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ON TILTING MODULES OVER CLUSTER-TILTED ALGEBRAS

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ABSTRACT. In this paper, we show that the tilting modules over a cluster-tilted algebra A lift to tilting objects in the associated cluster category $\mathcal{C}_H$. As a first application, we describe the induced exchange relation for tilting A-modules arising from the exchange relation for tilting object in $\mathcal{C}_H$. As a second application, we exhibit tilting A-modules having cluster-tilted endomorphism algebras.

Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [FZ02] in the context of canonical basis of quantized enveloping algebras and total positivity for algebraic groups, but quickly turned out to be related to many other fields in mathematics. In the representation theory of finite dimensional algebras, the so-called cluster categories were introduced in [BMR+06] (and also in [CCS06] for the $\mathbb{A}_n$ case) as a natural categorical model for the combinatorics of the corresponding cluster algebras of Fomin and Zelevinsky. The construction is as follows. Let Q be a quiver without oriented cycles. There is then, for a field k, an associated finite dimensional hereditary path algebra $H = kQ$. Since H has finite global dimension, its bounded derived category $D^b(H)$ of the finitely generated modules has almost split triangles [Hap88]. Let τ be the corresponding Auslander–Reiten translation functor. Denoting by F the composition $\tau^{-1}[1]$, where $[1]$ is the shift functor in $D^b(H)$, the cluster category $\mathcal{C}_H$ was defined as the orbit category $D^b(H)/F$, and was shown to be canonically triangulated [Kel05] and to have almost split triangles [BMR+06].

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In this model, the exceptional objects are associated with the cluster variables of [FZ02] while the tilting objects correspond to the clusters. Remarkably, one also defines an exchange relation on the tilting objects in $\mathcal{C}_H$, corresponding to the exchange relation on the clusters of [FZ02]. More precisely, an almost complete tilting object $\overline{T}$ in $\mathcal{C}_H$ has exactly two nonisomorphic indecomposable complements M and M^*, and these are related by exchange triangles

$$M^* \xrightarrow{g} B \xrightarrow{f} M \longrightarrow M^*[1] \qquad \text{and} \qquad M \xrightarrow{f^*} B^* \xrightarrow{g^*} M^* \longrightarrow M[1],$$

where f, g^* are minimal right $\operatorname{add}\overline{T}$-approximations and f^*, g are minimal left $\operatorname{add}\overline{T}$-approximations (see [BMR+06]).

In view of the importance of tilting theory in the representation theory of finite dimensional algebras, the (opposite) endomorphism algebras $\operatorname{End}_{\mathcal{C}_H}(T)^{op}$ of these tilting objects T, called cluster-tilted algebras, were then introduced and studied in [BMR07] (see also [CCS06]). Their module theory was shown to be to a large extent determined by the cluster categories in which they arise. Indeed, given a cluster category $\mathcal{C}_H$ and a tilting object T in $\mathcal{C}_H$, it was shown by Buan, Marsh, and Reiten [BMR07] that the functor $\operatorname{Hom}_{\mathcal{C}_H}(T, -)$ induces an equivalence $\mathcal{H}_T : \mathcal{C}_H/\operatorname{add} T[1] \longrightarrow \operatorname{mod}\operatorname{End}_{\mathcal{C}_H}(T)^{op}$.

Since then, cluster-tilted algebras have been studied by several authors, and revealed to have very nice properties, see for instance [ABS08a], [ABS08b], [BIRS08], [BMR08], [Kel08], [KR07]. In particular, they were shown in [KR07] to be Gorenstein algebras of Gorenstein dimension at most one, in [ABS08b] to be trivial extensions of tilted algebras and in [BIRS08], [Kel08] to be given by quivers with potentials.

In this paper, we are interested in the problem of identifying tilting modules over cluster-tilted algebras. Our motivation comes from two points of view. On one side, the nice exchange relation for tilting objects over cluster categories should carry over Buan–Marsh–Reiten's equivalence and result in a similar exchange relation for tilting modules over cluster-tilted algebras, allowing to identify many tilting modules. Of course, one then has to care about projective dimensions. On the other hand, as stressed above, cluster-tilted algebras enjoy some very nice properties. Tilting theory being intimately related to derived equivalences (under which many properties are known to be preserved) by Happel's and Rickard's theorems [Hap88], [Ric89], the study of tilting modules is then a natural question.

In what follows, we present two different methods to find tilting modules over cluster-tilted algebras, dividing the paper in two distinct parts.

The first approach follows the above discussion, in the sense that we study the exchange relation of tilting modules over cluster-tilted algebras coming from the exchange relation of tilting objects for cluster categories. As pointed out above, one then has to care about projective dimension in the following sense: if T and T' are two tilting objects over a cluster category $\mathcal{C}_H$ such that $\operatorname{add} T[1] \cap \operatorname{add} T' = \{0\}$, then it follows from Buan–Marsh–Reiten's equivalence (see also [KR07], [KZ]) that the image of T' under the equivalence $\mathcal{H}_T : \mathcal{C}_H/\operatorname{add} T[1] \longrightarrow \operatorname{mod} \operatorname{End}_{\mathcal{C}_H}(T)^{op}$ is exceptional and has the right number of indecomposable direct summands to be a tilting module, but a priori no one knows about its projective dimension, which generally turns out to be infinite; in other words $\mathcal{H}_T(T')$ is generally not a tilting $\operatorname{End}_{\mathcal{C}_H}(T)^{op}$-module. The situation is better in the other direction. Indeed, while lifting tilting modules over cluster-tilted algebras to objects in the cluster category obviously does not bring any projective dimension problems, one now has to care about the exceptionality of the resulting objects, since the cluster category contains more maps, namely those factoring through $\operatorname{add} T[1]$. The following theorem says that such problems do not occur.

Here, and in the sequel, we let $\mathcal{H}_T^{-1}$ be a quasi-inverse for the induced equivalence $\mathcal{H}_T : \mathcal{C}_H/\operatorname{add} T[1] \longrightarrow \operatorname{mod} \operatorname{End}_{\mathcal{C}_H}(T)^{op}$.

THEOREM 1. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. Let M, N be objects in $\mathcal{C}_H$. If $\mathcal{H}_T(M)$ and $\mathcal{H}_T(N)$ are A-modules of projective dimension at most one such that*

$$\operatorname{Ext}^1_A(\mathcal{H}_T(M), \mathcal{H}_T(N)) = 0 \quad \text{and} \quad \operatorname{Ext}^1_A(\mathcal{H}_T(N), \mathcal{H}_T(M)) = 0,$$

then

$$\operatorname{Ext}^1_{\mathcal{C}_H}(M, N) = 0 \quad \text{and} \quad \operatorname{Ext}^1_{\mathcal{C}_H}(N, M) = 0.$$

In particular, the tilting A-modules lift to tilting objects in $\mathcal{C}_H$.

From this, we get that the endomorphism algebras of tilting modules over cluster-tilted algebras are quotients of cluster-tilted algebras (Corollary 2.4).

On the other hand, the study of the possible complements for an almost complete tilting module has been the central point of many investigations during the past years. It is known that an almost complete tilting module of projective dimension at most one admits at most two nonisomorphic complements. Combining Theorem 1 with a result from [CHU94], [Hap95] (see Theorem 3.1) then allows to show that for a cluster-tilted algebra, these two complements are related by the exchange relation in $\mathcal{C}_H$.

THEOREM 2. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. Let $S = \overline{S} \oplus M$ be a (basic) tilting A-module, with M indecomposable. Also, let*

$$M^* \xrightarrow{g} B \xrightarrow{f} \mathcal{H}_T^{-1}(M) \longrightarrow M^*[1] \quad \text{and} \quad \mathcal{H}_T^{-1}(M) \xrightarrow{f^*} B^* \xrightarrow{g^*} M^* \longrightarrow \mathcal{H}_T^{-1}(M)[1]$$

be the corresponding exchange triangles in $\mathcal{C}_H$, where f, g^* are minimal right $\operatorname{add} \mathcal{H}_T^{-1}(\overline{S})$-approximations in $\mathcal{C}_H$ and f^*, g are minimal left $\operatorname{add} \mathcal{H}_T^{-1}(\overline{S})$-

approximations in $\mathcal{C}_H$. *The following are equivalent*:

(a) *There exists an indecomposable module* M', *not isomorphic to* M, *such that* $\overline{S} \oplus M'$ *is a tilting* A*-module*;
(b) $\overline{S} \oplus \mathcal{H}_T(M^*)$ *is a tilting* A*-module*;
(c) $\mathcal{H}_T(M^*) \neq 0$ *and* $\mathrm{pd}_A \mathcal{H}_T(M^*) \leq 1$.
(d) *Either* $\mathcal{H}_T(f)$ *is an epimorphism in* $\mathrm{mod}\, A$ *or* $\mathcal{H}_T(f^*)$ *is a monomorphism in* $\mathrm{mod}\, A$;
(e) $\overline{S}$ *is a faithful* A*-module.*

The second method deals with completely different tools. Given an algebra A, we consider the left part $\mathcal{L}_A$ and the right part $\mathcal{R}_A$ of its module category $\mathrm{mod}\, A$ (see [HRS96]). In [ACT04], Assem, Coelho, and Trepode studied the algebras A for which the subcategory $\mathrm{add}\, \mathcal{L}_A$ is functorially finite in $\mathrm{mod}\, A$ (in the sense of [AS80]) and called them *left supported*. Dually, they defined the *right supported* algebras (see Section 4 for details). They proved that A is left supported if and only if a specific A-module L is a tilting module, and similarly for the right supported algebras. As we shall see, the left and the right parts of a cluster-tilted not hereditary algebra are both finite, implying that any cluster-tilted algebra is left and right supported. The module L is the direct sum of the indecomposable Ext-injective modules in $\mathrm{add}\, \mathcal{L}_A$ and the indecomposable projective modules which are not in $\mathcal{L}_A$. Hence, L determines a “slice” in $\mathcal{L}_A$ given by the sum of all indecomposable Ext-injective modules in $\mathrm{add}\, \mathcal{L}_A$. Our results show that any basic object S in $\mathrm{add}\, \mathcal{L}_A$, which is maximal for the property that $\mathrm{Ext}^1_A(S, S) = 0$, gives rise to a tilting module. However, the ones given by slices in $\mathcal{L}_A$, called $\mathcal{L}_A$*-slices* (see Definition 5.12) give remarkable tilting modules, since their endomorphism algebra is still cluster-tilted.

THEOREM 3. *Let* $\mathcal{C}_H$ *be a cluster category*, T *be a tilting object in* $\mathcal{C}_H$ *and* A *be the cluster-tilted algebra* $\mathrm{End}_{\mathcal{C}_H}(T)^{op}$. *Assume that* A *is not hereditary and let* Σ *be an* $\mathcal{L}_A$*-slice. Also*, *let* $F = \bigoplus_{i=1}^m P_i$ *denote the direct sum of all indecomposable projective modules not in* $\mathcal{L}_A$. *Then*:

(a) $T_\Sigma = \Sigma \oplus F$ *is a tilting* A*-module*;
(b) *The algebra* $A_\Sigma = \mathrm{End}_A(T_\Sigma)^{op}$ *is isomorphic to* $\mathrm{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(T_\Sigma))^{op}$, *in particular* A_Σ *is cluster-tilted*;
(c) *The quiver of* A_Σ *is obtained from that of* A *with a finite number of reflections at sinks.*

This paper is organized as follows. In Section 1, we collect the necessary background concerning cluster categories and cluster-tilted algebras. Sections 2 and 3 are devoted to the proofs of Theorem 1 and Theorem 2, respectively. Finally, after some necessary preliminaries on supported algebras in Section 4, we prove Theorem 3 in Section 5.

