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THE RESIDUALS OF LEX PLUS POWERS IDEALS AND THE EISENBUD–GREEN–HARRIS CONJECTURE

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ABSTRACT. The n-type vectors introduced by Geramita, Harima, and Shin are in 1–1 correspondence with the Hilbert functions of Artinian lex ideals. Letting $\mathbb{A} = \{a_1, \ldots, a_n\}$ define the degrees of a regular sequence, we construct $\mathrm{lpp}_{\leq}(\mathbb{A})$-vectors which are in 1–1 correspondence with the Hilbert functions of certain lex plus powers ideals (depending on $\mathbb{A}$). This construction enables us to show that the residual of a lex plus powers ideal in an appropriate regular sequence is again a lex plus powers ideal. We then use this result to show that the Eisenbud–Green–Harris conjecture is equivalent to showing that lex plus powers ideals have the largest *last* graded Betti numbers (it is well known that the Eisenbud–Green–Harris conjecture is equivalent to showing that lex plus powers ideals have the largest first graded Betti numbers).

1. Introduction

Hilbert functions, in general, have been extensively studied. Let $R = k[x_1, \ldots, x_n]$, where each x_i has degree 1. Then Macaulay [11] characterized those sequences (called O-sequences) which occur as the Hilbert function of any k-algebra of the form R/I, where I is a homogeneous ideal. He showed that a sequence $S = \{c_i\}_{i \geq 0}$ is such a Hilbert function if and only if $c_{i+1} \leq c_i^{\langle i \rangle}$, where $-^{\langle i \rangle}$, known as Macaulay's function, is expressed in terms of the i-binomial expansion of an integer. In proving his result, Macaulay shows that lex ideals have the largest first graded Betti numbers among all ideals having a fixed Hilbert function. Bigatti [1] and Hulett [10] have independently generalized this by showing that, over fields of characteristic 0, lex

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ideals have the largest graded Betti numbers (not just the largest first graded Betti numbers) among all ideals having a fixed Hilbert function. Pardue [13] generalized this to fields of arbitrary characteristic.

At about the same time that Bigatti and Hulett proved their result, Eisenbud, Green, and Harris together conjectured that a generalization in a different direction of Macaulay's result should be true. Instead of restricting their attention to lex ideals, they look at ideals which, modulo appropriate powers of the variables, are lex ideals. These ideals have become known as lex plus powers ideals; letting $\mathbb{A} = \{a_1, \dots, a_n\}$ be a list of positive integers with $a_1 \le \dots \le a_n$, an ideal L containing $x_1^{a_1}, \dots, x_n^{a_n}$ as minimal generators is an $\mathbb{A}$-lex plus powers ideal if $\overline{L}$ is a lex ideal in $R/\langle x_1^{a_1}, \dots, x_n^{a_n}\rangle$. The conjecture states that as long as there is an $\mathbb{A}$-lex plus powers ideal attaining the Hilbert function H, then among all ideals with Hilbert function H that also contain a regular sequence of elements of degrees $a_1, a_2, \dots, a_n$, the $\mathbb{A}$-lex plus powers ideal has the largest first graded Betti numbers.

In light of both Bigatti and Hulett's result, and Eisenbud, Green, and Harris's conjecture, the following very natural conjecture was made by Charalambous and Evans: as long as there is an $\mathbb{A}$-lex plus powers ideal attaining the Hilbert function H, then among all ideals with Hilbert function H that also contain a regular sequence of elements of degrees $a_1, a_2, \dots, a_n$, the $\mathbb{A}$-lex plus powers ideal has the largest graded Betti numbers (not just the largest first graded Betti numbers).

As a result of Bigatti and Hulett's results, there has been much interest in studying lex ideals. One direction of study has led to the introduction of n-type vectors by Geramita, Harima, and Shin. These n-type vectors are in 1–1 correspondence with Artinian lex ideals. Since all lex plus powers ideals are by definition Artinian, it makes sense to look for an analogue to n-type vectors for lex plus powers ideals. We do this in Section 4. This enables us to prove our main result quite easily: that the residual of an $\mathbb{A}$-lex plus powers ideal in $\langle x_1^{a_1}, \dots, x_n^{a_n}\rangle$ is again a lex plus powers ideal. As a consequence of this, we show in Section 6 that the statement that lex plus powers ideals have largest first graded Betti numbers is equivalent to the statement that lex plus powers ideals have largest last graded Betti numbers (previously, it was shown in [14] that lex plus powers ideals having largest first graded Betti numbers implies having the largest last graded Betti numbers; we show the converse).

2. Background

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k with maximal ideal $m = (x_1, \dots, x_n)$, and fix an order on the monomials, $x_1 > \dots > x_n$. The following definition gives a notation for referring to the degrees of the elements of a regular sequence.

Definition 2.1. Let $\{a_1,\dots,a_n\}$ be a set of integers such that $1\le a_1\le \cdots\le a_n$. Then we call $\{f_1,\dots,f_n\}$ an $\{a_1,\dots,a_n\}$-regular sequence if $\{f_1,\dots, f_n\}$ is a regular sequence such that $\deg(f_i)=a_i$ for $i=1,\dots,n$.

Recall that the Hilbert function $H(R/I)$ of an ideal I is the sequence $\{\dim_k(R/I)_d\}_{d\ge 0}$. We denote $\dim_k(R/I)_d$ by $H(R/I,d)$. Then given a Hilbert function $\mathcal{H}$, and a list of degrees $\{a_1,\dots,a_n\}$, we can compare homogeneous ideals attaining $\mathcal{H}$ and containing an $\{a_1,\dots,a_n\}$-regular sequence. In this comparison, we will use as a fixed point a special ideal called an $\{a_1,\dots,a_n\}$-lex plus powers ideal.

Definition 2.2 (*Charalambous and Evans*). Suppose that $\mathbb{A}=\{a_1,\dots,a_n\}$ is a non-decreasing list of integers, $a_1\ge 1$. Then a monomial ideal L is a *lex plus powers ideal with respect to* $\mathbb{A}$, also called an $\mathbb{A}$-*lex plus powers ideal*, if L is minimally generated by monomials $x_1^{a_1},\dots,x_n^{a_n}$, $m_1,\dots,m_l$ such that for each $j=1,\dots,l$, all monomials of degree $\deg(m_j)$ which are larger than m_j in lex order are contained in L. We will abbreviate the terminology "lex plus powers with respect to $\mathbb{A}$" by saying that L is $LPP(\mathbb{A})$.

It is not difficult to construct (degenerative) examples of a Hilbert function $\mathcal{H}$, and a list of degrees $\mathbb{A}=\{a_1,\dots,a_n\}$ for which no $\mathbb{A}$-lex plus powers ideal L exists with $H(R/L)=\mathcal{H}$ (see [14]). Thus, we require the following technical definition.

Definition 2.3. Suppose that $\mathcal{H}$ is a Hilbert function and $\mathbb{A}=\{a_1,\dots,a_n\}$ is a nondecreasing list of integers, $a_1\ge 1$. We call $\mathcal{H}$ an $\mathbb{A}$-*lpp valid Hilbert function* if there exists an $LPP(\mathbb{A})$ ideal L such that $H(R/L)=\mathcal{H}$. Note that if an $LPP(\mathbb{A})$ ideal L attaining a given Hilbert function $\mathcal{H}$ exists, then it is clearly unique. We will sometimes refer to this ideal as $L_{\mathcal{H},\mathbb{A}}$.

Lex plus powers ideals are important because they are conjectured by Charalambous and Evans [2] to have extremal properties. In order to understand in what sense lex plus powers ideals should be extremal, we need to introduce some terminology. Recall that the i,jth graded Betti number of R/I is defined to be $\beta^I_{i,j} := (\operatorname{Tor}_i(R/I,k))_j$; by the usual abuse of notation, we also call this the i,jth graded Betti number of I. We will refer to the set of all graded Betti numbers of an ideal I as β^I. It is also convenient to make use of the notation of the computer algebra system Macaulay 2 [8], so we often refer to β^I as the Betti diagram of I (the Betti diagram of I is a table listing the graded Betti numbers of I—counting from zero, the entry in the i,jth position in this table is $\beta^I_{i,i+j}$).

Definition 2.4. Write $\mathcal{LP}^{\mathcal{H}}_{\mathbb{A}}$ to be the set of all sets of graded Betti numbers of all ideals $I\subset R$ containing an $\mathbb{A}$-regular sequence and attaining $\mathcal{H}$. Equivalently, this is the set of all Betti diagrams of such ideals.

There is an obvious partial order on $\mathcal{LP}^{\mathcal{H}}_{\mathbb{A}}$: for $\beta^I, \beta^J \in \mathcal{LP}^{\mathcal{H}}_{\mathbb{A}}$, we say that $\beta^I \geq \beta^J$ if $\beta^I_{i,j} \geq \beta^J_{i,j}$ for all i, j. With this we can describe the conjectured extremality of lex plus powers ideals.

Conjecture 2.5 (The lex plus powers conjecture). *If $\mathcal{H}$ is $\mathbb{A}$-lpp valid, then writing $L_{\mathcal{H},\mathbb{A}}$ to be the $\mathbb{A}$-lex plus powers ideal attaining $\mathcal{H}$, $\beta^{L_{\mathcal{H},\mathbb{A}}}$ is the unique largest element in $\mathcal{LP}^{\mathcal{H}}_{\mathbb{A}}$.*

There is a (on the face of it) weaker version of this conjecture due to Eisenbud, Green, and Harris, which claims that lex plus powers ideals should be capable of largest Hilbert function growth.

Conjecture 2.6 (The Eisenbud–Green–Harris conjecture). *Let $I \subset R$ contain an $\mathbb{A}$-regular sequence, and suppose there exists an $LPP(\mathbb{A})$ ideal L such that $H(R/I, d) = H(R/L, d)$. Then*

$$H(R/\langle L_d \rangle, d+1) \geq H(R/I, d+1),$$

where $\langle L_d \rangle$ is the ideal generated by the pure powers $x_1^{a_1}, \ldots, x_n^{a_n}$ and the forms in L of degree d.

That the lex plus powers conjecture (LPP) implies the Eisenbud–Green–Harris conjecture (EGH) is made clear by an equivalent formulation of the latter found in [14].

Conjecture 2.7. *Given an $\mathbb{A}$-lpp valid Hilbert function $\mathcal{H}$, then $\beta^{L_{\mathcal{H},\mathbb{A}}}_{1,i} \geq \beta^I_{1,i}$ for all i whenever $I \subset R$ attains $\mathcal{H}$ and contains an $\mathbb{A}$-regular sequence.*

It is an open question whether EGH implies LPP. Some progress was made on this question in [14] with the following theorem.

Theorem 2.8. *Let L be $LPP(\mathbb{A})$ for some $\mathbb{A} = \{a_1, \ldots, a_n\}$ and I be an ideal containing an $\mathbb{A}$-regular sequence such that $H(R/L) = H(R/I)$. If EGH holds, then* $\dim_k(\operatorname{soc}(L)_d) \geq \dim_k(\operatorname{soc}(I)_d)$ *for all d.*

That is, if the $\beta^{L_{\mathcal{H},\mathbb{A}}}_{1,j}$ are uniquely largest, then so are the $\beta^{L_{\mathcal{H},\mathbb{A}}}_{n,j}$. It was not decided in that paper whether the converse was true. We will show in this paper that the converse does hold. That is, we prove that the following conjecture and EGH are equivalent.

Conjecture 2.9. *Let L be $LPP(\mathbb{A})$ for some $\mathbb{A} = \{a_1, \ldots, a_n\}$ and I be an ideal containing an $\mathbb{A}$-regular sequence such that $H(R/L) = H(R/I)$. Then $\beta^L_{n,j} \geq \beta^I_{n,j}$, that is,* $\dim_k(\operatorname{soc}(L)_d) \geq \dim_k(\operatorname{soc}(I)_d)$ *for all d.*

This result will be a natural application of our main result, where we show that the residual of an $LPP(\mathbb{A})$-ideal in $\langle x_1^{b_1}, \ldots, x_n^{b_n} \rangle$ where $a_i \leq b_i$ for all i is again a lex plus powers ideal.

We recall here one further theorem, a result of Stanley.

THEOREM 2.10 (Stanley). *For every R-module M,*

$$\sum_{d=0}^{\infty} H(M,d)t^d = \frac{\sum_{d=0}^{\infty}\sum_{i=0}^{n}(-1)^i \beta_{i,d}^M t^d}{(1-t)^n}.$$

This theorem simply states that fixing a Hilbert function fixes the alternating sum of the graded Betti numbers of any ideal attaining it. In particular, if I and J have $H(R/I)=H(R/J)$, then $\sum_{i=0}^{n}(-1)^i\beta_{i,j}^I=\sum_{i=0}^{n}(-1)^i\beta_{i,j}^J$ for all j. This implies that for ρ the regularity of $H(R/I)$, $\beta_{n,\rho+n}^I=\beta_{n,\rho+n}^J$ and $\beta_{n-1,\rho+n-1}^I-\beta_{n,\rho+n-1}^I=\beta_{n-1,\rho+n-1}^J-\beta_{n,\rho+n-1}^J$. These last two facts will prove useful in Section 6.

3. The Hilbert function of lex plus powers ideals

In this section, we state a characterization of the Hilbert functions which can occur for $\{a_1,\dots,a_n\}$-lex plus powers ideals. This characterization follows from the work of Clements and Lindström and will be useful in the next section when we find an alternative to the Hilbert functions of lex plus powers ideals similar to the n-type vectors found by Geramita, Harima, and Shin in [7] for Hilbert functions of lex ideals. For more details then provided here, on the relationship between the work of Clements and Lindström and Macaulay's O-sequences, see [3].

DEFINITION 3.1. Let $\mathbb{A}=\{a_1,\dots,a_n\}$. Then a lex plus powers ideal L is said to be *lex plus powers with respect to* $\le\mathbb{A}$, or $\mathrm{lpp}_{\le}(\mathbb{A})$, if $L\subsetneq R$ contains the $\mathbb{A}$-regular sequence $\{x_1^{a_1},\dots,x_n^{a_n}\}$. Note that a $\mathbb{B}=\{b_1,\dots,b_n\}$-lex plus powers ideal is $\mathrm{lpp}_{\le}(\mathbb{A})$ if and only if $\mathbb{B}\le\mathbb{A}$, that is, if $b_i\le a_i$ for all $i=1,\dots,n$.

Although Clements and Lindström used different terminology, the following special case of the EGH conjecture can be found in their paper [3].

THEOREM 3.2. *Let $\mathbb{A}=\{a_1,\dots,a_n\}$, L be $LPP(\mathbb{A})$, and I be any monomial ideal in $R=k[x_1,\dots,x_n]$ containing $x_1^{a_1},\dots,x_n^{a_n}$ such that $H(R/I,d)=H(R/L,d)$. Then $H(R/I,d+1)\le H(R/\langle L_d\rangle,d+1)$.*

Since any $\mathrm{lpp}_{\le}(\mathbb{A})$-ideal is a monomial ideal containing $x_1^{a_1},\dots,x_n^{a_n}$, we obtain the following corollary.

COROLLARY 3.3. *Let $\mathbb{A}=\{a_1,\dots,a_n\}$, L be $LPP(\mathbb{A})$, and I be an* $\mathrm{lpp}_{\le}(\mathbb{A})$ *ideal such that $H(R/L,d)=H(R/I,d)$. Then*

$$H(R/I,d+1)\le H(R/\langle L_d\rangle,d+1).$$

Keeping in the Macaulayesque mindset, we introduce the following notation.

Definition 3.4. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$. Let L be an $LPP(\mathbb{A})$-ideal satisfying $H(R/L, d) = h$. Then define $h^{\langle d \rangle_{\mathbb{A}}} := H(R/\langle L_d \rangle, d+1)$. Furthermore, let $S = \{c_i\}_{i \geq 0}$ be a sequence satisfying $c_0 = 1$ and $c_{i+1} \leq c_i^{\langle i \rangle_{\mathbb{A}}}$ for all i. Then S is said to be an $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequence.

