P}{f}] \mid \nu(g) > 0\}$. We have that V dominates $R(1)$ and $R(1)$ dominates R.

THEOREM 3.1. *Suppose that k is a field, $k[x_1,\ldots,x_n]$, $k[y_1,\ldots,y_n]$ are polynomial rings and there exists a matrix (a_{ij}) of nonnegative integers satisfying*

$$x_i = \prod_{j=1}^{n} y_j^{a_{ij}}$$

for $1 \le i \le n$ with $\mathrm{Det}(a_{ij}) = \pm 1$. Let $R = k[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$ and $S = k[y_1,\ldots,y_n]_{(y_1,\ldots,y_n)}$. Suppose that ν is a rank 1 valuation of $k(y_1,\ldots,y_n)$ with valuation ring V which dominates S, such that $\nu(y_1),\ldots,\nu(y_n)$ are rationally independent. Then there exists a commutative diagram

$$\begin{array}{ccccc} & & T & & \\ & \nearrow & & \nwarrow & \\ R & & \rightarrow & & S \end{array} \tag{3.1}$$

such that T is a regular local ring dominated by V, and the northeast and northwest arrows are products of monoidal transforms along ν.

Proof. Let $A = (a_{ij})$, $\vec{v} = (\nu(x_1),\ldots,\nu(x_n))^t$ and $\vec{w} = (\nu(y_1),\ldots,\nu(y_n))$. By Theorem 2.1, there exists a sequence of permissible column additions and row interchanges

$$(A,\vec{v},\vec{w}) \to (A(1),\vec{v},\vec{w}(1)) \to \cdots \to (A(s),\vec{v},\vec{w}(s))$$

followed by a sequence of permissible row subtractions

$$(A(s),\vec{v},\vec{w}(s)) \to (A(s+1),\vec{v}(s+1),\vec{w}(s)) \to \cdots \to (A(t),\vec{v}(t),\vec{w}(t))$$

such that $A(t)$ is the $n \times n$ identity matrix.

We will construct a diagram (3.1), in which the northwest arrow is a product of monoidal transforms along ν,

$$S \to S(1) \to \cdots \to S(s) = T, \tag{3.2}$$

and the northeast arrow is a product of monoidal transforms along ν

$$R \to R(1) \to \cdots \to R(t-s) = T. \tag{3.3}$$

We inductively construct (3.2), with a system of regular parameters $(y_1(l),\ldots,y_n(l))$ in each $S(l)$, so that $x_i = \prod_j y_j(l)^{a_{ij}(l)}$ for $1 \le i \le n$, $\vec{v}(l) = (\nu(x_1),\ldots,\nu(x_n))^t = \vec{v}$ and $\vec{w}(l) = (\nu(y_1(l)),\ldots,\nu(y_n(l)))^t$ for $1 \le l \le s$.

Suppose that $A(l+1)$ is obtained from $A(l)$ by the row interchange T_{ij}. We define $S(l+1)$ to be $S(l)$, and we interchange the regular parameters x_i and x_j of R.

Suppose that $A(l+1)$ is obtained from $A(l)$ by the permissible column addition C_{ij}. We define $S(l+1)$ to be the local ring of the blow up of the prime ideal $(y_i(l), y_j(l))$ which is dominated by V. Since $\nu(y_j(l)) > \nu(y_i(l))$, we have that

$$S(l+1) = S(l)\left[\frac{y_j(l)}{y_i(l)}\right]_{(y_1(l+1),\ldots,y_n(l+1))},$$

where

$$y_k(l+1) = \begin{cases} y_k(l) & \text{if } k \neq j \\ \dfrac{y_j(l)}{y_i(l)} & \text{if } k = j \end{cases}$$

are regular parameters in $S(l+1)$.

We now inductively construct (3.3), with a system of regular parameters $(x_1(l), \ldots, x_n(l))$ in each $R(l)$, so that $x_i(l) = \prod_j y_j(s)^{a_{ij}(l+s)}$ for $1 \leq i \leq n$, $\vec{v}(l+s) = (\nu(x_1(l)), \ldots, \nu(x_n(l)))^t$ and $\vec{w}(l+s) = (\nu(y_1(s)), \ldots, \nu(y_n(s)))^t = \vec{w}(s)$ for $1 \leq l \leq t-s$.