1. Definitions and first preliminaries

In this section, we review some useful notions and results that will be used for the proofs of Theorem 1 and Theorem 2. More preliminaries concerning Theorem 3 are postponed to Section 4.

1.1. Basic notations. In this paper, all algebras are connected finite dimensional algebras over a field k, unless otherwise specified. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated (right) A-modules. For an A-module M, we, respectively, denote by $\operatorname{pd}_A M$ and $\operatorname{id}_A M$ the projective dimension and the injective dimension of M.

More generally, for an additive category $\mathcal{A}$ we let $\operatorname{ind}\mathcal{A}$ be a full subcategory whose objects are representatives of the isomorphism classes of indecomposable objects in $\mathcal{A}$. By an indecomposable object in $\mathcal{A}$, we therefore mean an object in $\operatorname{ind}\mathcal{A}$. In case $\mathcal{A}=\operatorname{mod} A$, for some algebra A, we write $\operatorname{ind} A$ instead of $\operatorname{ind}(\operatorname{mod} A)$. For an object T in $\mathcal{A}$, $\operatorname{add} T$ denotes the full subcategory of $\mathcal{A}$ with objects all direct summands of direct sums of copies of T.

1.2. Approximations. Let $\mathcal{A}$ be an additive category and $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$. For an object A in $\mathcal{A}$, a map $f: B\longrightarrow A$, with $B\in\mathcal{B}$ is called a *right $\mathcal{B}$-approximation* if any other map $f': B'\longrightarrow A$ with $B'\in\mathcal{B}$ factors through f, that is there exists $g: B'\longrightarrow B$ such that $f'=fg$. There is the dual notion of a *left $\mathcal{B}$-approximation.* If any object in $\mathcal{A}$ admits a right (left) $\mathcal{B}$-approximation, then $\mathcal{B}$ is said to be a *contravariantly (covariantly) finite* subcategory of $\mathcal{A}$. It is called *functorially finite* if it is both contravariantly finite and covariantly finite. Finally, a *minimal right $\mathcal{B}$-approximation* is a right $\mathcal{B}$-approximation $f: B\longrightarrow A$ such that for every $g: B\longrightarrow B$ such that $fg=f$, the map g is an isomorphism. The *minimal left $\mathcal{B}$-approximations* are defined dually. These notions were introduced in [AS80].

1.3. Almost complete tilting objects. Although the notions of tilting objects slightly differ according to the type of categories we consider (see Sections 1.4 and 1.5 for details), we will in any case say that an object $\overline{T}$ in an additive category $\mathcal{A}$ is an *almost complete tilting object* if it is not a tilting object but there exists an indecomposable object M in $\mathcal{A}$ such that $\overline{T}\oplus M$ is a tilting object. In this case, M is called a *complement* for $\overline{T}$. All (partial) tilting objects T we consider are assumed to be *basic*, that is, if $T=\bigoplus_{i=1}^n T_i$ is a decomposition in indecomposable direct summands of T, then $i\neq j$ implies $T_i\ncong T_j$.

1.4. Cluster categories and tilting objects. Let H be a hereditary algebra. As mentioned in the Introduction, the cluster category $\mathcal{C}_H$ is the orbit category $D^b(H)/F$, where $F=\tau^{-1}[1]$. Thus, the objects in $\mathcal{C}_H$ are the F-orbits $X=(F^i\widetilde{X})_{i\in\mathbb{Z}}$, where the $\widetilde{X}$ are objects in $D^b(H)$. The set of morphisms from $X=(F^i\widetilde{X})_{i\in\mathbb{Z}}$ to $Y=(F^i\widetilde{Y})_{i\in\mathbb{Z}}$ in $\mathcal{C}_H$ is given by

$$\mathrm{Hom}_{\mathcal{C}_H}(X,Y)=\coprod_{i\in\mathbb{Z}}\mathrm{Hom}_{D^b(H)}(\widetilde{X},F^i\widetilde{Y}).$$

It is shown in [Kel05] that $\mathcal{C}_H$ is a triangulated category and that the canonical functor $D^b(H)\longrightarrow\mathcal{C}_H$ is a triangle functor. Moreover, $\mathcal{C}_H$ has almost split triangles and the Auslander–Reiten translation $\tau_{\mathcal{C}_H}$ is equal to [1], the shift functor in $\mathcal{C}_H$.

Let $\mathcal{F}=\mathrm{ind}(\mathrm{mod}\,H\cup H[1])$ in $D^b(H)$, that is the set consisting of the indecomposable H-modules together with the objects $P[1]$ where P is an indecomposable projective H-module. It is easily seen that $\mathcal{F}$ contains exactly one representative from each F-orbit of indecomposable objects in $D^b(H)$. Hence, in many situations, like in the context of our proofs, we may (and will) identify $\mathcal{F}$ with $\mathrm{ind}\,\mathcal{C}_H$, and thus assume that any indecomposable object in $\mathcal{C}_H$ is a H-module or of the form $P[1]$. For two objects $\widetilde{M},\widetilde{N}$ in $\mathcal{F}$, we have $\mathrm{Hom}_{D^b(H)}(\widetilde{M},F^i\widetilde{N})=0$ for all $i\neq 0,1$ (see [BMR+06, (1.5)]).

Thus, more generally, the space $\mathrm{Hom}_{\mathcal{C}_H}(X,Y)=\coprod_{i\in\mathbb{Z}}\mathrm{Hom}_{D^b(H)}(\widetilde{X},F^i\widetilde{Y})$ is always finite dimensional. Also, by [BMR+06, (1.4), (1.7)],

$$D\,\mathrm{Ext}^1_{\mathcal{C}_H}(Y,X)\cong\mathrm{Ext}^1_{\mathcal{C}_H}(X,Y)\cong D\,\mathrm{Hom}_{\mathcal{C}_H}(Y,X[1]),$$

where $D=\mathrm{Hom}_k(-,k)$.

We recall the following definition from [BMR+06].

Definition 1.1. Let T be a basic object in $\mathcal{C}_H$. Then T is called a *cluster-tilting object*, or a *tilting object* for short, provided $\mathrm{Ext}^1_{\mathcal{C}_H}(T,T)=0$ and T has a maximal number of nonisomorphic direct summands (that is the same number as the number of nonisomorphic simple H-modules).

By [BMR+06, (3.3)], up to derived equivalence, one can always assume that a given tilting object T is induced by a tilting module over H. Also, an almost complete basic tilting object $\overline{T}$ in $\mathcal{C}_H$ has exactly two nonisomorphic complements M and M^*, and these are related by some exchange triangles

$$M^*\xrightarrow{g}B\xrightarrow{f}M\longrightarrow M^*[1]\quad\text{and}\quad M\xrightarrow{f^*}B^*\xrightarrow{g^*}M^*\longrightarrow M[1]$$

where f,g^* are minimal right $\mathrm{add}\,\overline{T}$-approximations and f^*,g are minimal left $\mathrm{add}\,\overline{T}$-approximations. The following particular case will be heavily exploited in Section 5. For more details on cluster categories, we refer to [BMR+06].

Remark 1.2. Let $\overline{T}$, M and B be as above and let $M\longrightarrow Q\longrightarrow\tau^{-1}_{\mathcal{C}_H}M\longrightarrow M[1]$ be the almost split triangle starting at M. If $Q\in\mathrm{add}\,\overline{T}$, then $Q=B$ and therefore $M^*=\tau^{-1}_{\mathcal{C}_H}M=M[-1]$. Hence, the exchange of M by M^* coincides with an almost split exchange in $\mathcal{C}_H$.

1.5. Cluster-tilted algebras and tilting modules. We start by recalling the following definition from [BMR07].

DEFINITION 1.3. Let H be a hereditary algebra, $\mathcal{C}_H$ be the associated cluster category and T be a tilting object in $\mathcal{C}_H$. The algebra $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$ is called *cluster tilted*.

In this case, the functor $\operatorname{Hom}_{\mathcal{C}_H}(T,-)$ induces an equivalence

$$\mathcal{H}_T : \mathcal{C}_H / \operatorname{add} T[1] \longrightarrow \operatorname{mod} A$$

under which the almost split sequences in $\operatorname{mod} A$ are induced by almost split triangles in $\mathcal{C}_H$ [BMR07]. Moreover, it was shown in [KR07] that any cluster-tilted algebra A is *Gorenstein of Gorenstein dimension at most one*, meaning that every projective module is of injective dimension at most one, and dually every injective module is of projective dimension at most one. As an important consequence, the projective dimension and the injective dimension of any A-module are either both infinite, or both are less than or equal to one (see [KR07, Section 2.1]). In particular, the tilting modules are of projective dimension at most one.

Therefore, in this context, a (basic) A-module S is a *tilting A-module* if:

- $\operatorname{pd}_A S \leq 1$ (equivalently $\operatorname{id}_A S \leq 1$);
- $\operatorname{Ext}^1_A(S,S) = 0$;
- The number of indecomposable direct summands of S equals the number of simple A-modules (equivalently the number of simple H-modules).

2. Proof of Theorem 1

In this section, we recall some useful features of the modules of projective or injective dimension at most one and prove Theorem 1. We start with the following well-known lemma (see [ASS06, (IV.2.13), (IV.2.14)] for instance).

LEMMA 2.1. *Let A be an algebra and M be an A-module.*

(a) $\operatorname{pd}_A M \leq 1$ *if and only if* $\operatorname{Hom}_A(DA, \tau M) = 0$. *Moreover, if* $\operatorname{pd}_A M \leq 1$, *then* $\operatorname{Ext}^1_A(M,N) \cong D\operatorname{Hom}_A(N, \tau M)$ *for each A-module N;*
(b) $\operatorname{id}_A M \leq 1$ *if and only if* $\operatorname{Hom}_A(\tau^{-1}M, A) = 0$. *Moreover, if* $\operatorname{id}_A M \leq 1$, *then* $\operatorname{Ext}^1_A(N,M) \cong D\operatorname{Hom}_A(\tau^{-1}M, N)$ *for each A-module N;*

where $D = \operatorname{Hom}_k(-,k) : \operatorname{mod} A^{op} \longrightarrow \operatorname{mod} A$ *denotes the usual duality.*

We note that if $\mathcal{C}_H$ is a cluster category and T is a tilting object in $\mathcal{C}_H$, with $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$, then the equivalence $\mathcal{C}_H / \operatorname{add} T[1] \longrightarrow \operatorname{mod} A$ takes the objects in $\operatorname{add} T$ to projective A-modules and the objects in $\operatorname{add} T[2]$ to injective A-modules. In view of this and the Gorenstein property of cluster-tilted algebras (see Section 1.5), the above lemma immediately implies the following result.