Remark 3.5. In the notation of Definition 3.4, Corollary 3.3 says that H is the Hilbert function of an $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideal if and only if H is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequence. Note that, to determine the Hilbert functions of $LPP(\mathbb{A})$-ideals, we cannot simply eliminate the sequences that are $\mathrm{lpp}_{\leq}(\mathbb{B})$-sequences for $\mathbb{B} \leq \mathbb{A}$, but $\mathbb{B} \neq \mathbb{A}$ from the set of $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequences. This is because of the possibility of overlap. For example, $I = \langle x^2, y^3, z^4, xy^2, xyz, xz^2, y^2z^2 \rangle$ and $J = \langle x^2, y^3, z^3, xy^2, xyz \rangle$ are, respectively $LPP(\{2,3,4\})$ and $LPP(\{2,3,3\})$-ideals, both having Hilbert function $H = 1\ 3\ 5\ 3\ 1\ 0 \rightarrow$.

Greene and Kleitman [3] found a Macaulayesque way of describing $h^{\langle i \rangle_{\mathbb{A}}}$, which we wish to consider in some detail, since we will be using their notation in later parts of this paper. Before doing so, we recall Macaulay's methods.

Let $d, h \in \mathbb{N}$ be given. Then it is well known that there are unique integers $k(d) > k(d-1) > \cdots > k(1) \geq 0$ such that $h = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1}$. Macaulay's theorem states that if h is the value of the Hilbert function of a graded module in degree d, then $H(M, d+1) \leq \binom{k(d)+1}{d+1} + \binom{k(d-1)+1}{d-1+1} + \cdots + \binom{k(1)+1}{1+1}$, and this bound is sharp. The process of obtaining the $k(i)$ and computing the bound can be beautifully visualized by writing Pascal's triangle as a rectangle:

$$
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
 & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 \ldots \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 3 & 6 & 10 & 15 & 21 \\
1 & 4 & 10 & 20 & 35 & 56 \\
1 & 5 & 15 & 35 & 70 & 126 \\
1 & 6 & 21 & 56 & 126 & 252 \\
\vdots & & & & & \ddots
\end{array}
$$

Example 3.6. Suppose that M is a graded module such that $H(M, 3) = 32$. Then to obtain an upper bound for $H(M, 4)$, one must first find the $k(i)$ which uniquely describe 32 in degree 3. First, look at the column numbered 3, and pick the largest number that is at most 32, namely 20. This is 3 rows down from the top, so we take $k(3) = 3 + 3 = 6$. Then look at the column numbered 2 and pick the largest number that is at most $32 - 20 = 12$, namely 10. This is again 3 rows down, so we take $k(2) = 2 + 3 = 5$. Finally, pick the 2 from the column numbered 1, which is 1 row down, so we take $k(1)=1+1=2$.

$$\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\\
1 & 1 & 1 & 1 & 1 & 1\ldots \\
1 & \boxed{2} & 3 & 4 & 5 & 6 \\
1 & 3 & 6 & 10 & 15 & 21 \\
1 & 4 & \boxed{10} & \boxed{20} & 35 & 56 \\
1 & 5 & 15 & 35 & 70 & 126 \\
1 & 6 & 21 & 56 & 126 & 252 \\
\vdots & & & & & \ddots
\end{array}$$

Recalling that the number in the ith row and jth column of Pascal's triangle is $\binom{i+j-1}{j}$, it is evident that we have just found $32=\binom{6}{3}+\binom{5}{2}+\binom{2}{1}$ (note that $\binom{6}{3}=20$, $\binom{5}{2}=10$, and $\binom{2}{1}=2$). Then to compute the bound for $H(M,4)$, we need $\binom{6+1}{3+1}+\binom{5+1}{2+1}+\binom{2+1}{1+1}$, and this is obtained by taking the number one column to the right of each of the boxed integers in the rectangular version of Pascal's triangle:

$$\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\\
1 & 1 & 1 & 1 & 1 & 1\ldots \\
1 & \boxed{2}\to & 3 & 4 & 5 & 6 \\
1 & 3 & 6 & 10 & 15 & 21 \\
1 & 4 & \boxed{10}\to & \boxed{20}\to & 35 & 56 \\
1 & 5 & 15 & 35 & 70 & 126 \\
1 & 6 & 21 & 56 & 126 & 252 \\
\vdots & & & & & \ddots
\end{array}$$

The result is $H(M,4)\leq 35+20+3=58$.

REMARK 3.7. There is a precise relationship between monomials of degree i and i-binomial expansions. Namely, if $h=\binom{m_i}{i}+\binom{m_{i-1}}{i-1}+\cdots+\binom{m_j}{j}$, then h is the codimension of a lex-segment in the vector space of polynomials of degree i in $n=m_i-i+2$ variables. Letting m be the smallest monomial of degree i in this lex-segment, we associate h to m. Namely, let $\alpha_r=\#\{t|m_t-t=n-1-r\}$ for $1\leq r\leq n-1$. Then the lex segment ending in the monomial $m=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{i-(\alpha_1+\cdots+\alpha_{n-1})}$ has codimension h in the vector space of degree i polynomials in $k[x_1,\ldots,x_n]$. (See [15] for details.)

Since $\alpha_1+\alpha_2+\cdots+\alpha_{n-1}$ is the number of terms in the i-binomial expansion of h, we see that $i-(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})=j-1$, so we can rewrite m as $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{j-1}$. In fact, this correspondence could have been used to define i-binomial expansions in the first place, and is the reason why they are so valuable in the study of Hilbert functions.

We now wish to state the growth bound for $\mathrm{lpp}_{\leq}(\mathbb{A})$ ideals in terms of the notation used by Greene and Kleitman in [9]. Let $d_1 \leq d_2 \leq \cdots \leq d_k$ and put $e_1 := d_k - 1, e_2 := d_{k-1} - 1, \ldots, e_k := d_1 - 1$. Then they used the notation $\binom{e_1,\ldots,e_k}{i}$ to be $\Delta H(R/I, i)$, where Δ represents the first difference function and $I \subset k[x_0, \ldots, x_n]$ is the ideal of a complete intersection of type $(d_1, \ldots, d_k)$. Note that $\binom{e_1}{i}$ is not the usual binomial coefficient; $\binom{e_1}{i}$ is 1 if $0 \leq i \leq e_1$ and is 0 if $i > e_1$. This will allow us to state the EGH conjecture using their Macaulayesque form, but first, we need a result stated in [9].

Definition/Proposition 3.8. *Let $\mathbb{A} = \{a_1, \ldots, a_n\}$ and d be given and let $0 < h \leq H(R/(x_1^{a_1}, \ldots, x_n^{a_n}), d)$. Let $a_i' = a_i - 1$. Then h can be written uniquely in the form*

$$h = \binom{a_n', a_{n-1}', \ldots, a_{n-(k(d)-d)}'}{d} + \binom{a_n', a_{n-1}', \ldots, a_{n-(k(d-1)-(d-1))}'}{d-1} + \cdots + \binom{a_n', a_{n-1}', \ldots, a_{n-(k(j)-j)}'}{j},$$

where $k(d) > k(d-1) > \cdots > k(j) \geq j \geq 1$ and $\#\{t | k(t) - t = i\} < a_{n-i-1}$ and the last term is non-zero.

We refer to this expression as the $d_{\mathbb{A}}$-Macaulay expansion for k. Furthermore,

$$h^{\langle d \rangle_{\mathbb{A}}} := \binom{a_n', a_{n-1}', \ldots, a_{n-(k(d)-d)}'}{d+1} + \binom{a_n', a_{n-1}', \ldots, a_{n-(k(d-1)-(d-1))}'}{d} + \cdots + \binom{a_n', a_{n-1}', \ldots, a_{n-(k(j)-j)}'}{j+1}.$$

One way to look at this proposition, is through the correspondence between monomials m and the codimension of the lex-segments ending in monomial m. Given a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, write the expansion for which $\alpha_i = \#\{t | k(t) - t = n - 1 - i\}$ for $1 \leq i \leq n - 1$, and then remove any zero terms at the end.

EXAMPLE 3.9. Let $R = k[x_1, x_2, x_3]$ and $\mathbb{A} = \{3, 4, 6\}$. The monomials of degree 7 in $k[x_1, x_2, x_3]/\langle x_1^3, x_2^4, x_3^6\rangle$ with their codimensions are listed below:

$$\begin{array}{ll}
x_1^2x_2^3x_3^2 & \binom{5,3}{7} + \binom{5,3}{6} + \binom{5}{5} + \binom{5}{4} + \binom{5}{3} = 8, \\
x_1^2x_2^2x_3^3 & \binom{5,3}{7} + \binom{5,3}{6} + \binom{5}{5} + \binom{5}{4} = 7, \\
x_1^2x_2x_3^4 & \binom{5,3}{7} + \binom{5,3}{6} + \binom{5}{5} = 6, \\
x_1^2x_3^5 & \binom{5,3}{7} + \binom{5,3}{6} = 5, \\
x_1x_2^3x_3^3 & \binom{5,3}{7} + \binom{5}{6} + \binom{5}{5} + \binom{5}{4} = 4, \\
x_1x_2^2x_3^4 & \binom{5,3}{7} + \binom{5}{6} + \binom{5}{5} = 3, \\
x_1x_2x_3^5 & \binom{5,3}{7} + \binom{5}{6} = 2, \\
x_2^3x_3^4 & \binom{5}{7} + \binom{5}{6} + \binom{5}{5} = 1, \\
x_2^2x_3^5 & \binom{5}{7} + \binom{5}{6} = 0.
\end{array}$$

CONJECTURE 3.10. (*Restatement of the EGH Conjecture*): *Let $I \subset R$ contain an $\mathbb{A}$-regular sequence and suppose there exists an $LPP(\mathbb{A})$-ideal L, such that $H(R/I, d) = H(R/L, d)$. Then $H(R/I, d+1) \leq H(R/I, d)^{\langle d\rangle_{\mathbb{A}}}$.*

EXAMPLE 3.11. Suppose for instance that $R = k[x_1, x_2, x_3]$, $\mathbb{A} = \{3, 4, 6\}$, and L is $\mathrm{lpp}_{\leq}(A)$ with $H(R/L, 4) = 9$. Then we consider the following rectangle

$$\begin{array}{rcccccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \dots \\
 \\
(1,1,6): & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \to \\
(1,4,6): & 1 & 2 & 3 & 4 & 4 & 4 & 3 & 2 & 1 & 0 & \to \\
(3,4,6): & 1 & 3 & 6 & 9 & 11 & 12 & 11 & 9 & 6 & 3 & 1 & 0 & \to,
\end{array}$$

where we have written (a_1, a_2, a_3) beside the row that consists of $\Delta H(R/I)$ for I a complete intersection of type (a_1, a_2, a_3). The top row is thus $\binom{5}{i}$ for $i \geq 0$, the second row is $\binom{5,3}{i}$ for $i \geq 0$, and the third row is $\binom{5,3,2}{i}$ for $i \geq 0$.

The largest number in the column numbered 4 which is at most 9 is 4. In the column numbered 3, we take the largest number that is at most $9 - 4 = 5$, which is 4. Finally, in the column numbered 2, we take 1. This expresses 9 as a $4_{\mathbb{A}}$-Macaulay expansion:

$$\begin{array}{lccccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \dots \\
 & & & & & & & & & & & & & \\
(1,1,6): & 1 & 1 & \boxed{1} & 1 & 1 & 1 & 0 & \to & & & & & \\
(1,4,6): & 1 & 2 & 3 & \boxed{4} & \boxed{4} & 4 & 3 & 2 & 1 & 0 & \to & & \\
(3,4,6): & 1 & 3 & 6 & 9 & 11 & 12 & 11 & 9 & 6 & 3 & 1 & 0 & \to .
\end{array}$$

Note that the number to the right of $\binom{e_1,\dots,e_k}{i}$ is just $\binom{e_1,\dots,e_k}{i+1}$. Thus, to calculate $9^{\langle 4\rangle_{\mathbb{A}}}$, the bound for $H(R/L,5)$, we again sum the numbers to the right of our boxed integers.

$$\begin{array}{lccccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \dots \\
 & & & & & & & & & & & & & \\
(1,1,6): & 1 & 1 & \boxed{1} & \to 1 & 1 & 1 & 0 & \to & & & & & \\
(1,4,6): & 1 & 2 & 3 & \boxed{4} & \to \boxed{4} & \to 4 & 3 & 2 & 1 & 0 & \to & & \\
(3,4,6): & 1 & 3 & 6 & 9 & 11 & 12 & 11 & 9 & 6 & 3 & 1 & 0 & \to .
\end{array}$$

Thus, we find that $H(R/L,5) \leq 4+4+1 = 9$.

Example 3.12. Suppose that L is an $\mathbb{A} = \{3,4,6\}$ lex plus powers ideal and $H(R/L,6) = 6$. The monomials of degree 6 not in L are

$$xy^2z^3,\ xyz^4,\ xz^5,\ y^3z^3,\ y^2z^4,\ \text{and } yz^5$$

and so in degree 7, at most the following monomials are not in L:

$$xy^2z^4,\ xyz^5,\ y^3z^4,\ \text{and } y^2z^5.$$

Then the diagram looks like

$$\begin{array}{lccccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \dots \\
 & & & & & & & & & & & & & \\
(1,1,6): & 1 & 1 & 1 & \boxed{1} & \to \boxed{1} & \to \boxed{1} & \to 0 & \to & & & & & \\
(1,4,6): & 1 & 2 & 3 & 4 & 4 & 4 & \boxed{3} & \to 2 & 1 & 0 & \to & & \\
(3,4,6): & 1 & 3 & 6 & 9 & 11 & 12 & 11 & 9 & 6 & 3 & 1 & 0 & \to
\end{array}$$

so that as expected $H(R/L,7) \leq 2+0+1+1 = 4$.

4. An analogue to n-type vectors for lex plus powers ideals

We wish to define a vector that will correspond in a natural way to lex plus powers ideals. This will be an analogue to the n-type vectors that correspond to lex ideals. Let $a \leq b$. Then any $LPP(a,b)$-ideal is of the form

$$L = \langle x^a, x^{a-1}y^{d_1}, x^{a-2}y^{d_2}, \dots, x^{a-s}y^{d_s}, y^b \rangle,$$

where $d_1 < d_2 < \cdots < d_s < b$. We associate to L the n-type vector $\mathcal{T} = (d_1, d_2, \dots, d_s, b, \dots, b)$ where there are $a-s$ b's and $a \leq b$. The condition that $a \leq b$ is crucial, for otherwise the ideal, would not be lex plus powers.

EXAMPLE 4.1. If we put $\mathcal{T} = (2,4,5,5,5,5)$, the associated ideal would be $I = \langle x^6, x^5y^2, x^4y^4, y^5\rangle$. Since this violates the condition that the powers of the variables be in nondecreasing order, the ideal is not $LPP(5,6)$. The $LPP(5,6)$-ideal with the same Hilbert function as I is $J = \langle x^5, x^4y^3, x^3y^5, y^6\rangle$ and this corresponds to the vector $(3,5,6,6,6)$. They both have the same graded Betti numbers, but for uniqueness purposes, we choose J as the $LPP(5,6)$-ideal.

REMARK 4.2. In three variables, it is easy to construct ([16, Remark 4.3]) many ideals which satisfy all the requirements of lex plus powers ideals except the condition that the powers of the variables are in nondecreasing order, and do not actually have the same graded Betti numbers as the lex plus powers ideal.

DEFINITION 4.3. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$.