Suppose that $A(l+1+s)$ is obtained from $A(l+s)$ by the permissible row subtraction R_{ji}. We define $R(l+1)$ to be the local ring of the blow up of the prime ideal $(x_i(l), x_j(l))$ which is dominated by V. Since $x_i(l)$ divides $x_j(l)$ in T, we have that

$$R(l+1) = R(l)\left[\frac{x_j(l)}{(x_i(l)}\right]_{(x_1(l+1),\ldots,x_n(l+1))},$$

where

$$x_k(l+1) = \begin{cases} x_k(l) & \text{if } k \neq j \\ \dfrac{x_j(l)}{x_i(l)} & \text{if } k = j \end{cases}$$

are regular parameters in $R(l+1)$.

Since $A(t) = Id$, we have that $x_i(t-s) = y_i(s)$ for $1 \leq i \leq n$, and thus T satisfies the conclusions of the theorem. $\square$

REMARK 3.2. If $S \to S(1)$ is a monoidal transform of a regular local ring S, then S is called an inverse monoidal transform of $S(1)$ (Chapter 6 of [C2]). With the notation of Theorem 3.1, a permissible column subtraction of $A = (a_{ij})$ induces an inverse monoidal transform $R \to S(1) \to S$ of S. We can use Theorem 2.9, instead of Theorem 2.1, to prove Theorem 3.1. Theorem 2.9 proves the equivalent statement that a diagram (3.1) can be constructed, where the northwest arrow is factored by a sequence of inverse monoidal transforms from T to S.

THEOREM 3.3. *Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, with a common quotient field K, such that S dominates R. Let V be a valuation ring of K which dominates S. Then there exists a regular local ring T, with quotient field K, such that T*

dominates S, V dominates T, and the inclusions $R \to T$ and $S \to T$ can be factored by sequences of monoidal transforms

$$
\begin{array}{ccccc}
 & & V & & \\
 & & \uparrow & & \\
 & & T & & \\
 & \nearrow & & \nwarrow & \\
R & & \to & & S.
\end{array}
\tag{3.4}
$$

Proof. Let $r = \mathrm{rank}(V)$. We can perform monoidal transforms on R and S so that the assumptions of [C2, Theorem 5.5] hold. By [C2, Theorem 5.5], there exists a commutative diagram of regular local rings

$$
\begin{array}{ccc}
R' & \to & S' \\
\uparrow & & \uparrow \\
R & \to & S
\end{array}
$$

such that R', S' have respective regular parameters $(z_1, \ldots, z_n)$, $(w_1, \ldots, w_n)$ satisfying the conclusions of [C2, Theorem 5.5]. In particular, there exists a matrix a_{ij} such that $z_i = \prod_j w_j^{a_{ij}}$ for $1 \le i \le n$, where $A = (a_{ij})$ has the block form

$$
A = \begin{pmatrix}
G_1 & & & & 0 & & \\
 & Id & & & & & \\
 & & G_2 & & & & \\
 & & & Id & & & \\
0 & & & & \cdots & & \\
 & & & & & Id & \\
 & & & & & & G_r
\end{pmatrix}.
$$

Here, $G_i = (g_{jk}(i))$ is an $s_i \times s_i$ matrix of determinant ± 1, so that we have

$$
\begin{aligned}
z_{t_1+\cdots+t_{i-1}+1} &= w_{t_1+\cdots+t_{i-1}+1}^{g_{11}(i)} \cdots w_{t_1+\cdots+t_{i-1}+s_i}^{g_{1s_i}(i)} \\
&\vdots \\
z_{t_1+\cdots+t_{i-1}+s_i} &= w_{t_1+\cdots+t_{i-1}+1}^{g_{s_i 1}(i)} \cdots w_{t_1+\cdots+t_{i-1}+s_i}^{g_{s_i s_i}(i)}
\end{aligned}
$$

for $1 \le i \le r$. We further have that $\nu(z_{t_1+\cdots+t_{i-1}+1}), \cdots, \nu(z_{t_1+\cdots+t_{i-1}+s_i})$ are rationally independent, and if $V_i = V \cap k(z_{t_1+\cdots+t_{i-1}+1}, \cdots, z_{t_1+\cdots+t_{i-1}+s_i})$, then V_i has rank 1 (and rational rank s_i).