Lemma 2.2. *Let $\mathcal{C}_H$ be a cluster category and T be a tilting object in $\mathcal{C}_H$. Let $A = \mathrm{End}_{\mathcal{C}_H}(T)^{op}$ and M be an A-module. The following conditions are equivalent*:

(a) $\mathrm{pd}_A M \le 1$;
(b) *In $\mathcal{C}_H$, any map from an object in* $\mathrm{add}\, T[2]$ *to* $\mathcal{H}_T^{-1}(M)[1]$ *factors through* $\mathrm{add}\, T[1]$;
(c) $\mathrm{id}_A M \le 1$;
(d) *In $\mathcal{C}_H$, any map from* $\mathcal{H}_T^{-1}(M)[-1]$ *to an object in* $\mathrm{add}\, T$ *factors through* $\mathrm{add}\, T[1]$.

We are now in position to prove Theorem 1.

Theorem 1. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \mathrm{End}_{\mathcal{C}_H}(T)^{op}$. Let M, N be objects in $\mathcal{C}_H$. If $\mathcal{H}_T(M)$ and $\mathcal{H}_T(N)$ are A-modules of projective dimension at most one such that*

$$\mathrm{Ext}^1_A(\mathcal{H}_T(M), \mathcal{H}_T(N)) = 0 \quad \textit{and} \quad \mathrm{Ext}^1_A(\mathcal{H}_T(N), \mathcal{H}_T(M)) = 0,$$

then

$$\mathrm{Ext}^1_{\mathcal{C}_H}(M, N) = 0 \quad \textit{and} \quad \mathrm{Ext}^1_{\mathcal{C}_H}(N, M) = 0.$$

In particular, the tilting A-modules lift to tilting objects in $\mathcal{C}_H$.

Proof. Clearly, it suffices to prove the theorem for M, N indecomposable. Moreover, we assume that T is induced by a tilting H-module.

We first assume that, in $\mathcal{C}_H$, M and N are two H-modules (see Section 1.4). Also, assume to the contrary that $\mathrm{Ext}^1_{\mathcal{C}_H}(M, N) \neq 0$. Then

$$\begin{aligned} 0 \neq \mathrm{Ext}^1_{\mathcal{C}_H}(M, N) &\cong D\,\mathrm{Hom}_{\mathcal{C}_H}(N, \tau_{\mathcal{C}_H} M) \\ &= D\,\mathrm{Hom}_{D^b(H)}(N, \tau M) \oplus D\,\mathrm{Hom}_{D^b(H)}(N, M[1]) \\ &\cong \mathrm{Hom}_{D^b(H)}(M, N[1]) \oplus D\,\mathrm{Hom}_{D^b(H)}(N, M[1]) \end{aligned}$$

and thus $\mathrm{Hom}_{D^b(H)}(M, N[1]) \neq 0$ or $\mathrm{Hom}_{D^b(H)}(N, M[1]) \neq 0$. Assume, without loss of generality, that $\mathrm{Hom}_{D^b(H)}(M, N[1]) \neq 0$. Also, we have

$$\begin{aligned} 0 &= D\,\mathrm{Ext}^1_A(\mathcal{H}_T(N), \mathcal{H}_T(M)) \\ &\cong \mathrm{Hom}_A(\mathcal{H}_T(M), \tau\mathcal{H}_T(N)) \\ &\cong \frac{\mathrm{Hom}_{\mathcal{C}_H}(M, N[1])}{\{f : M \longrightarrow N[1] \text{ factoring through } \mathrm{add}\, T[1]\}}, \end{aligned}$$

where the first isomorphism follows from Lemma 2.1. Therefore, any map in $\mathrm{Hom}_{\mathcal{C}_H}(M, N[1])$ factors through $\mathrm{add}\, T[1]$, and similarly for any map in $\mathrm{Hom}_{\mathcal{C}_H}(N, M[1])$. Lifting this property to $D^b(H)$ means, in particular, that any map in $\mathrm{Hom}_{D^b(H)}(M, N[1])$ factors through $\mathrm{add}(\tau T \oplus T[1])$. Now, let $\{f_1, \ldots, f_n\}$ be a basis for $\mathrm{Hom}_{D^b(H)}(M, N[1])$. For each i, there exist T_i', T_i'' in $\mathrm{add}\, T$ and maps $\binom{\alpha_i}{\beta_i} : M \longrightarrow \tau T_i' \oplus T_i''[1]$ and $(\gamma_i, \delta_i) : \tau T_i' \oplus T_i''[1] \longrightarrow N[1]$ such that $f_i = (\gamma_i, \delta_i) \circ \binom{\alpha_i}{\beta_i}$. Thus, taking $T' = \bigoplus_{i=1}^n T_i'$, $T'' = \bigoplus_{i=1}^n T_i''$, $\alpha =$

$\operatorname{diag}(\alpha_1,\ldots,\alpha_n)$ and $\beta = \operatorname{diag}(\beta_1,\ldots,\beta_n)$, we see that any map in $\operatorname{Hom}_{D^b(H)}(M,N[1])$ factors through $\binom{\alpha}{\beta} : M \longrightarrow \tau T' \oplus T''[1]$. In other words, we have a surjective map

$$\operatorname{Hom}_{D^b(H)}(\tau T',N[1]) \oplus \operatorname{Hom}_{D^b(H)}(T''[1],N[1]) \xrightarrow{\circ\binom{\alpha}{\beta}} \operatorname{Hom}_{D^b(H)}(M,N[1])\,.$$

Under the natural isomorphism

$$\operatorname{Hom}_{D^b(H)}(X,Y[1]) \cong \operatorname{Ext}^1_H(X,Y) \cong D\operatorname{Hom}_H(Y,\tau X) \quad \text{for } X,Y \in \operatorname{mod} H,$$

the map $\beta : M \longrightarrow T''[1]$ becomes an element of $D\operatorname{Hom}_H(T'',\tau M)$. More generally, the above surjective map becomes the surjective map

$$D\operatorname{Hom}_H(N,\tau^2T') \oplus \operatorname{Hom}_H(T'',N) \longrightarrow D\operatorname{Hom}_H(N,\tau M)$$

taking a pair (g,h) in $D\operatorname{Hom}_H(N,\tau^2T') \oplus \operatorname{Hom}_H(T'',N)$ to the morphism $\operatorname{Hom}_H(N,\tau M) \longrightarrow k$ sending an element $f \in \operatorname{Hom}_A(N,\tau M)$ onto the element $g(\tau(\alpha)f) + \beta(fh)$.

Applying the duality D yields an injective map

$$\operatorname{Hom}_H(N,\tau M) \longrightarrow \operatorname{Hom}_H(N,\tau^2T') \oplus D\operatorname{Hom}_H(T'',N)$$

taking an element $g \in \operatorname{Hom}_H(N,\tau M)$ to the pair $(\tau(\alpha)g,\overline{g})$, where $\overline{g}(h) = \beta(gh)$ for $h \in \operatorname{Hom}_H(T'',N)$.

Now, recall that by assumption $0 \neq \operatorname{Hom}_{D^b(H)}(M,N[1]) \cong \operatorname{Hom}_H(N,\tau M)$. Hence, let g be a nonzero morphism in $\operatorname{Hom}_H(N,\tau M)$. The injectivity of the above map gives $\tau(\alpha)g \neq 0$ or $gh \neq 0$ for some $h \in \operatorname{Hom}_H(T'',N)$. In other words, one of the two compositions

$$N \xrightarrow{g} \tau M \xrightarrow{\tau(\alpha)} \tau^2 T' \quad \text{and} \quad T'' \xrightarrow{h} N \xrightarrow{g} \tau M$$

is not zero. However, since any map in

$$\operatorname{Hom}_{\mathcal{C}_H}(N,M[1]) = \operatorname{Hom}_{D^b(H)}(N,\tau M) \oplus \operatorname{Hom}_{D^b(H)}(N,M[1])$$

factors through $\operatorname{add} T[1]$, g factors through $\operatorname{add} \tau T$ in $\operatorname{mod} H$, say through $\tau T'''$, with $T''' \in \operatorname{add} T$. The above compositions then yield a nonzero map of the form $\tau T''' \longrightarrow \tau^2 T'$ or $T'' \longrightarrow \tau T'''$, a contradiction to $\operatorname{Ext}^1_{\mathcal{C}_H}(T,T) = 0$. Hence, $\operatorname{Ext}^1_{\mathcal{C}_H}(M,N) = 0$, and dually $\operatorname{Ext}^1_{\mathcal{C}_H}(N,M) = 0$.

We now assume that $M \in \operatorname{mod} H$ and $N \in \operatorname{add} H[1]$. Let P be an indecomposable projective H-module such that $N = P[1]$. Then $\tau N = I$, where I is the indecomposable injective H-module satisfying $\operatorname{soc} I = \operatorname{top} P$. Now, assume that $\operatorname{Ext}^1_{\mathcal{C}_H}(M,N) \neq 0 \neq \operatorname{Ext}^1_{\mathcal{C}_H}(N,M)$. We have

$$0 \neq \operatorname{Ext}^1_{\mathcal{C}_H}(M,N) = \operatorname{Ext}^1_{\mathcal{C}_H}(M,P[1]) = D\operatorname{Hom}_{D^b(H)}(P[1],M[1])$$

and so $\operatorname{Hom}_H(P,M) \neq 0$. Similarly, $\operatorname{Ext}^1_{\mathcal{C}_H}(N,M)$ yields $\operatorname{Hom}_H(M,I) \neq 0$. Let $f \in \operatorname{Hom}_H(P,M)$ and $g \in \operatorname{Hom}_H(M,I)$ be nonzero morphisms. Since

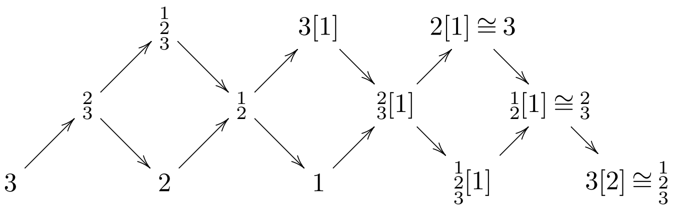
$\operatorname{soc} I = \operatorname{top} P$, we get a nonzero composition $P \xrightarrow{f} M \xrightarrow{g} I$ whose image in mod$A$ is a nonzero composition

$$\tau^{-1}\mathcal{H}_T(N) \xrightarrow{\mathcal{H}_T(f)} \mathcal{H}_T(M) \xrightarrow{\mathcal{H}_T(g)} \tau\mathcal{H}_T(N).$$

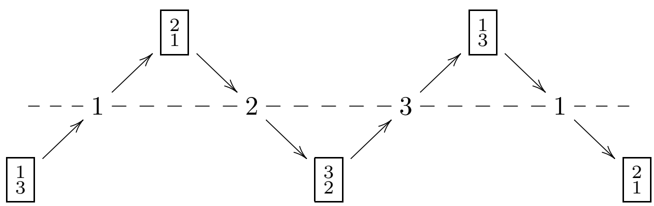
Since $\operatorname{pd}_A \mathcal{H}_T(M) \le 1$ and $\operatorname{pd}_A \mathcal{H}_T(N) \le 1$, it follows from the first part of the proof that f factors through $\operatorname{add} T$ while g factors through $\operatorname{add} \tau T$, contradicting $\operatorname{Ext}^1_H(T,T) = 0$. Hence, $\operatorname{Ext}^1_{\mathcal{C}_H}(M,N) = 0 = \operatorname{Ext}^1_{\mathcal{C}_H}(N,M)$. Finally, if M and N are both in $\operatorname{add} H[1]$, then $\operatorname{Ext}^1_{\mathcal{C}_H}(M,N) = 0 = \operatorname{Ext}^1_{\mathcal{C}_H}(N,M)$ and we are done. □

The following easy example shows that Theorem 1 is no longer true if we drop the assumption that $\operatorname{pd}_A \mathcal{H}_T(M) \le 1$ and $\operatorname{pd}_A \mathcal{H}_T(N) \le 1$.