If $n = 1$ and $\mathcal{T} = (d)$ for some $0 < d \le a_1$, we say that $\mathcal{T}$ is an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector. We say that $\mathcal{T} = \mathcal{T}_{c.i.(\mathbb{A})}$ if $\mathcal{T} = (a_1)$. We put $l(\mathcal{T}) = \sigma(\mathcal{T}) = \alpha_{\mathbb{A}}(\mathcal{T}) = d$ unless $\mathcal{T} = \mathcal{T}_{c.i.(\mathbb{A})}$, in which case we put $l(\mathcal{T}) = \sigma(\mathcal{T}) = a_1$ and $\alpha_{\mathbb{A}}(\mathcal{T}) = \infty$.

If $n > 1$, then $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_u)$ is an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector if the following conditions all hold: $u \le a_1$, $u \le l(\mathcal{T}_u)$, each $\mathcal{T}_i$ is an $\mathrm{lpp}_{\le}(\mathbb{A}_2)$-vector for $\mathbb{A}_2 = \{a_2, \ldots, a_n\}$ (in particular, $l(\mathcal{T}_u) \le a_2$), and $\sigma(\mathcal{T}_i) < \alpha_{\mathbb{A}_2}(\mathcal{T}_{i+1})$ for $1 \le i \le u-1$.

We define $l(\mathcal{T}) = u$ to be the length of $\mathcal{T}$, and $\sigma(\mathcal{T})$ and $\alpha_{\mathbb{A}}(\mathcal{T})$ as follows:

$$\sigma(\mathcal{T}) = \begin{cases} \sigma(\mathcal{T}_u) & \text{if } \mathcal{T}_u \ne \mathcal{T}_{c.i.(\mathbb{A}_2)}, \\ \sigma(\mathcal{T}_u) + s - 1 & \text{if } \mathcal{T}_u = \mathcal{T}_{c.i.(\mathbb{A}_2)}, \end{cases} \quad \text{where } s = \#i \text{ s.t. } \mathcal{T}_i = \mathcal{T}_u,$$

$$\alpha_{\mathbb{A}}(\mathcal{T}) = \begin{cases} l(\mathcal{T}) & \text{if } l(\mathcal{T}) < a_1, \\ l(\mathcal{T}) + \alpha_{\mathbb{A}_2}(\mathcal{T}_1) - 1 & \text{if } l(\mathcal{T}) = a_1. \end{cases}$$

Finally, we say that $\mathcal{T} = \mathcal{T}_{c.i.(\mathbb{A})}$ if $l(\mathcal{T}) = a_1$ and $\mathcal{T}_i = \mathcal{T}_{c.i.(\mathbb{A}_2)}$ for each i.

REMARK 4.4. $\alpha_{\mathbb{A}}(\mathcal{T}) < \infty$ unless $\mathcal{T} = \mathcal{T}_{c.i.(\mathbb{A})}$. Furthermore, $\alpha_{\mathbb{A}}(\mathcal{T}) \le \sigma(\mathcal{T})$ unless $\mathcal{T} = \mathcal{T}_{c.i.(\mathbb{A})}$.

NOTATION. For convenience, we will denote the vector $((d_1), \ldots, (d_m))$ by $(d_1, \ldots, d_m)$. Thus, for example, the vector $((1),(3),(4))$ will be written as $(1,3,4)$, and the vector $(((1),(2)),((1),(3),(4)))$ will be written as $((1,2),(1,3,4))$. This does however create some ambiguity since (d_1) could denote either the vector $((d_1))$ or the vector (d_1). If there is ever the possibility of any confusion, we will be explicit.

EXAMPLE 4.5. Let

$$\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5) = ((1,2),(1,3,4),(2,3,6,6),(5,6,6,6),(6,6,6,6)),$$

where each $\mathcal{T}_i$ is an $\mathrm{lpp}_{\le}(4,6)$-vector. Then both the vectors $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4)$ and $(\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5)$ are $\mathrm{lpp}_{\le}(\mathbb{A})$-vectors where $\mathbb{A} = \{4,4,6\}$ since $\sigma(\mathcal{T}_1) = 2 <$

$\alpha_{\mathbb{A}_2}(\mathcal{T}_2) = 3$, $\sigma(\mathcal{T}_2) = 4 < \alpha_{\mathbb{A}_2}(\mathcal{T}_3) = 4 + 2 - 1 = 5$, $\sigma(\mathcal{T}_3) = 6 + 2 - 1 = 7 < \alpha_{\mathbb{A}_2}(\mathcal{T}_4) = 4+5-1 = 8$ and $\sigma(\mathcal{T}_4) = 6+3-1 = 8 < \alpha_{\mathbb{A}_2}(\mathcal{T}_5) = \infty$. However, $\mathcal{T}$ is not an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector for any $\mathbb{A} = \{a_1, a_2, a_3\}$, for suppose it were. Then $a_1 \geq l(\mathcal{T}) = 5$. Since $\mathbb{A} = \{a_1, a_2, a_3\}$ must satisfy $a_1 \leq a_2 \leq a_3$, we also have $a_2 \geq 5$, while the repeating 6's in $\mathcal{T}_4$ force $a_3 = 6$. Then $\alpha_{\mathbb{A}_2}(\mathcal{T}_5) = 4$ and $\sigma(\mathcal{T}_4) = 8$, contradicting that $\sigma(\mathcal{T}_4) < \alpha_{\mathbb{A}_2}(\mathcal{T}_5)$. Notice also that $\mathcal{T}_3, \mathcal{T}_4$ and $\mathcal{T}_5$ are all $\mathrm{lpp}_{\leq}(5,6)$-vectors, but are not $\mathrm{lpp}_{\leq}(4,7)$-vectors.

To an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector $\mathcal{T}$, it is natural to associate an ideal $W_{\mathcal{T}}$ as follows:

DEFINITION 4.6. If $n = 1$ (so that $\mathbb{A} = \{a_1\}$) and $\mathcal{T}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector, say $\mathcal{T} = (d)$ with $d \leq a_1$, then define $W_{\mathcal{T}} := \langle x_1^d \rangle$ in $k[x_1]$.

If $n > 1$ and $\mathcal{T}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector, say $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_u)$ with $u \leq a_1$, then define

$$W_{\mathcal{T}} := \langle x_1^u, x_1^{u-1}\overline{W_{\mathcal{T}_1}}, \ldots, x_1\overline{W_{\mathcal{T}_{u-1}}}, \overline{W_{\mathcal{T}_u}} \rangle,$$

where $\overline{W_{\mathcal{T}_i}}$ is the image in $k[x_2, \ldots, x_n]$ under the isomorphism induced by $x_i \to x_{i+1}$ of the ideal $W_{\mathcal{T}_i} \subset k[x_1, \ldots, x_{n-1}]$ obtained by induction.

EXAMPLE 4.7. We compute $W_{\mathcal{T}}$ for the $\mathrm{lpp}_{\leq}(\{4,6,6\})$-vector

$$\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4) = ((1,2), (1,3,4), (2,3,6,6), (5,6,6,6)).$$

First, we have

$$\begin{aligned}
W_{\mathcal{T}_1} &= \langle x_1^2, x_1\overline{W_{(1)}}, \overline{W_{(2)}} \rangle = \langle x_1^2, x_1x_2, x_2^2 \rangle,\\
W_{\mathcal{T}_2} &= \langle x_1^3, x_1^2\overline{W_{(1)}}, x_1\overline{W_{(3)}}, \overline{W_{(4)}} \rangle = \langle x_1^3, x_1^2x_2, x_1x_2^3, x_2^4 \rangle,\\
W_{\mathcal{T}_3} &= \langle x_1^4, x_1^3\overline{W_{(2)}}, x_1^2\overline{W_{(3)}}, x_1\overline{W_{(6)}}, \overline{W_{(6)}} \rangle = \langle x_1^4, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^6, x_2^6 \rangle,\\
W_{\mathcal{T}_4} &= \langle x_1^4, x_1^3\overline{W_{(5)}}, x_1^2\overline{W_{(6)}}, x_1\overline{W_{(6)}}, \overline{W_{(6)}} \rangle = \langle x_1^4, x_1^3x_2^5, x_1^2x_2^6, x_1x_2^6, x_2^6 \rangle,
\end{aligned}$$

so

$$\begin{aligned}
W_{\mathcal{T}} &= \langle x_1^4, x_1^3\overline{W_{\mathcal{T}_1}}, x_1^2\overline{W_{\mathcal{T}_2}}, x_1\overline{W_{\mathcal{T}_3}}, \overline{W_{\mathcal{T}_4}} \rangle\\
&= \langle x_1^4,\ x_1^3\langle x_2^2, x_2x_3, x_3^2 \rangle,\ x_1^2\langle x_2^3, x_2^2x_3, x_2x_3^3, x_3^4 \rangle,\\
&\quad x_1\langle x_2^4, x_2^3x_3^2, x_2^2x_3^3, x_2x_3^6, x_3^6 \rangle,\ x_2^4, x_2^3x_3^5, x_2^2x_3^6, x_2x_3^6, x_3^6 \rangle\\
&= \langle x_1^4, x_1^3x_2^2, x_1^3x_2x_3, x_1^3x_3^2, x_1^2x_2^3, x_1^2x_2^2x_3\\
&\quad x_1^2x_2x_3^3, x_1^2x_3^4, x_1x_2^3x_3^2, x_1x_2^2x_3^3, x_2^4, x_2^3x_3^5, x_3^6 \rangle.
\end{aligned}$$

REMARK 4.8. If $\mathcal{T} = \mathcal{T}_{c.i.(\mathbb{A})}$, then $W_{\mathcal{T}} = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$. To see this, note that if $n = 1$ and $\mathcal{T} = (a_1)$, then $W_{\mathcal{T}} = \langle x_1^{a_1} \rangle$ and by induction, if $\mathcal{T} = (\mathcal{T}_{c.i.(\mathbb{A}_2)}, \ldots, \mathcal{T}_{c.i.(\mathbb{A}_2)})$ with $l(\mathcal{T}) = a_1$, then

$$W_{\mathcal{T}} = \langle x_1^{a_1}, \overline{W_{\mathcal{T}_{c.i.(\mathbb{A}_2)}}} \rangle = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle,$$

as required.

Before showing that $W_{\mathcal{T}}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideal if $\mathcal{T}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector, we first show that $\alpha(\mathcal{T})$ is the smallest degree of any element of $W_{\mathcal{T}}$ not in $\langle x_1^{a_1},\dots,x_n^{a_n}\rangle$ and that $\sigma(\mathcal{T})-1$ is the largest degree of any element of $k[x_1,\dots,x_n]$ not in $W_{\mathcal{T}}$. In fact, we give names to these parameters for any ideal containing $\langle x_1^{a_1},\dots,x_n^{a_n}\rangle$.

Definition 4.9. Let I be any ideal of $k[x_1,\dots,x_n]$ containing $\langle x_1^{a_1},\dots, x_n^{a_n}\rangle$. Then put

$$\alpha_{\mathbb{A}}(I)=\min\{i|f\in I\setminus\langle x_1^{a_1},\dots,x_n^{a_n}\rangle,\deg f=i\}\quad\text{and}$$
$$\sigma(I)=\min\{i|I_i=k[x_1,\dots,x_n]_i\}.$$

We use $\alpha_{\mathbb{A}}$ instead of α to distinguish it from the usual α, which is just $\alpha(I)=\min\{i|f\in I,\deg f=i\}$. $\sigma(I)$ is defined as usual.

Lemma 4.10. *Let $\mathcal{T}$ be an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector. Then $\alpha_{\mathbb{A}}(W_{\mathcal{T}})=\alpha_{\mathbb{A}}(\mathcal{T})$.*

Proof. The result is clear for $n=1$, so assume that $n>1$. Furthermore, the result is clear if $\mathcal{T}=\mathcal{T}_{c.i.(\mathbb{A})}$, so we assume this is not the case.

Let $\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_u,\mathcal{T}_u,\dots,\mathcal{T}_u)$, where $l(\mathcal{T})=u+v$, so there are $v+1$ $\mathcal{T}_u$'s. Then

$$W_{\mathcal{T}}=\langle x_1^{u+v},x_1^{u+v-1}\overline{W_{\mathcal{T}_1}},\dots,x_1^{v+1}\overline{W_{\mathcal{T}_{u-1}}},\overline{W_{\mathcal{T}_u}}\rangle.$$

There are four cases to consider, determined by whether or not $\mathcal{T}_u=\mathcal{T}_{c.i.(A)}$ and whether or not $u+v=a_1$. Each proof is similar, so we include only the case for which $\mathcal{T}_u=\mathcal{T}_{c.i.(A)}$ and $u+v=a_1$ as a representative.

We know by the induction hypothesis that the smallest degree of any element of $\overline{W_{\mathcal{T}_i}}$ not in $\langle x_2^{a_2},\dots,x_n^{a_n}\rangle$ is $\alpha_{\mathbb{A}_2}(\mathcal{T}_i)$. Now, $\overline{W_{\mathcal{T}_u}}=\langle x_2^{a_2},\dots,x_n^{a_n}\rangle$, so we can ignore it. Now for $i<u$, we have $\alpha_{\mathbb{A}_2}(\mathcal{T}_i)\leq\sigma(\mathcal{T}_i)<\alpha_{\mathbb{A}_2}(\mathcal{T}_{i+1})$, so $\alpha_{\mathbb{A}}(W_{\mathcal{T}})=u+v-1+\alpha_{\mathbb{A}_2}(\mathcal{T}_1)=a_1-1+\alpha_{\mathbb{A}_2}(\mathcal{T}_1)=l(\mathcal{T})+\alpha_{\mathbb{A}_2}(\mathcal{T}_1)-1=\alpha_{\mathbb{A}}(\mathcal{T})$. □

Lemma 4.11. *Let $\mathcal{T}$ be an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector. Then $\sigma(W_{\mathcal{T}})=\sigma(\mathcal{T})$.*

Proof. If $n=1$, the result is clear, so suppose that $n>1$. Let $\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_u,\dots,\mathcal{T}_u)$, where $l(\mathcal{T})=u+v$ and there are $v+1$ $\mathcal{T}_u$'s (if $v>0$ then $\mathcal{T}_u$ is necessarily $\mathcal{T}_{c.i.(\mathbb{A}_2)}$). Then we have

$$W_{\mathcal{T}}=\langle x_1^{u+v},x_1^{u+v-1}\overline{W_{\mathcal{T}_1}},\dots,x_1^{v+1}\overline{W_{\mathcal{T}_{u-1}}},\overline{W_{\mathcal{T}_u}}\rangle.$$

We know that there is an element of $x_1^v k[x_2,\dots,x_n]_{\sigma(\mathcal{T}_u)-1}$ that is not in $W_{\mathcal{T}}$. We claim that $(W_{\mathcal{T}})_{\sigma(\mathcal{T}_u)+v}=k[x_1,\dots,x_n]_{\sigma(\mathcal{T}_u)+v}$. So let f be a monomial of degree $\sigma(\mathcal{T}_u)+v$. If $x_1^{v+1}|f$, then we have that $f\in x_1^{v+i}k[x_2,\dots,x_n]_{\sigma(\mathcal{T}_u)-i}$ for some i. But $\sigma(\mathcal{T}_u)-i\geq\sigma(\mathcal{T}_{u-i})$, so $f\in W_{\mathcal{T}}$. If x_1^{v+1} does not divide f, then the part of f in $k[x_2,\dots,x_n]$ has degree at least $\sigma(\mathcal{T}_u)$, so $f\in W_{\mathcal{T}}$, as required. □

Theorem 4.12. *If $\mathcal{T}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector, then $W_{\mathcal{T}}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideal.*

Proof. If $n=1$, the result is clear. So, let $\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_u,\mathcal{T}_u,\dots,\mathcal{T}_u)$, where $l(\mathcal{T})=u+v$, so there are $v+1$ $\mathcal{T}_u$'s. Then

$$W_{\mathcal{T}}=\langle x_1^{u+v}, x_1^{u+v-1}\overline{W_{\mathcal{T}_1}},\dots,x_1^{v+1}\overline{W_{\mathcal{T}_{u-1}}},\overline{W_{\mathcal{T}_u}}\rangle.$$

By the induction hypothesis, each $W_{\mathcal{T}_i}$ is an $\mathrm{lpp}_{\le}(\mathbb{A}_2)$-ideal. Furthermore, since $l(\mathcal{T})\le a_1$, and $l(\mathcal{T})\le l(\mathcal{T}_u)\le a_2$, it is enough to show that any largest degree element of $x_1^{u+v-i}k[x_2,\dots,x_n]$ not in $x_1^{u+v-i}\overline{W_{\mathcal{T}_i}}$ has degree smaller than any smallest degree element of $x_1^{u+v-(i+1)}\overline{W_{\mathcal{T}_{i+1}}}$ not in $\langle x_1^{a_1},x_2^{a_2},\dots,x_n^{a_n}\rangle$. Thus, we need to show that $\sigma(\overline{W_{\mathcal{T}_i}})-1+u+v-i<\alpha_{\mathbb{A}_2}(\overline{W_{\mathcal{T}_{i+1}}})+u+v-(i+1)$ or in other words (from Lemmas 4.10 and 4.11) that $\sigma(\mathcal{T}_i)<\alpha_{\mathbb{A}_2}(\mathcal{T}_{i+1})$. Since $\mathcal{T}$ is an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector, we are done. ☐

To a given $\mathrm{lpp}_{\le}(\mathbb{A})$-vector, we associate a Hilbert function as follows.