Let

$$
\begin{aligned}
\overline{R}_i &= k\left[z_{t_1+\cdots+t_{i-1}+1}, \ldots, z_{t_1+\cdots+t_{i-1}+s_i}\right]_{(z_{t_1+\cdots+t_{i-1}+1}, \ldots, z_{t_1+\cdots+t_{i-1}+s_i})}, \\
\overline{S}_i &= k\left[w_{t_1+\cdots+t_{i-1}+1}, \ldots, w_{t_1+\cdots+t_{i-1}+s_i}\right]_{(w_{t_1+\cdots+t_{i-1}+1}, \ldots, w_{t_1+\cdots+t_{i-1}+s_i})}.
\end{aligned}
$$

By Theorem 3.1, there exists a regular local ring $\overline{T}_i$ which is dominated by V_i and a commutative diagram

$$\begin{tikzcd} & V_i & \\ & \overline{T}_i \arrow[u] & \\ \overline{R}_i \arrow[ur] \arrow[rr] & & \overline{S}_i \arrow[ul] \end{tikzcd}$$

such that the northeast and northwest arrows are products of monoidal transforms. By performing the corresponding sequences of monoidal transforms on R' and S' for $1 \le i \le r$, we obtain the conclusions of the theorem. $\square$

References

[Ab1] S. Abhyankar, *On the valuations centered in a local domain*, Amer. J. Math. **78** (1956), 321–348. MR 0082477 (18,556b)

[Ab] ———, *Algebraic geometry for scientists and engineers*, Mathematical Surveys and Monographs, vol. 35, American Mathematical Society, Providence, RI, 1990. MR 1075991 (92a:14001)

[AMR] D. Abramovich, K. Matsuki, and S. Rashid, *A note on the factorization theorem of toric birational maps after Morelli and its toroidal extension*, Tohoku Math. J. (2) **51** (1999), 489–537. MR 1725624 (2000i:14073)

[AMR1] ———, *Correction: "A note on the factorization theorem of toric birational maps after Morelli and its toroidal extension"*, Tohoku Math. J. (2) **52** (2000), 629–631. MR 1793939 (2001j:14069)

[Ch] C. Christensen, *Strong domination/weak factorization of three-dimensional regular local rings*, J. Indian Math. Soc. (N.S.) **45** (1981), 21–47 (1984). MR 828858 (88a:14010a)

[C1] S. D. Cutkosky, *Local factorization of birational maps*, Adv. Math. **132** (1997), 167–315. MR 1491444 (99c:14018)

[C2] ———, *Local monomialization and factorization of morphisms*, Astérisque **260** (1999). MR 1734239 (2001c:14027)

[C3] ———, *Monomialization of morphisms from 3-folds to surfaces*, Lecture Notes in Mathematics, vol. 1786, Springer-Verlag, Berlin, 2002. MR 1927974 (2004a:14014)

[C4] ———, *Local monomialization of transcendental extensions*, Ann. Inst. Fourier (Grenoble) **55** (2005), 1517–1586. MR 2172273

[C5] ———, *Toroidalization of birational morphism of projective 3-folds*, Mem. Amer. Math. Soc., to appear.

[HHS] W. Heinzer, C. Huneke, and J. D. Sally, *A criterion for spots*, J. Math. Kyoto Univ. **26** (1986), 667–671. MR 864469 (87k:13034)

[H] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I*, Ann. of Math. (2) **79** (1964), 109–203; *II*, ibid. (2) **79** (1964), 205–326. MR 0199184 (33 #7333)

[KM] Y. Kachi and S. B. Mulay, *Local-to-global correspondence in algebraic geometry*, The Fano Conference, Univ. Torino, Turin, 2004, pp. 485–514. MR 2112588 (2006a:14018)

[K] K. Karu, *Local strong factorization of toric birational maps*, J. Algebraic Geom. **14** (2005), 165–175. MR 2092130 (2005j:14015)

[Mo] R. Morelli, *The birational geometry of toric varieties*, J. Algebraic Geom. **5** (1996), 751–782. MR 1486987 (99b:14056)

[Sa] J. Sally, *Regular overrings of regular local rings*, Trans. Amer. Math. Soc. **171** (1972), 291–300. MR 0309929 (46 #9033)

[Sh] D.L. Shannon, *Monoidal transforms*, Amer. J. Math **45** (1973), 294–320. MR0330154 (48 #8492)

[Z] O. Zariski, *Introduction to the problem of minimal models in the theory of algebraic surfaces*, Publications of the Mathematical Society of Japan, no. 4, The Mathematical Society of Japan, Tokyo, 1958. MR 0097403 (20 #3872)

Steven Dale Cutkosky, Department of Mathematics, University of Missouri, Columbia MO 65201, USA

E-mail address: cutkoskys@missouri.edu

Hema Srinivasan, Department of Mathematics, University of Missouri, Columbia MO 65201, USA

E-mail address: srinivasanh@missouri.edu,