EXAMPLE 2.3. Let Q be the quiver $1 \to 2 \to 3$ and $H = kQ$ be the path algebra. The AR-quiver of the corresponding cluster category $\mathcal{C}_H$ is given by



where the indecomposable objects are represented by the Loewy series of the corresponding H-modules. Let T be the tilting object $T = 3 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 1$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$ be the corresponding (self-injective) cluster-tilted algebra. Its AR-quiver is given by



where the dashed lines represent the Auslander–Reiten translates and the identified modules are the projective–injective modules. In $\mathcal{C}_H$, let $M = 3$ and $N = \begin{smallmatrix}1\\2\end{smallmatrix}$. We have

$$\operatorname{Ext}^1_{\mathcal{C}_H}(M,N) = \operatorname{Hom}_{\mathcal{C}_H}(M, N[1]) = \operatorname{Hom}_{\mathcal{C}_H}(M, \tau_{\mathcal{C}_H} N) \neq 0.$$

However, $\mathcal{H}_T(M) = \begin{smallmatrix}1\\3\end{smallmatrix}$ and $\mathcal{H}_T(N) = 2$, and since $\mathcal{H}_T(M)$ is projective–injective, we have

$$\operatorname{Ext}^1_A(\mathcal{H}_T(M), \mathcal{H}_T(N)) = \operatorname{Ext}^1_A(\mathcal{H}_T(N), \mathcal{H}_T(M)) = 0.$$

Since $\operatorname{pd}_A \mathcal{H}_T(M) \leq 1$ but $\operatorname{pd}_A \mathcal{H}_T(N) = \infty$, this shows that Theorem 1 is no longer true when we drop the assumption that $\operatorname{pd}_A \mathcal{H}_T(M) \leq 1$ and $\operatorname{pd}_A \mathcal{H}_T(N) \leq 1$.

Also, consider $N' = \frac{2}{3}$ in $\mathcal{C}_H$. Then $\operatorname{Ext}^1_{\mathcal{C}_H}(M, N') = \operatorname{Ext}^1_{\mathcal{C}_H}(N', M) = 0$ but $\operatorname{pd}_A \mathcal{H}_T(N') = \infty$, showing moreover that the converse of Theorem 1 generally fails.

As a consequence of Theorem 1, we obtain the following nice result.

Corollary 2.4. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. Let S be a tilting A-module. Then $\operatorname{End}_A(S)^{op}$ is a quotient of the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}_H}(S)^{op}$.*

Proof. By Theorem 1, $\mathcal{H}_T^{-1}(S)$ is a tilting object in $\mathcal{C}_H$, thus $\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(S))^{op}$ is cluster-tilted. The result then follows from the Buan–Marsh–Reiten equivalence $\mathcal{H}_T : \mathcal{C}_H / \operatorname{add} T[1] \longrightarrow \operatorname{mod} A$. □

In Section 5, we discuss examples where $\operatorname{End}_A(S)^{op} \cong \operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(S))^{op}$.

3. Exchange relation for cluster-tilted algebras

Here, we discuss the induced exchange relation of tilting modules over cluster-tilted algebras in view of Theorem 1 and the exchange relation for tilting objects in the cluster categories. For clear reasons (for instance when a cluster-tilted algebra has projective–injective modules), it is not always possible to exchange an indecomposable direct summand M of a tilting module $\overline{S} \oplus M$ by another indecomposable M^* such that $\overline{S} \oplus M^*$ is a tilting module. In this section, we give sufficient and necessary conditions for the existence of such a complement M^* for cluster-tilted algebras. Basically, we show that if such a M^* exists, then it is given by the exchange triangles in $\mathcal{C}_H$.

More generally, complements of almost complete tilting modules (of arbitrary finite projective dimension) over artin algebras have been studied by several authors, in particular by Coelho, Happel and Unger (see [CHU94], [Hap95] for instance). A very weak version of one of their main results, but sufficient for our purpose, goes as follows. Recall the the *finitistic dimension* of an algebra is the supremum of the projective dimensions of its (finitely generated) modules of finite projective dimension.

Theorem 3.1 ([CHU94], [Hap95]). *Let A be an artin algebra with finite finitistic dimension. Let $\overline{S}$ be an almost complete tilting module with $\operatorname{pd}_A \overline{S} \leq 1$.*

(a) *If $\overline{S}$ is not faithful, then $\overline{S}$ admits a unique complement.*
(b) *If $\overline{S}$ is faithful, then $\overline{S}$ admits exactly two complements M and M' and there exists a short exact sequence*

$$0 \longrightarrow M \xrightarrow{f} C \xrightarrow{g} M' \longrightarrow 0$$

where f is a minimal left $\operatorname{add}\overline{S}$*-approximation and g is a minimal right* $\operatorname{add}\overline{S}$*-approximation.*

Below, we show that Theorem 2 is obtained by combining Theorem 3.1 with Theorem 1. We need to recall one further result, borrowed from [KZ, (2.3)].

LEMMA 3.2. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. Let $L \xrightarrow{g} M \xrightarrow{f} N \xrightarrow{h} L[1]$ be a triangle in $\mathcal{C}_H$. Then, in* $\operatorname{mod} A$:

(a) *$\mathcal{H}_T(f)$ is a monomorphism if and only if $\mathcal{H}_T(g) = 0$.*
(b) *$\mathcal{H}_T(f)$ is an epimorphism if and only if $\mathcal{H}_T(h) = 0$.*

We are now able to prove Theorem 2. Recall from Section 1.5 that cluster-tilted algebras are Gorenstein of Gorenstein dimension at most one, forcing them to have finitistic dimension at most one.

THEOREM 2. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. Let $S = \overline{S} \oplus M$ be a (basic) tilting A-module, with M indecomposable. Also, let*

$$M^* \xrightarrow{g} B \xrightarrow{f} \mathcal{H}_T^{-1}(M) \xrightarrow{h} M^*[1] \quad \textit{and} \quad \mathcal{H}_T^{-1}(M) \xrightarrow{f^*} B^* \xrightarrow{g^*} M^* \xrightarrow{h^*} \mathcal{H}_T^{-1}(M)[1]$$

be the corresponding exchange triangles in $\mathcal{C}_H$, where f, g^* are minimal right $\operatorname{add}\mathcal{H}_T^{-1}(\overline{S})$*-approximations in $\mathcal{C}_H$ and f^*, g are minimal left* $\operatorname{add}\mathcal{H}_T^{-1}(\overline{S})$*-approximations in $\mathcal{C}_H$. The following are equivalent:*

(a) *There exists an indecomposable module M', not isomorphic to M, such that $\overline{S} \oplus M'$ is a tilting A-module;*
(b) *$\overline{S} \oplus \mathcal{H}_T(M^*)$ is a tilting A-module;*
(c) *$\mathcal{H}_T(M^*) \neq 0$ and $\operatorname{pd}_A \mathcal{H}_T(M^*) \leq 1$.*
(d) *Either $\mathcal{H}_T(f)$ is an epimorphism in* $\operatorname{mod} A$ *or $\mathcal{H}_T(f^*)$ is a monomorphism in* $\operatorname{mod} A$;
(e) *$\overline{S}$ is a faithful A-module.*

Proof. We mention that the existence of the exchange triangles in the statement follows from Theorem 1.

Clearly, the equivalence of (a) and (e) follows from Theorem 3.1. The same theorem also shows that (b) implies (e), while trivially (b) implies (c).

We now show that (c) implies (b) and (d). By the exchange relation in $\mathcal{C}_H$, we know that $\mathcal{H}_T^{-1}(\overline{S}) \oplus M^*$ is a tilting object in $\mathcal{C}_H$. In particular, $\operatorname{Ext}^1_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(\overline{S}) \oplus M^*, \mathcal{H}_T^{-1}(\overline{S}) \oplus M^*) = 0$, and so

$$\operatorname{Ext}^1_A\big(\overline{S} \oplus \mathcal{H}_T(M^*), \overline{S} \oplus \mathcal{H}_T(M^*)\big) = 0$$

(see [KZ, (4.9)]). Since $\operatorname{pd}_A \mathcal{H}_T(M^*) \leq 1$ by assumption, $\overline{S} \oplus \mathcal{H}_T(M^*)$ is a tilting A-module. This shows (b). Now, by Theorem 3.1, there exists a short exact sequence of the form

$$0 \longrightarrow \mathcal{H}_T(M^*) \longrightarrow C \xrightarrow{\underline{j}} M \longrightarrow 0 \quad \text{or} \quad 0 \longrightarrow M \xrightarrow{\underline{j}^*} C^* \longrightarrow \mathcal{H}_T(M^*) \longrightarrow 0,$$

where $C, C^* \in \operatorname{add} \overline{S}$. Assume that the first exact sequence exists, and let $j : \mathcal{H}_T^{-1}(C) \longrightarrow \mathcal{H}_T^{-1}(M)$ be a morphism in $\mathcal{C}_H$ such that $\mathcal{H}_T(j) = \underline{j}$. Now, since $f : B \longrightarrow \mathcal{H}_T^{-1}(M)$ is a right $\operatorname{add} \mathcal{H}_T^{-1}(\overline{S})$-approximation, there exists $f' : \mathcal{H}_T^{-1}(C) \longrightarrow \mathcal{H}_T^{-1}(B)$ such that $j = ff'$. Then $\underline{j} = \mathcal{H}_T(j) = \mathcal{H}_T(f) \circ \mathcal{H}_T(f')$, showing that $\mathcal{H}_T(f)$ is an epimorphism. Similarly, if the second short exact sequence exists, then $\mathcal{H}_T(f^*)$ is a monomorphism. This shows (d).