Definition 4.13. If $n=1$, so that $\mathcal{T}=(d)$ is an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector, then define $H_{\mathcal{T}}$ to be the sequence $H_{\mathcal{T}}:=1\ 1\ 1\ \cdots\ 1\ 0\to$ with d 1's.

If $\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_u)$, then define $H_{\mathcal{T}}$ to be the sequence

$$H_{\mathcal{T}}(i):=\sum_{j=1}^{u}H_{\mathcal{T}_j}(i-u+j).$$

We want to show that if $\mathcal{T}$ is an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector, then $H(R/W_{\mathcal{T}})=H_{\mathcal{T}}$. We need the following lemmas.

Lemma 4.14. *Let $\mathcal{T}$ be an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector. Then $\alpha_{\mathbb{A}_2}(\mathcal{T}_1)+j\le\alpha_{\mathbb{A}_2}(\mathcal{T}_{j+1})$ for all $0\le j\le l(\mathcal{T})-1$.*

Proof. The proof is easy, and hence omitted. ☐

Lemma 4.15. *Let $\mathcal{T}$ be an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector. Let $0\le j\le l(\mathcal{T})-1$. Then $\sigma(\mathcal{T})-j\ge\sigma(\mathcal{T}_{l(\mathcal{T})-j})$.*

Proof. The proof is easy, and hence omitted. ☐

Lemma 4.16. *Let $\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_u,\mathcal{T}_u,\dots,\mathcal{T}_u)$ be an $\mathrm{lpp}_{\le}(\mathbb{A})$-type vector. Then $W_{\mathcal{T}_i}\supsetneq W_{\mathcal{T}_{i+1}}$ for all $i=1,\dots,u-1$.*

Proof. For notational convenience, we leave out the bar notation and assume it to be understood, so we write $\overline{W_{\mathcal{T}_1}}$ as $W_{\mathcal{T}_1}$ and $\overline{\overline{W_{(\mathcal{T}_1)_1}}}$ as $W_{(\mathcal{T}_1)_1}$.

We use induction on n, where n is the length of $\mathbb{A}$.

$n=2$: $\mathcal{T}=(e_1,\dots,e_u,e_u,\dots,e_u)$. We need to show that $\langle x_1^{e_i}\rangle\supsetneq\langle x_1^{e_{i+1}}\rangle$ for $i<u$, but this is true since $e_{i+1}>e_i$.

$n>2$: We first show that $((\mathcal{T}_i)_{l(\mathcal{T}_i)-j},(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})-j})$ is an $\mathrm{lpp}_{\le}(\mathbb{A}_2)$-type vector for $0\le j\le l(\mathcal{T}_i)-1$. Let $\mathcal{T}_i=((\mathcal{T}_i)_1,(\mathcal{T}_i)_2,\dots,(\mathcal{T}_i)_{l(\mathcal{T}_i)})$ and $\mathcal{T}_{i+1}=((\mathcal{T}_{i+1})_1,\dots,(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})})$. Now,

$$\begin{aligned}\sigma\big((\mathcal{T}_i)_{l(\mathcal{T}_i)-j}\big)&\le\sigma(\mathcal{T}_i)-j\quad\text{by Lemma 4.15}\\&<\alpha_{\mathbb{A}_2}(\mathcal{T}_{i+1})-j\end{aligned}$$

$$\begin{aligned}&\leq \alpha_{\mathbb{A}_3}((\mathcal{T}_{i+1})_1)+l(\mathcal{T}_{i+1})-j-1\\&\leq \alpha_{\mathbb{A}_3}\big((\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})-j}\big) \quad \text{by Lemma 4.14.}\end{aligned}$$

Thus, each $((\mathcal{T}_i)_{l(\mathcal{T}_i)-j},(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})-j})$ is an $\mathrm{lpp}_{\leq}(\mathbb{A}_2)$-type vector for $0\leq j\leq l(\mathcal{T}_i)-1$.

Thus, by the induction hypothesis (and since $l(\mathcal{T}_i)\leq l(\mathcal{T}_{i+1})$),

$$\begin{aligned}W_{(\mathcal{T}_i)_{l(\mathcal{T}_i)}}&\supsetneq W_{(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})}};\\W_{(\mathcal{T}_i)_{l(\mathcal{T}_i)-1}}&\supsetneq W_{(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})-1}};\\&\vdots\\W_{(\mathcal{T}_i)_1}&\supsetneq W_{(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})-l(\mathcal{T}_i)+1}}.\end{aligned}$$

Thus,

$$\begin{aligned}W_{\mathcal{T}_{i+1}}:=&\big\langle x_2^{l(\mathcal{T}_{i+1})},x_2^{l(\mathcal{T}_{i+1})-1}\big(W_{(\mathcal{T}_{i+1})_1}\big),\dots,x_2^{l(\mathcal{T}_i)}W_{(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})-l(\mathcal{T}_i)}},\\&\quad x_2^{l(\mathcal{T}_i)-1}W_{(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})-l(\mathcal{T}_i)+1}},\dots,W_{(\mathcal{T}_{i+1})_{l(\mathcal{T}_{i+1})}}\big\rangle\\&\subsetneq\big\langle x_2^{l(\mathcal{T}_i)},x_2^{l(\mathcal{T}_i)-1}W_{(\mathcal{T}_i)_1},\dots,W_{(\mathcal{T}_i)_{l(\mathcal{T}_i)}}\big\rangle\\&=W_{\mathcal{T}_i}.\end{aligned}$$

□

THEOREM 4.17. *Let $\mathcal{T}$ be an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector. Then $H(R/W_{\mathcal{T}})=H_{\mathcal{T}}$.*

Proof. We use induction on n, the length of $\mathbb{A}$. If $n=1$, the result is clear. So suppose that $n>1$. Let $\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_s)$. Let $R=k[x_1,\dots,x_n]$. Then $W_{\mathcal{T}}=\langle x_1^s,x_1^{s-1}\overline{W_{\mathcal{T}_1}},\dots,\overline{W_{\mathcal{T}_s}}\rangle$. It is enough to show that

$$\operatorname{codim}(W_{\mathcal{T}})_d=\sum_{e=1}^{s}\operatorname{codim}(\overline{W_{\mathcal{T}_e}})_{d-s+e}.$$

Now,

$$\operatorname{codim}(W_{\mathcal{T}})_d=\#\{\text{monomials in } R_d \text{ not in } W_{\mathcal{T}}\}.$$

Let M be the set of all monomials of R not in $W_{\mathcal{T}}$, and let $T=k[x_2,\dots,x_n]$. Then,

$$\begin{aligned}M\subseteq&\{\text{monomials in } T \text{ not in } \overline{W_{\mathcal{T}_s}}\}\\&\dot\cup\{x_1\cdot(\text{monomials in } T \text{ not in } \overline{W_{\mathcal{T}_{s-1}}})\}\\&\vdots\\&\dot\cup\{x_1^{s-1}\cdot(\text{monomials in } T \text{ not in } \overline{W_{\mathcal{T}_1}})\}.\end{aligned}$$

We will show equality. Certainly, any monomial of T that is not in $\overline{W_{\mathcal{T}_s}}$ cannot be in $W_{\mathcal{T}}$. Consider any monomial m of $x_1^{s-i}T$ that is not in $x_1^{s-i}\overline{W_{\mathcal{T}_i}}$. By Lemma 4.16, $\overline{W_{\mathcal{T}_j}}\subseteq\overline{W_{\mathcal{T}_i}}$ for all $j\geq i$. Write $m=x_1^{s-i}p$, where $p\in k[x_2,\dots,x_n]$. Now, if we had $m\in W_{\mathcal{T}}$, then we would have $\frac{m}{x_1^{j-i}}\in x_1^{s-j}\overline{W_{\mathcal{T}_j}}$

for some $j > i$. In other words, $m = x_1^{s-i}p$ for some $p \in \overline{W_{\mathcal{T}_j}}$, and some $j > i$. This contradicts that $\overline{W_{\mathcal{T}_j}} \subseteq \overline{W_{\mathcal{T}_i}}$ for all $j \geq i$. $\square$

So far, we have seen that if $\mathcal{T}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector, then $W_{\mathcal{T}}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideal with $H(R/W_{\mathcal{T}}) = H_{\mathcal{T}}$, $\alpha_{\mathbb{A}}(H) = \alpha_{\mathbb{A}}(\mathcal{T})$ and $\sigma(H) = \sigma(\mathcal{T})$. In particular, $H(R/W_{\mathcal{T}})$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequence. We now wish to show that given any $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequence H, we can obtain an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector $\mathcal{T}$, and furthermore that the function $H \to \mathcal{T}$ and the function $\mathcal{T} \to H_{\mathcal{T}}$ are inverses of each other.

We begin by decomposing a given $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequence S into two "smaller" such sequences S_1 and S_1' by using a decomposition similar to that used by Geramita, Maroscia, and Roberts in [5]. Suppose $S = 1\ b_1\ b_2\ b_3 \cdots$, where $b_1 \geq 2$.

Put $e_i = \binom{a_n-1, a_{n-1}-1, \ldots, a_{n-(b_1-2)}-1}{i}$ and $c_i = b_{i+1} - e_{i+1}$. Define S_1 as follows:

(1) if $c_i \geq 0$ for all i, set $S_1(i) = c_i$ for all i;
(2) if $c_i \geq 0$ for all $i \leq h-1$ and $c_h < 0$, then set $S_1 = c_0\ c_1 \cdots c_{h-1} 0 \to$.

In any case, we let h (possibly infinite) be the smallest integer for which $c_h < 0$. Then define S_1' as follows:

$$S_1'(i) = \begin{cases} e_i & \text{if } i \leq h, \\ b_i & \text{if } i \geq h+1. \end{cases}$$

From the definition of S_1 and S_1', it is clear that $S(i) = S_1'(i) + S_1(i-1)$.

THEOREM 4.18. *Let $S = \{b_i\}_{i \geq 0}$ be an $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequence. Let S_1 and S_1' be constructed as above. Then S_1 and S_1' are $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequences.*

Proof. Using the Macaulayesque notation for the generalized binomial coefficients, the proof of this statement follows word for word the proof of [5, Theorem 3.2], so we omit it. $\square$

Before showing the correspondence between $\mathrm{lpp}_{\leq}(\mathbb{A})$-vectors and Hilbert functions of $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideals, we need the following lemma.

LEMMA 4.19. *Let $\mathbb{A} = \{a_1, \ldots, a_n\}$ and let S be an $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequence, and S_1 obtained from S as above. Suppose that $S(1) = n$. Then $\alpha_{\mathbb{A}}(S_1) < \alpha_{\mathbb{A}}(S)$.*

Proof. If $S_1(1) < S(1)$, then $\alpha_{\mathbb{A}}(S_1) = 1 < \alpha_{\mathbb{A}}(S)$, so suppose that $S_1(1) = S(1)$. We consider three cases.

Case 1: $\alpha_{\mathbb{A}}(S) \leq h$. We again use the notation that $a_i' = a_i - 1$. Then

$$\begin{aligned} S_1\big(\alpha_{\mathbb{A}}(S) - 1\big) &= b_{\alpha_{\mathbb{A}}(S)} - e_{\alpha_{\mathbb{A}}(S)} \\ &< \binom{a_n', a_{n-1}', \ldots, a_1'}{\alpha_{\mathbb{A}}(S)} - \binom{a_n', a_{n-1}', \ldots, a_2'}{\alpha_{\mathbb{A}}(S)} \\ &= \binom{a_n', a_{n-1}', \ldots, a_2'}{\alpha_{\mathbb{A}}(S) - 1} + \binom{a_n', a_{n-1}', \ldots, a_2'}{\alpha_{\mathbb{A}}(S) - 2} + \cdots \end{aligned}$$

$$
\begin{aligned}
&+\binom{a'_n, a'_{n-1}, \ldots, a'_2}{\alpha_{\mathbb{A}}(S) - a'_1} \\
\leq &\binom{a'_n, a'_{n-1}, \ldots, a'_2}{\alpha_{\mathbb{A}}(S) - 1} + \cdots + \binom{a'_n, a'_{n-1}, \ldots, a'_2}{\alpha_{\mathbb{A}}(S) - a'_1} \\
&+\binom{a'_n, a'_{n-1}, \ldots, a'_2}{\alpha_{\mathbb{A}}(S) - 1 - a'_1} \\
= &\binom{a'_n, a'_{n-1}, \ldots, a'_1}{\alpha_{\mathbb{A}}(S) - 1}.
\end{aligned}
$$

So, $\alpha_{\mathbb{A}}(S_1) \leq \alpha_{\mathbb{A}}(S) - 1$.

Case 2: $h+1 \leq \alpha_{\mathbb{A}}(S) < \infty$. Then $S_1(\alpha_{\mathbb{A}}(S)-1) = 0 < \binom{a'_n, a'_{n-1}, \ldots, a'_1}{\alpha_{\mathbb{A}}(S)-1}$, so $\alpha_{\mathbb{A}}(S_1) < \alpha_{\mathbb{A}}(S)$.

Case 3: $\alpha_{\mathbb{A}}(S) = \infty$. Then $S(i) = b_i = \binom{a'_n, \ldots, a'_1}{i}$ and in particular, $b_1 = \binom{a'_n, \ldots, a'_1}{1} = n$, so $e_i = \binom{a'_n, \ldots, a'_2}{i}$. Then

$$
\begin{aligned}
S_1(i) &= b_{i+1} - e_{i+1} \\
&= \binom{a'_n, \ldots, a'_1}{i+1} - \binom{a'_n, \ldots, a'_2}{i+1} \\
&= \binom{a'_n, \ldots, a'_2}{i} + \cdots + \binom{a'_n, \ldots, a'_2}{i+1-a'_1} \\
&= \binom{a'_n, \ldots, a'_2, a'_1 - 1}{i}
\end{aligned}
$$

and hence $\alpha_{\mathbb{A}}(S_1) < \infty = \alpha_{\mathbb{A}}(S)$. $\square$

THEOREM 4.20. *There is a* 1–1 *correspondence between* $\mathrm{lpp}_{\leq}(\mathbb{A})$*-vectors and Hilbert functions of* $\mathrm{lpp}_{\leq}(\mathbb{A})$*-ideals, where if* $\mathcal{T}$ *corresponds to* H *(we write* $\mathcal{T} \leftrightarrow H$*), then* $\alpha_{\mathbb{A}}(\mathcal{T}) = \alpha_{\mathbb{A}}(H)$ *and* $\sigma(\mathcal{T}) = \sigma(H)$.