Conversely, (d) implies (c). Indeed, assume for instance that $\mathcal{H}_T(f)$ is an epimorphism. By Lemma 3.2, we have $\mathcal{H}_T(h) = 0$. Hence, $h[-1]$ factors through $\operatorname{add} T$. Since $\operatorname{pd}_A M \leq 1$, Lemma 2.2 (d) implies that $h[-1]$ factors through $\operatorname{add} T[1]$. Thus, by [KZ, (3.4)], we get a short exact sequence

$$0 \longrightarrow \mathcal{H}_T(M^*) \xrightarrow{\mathcal{H}_T(g)} \mathcal{H}_T(B) \xrightarrow{\mathcal{H}_T(f)} M \longrightarrow 0$$

in $\operatorname{mod} A$. Since $\operatorname{pd}_A M \leq 1$ and $\operatorname{pd}_A \mathcal{H}_T(B) \leq 1$, we get $\operatorname{pd}_A \mathcal{H}_T(M^*) < \infty$, and so $\operatorname{pd}_A \mathcal{H}_T(M^*) \leq 1$. Moreover, $\mathcal{H}_T(M^*) \neq 0$ since $M \notin \operatorname{add} \overline{S}$. Since a similar proof holds when $\mathcal{H}_T(f^*)$ is a monomorphism, (d) implies (c).

Finally, we show that (e) implies (b). Assume that $\overline{S}$ is faithful. By Theorem 3.1, there exists an indecomposable module M', not isomorphic to M, such that $\overline{S} \oplus M'$ is a tilting module. By Theorem 1, $\mathcal{H}_T^{-1}(\overline{S}) \oplus \mathcal{H}_T^{-1}(M')$ is a tilting object in $\mathcal{C}_H$, and since $M' \neq M$, we infer that $\mathcal{H}_T^{-1}(M') = M^*$. So $M' = \mathcal{H}_T(M^*)$, and consequently, $\overline{S} \oplus \mathcal{H}_T(M^*)$ is a tilting module. □

4. More preliminaries: the left and right parts

Here starts the second part of the paper, whose objective is to exhibit some tilting modules over cluster-tilted algebras whose endomorphism algebras are again cluster-tilted. This is achieved with the help of Theorem 1, but also with the property of cluster-tilted algebras of being left and right supported. Here, we gather the necessary terminology for the rest of the paper.

4.1. Paths and cycles. Let A be an algebra. A *path in* $\operatorname{ind} A$, or simply a *path*, is a sequence $\delta : M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = N \quad (t \geq 0)$ where $M_i \in \operatorname{ind} A$ and f_i is a nonzero morphism for each i. In this case, we write $M \rightsquigarrow N$ and we say that M is a *predecessor* of N and N is a *successor* of M. If each f_i is irreducible, then δ is *sectional* if it contains no triples (M_{i-1}, M_i, M_{i+1}) such that $\tau_A M_{i+1} = M_{i-1}$. A *refinement* of δ is a path $M = M'_0 \longrightarrow M'_1 \longrightarrow \cdots \longrightarrow M'_s = N$, with $s \geq t$, with an injective order-preserving function $\sigma : \{1, \ldots, t-1\} \longrightarrow \{1, \ldots, s-1\}$ such that $M_i = M'_{\sigma(i)}$

when $1 \le i \le t-1$. Finally, a path δ is a *cycle* if $M = N$ and at least one f_i is not an isomorphism. A subquiver Σ of a connected component Γ of the AR-quiver of A is *acyclic* if it contains no cycles and *convex* if any path in Γ starting and ending at modules in Σ consists only of modules in Σ.

4.2. The left and right parts of a module category. For an algebra A, we define the *left part* $\mathcal{L}_A$ and the *right part* $\mathcal{R}_A$ of $\operatorname{mod} A$ as follows (see [HRS96]):

(1) $\mathcal{L}_A = \{M \in \operatorname{ind} A : \operatorname{pd}_A N \le 1 \text{ for each predecessor } N \text{ of } M\}$,

(2) $\mathcal{R}_A = \{M \in \operatorname{ind} A : \operatorname{id}_A N \le 1 \text{ for each successor } N \text{ of } M\}$.

Clearly, $\mathcal{L}_A$ is closed under predecessors and $\mathcal{R}_A$ is closed under successors. The left and the right parts have been used in recent years to describe many classes of algebras, amongst them the quasitilted [HRS96] and the laura algebras [AC03]. The next result is helpful to detect the modules lying in these parts.

LEMMA 4.1 ([AC03, (1.6)]). *Let A be an algebra.*

(a) *$\mathcal{L}_A$ consists of the modules $M \in \operatorname{ind} A$ such that, if there exists a path from an indecomposable injective module to M, then this path can be refined to a path of irreducible morphisms, and any such refinement is sectional.*
(b) *$\mathcal{R}_A$ consists of the modules $N \in \operatorname{ind} A$ such that, if there exists a path from N to an indecomposable projective module, then this path can be refined to a path of irreducible morphisms, and any such refinement is sectional.*

4.3. Left and right supported algebras. The study of the left and right parts lead in [ACT04] to the introduction of the left and right supported algebras.

DEFINITION 4.2 ([ACT04]). Let A be an algebra. Then A is called *left supported* provided the subcategory $\operatorname{add} \mathcal{L}_A$ is functorially finite in $\operatorname{mod} A$, and *right supported* provided the subcategory $\operatorname{add} \mathcal{R}_A$ is functorially finite in $\operatorname{mod} A$ (see Section 1.2).

Since $\mathcal{L}_A$ is closed under predecessors, $\operatorname{add} \mathcal{L}_A$ is clearly covariantly finite. Thus, an algebra A is left supported if $\operatorname{add} \mathcal{L}_A$ is contravariantly finite, and dually right supported if $\operatorname{add} \mathcal{R}_A$ is covariantly finite. For instance, any hereditary algebra is trivially left and right supported.

In what follows, we mainly focus on left supported algebras, instead of right supported algebras, and leave the primal-dual translation to the reader.

When dealing with left supported algebras, the Ext-injective modules in $\operatorname{add} \mathcal{L}_A$ play a prominent role since they determine whether the algebra is left supported or not. Recall that a module $M \in \mathcal{L}_A$ is Ext-*injective in* $\operatorname{add} \mathcal{L}_A$ if $\operatorname{Ext}^1_A(N, M) = 0$ for each $N \in \mathcal{L}_A$, or equivalently if $\tau^{-1} M \notin \mathcal{L}_A$. Then, by

[ACT04, (3.1)], the class $\mathcal{E}$ of indecomposable Ext-injective modules in $\mathcal{L}_A$ is the union of two disjoint subclasses:

$$\mathcal{E}_1 = \{M \in \mathcal{L}_A : \text{there exists an injective } I \text{ in } \operatorname{ind} A \text{ and a path } I \rightsquigarrow M\},$$
$$\mathcal{E}_2 = \{M \in \mathcal{L}_A \backslash \mathcal{E}_1 : \text{there exists a projective } P \in \operatorname{ind} A \backslash \mathcal{L}_A \text{ and a sectional path } P \rightsquigarrow \tau^{-1} M\}.$$

Hence, $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ and we denote by E (or E_1, or E_2) the direct sum of all indecomposable A-modules lying in $\mathcal{E}$ (or $\mathcal{E}_1$, or $\mathcal{E}_2$ respectively). We also denote by F the direct sum of a full set of representatives of the isomorphism classes of indecomposable projective A-modules not lying in $\mathcal{L}_A$. We let $L = E \oplus F$ and $U = E_1 \oplus \tau^{-1} E_2 \oplus F$. With these notations, we have the following reformulation of [ACT04, (3.3), (4.2)] and [ACPT07, (5.4)].

THEOREM 4.3. *An algebra A is left supported if and only if L is a tilting A-module, and this occurs if and only if U is a tilting A-module.*

As we will see, any cluster-tilted algebra is left supported, and so the above provides canonical tilting modules, whose endomorphism algebras will turn out to be again cluster-tilted. For instance, in the easiest (but unfortunately degenerate and not interesting) case where $\mathcal{L}_A = \emptyset$, we get the trivial tilting module $L = U = A$, whose endomorphism algebra is obviously cluster-tilted. Notice that we often get $\mathcal{L}_A \neq \emptyset$. In fact, it is easily verified that for an algebra A, we have $\mathcal{L}_A \neq \emptyset$ if and only if the ordinary quiver of A has a sink.

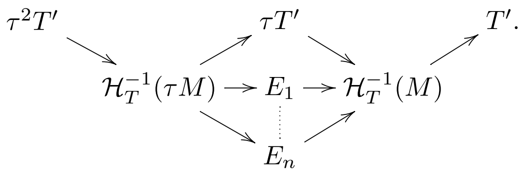
5. Special tilting modules

In this section, we prove Theorem 3. This is made in several steps. We start by proving that any cluster-tilted is left (and right) supported.

5.1. Cluster-tilted algebras are left supported. Let A be a cluster-tilted algebra. If A is hereditary, then $\operatorname{add} \mathcal{L}_A = \operatorname{mod} A$, and so A is trivially left (and dually right) supported. Our first aim is to show that this property still holds for cluster-tilted not hereditary algebras. We need the following lemma.

LEMMA 5.1. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. Assume that A is not hereditary. Then any connected component of the AR-quiver of A either contains no projective modules and no injective modules, or contains at least one projective module and at least one injective module.*

Proof. Let Γ_A denote the AR-quiver of A. Let P be an indecomposable projective A-module and Γ be the connected component of Γ_A containing P. Also, let Σ be the maximal full, connected and convex subquiver of Γ containing only indecomposable projective modules, including P. Since A is not hereditary, it follows from the shape of the AR-quivers of cluster-tilted algebras that Σ has less vertices then the number of τ-orbits in Γ. Hence, there exists P' in Σ together with an irreducible morphism $M \longrightarrow P'$ in Γ, where M is indecomposable not projective. By construction, M belongs to Γ. Let T' be the indecomposable direct summand of T such that $\mathcal{H}_T(T') = P'$. In $\mathcal{C}_H$, we have the following diagram of irreducible morphisms



Since M is not projective, we have $\tau M \neq 0$. Then $n \geq 1$ and there is in Γ an irreducible morphism from the indecomposable injective A-module $I = \mathcal{H}_T(\tau^2 T')$ to τM. Hence, Γ contains at least one injective module. Dually, any connected component containing an injective module also contains a projective module. $\square$

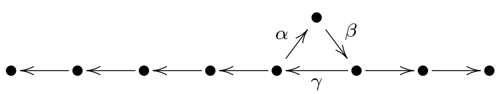
Proposition 5.2. *Let A be a cluster-tilted not hereditary algebra. Then $\mathcal{L}_A$ and $\mathcal{R}_A$ are finite sets. In particular, A is left and right supported.*

Proof. Assume that $\mathcal{L}_A \neq \emptyset$. Since $\mathcal{L}_A$ is closed under predecessors, $\mathcal{L}_A$ contains projective modules. Let P be such a module. By [CL02, (1.1)] and Lemma 5.1, there exists an integer $m \geq 0$ such that $\tau^{-m}P$ is a successor of an injective module. Let m be minimal for this property. Then, by Lemma 4.1, we have $\tau^{-m-1}P \notin \mathcal{L}_A$ and so $\tau^{-m}P \in \mathcal{E}$. Since this holds for any projective in $\mathcal{L}_A$, this shows that A is left supported by [ACT04, (3.3)], and that $\mathcal{L}_A$ is finite by [ACT04, (5.4)]. Dually, $\mathcal{R}_A$ is finite. $\square$

As a consequence, we get a straightforward characterization of the cluster-tilted algebras which are laura. Recall from [AC03] that an algebra A is *laura* provided the set $\operatorname{ind} A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ is finite.

Corollary 5.3. *A cluster-tilted algebra is laura if and only if it is hereditary or representation finite.*

Example 5.4. Let A be the cluster-tilted algebra (of type $\mathbb{A}_8$) given by the quiver



with the relations $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$. Its AR-quiver is given in Figure 1, in which the projective modules are identified with circles and the injective modules are identified with squares. The left part $\mathcal{L}_A$ has two clearly identified connected components, that one can compute using Lemma 4.1.

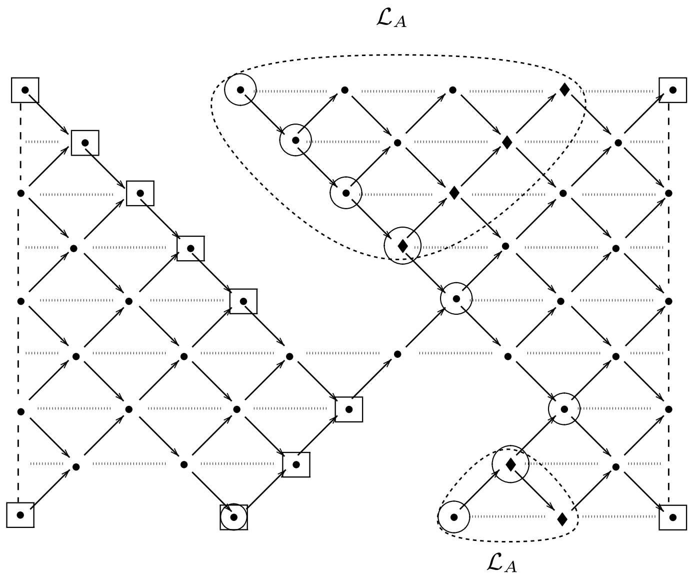


FIGURE 1. AR-quiver of the algebra of Example 5.4.

Both ends are identified along the vertical dotted lines, in the inverse order like a Mobiüs band. Finally, the black diamonds represent the (indecomposable) Ext-injective modules in $\operatorname{add}\mathcal{L}_A$.

DEFINITION 5.5. Let A be an artin algebra and $\overline{P}$ denote the direct sum of all indecomposable projective modules in $\mathcal{L}_A$. The algebra $A_\lambda = \operatorname{End}_A(\overline{P})^{op}$ is called the *left support algebra* of A.

In [ACT04], [Sko03], the left support algebra was studied and shown to be a direct product of quasitilted algebras, hence not connected in general. In Example 5.4, one can observe that A_λ is a direct product of (two) hereditary algebras, and also that $\mathcal{E}_1 = \emptyset$ since, equivalently, $\mathcal{L}_A$ contains no injective module. Also, the left part is given by the modules which are not successors of any injective module. This is not a coincidence as the following results show.

PROPOSITION 5.6. *Let A be a Gorenstein algebra of Gorenstein dimension at most one. The left support algebra A_λ is a direct product of hereditary algebras.*

Proof. Since $\mathcal{L}_A \subseteq \operatorname{mod} A_\lambda$ by [ACT04], it suffices to show that if P is a projective module in $\mathcal{L}_A$ and $M \longrightarrow P$ is an irreducible morphism, then M is projective. If M is not projective, then $\tau M \neq 0$, and thus $\operatorname{Hom}_A(\tau^{-1}(\tau M), P) \neq 0$.

By Lemma 2.1, this gives $\mathrm{id}_A \tau M > 1$. The Gorenstein property then implies $\mathrm{pd}_A \tau M > 1$ (see Section 1.4), a contradiction to $\tau M \in \operatorname{add}\mathcal{L}_A$. Thus, M is projective. □

Corollary 5.7. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. If A is not hereditary, then $\mathcal{E}_1 = \emptyset$.*

Proof. We first show that if I' is an injective module in $\mathcal{L}_A$ and $f : I' \longrightarrow M$ is an irreducible morphism, with M indecomposable, then $M \in \mathcal{L}_A$. Indeed, assume that $M \notin \mathcal{L}_A$. Then τM is Ext-injective in $\mathcal{L}_A$ and $\tau M \in \mathcal{E}_1 \cup \mathcal{E}_2$ (observe that M is not projective otherwise f would be an isomorphism). If $M \in \mathcal{E}_1$, then there exists an injective module I'' in $\mathcal{L}_A$ together with a path $I'' = N_0 \longrightarrow \cdots \longrightarrow N_m = \tau M \longrightarrow I'$ in $\mathcal{L}_A$. Now, since A_λ is hereditary by Proposition 5.6, τM is injective, and so $M = 0$, a contradiction. Hence, $\tau M \in \mathcal{E}_2$. Then, there exists an indecomposable projective module $P \notin \mathcal{L}_A$ and a sectional path $\delta : P \rightsquigarrow M$. Let T' be the direct summand of T such that $\mathcal{H}_T(T') = P$ and T'' be the direct summand of T such that $\mathcal{H}_T(T''[2]) = I'$. Then, lifting the path δ in $\mathcal{C}_H$, and using the fact that δ does not factor through I' (since $I' \in \mathcal{L}_A$), yields a sectional (thus nonzero) path from T' to $T''[1]$, a contradiction to $\operatorname{Hom}_{\mathcal{C}_H}(T, T[1]) \neq 0$. Therefore $M \in \mathcal{L}_A$.

Now, assume that I is an injective module in $\mathcal{L}_A$. Let Γ be the connected component of the AR-quiver of A containing I and Σ be the maximal full, connected and convex subquiver of Γ containing only indecomposable injective modules, including I. Observe that since $\mathcal{L}_A$ is closed under predecessors, and in view of the first part of the proof, any injective module in Σ lies in $\mathcal{L}_A$. Now, dualizing the arguments in the proof of Lemma 5.1 yields an injective module I' in Σ together with an irreducible morphism $I' \longrightarrow M$, where M is not injective. But since $I' \in \mathcal{L}_A$, we get $M \in \mathcal{L}_A$ by the first part of the proof, a contradiction to the fact that A_λ is a direct product of hereditary algebras (since M is not injective). □

Thus, the left part of a cluster-tilted not hereditary algebra contains no injective modules. We get the following easy consequence of Lemma 4.1.

Corollary 5.8. *Let A be a cluster-tilted algebra. If A is not hereditary, then*

$$\mathcal{L}_A = \{M \in \operatorname{ind} A : M \textit{ is not a successor of an injective module}\},$$
$$\mathcal{R}_A = \{M \in \operatorname{ind} A : M \textit{ is not a predecessor of a projective module}\}.$$

The following lemma, whose proof follows directly from the above corollary, will be useful in the next section.

Lemma 5.9. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. The functor $\operatorname{Hom}_{\mathcal{C}_H}(T, -)$ induces an equivalence $\mathcal{L}_T \longrightarrow \mathcal{L}_A$, where $\mathcal{L}_T$ is the set of all indecomposable objects M in $\mathcal{C}_H \setminus$*

$\operatorname{add} T[1]$ *such that if there exists a path from an indecomposable object in* $\operatorname{add} T[2]$ *to* M, *then* (*at least one morphism in*) *this path factors through* $\operatorname{add} T[1]$.

Proposition 5.6 has another nice direct consequence.

COROLLARY 5.10. *Let* A *be a Gorenstein algebra of Gorenstein dimension at most one. Then* A *is hereditary if and only if* $A \in \operatorname{add} \mathcal{L}_A$, *and this occurs if and only if* A *is quasitilted* (*see* [HRS96, (II.1.14)]).

5.2. Endomorphism algebras of $\mathcal{L}_A$-slices. Here, we introduce the concept of $\mathcal{L}_A$-slices and show that if A is cluster-tilted, then these $\mathcal{L}_A$-slices induce tilting modules whose endomorphism algebras are again cluster-tilted.

We first recall the following definition.

DEFINITION 5.11. Let (Γ, τ) be a connected translation quiver. A connected full subquiver Σ of Γ is a *section* in Γ if:

(S1) Σ is acyclic;
(S2) For each $x \in \Gamma$, there exists a unique $n \in \mathbb{Z}$ such that $\tau^n x \in \Sigma$;
(S3) Σ is convex in Γ.

In case Γ is a connected component of the AR-quiver of an algebra A, then a section Σ is called a *complete slice* provided it is faithful and $\operatorname{Hom}_A(X, \tau Y) = 0$ for each $X, Y \in \Sigma$.

The well-known criterion of Liu and Skowroński (see [ASS06, (VIII.5.6)] for instance) asserts that an algebra A is tilted if and only if its AR-quiver has a connected component containing a complete slice.

By [ACT04, Theorem B], an algebra A is left supported if and only if each connected component of A_λ is a tilted algebra and the restriction of E (see Section 4.3) to this component is a complete slice. Since, by construction, we have $\mathcal{L}_A \subseteq \operatorname{mod} A_\lambda \subseteq \operatorname{mod} A$, this motivates the following definition.

DEFINITION 5.12. Let A be an algebra and $A_\lambda = A_1 \times \cdots \times A_m$ be its left support algebra. An $\mathcal{L}_A$*-slice* is a direct product $S = S_1 \times \cdots \times S_m$, with each S_i a complete slice in $\operatorname{mod} A_i \cap \mathcal{L}_A$.

Such $\mathcal{L}_A$-slices do not always exist, for instance when $A = A_\lambda$ is a quasitilted not tilted algebra, or worse when $\mathcal{L}_A = \emptyset$. Here, we give two canonical examples of $\mathcal{L}_A$-slices when A is cluster-tilted.

EXAMPLE 5.13. Let A be a cluster-tilted algebra such that $\mathcal{L}_A \neq \emptyset$.

(a) By Proposition 5.6, A_λ is a direct product of hereditary algebras. Then the full subquiver generated by the set $\Sigma_P = \{P_1, \ldots, P_n\}$ of indecomposable projective modules in $\mathcal{L}_A$ is an $\mathcal{L}_A$-slice.
(b) By Proposition 5.2, A is left supported. Hence, by [ACT04, Theorem B], the direct sum E of the indecomposable Ext-injective modules in $\operatorname{add} \mathcal{L}_A$ is an $\mathcal{L}_A$-slice (compare with Example 5.4).

Clearly, these two examples are extremal, in the sense that any $\mathcal{L}_A$-slice lies between these two. Moreover, we get the following lemma.