Proof. We first show that the map $\mathcal{T} \to H_{\mathcal{T}}$ is 1–1. We already know that it preserves σ and $\alpha_{\mathbb{A}}$ and that it does map $\mathrm{lpp}_{\leq}(\mathbb{A})$-vectors to $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequences. We use induction on n, the base case $n=1$ being trivial.

So suppose that $\mathcal{T} \to H$ and $\mathcal{T}' \to H$. We first reduce to the case where $\mathcal{T}_{l(\mathcal{T})} \neq \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$ and $\mathcal{T}'_{l(\mathcal{T}')} \neq \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$.

Suppose that $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_u, \mathcal{T}_u, \ldots, \mathcal{T}_u)$ and $\mathcal{T}' = (\mathcal{T}'_1, \ldots, \mathcal{T}'_v, \mathcal{T}'_v, \ldots, \mathcal{T}'_v)$ where $\mathcal{T}_u = \mathcal{T}_{c.i.(\mathbb{A}_2)}$. Then $\sigma(\mathcal{T}) = \sigma(c.i.(a_2, \ldots, a_n)) + \#\mathcal{T}_u\text{'s} - 1$.

If $\mathcal{T}'_v \neq \mathcal{T}_{c.i.(\mathbb{A}_2)}$, then $\sigma(\mathcal{T}') = \sigma(\mathcal{T}'_v) < \sigma(c.i.(a_2, \ldots, a_n))$, contradicting that $\sigma(\mathcal{T}') = \sigma(\mathcal{T})$. So, $\mathcal{T}'_v = \mathcal{T}_{c.i.(a\mathbb{A}_2)}$ and $\sigma(\mathcal{T}') = \sigma(c.i.(a_2, \ldots, a_n)) + \#\mathcal{T}'_v\text{'s} - 1$. Then $\#\mathcal{T}_u\text{'s} = \#\mathcal{T}'_v\text{'s}$. So, we also have $(\mathcal{T}_1, \ldots, \mathcal{T}_{u-1})$ and $(\mathcal{T}'_1, \ldots, \mathcal{T}'_{v-1})$ get mapped to the same Hilbert function. Thus, we may assume that $\mathcal{T}_{l(\mathcal{T})} \neq \mathcal{T}_{c.i.(\mathbb{A}_2)}$ and $\mathcal{T}'_{l(\mathcal{T}')} \neq \mathcal{T}_{c.i.(\mathbb{A}_2)}$.

So let $\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_u)$ and $\mathcal{T}'=(\mathcal{T}_1',\dots,\mathcal{T}_v')$. Since $\alpha_{\mathbb{A}}(\mathcal{T})=\alpha_{\mathbb{A}}(H)=\alpha_{\mathbb{A}}(\mathcal{T}')$, we have $u=v$. From here, the argument that $\mathcal{T}=\mathcal{T}'$ follows word for word the argument in [7, Theorem 2.6], so we omit it.

Now, we define the map $H\to\mathcal{T}$ inductively as follows:

If $n=1$, then $H=1\ 1\cdots 1\ 0\to$ where there are d 1's, for some $d\le a_1$. So put $H\to\mathcal{T}=(d)$.

If $n>1$, we may as well assume that $a_i\ge 2$ for all i, and that $H(1)=n$, for if $H(1)<n$, then we claim that H is also an $\mathrm{lpp}_{\le}(\mathbb{A}_2)$-sequence. To see this, consider the Macaulayesque rectangle used to construct $\mathrm{lpp}_{\le}(\mathbb{A})$-sequences, where the ith row consists of $\binom{a_n-1,\dots,a_{n-i+1}-1}{j}$ for $j\ge 0$. So if $H(1)\le n-1$, then H cannot in any degree occur below the row consisting of $\binom{a_n-1,\dots,a_2-1}{j}$ for $j\ge 0$. So H is also an $\mathrm{lpp}_{\le}(\mathbb{A}_2)$-sequence. Thus, in this case we may use induction on n.

Now, decompose H into H_1 and H_1'. By induction on n, send $H_1'\to\mathcal{T}_1'$. By Lemma 4.19, $\alpha_{\mathbb{A}}(H_1)<\alpha_{\mathbb{A}}(H)$, so by induction on $\alpha_{\mathbb{A}}$ (the base case $\alpha_{\mathbb{A}}=1$ being the induction hypothesis on n), we send $H_1\to\mathcal{T}_1=((\mathcal{T}_1)_1,\dots,(\mathcal{T}_1)_{l(\mathcal{T}_1)})$. Then send $H\to((\mathcal{T}_1)_1,\dots,(\mathcal{T}_1)_{l(\mathcal{T}_1)},\mathcal{T}_1')$. This is an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector, since

$$\begin{aligned}\sigma\big((\mathcal{T}_1)_{l(\mathcal{T}_1)}\big)&\le\sigma(\mathcal{T}_1)=\sigma(H_1)\text{ by induction}\\&\le h\text{ by construction of }H_1\\&<\alpha_{\mathbb{A}_2}(H_1')\text{ by construction of }H_1'.\end{aligned}$$

Next, we claim that $H\to\mathcal{T}\to H$ is the identity map. This is clearly true when $n=1$, so we use induction on n and assume that $n>1$. Note that if $H\to\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_u)$, we must have $H_1\to(\mathcal{T}_1,\dots,\mathcal{T}_{u-1})$ and $H_1'\to\mathcal{T}_u$, by definition. Then

$$\begin{aligned}H\to\mathcal{T}=(\mathcal{T}_1,\dots,\mathcal{T}_u)&\to H_{\mathcal{T}_u}(i)+H_{(\mathcal{T}_1,\dots,\mathcal{T}_{u-1})}(i-1)\quad\text{by definition}\\&=H_1'(i)+H_1(i-1)\quad\text{by induction since}\\&\qquad\qquad H_1'\to\mathcal{T}_u\text{ and }H_1\to(\mathcal{T}_1,\dots,\mathcal{T}_{u-1})\\&=H(i).\end{aligned}$$

This, together with $\mathcal{T}\to H_{\mathcal{T}}$ being 1–1 shows that $\mathcal{T}\to H_{\mathcal{T}}$ and $H\to\mathcal{T}$ are inverses of each other. □

COROLLARY 4.21. *Given an* $LPP(\mathbb{A})$ *ideal* L, *there is an* $\mathrm{lpp}_{\le}(\mathbb{A})$-*vector* $\mathcal{T}$ *such that* $W_{\mathcal{T}}=L$.

Proof. We show by induction on n that if $\mathcal{T}$ is the $\mathrm{lpp}_{\le}(\mathbb{A})$-vector (from Theorem 4.20) with $H_{\mathcal{T}}=H(R/L)=H$ for an $LPP(\mathbb{A})$ ideal L, then $W_{\mathcal{T}}$ minimally contains each of $x_1^{x_1},\dots,x_n^{a_n}$. That is, we show that $W_{\mathcal{T}}$ is $LPP(\mathbb{A})$ (we already know by Theorem 4.12 that $W_{\mathcal{T}}$ is $\mathrm{lpp}_{\le}(\mathbb{A})$). This is sufficient because $H(R/W_{\mathcal{T}})=H_{\mathcal{T}}=H$ (by Theorem 4.17), but the $LPP(\mathbb{A})$ ideal attaining H is unique.

If $n = 1$, it is easy to see that $W_{\mathcal{T}} = \langle x_1^{a_1} \rangle$, so clearly $W_{\mathcal{T}}$ minimally contains $x_1^{a_1}$.

Now, suppose that $n > 1$. By Theorem 4.20, we have that $\alpha_{\mathbb{A}}(\mathcal{T}) = \alpha_{\mathbb{A}}(H)$ and $\alpha_{\mathbb{A}}(H) \geq a_1$ since L is $LPP(\mathbb{A})$, so $l(\mathcal{T}) = a_1$ and $x_1^{a_1}$ is a minimal generator of $W_{\mathcal{T}}$. By the definition of $W_{\mathcal{T}}$, $x_2^{a_2}, \dots, x_n^{a_n}$ are minimal generators of $W_{\mathcal{T}}$ if and only if they are minimal monomial generators of $\overline{W_{\mathcal{T}_{l(\mathcal{T})}}}$. Following the proof of Theorem 4.20, we have that $\mathcal{T}_{l(\mathcal{T})} = \mathcal{T}_1'$ where $\mathcal{T}_1'$ corresponds to H_1'. Thus, we are done by induction if H_1' is the Hilbert function of a $LPP(a_2, \dots, a_n)$ ideal in $k[x_2, \dots, x_n]$. Writing h to be the smallest (possibly infinite) degree of a nonpure power monomial generator of L which is not divisible by x_1, then because L is $LPP(\mathbb{A})$, we have

$$\langle x_1, x_2^{a_2}, \dots, x_n^{a_n} \rangle_d = \langle L, x_1 \rangle_d$$

for all $d < h$ and $\langle L, x_1 \rangle_d = L_d$ for all $d \geq h$. Hence, it follows directly from the definition of H_1' that $H_1' = H(\overline{R}/\overline{L})$ where $\overline{L}$ is the image of L in $\overline{R} = k[x_2, \dots x_n]$ (here we assume that $a_1 > 1$, else we are done by induction, and hence that $e_i = H(R/\langle x_1, x_2^{a_2}, \dots, x_n^{a_n} \rangle, i))$. This completes the proof. $\square$

EXAMPLE 4.22. Consider the $\mathrm{lpp}_{\leq}(\{4,4,6\})$-vector

$$\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4) = ((1,2),(1,3,4),(2,3,6,6),(5,6,6,6)).$$

Then letting $\mathcal{T} \to H$ and $\mathcal{T}_i \to H_i$, we have

$$\begin{array}{lcccccccccc} H_4: & 1 & 2 & 3 & 4 & 4 & 4 & 3 & 2 & 0 & \to \\ H_3: & & 1 & 2 & 3 & 4 & 4 & 2 & 1 & 0 & \to \\ H_2: & & & 1 & 2 & 3 & 2 & 0 & \to & & \\ H_1: & & & & 1 & 2 & 0 & \to & & & \\ \hline H: & 1 & 3 & 6 & 10 & 13 & 10 & 5 & 3 & 0 & \to . \end{array}$$

Now, beginning with $H = 1\ 3\ 6\ 10\ 13\ 10\ 5\ 3\ 0 \to$, an $\mathrm{lpp}_{\leq}(\mathbb{A})$-sequence, we have:

$$\begin{array}{lcccccccccc} b_i: & 1 & 3 & 6 & 10 & 13 & 10 & 5 & 3 & 0 & \to \\ e_i: & 1 & 2 & 3 & 4 & 4 & 4 & 3 & 2 & 1 & 0 \ \to \\ \hline c_i: & 1 & 3 & 6 & 9 & 6 & 2 & 1 & -1 & 0 & \to . \end{array}$$

So, $S_1 = 1\ 3\ 6\ 9\ 6\ 2\ 1\ 0 \to$ and $S_1' = 1\ 2\ 3\ 4\ 4\ 4\ 3\ 2\ 0 \to$.
Continuing, we decompose S as

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 4 & 4 & 3 & 2 & 0 & \to \\ & 1 & 2 & 3 & 4 & 4 & 2 & 1 & 0 & \to \\ & & 1 & 2 & 3 & 2 & 0 & \to & & \\ & & & 1 & 2 & 0 & \to . & & & \end{array}$$

We decompose each of these further to obtain:

$$\begin{array}{lcl} 1\ 2\ 3\ 4\ 4\ 4\ 4\ 3\ 2\ 0\ \rightarrow & \longleftrightarrow & (5,6,6,6) \\ 1\ 2\ 3\ 4\ 4\ 2\ 1\ 0\ \rightarrow & \longleftrightarrow & (2,3,6,6) \\ 1\ 2\ 3\ 2\ 0\ \rightarrow & \longleftrightarrow & (1,3,4) \\ 1\ 2\ 0\ \rightarrow & \longleftrightarrow & (1,2). \end{array}$$

So we indeed obtain $\mathcal{T}$ back from H.

5. Ideal colon

In this section, our goal is to show that the residual of an $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideal in the complete intersection of type $(a_1, \ldots, a_n)$ is again an $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideal.

In two variables, where $\mathbb{A} = \{a, b\}$, the residual of an $LPP(\mathbb{A})$-ideal inside the $c.i.(a, b)$ is always a lex ideal, namely

$$\begin{aligned} &\langle x^a, y^b\rangle : \langle x^a, x^{a-1}y^{d_1}, x^{a-2}y^{d_2}, \ldots, x^{a-s}y^{d_s}, y^b\rangle \\ &\quad = \langle x^s, x^{s-1}y^{b-d_s}, \ldots, xy^{b-d_2}, y^{b-d_1}\rangle. \end{aligned}$$

As before, we associate to the $LPP(\mathbb{A})$-ideal

$$\langle x^a, x^{a-1}y^{d_1}, x^{a-2}y^{d_2}, \ldots, x^{a-s}y^{d_s}, y^b\rangle$$

the $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector $\mathcal{T} = (d_1, \ldots, d_s, b, , \ldots, b)$, where there are $a - s$ b's, so that the length of $\mathcal{T}$ is a. Then we associate to the residual lex ideal the 2-type vector $(b - d_s, \ldots, b - d_1)$. We can use monomial lifting (see [4, Theorem 2.2]) to associate a finite set of points to each of these ideals. The set of points obtained from the lex ideal in this way is an example of a k-configuration. From the lpp ideal, we obtain the complement of the k-configuration in the $c.i.(a, b)$; this complementary set of points is an example of a weak k-configuration, as defined in [6, Definition 2.8]. In fact, $\mathrm{lpp}_{\leq}(\{a, b\})$-vectors are exactly the "types" of weak k-configurations that occur in Theorem 2.10 of their paper. It was this fact that motivated the definition of $\mathrm{lpp}_{\leq}(\{a, b\})$-vectors and the generalization to larger numbers of variables.

EXAMPLE 5.1. The following ideal is LPP(5,7): $I = \langle x^5, x^4y, x^3y^3, x^2y^4, y^7\rangle$. We associate to I the $\mathrm{lpp}_{\leq}(\{5, 7\})$-vector $(1, 3, 4, 7, 7)$. Then inside a $c.i.(5, 7)$, we draw a weak k-configuration of type $(1, 3, 4, 7, 7)$:

$$\begin{array}{ccccccc} \bullet & \circ & \circ & \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

In this case, the complement of the weak k-configuration is a k-configuration of type $(3, 4, 6)$.

The fact that the residual of an $LPP\{a,b\}$-ideal in the $c.i.(a,b)$ is a lex ideal provides a proof of the LPP conjecture in two variables (for another proof, see [14, Theorems 5.1 and 5.2]). Since x^a, y^b are never minimal generators of the residual lex ideal, the resolution of the lex plus powers ideal obtained from dualizing the minimal free resolution of the lex ideal is in fact minimal (see page 154 of [12]). Hence, since lex ideals have extremal resolutions, it follows that the lex plus powers ideals have extremal resolutions among all ideals containing an $\{a,b\}$-regular sequence.

In more than two variables, this argument does not work for several reasons. First, the generators of the complete intersection might be generators of the residual ideal; second, even if they were not, we would not be guaranteed that the resolution obtained by dualizing was minimal, and third, the residual of an $LPP(\mathbb{A})$-ideal is no longer necessarily a lex ideal.

In this section, however, we show that the residual of an $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideal is necessarily another $\mathrm{lpp}_{\leq}(\mathbb{A})$-ideal. Given an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector $\mathcal{T}$, we want to define a residual $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector $\mathcal{T}^*$. As the original definition was motivated by considering points, we observe the following in passing: we can associate to $W_{\mathcal{T}}$, and hence to $\mathcal{T}$, a natural set of points $\mathbb{X}$ in $\mathbb{P}^n$ contained in a complete intersection of type $(a_1,\dots,a_n)$ obtained by lifting the monomial ideal $W_{\mathcal{T}}$. It will turn out that $\mathcal{T}^*$ is defined such that lifting $W_{\mathcal{T}^*}$ yields $\mathbb{X}^c$ where the complement is taken in $c.i.(a_1,\dots,a_n)$.