LEMMA 5.14. *Let A be a cluster-tilted not hereditary algebra. Let Σ_P be the $\mathcal{L}_A$-slice generated by the projective modules in $\mathcal{L}_A$. Then any $\mathcal{L}_A$-slice Σ can be reached from Σ_P by a finite number of almost split exchanges.*

Proof. Let Σ be an $\mathcal{L}_A$-slice and $P_1, \dots, P_n$ be the vertices of Σ_P. Assume that Σ_P has a source P_i which is not in Σ. Then replacing in Σ_P the module P_i by $\tau^{-1}P_i$, and all arrows $P_i \longrightarrow P_j$ by their corresponding arrows $P_j \longrightarrow \tau^{-1}P_i$ yields a new $\mathcal{L}_A$-slice Σ'_P. By iterating this procedure and invoking that $\mathcal{L}_A$ is finite by Proposition 5.2, we get after finitely many steps the $\mathcal{L}_A$-slice Σ. □

Clearly, by using the above procedure, the number of needed almost split exchanges to reach the $\mathcal{L}_A$-slice Σ is uniquely determined. Indeed, if $\Sigma = \{M_1, \dots, M_n\}$ with $M_i = \tau^{-t_i}P_i$ for each i, then the number of required exchanges is given by $t_\Sigma = \sum_{i=1}^n t_i$. In particular, when $\Sigma = E$ (see Section 4.3), then $t_\Sigma = |\mathcal{L}_A| - n$, where n denotes the number of indecomposable projective modules in $\mathcal{L}_A$.

We can now prove Theorem 3.

THEOREM 3. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and A be the cluster-tilted algebra* $\operatorname{End}_{\mathcal{C}_H}(T)^{op}$. *Assume that A is not hereditary and let Σ be an $\mathcal{L}_A$-slice. Also, let $F = \bigoplus_{i=1}^m P_i$ denote the direct sum of all indecomposable projective modules not in $\mathcal{L}_A$. Then*

(a) $T_\Sigma = \Sigma \oplus F$ *is a tilting A-module*;
(b) *The algebra* $A_\Sigma = \operatorname{End}_A(T_\Sigma)^{op}$ *is isomorphic to* $\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(T_\Sigma))^{op}$, *in particular A_Σ is cluster-tilted*;
(c) *The quiver of A_Σ is obtained from that of A with a finite number of reflections at sinks.*

Proof. (a) We prove a more general fact. Let n be the number of indecomposable projective modules in $\mathcal{L}_A$ and $M_1, \dots, M_n$ be A-modules in $\mathcal{L}_A (\subseteq \operatorname{mod} A_\lambda)$ such that $\operatorname{Hom}_{A_\lambda}(M_i, \tau M_j) = 0$ for all i, j. Since $\mathcal{L}_A$ is closed under predecessors, we get $0 = \operatorname{Hom}_A(M_i, \tau M_j) = \operatorname{Ext}^1_A(M_j, M_i)$ for all i, j. Let $\Sigma = \bigoplus_{i=1}^n M_i$ and $T_\Sigma = M \oplus F$. Then $\operatorname{Ext}^1_A(\Sigma, F) \cong D\operatorname{Hom}_A(F, \tau\Sigma) = 0$, and since $\operatorname{pd}_A T_\Sigma \leq 1$ by construction, T_Σ is a tilting A-module.

(b) By Theorem 1, $\mathcal{H}_T^{-1}(T_\Sigma)$ is a tilting object in $\mathcal{C}_H$. So $\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(T_\Sigma))^{op}$ is cluster-tilted. In view of the equivalence $\mathcal{H}_T : \mathcal{C}_H / \operatorname{add} T[1] \longrightarrow \operatorname{mod} A$, it then suffices to show that no morphisms between two direct summands of $\mathcal{H}_T^{-1}(T_\Sigma)$ in $\mathcal{C}_H$ factors through $\operatorname{add} T[1]$. We prove this by induction on number t_Σ of necessary almost split exchanges to reach Σ from the $\mathcal{L}_A$-slice Σ_P generated by the set of indecomposable projective modules in $\mathcal{L}_A$ (see Lemma 5.14).

Let $\Sigma = \{M_1, \ldots, M_n\}$. Also, let $S_1, \ldots, S_n$ be the indecomposable objects in $\mathcal{C}_H$ such that $\mathcal{H}_T(S_i) = M_i$ for all i, and let $T_1, \ldots, T_m$ be the indecomposable direct summands of T corresponding to the indecomposable direct summands $P_1, \ldots, P_m$ of F, that is $\mathcal{H}_T(T_j) = P_j$ for all j. In other words,

$$\mathcal{H}_T^{-1}(T_\Sigma) = \left(\bigoplus_{i=1}^{n} S_i\right) \oplus \left(\bigoplus_{j=1}^{m} T_j\right).$$

If $t_\Sigma = 0$, then $\Sigma = \Sigma_P$ and $\mathcal{H}_T^{-1}(T_\Sigma) = T$. The claim then follows from $\mathrm{Hom}_{\mathcal{C}_H}(T, T[1]) = 0$. Assume that $t_\Sigma > 0$. Since each connected component of Σ is acyclic, Σ contains some sinks. Also, since A_λ is hereditary, some of these sinks are not projective. Assume that M_1 is a nonprojective sink in Σ and consider the $\mathcal{L}_A$-slice Σ' obtained by replacing in Σ the module M_1 by τM_1 and all arrows $M_i \longrightarrow M_1$ by their corresponding arrows $\tau M_1 \longrightarrow M_i$. So $\Sigma' = \{\tau M_1, M_2, \ldots, M_n\}$. We have $t_{\Sigma'} < t_\Sigma$, and thus, by induction, no morphisms between two direct summands of

$$\mathcal{H}_T^{-1}(T_{\Sigma'}) = \mathcal{H}_T^{-1}(\Sigma') \oplus \mathcal{H}_T^{-1}(F) = (\tau S_1 \oplus S_2 \oplus \cdots \oplus S_n) \oplus \left(\bigoplus_{j=1}^{m} T_j\right)$$

in $\mathcal{C}_H$ factors through $\mathrm{add}\, T[1]$.

To prove our claim, we then have to show that no morphisms in one of the Hom-spaces: (i) $\mathrm{Hom}_{\mathcal{C}_H}(S_1, S_i)$, (ii) $\mathrm{Hom}_{\mathcal{C}_H}(S_i, S_1)$, (iii) $\mathrm{Hom}_{\mathcal{C}_H}(S_1, T_j)$, and (iv) $\mathrm{Hom}_{\mathcal{C}_H}(T_j, S_1)$, for $1 \le i \le n$ and $1 \le j \le m$, factors through $\mathrm{add}\, T[1]$.

(i) For each $i = 2, \ldots, n$, we have $\mathrm{Hom}_{\mathcal{C}_H}(S_1, S_i) \cong \mathrm{Hom}_{\mathcal{C}_H}(\tau S_1, \tau S_i) = 0$ because $\mathcal{H}_T^{-1}(T_{\Sigma'})$ is a tilting object in $\mathcal{C}_H$. This is sufficient. The case $i = 1$ is proven in (ii).

(ii) Let $i \in \{1, 2, \ldots, n\}$ and $f : S_i \longrightarrow S_1$ be a nonzero morphism. If f is an isomorphism, then $i = 1$ and f does not factor through $\mathrm{add}\, T[1]$ since $\mathcal{H}_T(S_1) = M_1 \neq 0$. Assume now that f is not an isomorphism and let

$$\tau_{\mathcal{C}_H} S_1 \xrightarrow{g} \bigoplus_{k=1}^{q} S_{1,k} \xrightarrow{h} S_1 \longrightarrow \tau_{\mathcal{C}_H} S_1[1]$$

be the almost split triangle ending at S_1. Observe that since M_1 is a sink in Σ, it follows from the construction of Σ by Σ_P (see Lemma 5.14) that $S_{1,k}$ is a vertex in $\mathcal{H}_T^{-1}(\Sigma) \cap \mathcal{H}_T^{-1}(\Sigma')$ for each k. Since f is not an isomorphism, it factors through h, namely there exists a nonzero morphism $f' : S_i \longrightarrow \bigoplus_{k=1}^{q} S_{1,k}$ such that $f = hf'$. Now, assume that f factors through $\mathrm{add}\, T[1]$. Then so does f'. Let

$$f'' = \begin{cases} f'h, & \text{if } i = 1, \\ f', & \text{if } i \in \{2, \ldots, n\}. \end{cases}$$

In any case, f'' is a nonzero morphism in $\operatorname{add}\mathcal{H}_T^{-1}(T_{\Sigma'})$ factoring through $\operatorname{add}T[1]$, a contradiction to the induction hypothesis. Hence, f does not factor through $\operatorname{add}T[1]$.

(iii) For $j=1,\dots,m$, we have $\operatorname{Hom}_{\mathcal{C}_H}(S_1,T_j)\cong\operatorname{Hom}_{\mathcal{C}_H}(\tau S_1,\tau T_j)=0$ because $T_{\Sigma'}$ is a tilting object in $\mathcal{C}_H$. This is sufficient.

(iv) Finally, since $\operatorname{Hom}_{\mathcal{C}_H}(T,T[1])=0$, no morphisms from some T_j to S_1 factors through $\operatorname{add}T[1]$.

Consequently, $\operatorname{End}_A(T_\Sigma)^{op}\cong\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(T_\Sigma))^{op}$ is cluster-tilted.

(c) We first recall a general fact: let $A=\operatorname{End}_{\mathcal{C}_H}(T)^{op}$ be a cluster-tilted algebra. Also, let $T=\overline{T}\oplus M$, with M indecomposable, and M^* be the other complement for $\overline{T}$. Finally, let $T^*=\overline{T}\oplus M^*$ and $A^*=\operatorname{End}_{\mathcal{C}_H}(T^*)^{op}$. By a result of Buan, Marsh and Reiten [BMR08] the quivers Q_A of A and Q_{A^*} of A^* are related by the quiver mutation formula of Fomin and Zelevinsky. In particular, when M corresponds to a sink in Q_A, then Q_{A^*} is obtained from Q_A by performing a reflection at this sink.

In our case, because A_λ is hereditary, each almost split exchange performed in the proof of Lemma 5.14 (in order to reach Σ from Σ_P) coincides in $\mathcal{C}_H$ with an almost split exchange of an indecomposable direct summand M of a certain tilting object, say $T_{\Sigma'}=\overline{T_{\Sigma'}}\oplus M$, by the other complement $M^*=\tau^{-1}M$ of $\overline{T_{\Sigma'}}$ (see Remark 1.2). Moreover, M corresponds to a sink in the quiver associated with $\operatorname{End}_A(T_{\Sigma'})^{op}\cong\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(T_{\Sigma'}))^{op}$. Therefore, by [BMR08], this almost split exchange coincides with a reflection at a sink in the quiver of $\operatorname{End}_A(T_{\Sigma'})^{op}$. Now, since, in the notations of (b), $A_{\Sigma_P}=A$ and Σ can be reached from Σ_P with t_Σ almost split exchanges, this means that the quiver of A_Σ can be obtained from that of A by performing t_Σ reflections at sinks. □

Recall from Theorem 4.3 that A is left supported if and only if the A-modules $L=E\oplus F$ and $U=E_1\oplus E_2\oplus F$ are tilting modules. Since L is induced by the Ext-injective modules in $\operatorname{add}\mathcal{L}_A$, it follows from the above theorem that $\operatorname{End}_A(L)^{op}$ is cluster-tilted. We now show that the same holds for $\operatorname{End}_A(U)^{op}$ although U does not arise from an $\mathcal{L}_A$-slice. At this point, we stress that since $E_1=0$ by Corollary 5.7, we have $U=\tau^{-1}E_2\oplus F=\tau^{-1}E\oplus F$.

We need the following lemma (compare with Example 5.4).

Lemma 5.15. *Let A be an algebra and $\mathcal{E}$ be the set of all indecomposable Ext-injective modules in* $\operatorname{add}\mathcal{L}_A$. *If M is a source in $\mathcal{E}$ and $f:M\longrightarrow N$ is an irreducible morphism, with N indecomposable, then $N\in\mathcal{E}$ or N is projective.*

Proof. Indeed, if $N\notin\mathcal{E}$ and N is not projective, then τN exists and belongs to $\mathcal{L}_A$ (since it is a predecessor of M). Moreover, $N\notin\mathcal{E}$ implies $N\notin\mathcal{L}_A$ since $\mathcal{E}$ is closed under successors in $\mathcal{L}_A$ by [ACT04, (3.4)]. So $\tau N\in\mathcal{E}$. But this contradicts the fact that M is a source in $\mathcal{E}$. So $N\in\mathcal{E}$ or N is projective. □

Proposition 5.16. *Let A be a cluster-tilted algebra which is not hereditary and $U=\tau^{-1}E\oplus F$ be as above. Then,*

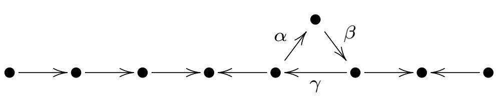
(a) *U is a tilting A-module*;
(b) *The algebra $A_U = \operatorname{End}_A(U)^{op}$ is isomorphic to $\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(U))^{op}$, in particular A_U is cluster-tilted*;
(c) *The quiver of A_U is obtained from that of A with $|\mathcal{L}_A|$ reflections at sinks.*

Proof. (a) This follows from Theorem 4.3.

(b) and (c) By Theorem 1, $\mathcal{H}_T^{-1}(U)$ is a tilting object in $\mathcal{C}_H$. Also, by continuing the procedure in the proof of Lemma 5.14, $\tau^{-1}E$ is obtained from E by performing n almost split exchanges in $\operatorname{mod} A$, where n denotes the number of projective modules in $\mathcal{L}_A$. By Lemma 5.15 and Remark 1.2, these exchanges correspond in $\mathcal{C}_H$ to (almost split) exchanges of tilting object. So, the quiver of $\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(U))^{op}$ is obtained from that of $\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(L))^{op}$ with n reflections at sinks. Since, by Theorem 3, the quiver of $\operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(L))^{op}$ is obtained from that of A with $|\mathcal{L}_A| - n$ reflections at sinks, this proves (c). Also, as in the proof of Theorem 3, one can show by induction that $\operatorname{End}_A(U)^{op} \cong \operatorname{End}_{\mathcal{C}_H}(\mathcal{H}_T^{-1}(U))^{op}$, proving (b). □

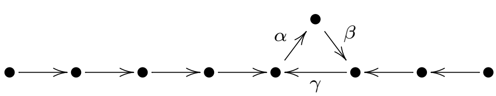
EXAMPLE 5.17. Let A be the cluster-tilted not hereditary algebra of Example 5.4. Let E be the direct sum of the indecomposable Ext-injective modules in $\operatorname{add}\mathcal{L}_A$ (those identified with black diamonds) and F be the direct sum of the three indecomposable projective modules not lying in $\mathcal{L}_A$. As usual, let $L = E \oplus F$ and $U = \tau^{-1}E \oplus F$.

(a) The algebra $\operatorname{End}_A(L)^{op}$ is the cluster-tilted algebra given by the quiver



with the relations $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$.

(b) The algebra $\operatorname{End}_A(U)^{op}$ is the cluster-tilted algebra given by the quiver



with the relations $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$.

In the above example, one can observe that the quiver of the algebra $A_U = \operatorname{End}_A(U)^{op}$ has no sinks, meaning that $\mathcal{L}_{A_U} = \emptyset$.

This phenomenon is explained by the following results, whose straightforward, but tedious proofs are left to the reader. Here, the notation $\mathcal{L}_T$ refers to the subcategory of $\mathcal{C}_H$ introduced in Lemma 5.9 and $\mathcal{R}_T$ refers to its analogue for the right part.

PROPOSITION 5.18. *Let $\mathcal{C}_H$ be a cluster category, T be a tilting object in $\mathcal{C}_H$ and $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$ be cluster-tilted not hereditary. Assume that $\Sigma = \{M_1, \ldots, M_n\}$ is an $\mathcal{L}_A$-slice having a source M_1 such that $\tau^{-1}M_1 \in \mathcal{L}_A$. Let*

$\Sigma' = \{\tau^{-1}M_1, M_2, \ldots, M_n\}$ *be the* $\mathcal{L}_A$*-slice obtained from* Σ *by performing an almost split exchange at* M_1. *Let* $T_\Sigma = \Sigma \oplus F$ *and* $T_{\Sigma'} = \Sigma' \oplus F$. *Then, in* $\mathcal{C}_H$:

(a) $\mathcal{L}_{\mathcal{H}_T^{-1}(T_{\Sigma'})} = \mathcal{L}_{\mathcal{H}_T^{-1}(T_\Sigma)} \setminus \{\mathcal{H}_T^{-1}(M_1)\}$.

(b) $\mathcal{R}_{\mathcal{H}_T^{-1}(T_{\Sigma'})} = \mathcal{R}_{\mathcal{H}_T^{-1}(T_\Sigma)} \cup \{\mathcal{H}_T^{-1}(\tau M_1)\}$.

In particular, $|\mathcal{L}_{\mathcal{H}_T^{-1}(T_{\Sigma'})}| + |\mathcal{R}_{\mathcal{H}_T^{-1}(T_{\Sigma'})}| = |\mathcal{L}_{\mathcal{H}_T^{-1}(T_\Sigma)}| + |\mathcal{R}_{\mathcal{H}_T^{-1}(T_\Sigma)}|$.

Corollary 5.19. *Let* $\mathcal{C}_H$ *be a cluster category,* T *be a tilting object in* $\mathcal{C}_H$ *and* $A = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. *Assume that* A *is not hereditary and let* $\Sigma_P = \{P_1, \ldots, P_n\}$ *be the* $\mathcal{L}_A$*-slice generated by the indecomposable projective modules in* $\mathcal{L}_A$. *Also, let* Σ *be an* $\mathcal{L}_A$*-slice or* $\tau^{-1}E$, *and assume that* Σ *can be reached from* Σ_P *with* t_Σ *almost split exchanges* (*as in Lemma* 5.14). *Finally, let* $T_\Sigma = \Sigma \oplus F$.

(a) $|\mathcal{L}_{\mathcal{H}_T^{-1}(T_\Sigma)}| = |\mathcal{L}_T| - t_\Sigma$.

(b) $|\mathcal{R}_{\mathcal{H}_T^{-1}(T_\Sigma)}| = |\mathcal{R}_T| + t_\Sigma$.

In particular, for $U = \tau^{-1}E \oplus F$, *we get* $|\mathcal{L}_{\mathcal{H}_T^{-1}(U)}| = 0$ *and* $|\mathcal{R}_{\mathcal{H}_T^{-1}(U)}| = |\mathcal{R}_T| + |\mathcal{L}_T|$.

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References

[ABS08a] I. Assem, T. Brüstle and R. Schiffler, *Cluster-tilted algebras and slices*, J. Algebra **319** (2008), 3464–3479. MR 2408327

[ABS08b] ———, *Cluster-tilted algebras as trivial extensions*, Bull. Lond. Math. Soc. **40** (2008), 151–162. MR 2409188

[ACPT07] I. Assem, J. A. Cappa, M. I. Platzeck and S. Trepode, *Some characterisations of supported algebras*, J. Pure Appl. Algebra **208** (2007), 1121–1135. MR 2283451

[AC03] I. Assem and F. U. Coelho, *Two-sided gluings of tilted algebras*, J. Algebra **269** (2003), 456–479. MR 2015287

[ACT04] I. Assem, F. U. Coelho and S. Trepode, *The left and the right parts of a module category*, J. Algebra **281** (2004), 518–534. MR 2098382

[ASS06] I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras, 1: Techniques of representation theory*, London Mathematical Society Student Texts, vol. 65, Cambridge Univ. Press, Cambridge, 2006. MR 2197389

[AS80] M. Auslander and S. O. Smalø, *Preprojective modules over artin algebras*, J. Algebra **66** (1980), 61–122. MR 0591246

[BIRS08] A. Buan, O. Iyama, I. Reiten and D. Smith, *Mutations of cluster-tilting objects and potentials*, preprint, available at arXiv:0804.3813, 2008.

[BMR$^+$06] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), 572–618. MR 2249625

[BMR07] A. Buan, R. Marsh and I. Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc. **359** (2007), 323–332. MR 2247893

[BMR08] ———, *Cluster mutation via quiver representations*, Comment. Math. Helv. **83** (2008), 143–177. MR 2365411

[CCS06] P. Caldero, F. Chapoton and R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. **358** (2006), 1347–1364. MR 2187656

[CHU94] F. U. Coelho, D. Happel and L. Unger, *Complements to partial tilting modules*, J. Algebra **170** (1994), 184–205. MR 1302837

[CL02] F. U. Coelho and M. Lanzilotta, *On non-semiregular components containing paths from injective to projective modules*, Comm. Algebra **30** (2002), 4837–4849. MR 1940467

[FZ02] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529. MR 1887642

[FL09] C. Fu and P. Liu, *Lifting to cluster-tilting objects in 2-Calabi-Yau triangulated categories*, Comm. Algebra **37** (2009), 2410–2418.

[Hap88] D. Happel, *Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge Univ. Press, Cambridge, 1988. MR 0935124

[Hap95] ———, *Selforthogonal modules*, Abelian groups and modules: Proceedings of the Padova conference (Dordrecht), Math. Appl., vol. 343, Kluwer Academic, Dordrecht, 1995, pp. 257–276. MR 1378204

[HRS96] D. Happel, I. Reiten and S. O. Smalø, *Tilting in abelian categories and quasi-tilted algebras*, Mem. Amer. Math. Soc. **120** (1996), no. 575, viii+ 88 pp. MR 1327209

[HJ08] T. Holm and P. Jørgensen, *On the relation between cluster and classical tilting*, preprint, available at arXiv:0810.0411, 2008.

[Kel05] B. Keller, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581. MR 2184464

[Kel08] ———, *Deformed CY-completions and their duals*, to appear (2008).

[KR07] B. Keller and I. Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), 123–151. MR 2313531

[KZ] S. Koenig and B. Zhu, *From triangulated categories to Abelian categories–cluster tilting in a general framework*, Math. Z. **258** (2008), 143–160. MR 2350040

[Ric89] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), 436–456. MR 1002456

[Sko03] A. Skowroński, *On artin algebras with almost all indecomposable modules of projective or injective dimension at most one*, Cent. Eur. J. Math. **1** (2003), 108–122. MR 1953044

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