DEFINITION 5.2. If $n=1$, so that $\mathcal{T}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector (d), $d \leq a_1$, then $\mathcal{T}^* := (a_1 - d)$ if $d < a_1$; otherwise, we define $\mathcal{T}^* = \emptyset$.

If $\mathcal{T}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector $(\mathcal{T}_1,\dots,\mathcal{T}_u)$ and if $u < a_1$, then

$$\mathcal{T}^* := \big((\mathcal{T}_u)^*, \dots, (\mathcal{T}_1)^*, \mathcal{T}_{c.i.(\mathbb{A}_2)}, \dots, \mathcal{T}_{c.i.(\mathbb{A}_2)}\big),$$

where there are $a_1 - u$ $\mathcal{T}_{c.i.(\mathbb{A}_2)}$'s; otherwise, $\mathcal{T}^* = ((\mathcal{T}_u)^*, \dots, (\mathcal{T}_1)^*)$. In particular, $(\mathcal{T}_u)^* = (\mathcal{T}^*)_1$ unless $(\mathcal{T}_u)^* = \emptyset$.

REMARK 5.3. Note that we can define $l(\mathcal{T}^*)$, $\alpha_{\mathbb{A}}(\mathcal{T}^*)$, and $\sigma(\mathcal{T}^*)$, just as we defined these parameters for $\mathcal{T}$, even before knowing that $\mathcal{T}^*$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector; we also put $\alpha(\emptyset) = \sigma(\emptyset) = 0$. Furthermore, if $\mathcal{T} \neq \mathcal{T}_{c.i.(\mathbb{A})}$, then we can perform the same operation on $\mathcal{T}^*$ as we did on $\mathcal{T}$ to obtain $\mathcal{T}^*$, and we get $\mathcal{T}$ back. We write this as $(\mathcal{T}^*)^* = \mathcal{T}$. As well, it is clear that $l(\mathcal{T}) < a_1 \Leftrightarrow (\mathcal{T}^*)_{l(\mathcal{T}^*)} = \mathcal{T}_{c.i.(\mathbb{A}_2)}$.

We want to show that if $\mathcal{T}$ is an $\mathrm{lpp}_{\leq}(\mathbb{A})$-vector, then so is $\mathcal{T}^*$.

NOTATION. In what follows, we remove the subscript $\mathbb{A}$ from the α notation and assume it to be understood. So we write $\alpha(\mathcal{T})$ for $\alpha_{\mathbb{A}}(\mathcal{T})$, $\alpha(\mathcal{T}_i)$ for $\alpha_{\mathbb{A}_2}(\mathcal{T}_i)$, $\alpha((\mathcal{T}_i)_j)$ for $\alpha_{\mathbb{A}_3}((\mathcal{T}_i)_j)$, etc., assuming the subscript is understood.

LEMMA 5.4. *Let $\mathcal{T}$ be an* $\mathrm{lpp}_{\leq}(\mathbb{A})$*-vector, $\mathcal{T} \neq \mathcal{T}_{c.i.(\mathbb{A})}$. Then*

$$\alpha(\mathcal{T}) + \sigma(\mathcal{T}^*) = \sigma(c.i.(a_1,\dots,a_n)) = \sigma(\mathcal{T}) + \alpha(\mathcal{T}^*).$$

Proof. When $n = 1$, the result is trivial. To show that $\alpha(\mathcal{T}) + \sigma(\mathcal{T}^*) = \sigma(c.i.(\mathbb{A}))$, we consider the two cases $l(\mathcal{T}) < a_1$ and $l(\mathcal{T}) = a_1$.

If $l(\mathcal{T}) < a_1$, then $(\mathcal{T}^*)_{l(\mathcal{T}^*)} = \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$. Then,

$$\alpha(\mathcal{T}) + \sigma(\mathcal{T}^*) = l(\mathcal{T}) + \sigma(c.i.(a_2,\ldots,a_n)) + s - 1,$$

where $s = \#\{i | (\mathcal{T}^*)_i = \mathcal{T}_{c.i.(\mathbb{A}_2)}\}$. But $l(\mathcal{T}) + s = a_1$, so

$$\alpha(\mathcal{T}) + \sigma(\mathcal{T}^*) = a_1 + \sigma(c.i.(a_2,\ldots,a_n)) - 1 = \sigma(c.i.(a_1,\ldots,a_n)).$$

If $l(\mathcal{T}) = a_1$, then $(\mathcal{T}^*)_{l(\mathcal{T}^*)} \neq \mathcal{T}_{c.i.(\mathbb{A}_2)}$. So

$$\alpha(\mathcal{T}) + \sigma(\mathcal{T}^*) = a_1 + \alpha(\mathcal{T}_1) - 1 + \sigma\big((\mathcal{T}^*)_{l(\mathcal{T}^*)}\big).$$

But $(\mathcal{T}^*)_{l(\mathcal{T}^*)} = (\mathcal{T}_1)^*$, so

$$\alpha(\mathcal{T}_1) + \sigma\big((\mathcal{T}^*)_{l(\mathcal{T}^*)}\big) = \sigma(c.i.(a_2,\ldots,a_n))$$

by induction, so

$$\alpha(\mathcal{T}) + \sigma(\mathcal{T}^*) = a_1 - 1 + \sigma(c.i.(a_2,\ldots,a_n)) = \sigma(c.i.(a_1,\ldots,a_n)).$$

To show that $\sigma(\mathcal{T}) + \alpha(\mathcal{T}^*) = \sigma(c.i.(a_1,\ldots,a_n))$, we consider the two cases $\mathcal{T}_{l(\mathcal{T})} = \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$ and $\mathcal{T}_{l(\mathcal{T})} \neq \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$.

If $\mathcal{T}_{l(\mathcal{T})} = \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$, then $l(\mathcal{T}^*) < a_1$, so $\alpha(\mathcal{T}^*) = l(\mathcal{T}^*)$. Furthermore, $\sigma(\mathcal{T}) = \sigma(c.i.(\mathbb{A}_2)) + s - 1$, where s is the number of integers i such that $\mathcal{T}_i = \mathcal{T}_{c.i.(\mathbb{A}_2)}$. But $l(\mathcal{T}^*) + s = a_1$, so

$$\alpha(\mathcal{T}^*) + \sigma(\mathcal{T}) = a_1 + \sigma(c.i.(\mathbb{A}_2)) - 1 = \sigma(c.i.(\mathbb{A})).$$

If $\mathcal{T}_{l(\mathcal{T})} \neq \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$, then $l(\mathcal{T}^*) = a_1$. So

$$\alpha(\mathcal{T}^*) = a_1 + \alpha_{\mathbb{A}_2}((\mathcal{T}^*)_1) - 1 = a_1 + \alpha_{\mathbb{A}_2}\big(\big(\mathcal{T}_{l(\mathcal{T})}\big)^*\big) - 1,$$

since $(\mathcal{T}_{l(\mathcal{T})})^* \neq \emptyset$. Furthermore, $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_{l(\mathcal{T})})$. By the induction hypothesis, $\alpha_{\mathbb{A}_2}((\mathcal{T}_{l(\mathcal{T})})^*) + \sigma(\mathcal{T}_{l(\mathcal{T})}) = \sigma(c.i.(\mathbb{A}_2))$, so

$$\sigma(\mathcal{T}) + \alpha(\mathcal{T}^*) = a_1 + \sigma(c.i.(\mathbb{A}_2) - 1 = \sigma(c.i.(\mathbb{A})). \qquad \square$$

THEOREM 5.5. *Let $\mathcal{S}$ and $\mathcal{T}$ be* $\mathrm{lpp}_{\leq}(\mathbb{A})$*-vectors, $\mathcal{S} \neq \mathcal{T}_{c.i.(\mathbb{A})}$. Then $\sigma(\mathcal{S}) < \alpha(\mathcal{T}) \Rightarrow \sigma(\mathcal{T}^*) < \alpha(\mathcal{S}^*)$.*

Proof. We consider several cases below.

Case 1: Suppose $\mathcal{S}_{l(\mathcal{S})} = \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$ and $l(\mathcal{T}) < a_1$. Then $l(\mathcal{S}^*) < a_1$ and $(\mathcal{T}^*)_{l(\mathcal{T}^*)} = \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$. Then since $\sigma(\mathcal{S}) < \alpha(\mathcal{T})$, we have $\sigma(\mathcal{S}_{l(\mathcal{S})}) + s - 1 < l(\mathcal{T})$ where $s = \#\{i | \mathcal{S}_i = \mathcal{S}_{l(\mathcal{S})}\}$. Now,

$$\sigma(\mathcal{T}^*) < \alpha(\mathcal{S}^*) \quad \Leftrightarrow \quad \sigma\big((\mathcal{T}^*)_{l(\mathcal{T}^*)}\big) + t - 1 < l(\mathcal{S}^*),$$

where $t = \#\{i | (\mathcal{T}^*)_i = (\mathcal{T}^*)_{l(\mathcal{T}^*)}\}$. But,

$$\sigma\big(\mathcal{S}_{l(\mathcal{S})}\big) = \sigma\big(\mathcal{T}^*_{l(\mathcal{T}^*)}\big) = \sigma(c.i.(a_2,\ldots,a_n)),$$

so it is enough to show that $l(\mathcal{T}) - s = l(\mathcal{S}^*) - t$. But, $l(\mathcal{T}) + t = a_1 = l(\mathcal{S}^*) + s$, so we are done in this case.

Case 2: Suppose $\mathcal{S}_{l(\mathcal{S})} = \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$ and $l(\mathcal{T}) = a_1$. Then $l(\mathcal{S}^*) < a_1$ and $(\mathcal{T}^*)_{l(\mathcal{T}^*)} \neq \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$. So, $\sigma(\mathcal{T}^*) = \sigma((\mathcal{T}^*)_{l(\mathcal{T}^*)})$ and $\alpha(\mathcal{S}^*) = l(\mathcal{S}^*)$. We need to show that $\sigma((\mathcal{T}^*)_{l(\mathcal{T}^*)}) < l(\mathcal{S}^*)$. If $\mathcal{T} = \mathcal{T}_{c.i.(\mathbb{A})}$ this is obvious, so we may assume that $\mathcal{T} \neq \mathcal{T}_{c.i.(\mathbb{A})}$. Note that $l(\mathcal{S}^*) = a_1 - s$ where $s = \#\{i | \mathcal{S}_i = \mathcal{S}_{l(\mathcal{S})}\}$. Since $\sigma(\mathcal{S}) < \alpha(\mathcal{T})$, we have $\sigma(\mathcal{S}_{l(\mathcal{S})}) + s - 1 < l(\mathcal{T}) + \alpha(\mathcal{T}_1) - 1$. Note that $\alpha(\mathcal{T}_1) < \infty$ since $\mathcal{T} \neq \mathcal{T}_{c.i.(\mathbb{A})}$ implies $\mathcal{T}_1 \neq \mathcal{T}_{c.i.(\mathbb{A}_2)}$ when $l(\mathcal{T}) = a_1$. Thus, we may rewrite the inequality to obtain $\sigma(\mathcal{S}_{l(\mathcal{S})}) - \alpha(\mathcal{T}_1) < l(\mathcal{T}) - s = a_1 - s$. But $(\mathcal{T}_1)^* = (\mathcal{T}^*)_{l(\mathcal{T}^*)}$ since $l(\mathcal{T}) = a_1$ hence by Lemma 5.4, $\sigma((\mathcal{T}^*)_{l(\mathcal{T}^*)}) = \sigma(c.i.(\mathbb{A}_2)) - \alpha(\mathcal{T}_1) < a_1 - s = l(S^*)$, as required.

Case 3: Suppose $\mathcal{S}_{l(\mathcal{S})} \neq \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$ and $l(\mathcal{T}) < a_1$. Then $l(\mathcal{S}^*) = a_1$ and $(\mathcal{T}^*)_{l(\mathcal{T}^*)} = \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$. Let $s = \#\{i | (\mathcal{T}^*)_i = (\mathcal{T}^*)_{l(\mathcal{T}^*)}\}$. We need to show that $\sigma((\mathcal{T}^*)_{l(\mathcal{T}^*)}) + s - 1 < a_1 + \alpha((\mathcal{S}^*)_1) - 1$ or in other words, $\sigma(c.i.(a_2,\ldots,a_n)) - \alpha((\mathcal{S}^*)_1) < a_1 - s = l(\mathcal{T})$. But $l(\mathcal{S}^*) = a_1$, so $((\mathcal{S}^*)_1)^* = \mathcal{S}_{l(\mathcal{S})}$ and hence $(\mathcal{S}^*)_1 = (\mathcal{S}_{l(\mathcal{S})})^*$. So by Lemma 5.4 applied to $\mathcal{S}_{l(\mathcal{S})}$, the left-hand side of this last inequality is $\sigma(\mathcal{S}_{l(\mathcal{S})})$. But $\sigma(\mathcal{S}_{l(\mathcal{S})}) = \sigma(\mathcal{S}) < \alpha(\mathcal{T}) = l(\mathcal{T})$, so we are done in this case.

Case 4: Suppose $\mathcal{S}_{l(\mathcal{S})} \neq \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$ and $l(\mathcal{T}) = a_1$. Then, $l(\mathcal{S}^*) = a_1$ and $(\mathcal{T}^*)_{l(\mathcal{T}^*)} \neq \mathcal{T}_{c.i.(a_2,\ldots,a_n)}$. Since $\sigma(\mathcal{S}) < \alpha(\mathcal{T})$, we have $\sigma(\mathcal{S}_{l(\mathcal{S})}) < a_1 + \alpha(\mathcal{T}_1) - 1$. We need to show that $\sigma(\mathcal{T}^*) < \alpha(\mathcal{S}^*)$, in other words, $\sigma((\mathcal{T}^*)_{l(\mathcal{T}^*)}) < a_1 + \alpha((\mathcal{S}^*)_1) - 1$. Note that if $\mathcal{T}_1 = \mathcal{T}_{c.i.(\mathbb{A}_2)}$, then $\sigma((\mathcal{T}^*)_{l(\mathcal{T}^*)}) = 0$ while $\alpha((\mathcal{S}^*)_1) > 0$, so we may assume $\mathcal{T}_1 \neq \mathcal{T}_{c.i.(\mathbb{A}_2)}$. It is enough to show that

$$\sigma\big(\mathcal{S}_{l(\mathcal{S})}\big) - \alpha(\mathcal{T}_1) = \sigma\big((\mathcal{T}^*)_{l(\mathcal{T}^*)}\big) - \alpha((\mathcal{S}^*)_1),$$

in other words, that

$$\sigma\big(\mathcal{S}_{l(\mathcal{S})}\big) + \alpha((\mathcal{S}^*)_1) = \sigma\big((\mathcal{T}^*)_{l(\mathcal{T}^*)}\big) + \alpha(\mathcal{T}_1).$$

But $l(\mathcal{S}^*) = a_1$, so $(\mathcal{S}^*)_1 = (\mathcal{S}_{l(\mathcal{S})})^*$ and $l(\mathcal{T}) = a_1$, so $(\mathcal{T}_1)^* = (\mathcal{T}^*)_{l(\mathcal{T}^*)}$. Thus by Lemma 5.4 applied to $\mathcal{S}_{l(\mathcal{S})}$ and $\mathcal{T}_1$,

$$\sigma\big(\mathcal{S}_{l(\mathcal{S})}\big) + \alpha((\mathcal{S}^*)_1) = \sigma(c.i.(a_2,\ldots,a_n)) = \sigma\big((\mathcal{T}^*)_{l(\mathcal{T}^*)}\big) + \alpha(\mathcal{T}_1),$$

as required. □

Corollary 5.6. *If $\mathcal{T}$ is an* $\mathrm{lpp}_{\le}(\mathbb{A})$*-vector, $\mathcal{T} \neq \mathcal{T}_{c.i.(\mathbb{A})}$, then so is $\mathcal{T}^*$.*

Proof. Let $\mathbb{A} = \{a_1,\ldots,a_n\}$. If $n = 1$, the result is obvious, so assume $n > 1$ and let $\mathcal{T} = (\mathcal{T}_1,\ldots,\mathcal{T}_u)$ be an $\mathrm{lpp}_{\le}(\mathbb{A})$-vector so that $u \le a_1$, $u \le l(\mathcal{T}_u)$, each $\mathcal{T}_i$ is an $\mathrm{lpp}_{\le}(\mathbb{A}_2)$-vector and $\sigma(\mathcal{T}_i) < \alpha_{\mathbb{A}_2}(\mathcal{T}_{i+1})$ for $1 \le i \le u-1$. Then

$$\mathcal{T}^* = \big((\mathcal{T}_u)^*,\ldots,(\mathcal{T}_1)^*,\mathcal{T}_{c.i.(\mathbb{A}_2)},\ldots,\mathcal{T}_{c.i.(\mathbb{A}_2)}\big),$$

where there are $a_1 - u$ (possibly 0) $\mathcal{T}_{c.i.(\mathbb{A}_2)}$'s. By the induction hypothesis, each $\mathcal{T}_i^*$ is an $\mathrm{lpp}_{\le}(\mathbb{A}_2)$-vector and $l(\mathcal{T}^*) \le a_1$ by construction. To see that $l(\mathcal{T}^*) \le l((\mathcal{T}^*)_{l(\mathcal{T}^*)})$, we consider two cases.

Case 1: $u < a_1$. Then $(\mathcal{T}^*)_{l(\mathcal{T}^*)} = \mathcal{T}_{c.i.(\mathbb{A}_2)}$ and $l((\mathcal{T}^*)_{l(\mathcal{T}^*)}) = a_2 \ge a_1 \ge l(\mathcal{T}^*)$.

Case 2: $u=a_1$. Let $\mathcal{T}=(\mathcal{T}_1,\ldots,\mathcal{T}_u)=(\mathcal{T}_1,\ldots,\mathcal{T}_s,\mathcal{T}_{c.i.(\mathbb{A}_2)},\ldots,\mathcal{T}_{c.i.(\mathbb{A}_2)})$ where $\mathcal{T}_s\neq\mathcal{T}_{c.i.(\mathbb{A}_2)}$ and $s\leq u=a_1$. Then $l(\mathcal{T}^*)=s$ and $(\mathcal{T}^*)_{l(\mathcal{T}^*)}=(\mathcal{T}_1)^*$, so we need to show that $l((\mathcal{T}_1)^*)\geq s$. First note that since $\sigma(\mathcal{T}_1)<\alpha_{\mathbb{A}_2}(\mathcal{T}_2)\leq\sigma(\mathcal{T}_2)<\cdots<\alpha_{\mathbb{A}_2}(\mathcal{T}_s)\leq\sigma(\mathcal{T}_s)$, we have $\sigma(\mathcal{T}_1)\leq\sigma(\mathcal{T}_s)-s+1\leq\sigma(c.i.(\mathbb{A}_2))-1-s+1=\sigma(c.i.(\mathbb{A}_2))-s$.

If $n=2$, then $\mathcal{T}_1=\sigma(\mathcal{T}_1)\leq\sigma(c.i.(\mathbb{A}_2))-s=a_2-s$, so $s\leq a_2-\mathcal{T}_1=(\mathcal{T}_1)^*=l((\mathcal{T}_1)^*)$ as required. If $n>2$, then let t be the number of $\mathcal{T}_{c.i.(\mathbb{A}_3)}$'s in $\mathcal{T}_1$. If $t=0$, then $l((\mathcal{T}_1)^*)=a_2$, and $a_2\geq a_1\geq s$, so $l((\mathcal{T}_1)^*)\geq s$ as required. Otherwise, $\sigma(\mathcal{T}_1)=\sigma(c.i.(\mathbb{A}_3))+t-1\leq\sigma(c.i.(\mathbb{A}_2))-s=\sigma(c.i.(\mathbb{A}_3))+a_2-1-s$. So, $t\leq a_2-s$; that is, $s\leq a_2-t=l((\mathcal{T}_1)^*)$, as required.

Thus, it only remains to prove that $\sigma((\mathcal{T}^*)_i)<\alpha((\mathcal{T}^*)_{i+1})$ for all $i=1,\ldots,l(\mathcal{T}^*)-1$, that is, that $\sigma((\mathcal{T}_i)^*)<\alpha((\mathcal{T}_{i-1})^*)$ for $i=2,\ldots,m$, where m is the largest index such that $T_m\neq\mathcal{T}_{c.i.(\mathbb{A})}$, but this is the content of Theorem 5.5. $\square$

THEOREM 5.7. *Suppose that $L\subsetneq\langle x_1^{a_1},\ldots,x_n^{a_n}\rangle$ is an LPP($\mathbb{A}$)-ideal. Then $\langle x_1^{a_1},\ldots,x_n^{a_n}\rangle:L$ is* $\text{lpp}_{\leq}(\mathbb{A})$.

REMARK 5.8. Chris Francisco has also discovered a (quite different) proof of this result.

Proof. By Corollary 4.21, there is an $\text{lpp}_{\leq}(\mathbb{A})$-vector $\mathcal{T}$ such that $W_{\mathcal{T}}=L$. By Corollary 5.6 and Theorem 4.12, it is enough to show that $W_{\mathcal{T}^*}=\langle x_1^{a_1},\ldots,x_n^{a_n}\rangle:W_{\mathcal{T}}$. Proceed by induction on n. If $n=1$, then $\mathcal{T}=(d)$ for $d<a$, $\mathcal{T}^*=(a_1-d)$, $W_{\mathcal{T}}=(x_1^d)$, and $W_{\mathcal{T}^*}=(x_1^{a_1-d})$, so clearly $W_{\mathcal{T}^*}=\langle x_1^{a_1}\rangle:W_{\mathcal{T}}$.

Now let $n>1$. There are four cases to consider: $L(\mathcal{T})=a_1$ and $\mathcal{T}_{l(\mathcal{T})}=\mathcal{T}_{c.i.(\mathbb{A}_2)}$, $L(\mathcal{T})=a_1$ and $\mathcal{T}_{l(\mathcal{T})}\neq\mathcal{T}_{c.i.(\mathbb{A}_2)}$, $L(\mathcal{T})<a_1$ and $\mathcal{T}_{l(\mathcal{T})}=\mathcal{T}_{c.i.(\mathbb{A}_2)}$, and $L(\mathcal{T})<a_1$ and $\mathcal{T}_{l(\mathcal{T})}\neq\mathcal{T}_{c.i.(\mathbb{A}_2)}$. The proof is similar in each case, so we provide only the first instance as a representative.

Suppose

$$\mathcal{T}=(\mathcal{T}_1,\ldots,\mathcal{T}_u,\ldots,\mathcal{T}_u),$$

where there are $v+1$ $\mathcal{T}_u=\mathcal{T}_{c.i.(\mathbb{A}_2)}$'s and $l(\mathcal{T})=u+v=a_1$. Then

$$\begin{aligned}\mathcal{T}^*&=((\mathcal{T}_{u-1})^*,\ldots,(\mathcal{T}_1)^*),\\ W_{\mathcal{T}}&=(x_1^{u+v},x_1^{u+v-1}\overline{W_{\mathcal{T}_1}},\ldots,x_1^{v+1}\overline{W_{\mathcal{T}_{u-1}}},\overline{W_{\mathcal{T}_u}})\\ &=\big(x_1^{u+v},x_1^{u+v-1}\overline{W_{\mathcal{T}_1}},\ldots,x_1^{v+1}\overline{W_{\mathcal{T}_{u-1}}},\overline{W_{\mathcal{T}_{c.i.(\mathbb{A}_2)}}}\big),\end{aligned}$$

and

$$W_{\mathcal{T}^*}=\big(x_1^{u-1},x_1^{u-2}\overline{W_{(\mathcal{T}_{u-1})^*}},\ldots,x_1\overline{W_{(\mathcal{T}_2)^*}},\overline{W_{(\mathcal{T}_1)^*}}\big).$$

We first show that $W_{\mathcal{T}^*}\subseteq\langle x_1^{a_1},\ldots,x_n^{a_n}\rangle:W_{\mathcal{T}}$. Let let m be a minimal monomial generator of $W_{\mathcal{T}^*}$. If $m=x_1^{u-1}$, then the result is clear, so suppose that $m=x_1^{u-j}m'$ where $m'\in\overline{W_{(\mathcal{T}_{u-j+1})^*}}$. We need to show that $mx_1^{v+j-i}\times$

$\overline{W_{\mathcal{T}_{u-(j-i)}}} \subseteq \langle x_1^{a_1}, \ldots, x_n^{a_n}\rangle$ for $i = 1, \ldots, j-1$. By the induction hypothesis, $\overline{W_{(\mathcal{T}_{v-j+1})^*}} = \langle x_2^{a_2}, \ldots, x_n^{a_n}\rangle : \overline{W_{\mathcal{T}_{v-j+1}}}$, so applying Lemma 4.16, we have $\langle x_1^{a_1}, \ldots, x_1^{a_n}\rangle \supseteq m x_1^{v+j-1}\overline{W_{\mathcal{T}_{u-j+1}}} \supseteq m x_1^{v+j-2}\overline{W_{\mathcal{T}_{u-j+2}}} \supseteq \cdots \supseteq m x_1^{v+1}\overline{W_{\mathcal{T}_{u-1}}}$ as required.

Now suppose that m is a minimal monomial generator in $\langle x_1^{a_1}, \ldots, x_n^{a_n}\rangle :$ $W_{\mathcal{T}}$. If $m = x_1^j$, then $m x_1^{v+1}\overline{W_{\mathcal{T}_{u-1}}}$ forces $j = u-1$. So write $m = x_1^{u-1-j}m'$ where $1 \le j \le u-1$ and $m' \in k[x_2, \ldots, x_n]$. Since

$$m x_1^{v+j}\overline{W_{\mathcal{T}_{u-j}}} \subseteq \langle x_1^{a_1}, \ldots, x_n^{a_n}\rangle,$$

it must be that $m' \in \langle x_2^{a_2}, \ldots, x_n^{a_n}\rangle : \overline{W_{\mathcal{T}_{u-j}}} = \overline{W_{(\mathcal{T}_{u-j})^*}}$ and hence $m \in W_{\mathcal{T}^*}$ as required. □

6. Applications of the theorem for colon ideals

The fact that the residual of a lex plus powers ideal is again lex plus powers allows us to prove the (moral) converse to the following theorem in [14].

Theorem 6.1. *Let L be $LPP(\mathbb{A})$ for some $\mathbb{A} = \{a_1, \ldots, a_n\}$, and I be an ideal containing an $\mathbb{A}$-regular sequence such that $H(R/L) = H(R/I)$. If EGH holds, then* $\dim_k(\operatorname{soc}(L)_d) \ge \dim_k(\operatorname{soc}(I)_d)$ *for all d, where* $\operatorname{soc}(L)_d$ *refers to the dth graded piece of the socle of R/L (and similarly for I).*

We will here demonstrate that if lex plus powers ideals can be shown to have always largest socles, the EGH must be true. More precisely, we will prove that EGH is equivalent to the following conjecture.

Conjecture 6.2. *Suppose that L is $LPP(\mathbb{A})$ for some $\mathbb{A} = \{a_1, \ldots, a_n\}$, and I is an ideal containing an $\mathbb{A}$-regular sequence such that $H(R/L) = H(R/I)$. Then $\beta^L_{n,j} \ge \beta^I_{n,j}$ for all j.*

The proof of the equivalence will require a few lemmas and a proposition. We give the following comments to motivate these preliminary results. Suppose that L is $LPP(\mathbb{A})$ with $\underline{\mathbf{x}} = \{x_1^{a_1}, \ldots, x_n^{a_n}\}$, I contains an $\mathbb{A}$-regular sequence $\underline{\mathbf{y}} = \{y_1^{a_1}, \ldots, y_n^{a_n}\}$, and $H(R/L) = H(R/I)$. Our goal is to compare the socles of $(\underline{\mathbf{x}} : L)$ and $(\underline{\mathbf{y}} : I)$ (via Conjecture 6.2) and transfer this comparison to a comparison of the first graded Betti numbers of L and I. By Theorem 5.7, we know that $(\underline{\mathbf{x}} : L)$ is again a lex plus powers ideal, so Conjecture 6.2 will apply if we can demonstrate that $(\underline{\mathbf{y}} : I)$ contains a regular sequence in the same degrees as those of the minimal monomial regular sequence in $(\underline{\mathbf{x}} : L)$ (note that the Hilbert functions of the two colon ideals are obviously equal). This follows from the lemmas below. We first prove (Lemma 6.3) that if L is $LPP(\mathbb{A})$, then the degrees of the minimal monomial regular sequence in the residual can only drop in degrees for which the colon consists of a lex segment. We then use this fact to show (Lemma 6.4) that $(\underline{\mathbf{y}} : I)$ contains a regular sequence in the degrees of the minimal monomial regular sequence in $(\underline{\mathbf{x}} : L)$. Proposition 6.5 then allows us to compute the first graded Betti numbers of L and I from the socle degrees of $(\underline{\mathbf{x}} : L)$ and $(\underline{\mathbf{y}} : I)$, respectively. After these preparations, we will be able to prove the theorem.

LEMMA 6.3. *Let L be an $\{a_1, \dots, a_n\}$-lex plus powers ideal and $\underline{\mathbf{x}}$ be the ideal generated by $\{x_1^{a_1}, \dots, x_n^{a_n}\}$. If $(\underline{\mathbf{x}} : L)$ is an $\{a'_1, \dots, a'_n\}$-lex plus powers ideal with $a'_s < a_s$ for some $1 \le s \le n$, then $(\underline{\mathbf{x}} : L)_{a'_s}$ is a lex segment.*

Proof. Note that if $a'_i = a'_s$ for some $i > s$, then $a'_i = a'_s < a_s \le a_i$, so we can assume without harm that $a'_s < a'_{s+1}$ or $s = n$. It follows that if $m \in (\underline{\mathbf{x}} : L)_{a'_s}$ and $m < x_s^{a'_s}$, then m is not a pure power. Because $x_s^{a'_s}$ is a minimal generator, m must be a minimal generator as well, and thus it is part of the lex segment of $(\underline{\mathbf{x}} : L)_{a'_s}$.

So it is enough to show that if $m \in R_{a'_s}$ and $m > x_s^{a'_s}$, then $m \in (\underline{\mathbf{x}} : L)$. Note that $s > 1$ (otherwise we are finished). If $m \notin (\underline{\mathbf{x}} : L)$, then there is a minimal monomial generator $\lambda \in L$ such that $m\lambda \notin \underline{\mathbf{x}}$. It follows that $m(i) + \lambda(i) < a_i$ for all $i = 1, \dots, n$. Now, since $x_s^{a'_s} \in (\underline{\mathbf{x}} : L)$, we have $\lambda x_s^{a'_s} \in \langle \underline{\mathbf{x}} \rangle$, and so $\lambda(s) + a'_s \ge a_s$. In particular, this implies that $\lambda(s) > 0$. If $\deg \lambda = d$, then since $\lambda(i) < a_i$ for all i, λ is part of the lex segment of L_d, and thus if $\lambda' \in R_d$ and $\lambda' > \lambda$, then $\lambda' \in L_d$ as well.

Now let $t < s$ be such that $m(t) > 0$ (such an element exists because $m > x_s^{a'_s}$) and consider the element

$$\lambda' = x_1^{\lambda(1)+\gamma(1)} \cdots x_{s-1}^{\lambda(s-1)+\gamma(s-1)} x_{s+1}^{\lambda(s+1)+\gamma(s+1)} \cdots x_n^{\lambda(n)+\gamma(n)},$$

where the $\gamma(i)$ for $i \ne s$ are any choice of elements of $\mathbb{N}$ such that $\sum_{i \ne s} \gamma(i) = \lambda(s)$, $\gamma(t) \ge 1$, and $\gamma(i) \le m(i)$ for all $i \ne s$. Such a choice of $\gamma(i)$ is possible unless $\lambda(s) = \sum_{i \ne s} \gamma(i) > \sum_{i \ne s} m(i) = \deg(m) - m(s) = a'_s - m(s)$ in which case $\lambda(s) + m(s) > a'_s$, a contradiction. The existence of such a γ, however, also gives a contradiction. Because $\lambda' > \lambda$, we have that $x_s^{a'_s} \lambda' \in \underline{\mathbf{x}}$. But $\lambda'(s) = 0$ and $a'_s < a_s$, so for some $i \ne s$, $a_i \le \lambda(i) + \gamma(i) \le \lambda(i) + m(i)$. $\square$

LEMMA 6.4. *Suppose that I minimally contains an $\mathbb{A} = \{a_1, \dots, a_n\}$-regular sequence $\underline{\mathbf{y}}$, $H(R/I)$ is $\mathbb{A}$-lpp valid, and let L be the $\mathbb{A}$-lex plus powers ideal such that $H(R/I) = H(R/L)$. If $(\underline{\mathbf{x}} : L)$ is $\{a'_1, \dots, a'_n\}$-lex plus powers, then $(\underline{\mathbf{y}} : I)$ contains an $\{a'_1, \dots, a'_n\}$-regular sequence.*

Proof. Let t be the smallest integer such that $(\underline{\mathbf{y}} : I)$ fails to contain an $\{a'_1, \dots, a'_t\}$-regular sequence. Thus, there is a $\{b_1, \dots, b_n\}$-regular sequence in $(\underline{\mathbf{x}} : I)$, such that $b_i \le a'_i$ for $1 \le i < t$, and $a'_t < b_t \le a_t$. We can choose $\{b_1, \dots, b_n\}$ such that b_t satisfies the second inequality because $(\underline{\mathbf{y}} : I)$ contains an $\{a_1, \dots, a_n\}$-regular sequence by construction and thus certainly contains an $\{a'_1, \dots, a'_{t-1}, a_t, a_{t+1}, \dots, a_n\}$-regular sequence. By Lemma 6.3, $a'_t < a_t$

implies that $(\underline{\mathbf{x}} : L)_{a'_t}$ is a lex segment. Consider then the ideals $(\underline{\mathbf{x}} : L)_{a'_t} + \langle x_1, \ldots, x_n\rangle^{a'_t+1}$ and $(\underline{\mathbf{y}} : I)_{a'_t} + \langle x_1, \ldots, x_n\rangle^{a'_t+1}$. Both of these ideals attain the same Hilbert function, and the former is a lex ideal containing a regular sequence of length at least t in degree a'_t. It is not difficult to show (see, i.e., Corollary 2.13 in [14]) that all ideals attaining a given Hilbert function contain a regular sequence in the degrees of the minimal monomial regular sequence in the lex ideal with that Hilbert function. Thus, $(\underline{\mathbf{y}} : I)_{a'_t} + \langle x_1, \ldots, x_n\rangle^{a'_t+1}$ must also contain a regular sequence of length at least t by degree a'_t, that is, $(\underline{\mathbf{y}} : I)$ must contain a regular sequence in degrees $a_1, \ldots, a_t$, a contradiction. □

COROLLARY 6.5. *Let* $\mathbb{A} = \{a_1, \ldots, a_n\}$ *be a list of degrees, write* $|j|$ *to denote the number of elements of* $\mathbb{A}$ *equal to* j, *and suppose that* $\underline{\mathbf{y}}$ *is an* $\mathbb{A}$*-regular sequence in an ideal* $I \subset R$. *Then for all* j *there exist* $0 \le t_j \le |j|$ *such that* $\beta^{(\underline{\mathbf{y}}:I)}_{n,\omega-j} = \beta^I_{1,j} - t_j$. *Furthermore, if* $\underline{\mathbf{y}}$ *is minimally contained in* I, *then* $t_j = |j|$ *for all* j.

Proof. We suppose first that $\underline{\mathbf{y}}$ is minimally contained in I. Let $\mathcal{F}_\bullet$ be a minimal free resolution of R/I

$$\mathcal{F}_\bullet := 0 \to \sum_j R^{\beta^I_{n,j}}[-j] \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} \sum_j R^{\beta^I_{1,j}}[-j] \xrightarrow{\delta_1} R \to 0,$$

and $\mathcal{K}_\bullet$ be the Koszul complex

$$\mathcal{K}_\bullet = 0 \to R[-\omega] \xrightarrow{\partial_n} \sum_j R^{\beta^K_{n-1,j}}[-j] \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \sum_j R^{\beta^K_{1,j}}[-j] \xrightarrow{\partial_1} R \to 0$$

resolving $R/\underline{\mathbf{y}}$, where the $\beta^K_{i,j}$ are the Betti numbers of the Koszul complex resolving $R/\underline{\mathbf{y}}$ and $\omega = \sum a_i$. Note that $|j| = \beta^K_{1,j}$. The map $\phi : R/\underline{\mathbf{y}} \to R/I$ induces a chain map

$$\begin{tikzcd}
0 \arrow[r] & \sum_j R^{\beta^I_{n,j}}[-j] \arrow[r, "\delta_n"] & \sum_j R^{\beta^I_{n-1,j}}[-j] \arrow[r, "\delta_{n-1}"] & \cdots \\
0 \arrow[r] & R[-\omega] \arrow[u, "\phi_n"] \arrow[r, "\partial_n"] & \sum_j R^{\beta^K_{n-1,j}}[-j] \arrow[u, "\phi_{n-1}"] \arrow[r, "\partial_{n-1}"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_2"] & \sum_j R^{\beta^I_{1,j}}[-j] \arrow[r, "\delta_1"] & R \arrow[r] & 0 \\
\cdots \arrow[r, "\partial_2"] & \sum_j R^{\beta^K_{1,j}}[-j] \arrow[u, "\phi_1"] \arrow[r, "\partial_1"] & R \arrow[u, "\phi_0"] \arrow[r] & 0.
\end{tikzcd}$$

We know that $\phi_0 = 1_R$ by construction and that ϕ_1 is a rank n matrix (over k) all of whose entries are in k because $\underline{\mathbf{y}}$ is minimally contained in I. Let $\mathcal{E}_\bullet$ denote the mapping cone on the diagram induced by ϕ,

$$\mathcal{E}_\bullet := \ 0 \to R \xrightarrow{\psi_{n+1}} R^{\alpha_n^I} \oplus R^n \xrightarrow{\psi_n} \cdots \xrightarrow{\psi_2} R^{\alpha_1^I} \oplus R \xrightarrow{\psi_1} R \to 0,$$

where we have used α_j^I to denote the jth Betti number of R/I and have suppressed the graded notation at this step so that the resolution is more legible. The dual of $\mathcal{E}_\bullet$ is

$$\mathcal{E}_\bullet^* := \ 0 \to R \xrightarrow{\psi_1^*} R^{\alpha_1^I} \oplus R \xrightarrow{\psi_2^*} R^{\alpha_2^I} \oplus R^n \xrightarrow{\psi_3^*} \cdots \xrightarrow{\psi_4^*} R^{\alpha_n^I} \oplus R \xrightarrow{\psi_{n+1}^*} R \to 0,$$

and it is not difficult to show that $\mathcal{E}_\bullet^*$ is a free resolution of $R/(\underline{\mathbf{y}} : I)$. This resolution is never minimal, but we are able to identify the cause of the non-minimality in the $(n-1)$st, the nth, and the $(n+1)$st terms of $\mathcal{E}_\bullet^*$. In fact, the map ψ_1^* is just multiplication by 1_R (actually, 1_{R^*}) in the right coordinate, $\psi_1^*(m) = (0, m)$. This implies that the copy of R constituting $\mathcal{E}_{n+1}$ maps isomorphically onto the copy of R belonging to $\mathcal{F}_n$ in $\mathcal{E}_n^*$, and we may remove both from the resolution. So

$$\mathcal{E}_\bullet' := 0 \to R^{\alpha_1^I} \xrightarrow{\psi_2^*} R^{\alpha_2^I} \oplus R^n \xrightarrow{\psi_3^*} \cdots \xrightarrow{\psi_4^*} R^{\alpha_n^I} \oplus R \xrightarrow{\psi_{n+1}^*} R \to 0,$$

is a free resolution of $R/(\underline{\mathbf{y}} : I)$ where we abuse notation and reuse ψ_2^* to denote the restriction of ψ_2^* to $R^{\alpha_1^I}$.

Now, for $m \in R^{\alpha_1^I}$, $\psi_2^*(m) = (\delta_2^*(m), -\phi_1^*(m))$, and as we noted above ϕ_1 (and hence also ϕ_1^*) is a rank n matrix consisting of degree zero elements. Thus for each i, a copy of $R[-\omega + a_i]$ in $\mathcal{E}_n'$ maps isomorphically onto the copy of $R[-\omega + a_i]$ in $\mathcal{E}_{n-1}'$ (we remember the grading at this step). These pairs may be removed from $\mathcal{E}_\bullet'$, so write $\overline{\psi}_2^*$ to be the map given by restriction of ψ_2^* to $\sum_j R^{\beta_{1,j}^I - |j|}[-\omega + j]$, and $\overline{\psi}_3^*$ to be the restriction of ψ_3^* to $\sum_j R^{\beta_{2,j}^I}[-\omega + j]$. Thus

$$\mathcal{E}_\bullet'' := 0 \to \sum_j R^{\beta_{1,j}^I - |j|}[-\omega + j] \xrightarrow{\overline{\psi}_2^*} \sum_j R^{\beta_{2,j}^I}[-\omega + j] \xrightarrow{\overline{\psi}_3^*} \cdots \xrightarrow{\psi_{n+1}^*} R \to 0,$$

is a free resolution of R/I, and although it may fail to be minimal, no further cancellation can occur between $\sum_j R^{\beta_{1,j}^I - |j|}[-\omega + j]$ and $\sum_j R^{\beta_{2,j}^I}[-\omega + j]$. We conclude that $\beta_{n,\omega-j}^{(\underline{\mathbf{y}}:I)} = \beta_{1,j}^I - |j|$ as required.

In the case that I fails to minimally contain an $\mathbb{A}$-regular sequence, this argument needs only a small modification. The cyclic module $R/(\underline{\mathbf{y}} : I)$ can again be resolved using the dual of the mapping cone on $R/\underline{\mathbf{y}} \to R/I$, yielding

$$\mathcal{E}_\bullet^* := \ 0 \to R \xrightarrow{\psi_1^*} R^{\alpha_1^I} \oplus R \xrightarrow{\psi_2^*} R^{\alpha_2^I} \oplus R^n \xrightarrow{\psi_3^*} \cdots \xrightarrow{\psi_4^*} R^{\alpha_n^I} \oplus R \xrightarrow{\psi_{n+1}^*} R \to 0,$$

and we can again remove the extra copy of R which constitutes $\mathcal{E}_{n+1}^*$. The result then follows after noting that there are at most $|j|$ copies of $R[-\omega + j]$

in $\mathcal{E}'_{n-1} = \sum_j R^{\beta^I_{2,j}}[-\omega+j] \oplus \sum_j R^{\beta^K_{1,j}}[-\omega+j]$ which can cancel with copies of $R[-\omega+j]$ in $\mathcal{E}'_n = \sum_j R^{\beta^I_{1,j}}[-\omega+j]$. We conclude that $\beta^{(\underline{\mathbf{y}}:I)}_{n,\omega-j} = \beta^I_{1,j} - t_j$ for $0 \le t_j \le |j|$ as required. □

Proving the main theorem of this section is now easily accomplished.

THEOREM 6.6. *Suppose that L is lex plus powers with respect to the degree sequence $\mathbb{A} = \{a_1, \dots, a_n\}$, $I \subset R$, both share the same Hilbert function, and I contains an $\mathbb{A}$-regular sequence. If the lex plus powers conjecture for socles (Conjecture 6.2) holds, then $\beta^L_{1,j} \ge \beta^I_{1,j}$ for all j.*

Proof. Let $\underline{\mathbf{x}} = \{x_1^{a_1}, \dots, x_n^{a_n}\} \subset L$ and let $\underline{\mathbf{y}}$ be an $\{a_1, \dots, a_n\}$-regular sequence in I. We know that $(\underline{\mathbf{x}} : L)$ and $(\underline{\mathbf{y}} : I)$ share the same Hilbert function, the former is $\{a'_1, \dots, a'_n\}$-lex plus powers, and the latter contains an $\{a'_1, \dots, a'_n\}$-regular sequence (by Lemma 6.4). By Proposition 6.5, $\beta^{(\underline{\mathbf{x}}:L)}_{n,\omega-j} = \beta^L_{1,j} - |j|$, and $\beta^{(\underline{\mathbf{y}}:I)}_{n,\omega-j} = \beta^I_{1,j} - t_j$. But by hypothesis, $\beta^{(\underline{\mathbf{x}}:L)}_{n,j} \ge \beta^{(\underline{\mathbf{y}}:I)}_{n,j}$, and as $|j| \ge t_j$, we conclude that $\beta^L_{1,j} \ge \beta^I_{1,j}$ for all j as required. □

We conclude by noting that in order to prove Conjecture 6.2, it is enough to demonstrate that lex plus powers ideals have largest socles in a single degree. In particular, Conjecture 6.2, and hence EGH, is equivalent to the following.

CONJECTURE 6.7. *Let L be $LPP(\mathbb{A})$ for some $\mathbb{A} = \{a_1, \dots, a_n\}$ and let $\rho_{\mathcal{H}}$ be the regularity of $\mathcal{H} = H(R/L)$. Then $\beta^L_{n,\rho_{\mathcal{H}}+n-1} \ge \beta^I_{n,\rho_{\mathcal{H}}+n-1}$ for any ideal $I \subset R$ containing an $\mathbb{A}$-regular sequence and attaining $\mathcal{H}$.*

THEOREM 6.8. *Conjecture 6.7 and Conjecture 6.2 are equivalent.*

Proof. It is obvious that Conjecture 6.2 implies Conjecture 6.7. So suppose that Conjecture 6.7 holds, L is $LPP(\mathbb{A})$ for some $\mathbb{A} = \{a_1, \dots, a_n\}$, $I \subset R$ contains an $\mathbb{A}$-regular sequence, and $H(R/L) = H(R/I) = \mathcal{H}$ has regularity $\rho_{\mathcal{H}}$. Now $\beta^L_{n,\rho_{\mathcal{H}}+n-1} \ge \beta^I_{n,\rho_{\mathcal{H}}+n-1}$ by hypothesis, and $\beta^L_{n,\rho_{\mathcal{H}}+n} = \beta^I_{n,\rho_{\mathcal{H}}+n}$ because L and I attain the same Hilbert function. Thus, it remains to show that $\beta^L_{n,j} \ge \beta^I_{n,j}$ for all $j \le \rho_{\mathcal{H}} + n - 2$. This is easily accomplished. Let $\overline{L}$ and $\overline{I}$ be the ideals $L + \langle x_1, \dots, x_n\rangle^{\rho_{\mathcal{H}}}$ and $I + \langle x_1, \dots, x_n\rangle^{\rho_{\mathcal{H}}}$, respectively. Then $H(R/\overline{L}) = H(R/\overline{I})$ and $\rho_{H(R/\overline{L})} = \rho_{\mathcal{H}} - 1$; by induction on ρ we have that $\beta^L_{n,j} = \beta^{\overline{L}}_{n,j} \ge \beta^{\overline{I}}_{n,j} = \beta^I_{n,j}$ for $j \le \rho_{\mathcal{H}} + n - 2$ as required (where we make use of the fact that adding $\langle x_1, \dots, x_n\rangle^{\rho_{\mathcal{H}}}$ to L and I only perturbs the last two rows of their Betti diagrams). □

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