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ON NULL SETS OF SOBOLEV–ORLICZ CAPACITIES

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ABSTRACT. This paper considers sufficient conditions for a Young function of type $t^p\varphi(t)$, with p greater than one, so that certain Sobolev–Orlicz capacities have the same null sets. Examples of such Young functions are given too.

1. Introduction

In [2], the authors studied Sobolev–Orlicz capacities, in particular, the null sets of these capacities. As an application, they proved vanishing exponential integrability results in [3]. It was shown in [2, Theorem 2.3 and Corollary 2.6, p. 1160] that if a Young function $\Phi\colon [0,\infty) \to [0,\infty)$ is defined by $\Phi(t) = t^p \log(e+t)^\theta$, with $p \in (1,\infty)$ and $\theta \in [0, p-1]$, then capacities $\mathcal{B}_{\alpha,\Phi}$ and $\mathcal{P}_{\alpha,\Phi}$ have the same null sets in a fixed ball $B^n(0,R_0)$, when $\alpha = n/p$. The capacities $\mathcal{B}_{\alpha,\Phi}$ and $\mathcal{P}_{\alpha,\Phi}$, with $\alpha = n/p$, are defined for all subsets of $B^n(0,R_0)$ by

$$\mathcal{B}_{\alpha,\Phi}(E) = \inf\left\{ \int_{B^n(0,R_0)} \Phi(f(x))\,dx \,\middle|\, f \text{ nonnegative, } (G_\alpha * f)(x) \ge 1 \text{ on } E \right\}$$

and

$$\mathcal{P}_{\alpha,\Phi}(E) = \inf\big\{ \|f\|_{L^\Phi(B^n(0,R_0))} \big| f \text{ nonnegative, } (G_\alpha * f)(x) \ge 1 \text{ on } E \big\}.$$

Here, $\|\cdot\|_{L^\Phi(B^n(0,R_0))}$ is the Luxemburg norm, and G_α is the Bessel kernel for $\alpha > 0$, and $G_\alpha * f$ is the convolution of G_α and f. The purpose of this note is to generalize this result to the case when a Young function Φ is of type $t^p\varphi(t)$, where φ is a more general function than the function $\log(e+t)^\theta$. The conditions for Φ are given in Section 6.

We call capacities $\mathcal{P}_{\alpha,\Phi}$ and $\mathcal{B}_{\alpha,\Phi}$ Sobolev–Orlicz capacities, since it follows from [4, Remark 3.11, p. 243] that if a Young function Φ and its complementary Young function satisfy Δ_2-condition, and k is a positive integer, then a function $u\colon \mathbb{R}^n \to \mathbb{R}$ belongs to the Sobolev–Orlicz space $W^{k,\Phi}(\mathbb{R}^n)$ if and

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only if u belongs to the Orlicz potential space $L^{k,\Phi}(\mathbb{R}^n)$. For any positive integer k, the Sobolev–Orlicz space $W^{k,\Phi}(\mathbb{R}^n)$ is defined to be the set of functions $u:\mathbb{R}^n\to\mathbb{R}$ such that u and its weak derivatives up to order k belong to the Orlicz space $L^\Phi(\mathbb{R}^n)$. The Orlicz potential space $L^{\alpha,\Phi}(\mathbb{R}^n)$ is defined for $\alpha>0$ by

$$L^{\alpha,\Phi}(\mathbb{R}^n)=\{u:\mathbb{R}^n\to\mathbb{R}|u=G_\alpha * f, f\in L^\Phi(\mathbb{R}^n)\}.$$

We are going to show in Theorem 8.1 that if Φ satisfies the conditions given in 6.1, and $p\in(1,\infty)$, and E is a subset of a fixed ball $B^n(0,R_0)$, and $\alpha=n/p$, then there is a positive constant C, depending on n, p, R_0, and φ, such that

$$\mathcal{B}_{\alpha,\Phi}(E)\le C\varphi(\mathcal{P}_{\alpha,\Phi}(E))\mathcal{P}_{\alpha,\Phi}(E)^p.$$

Further, if $P_{\alpha,\Phi}(E)>0$, then

$$C^{-1}\mathcal{P}_{\alpha,\Phi}(E)^p\left[\varphi\left(\frac{1}{\mathcal{P}_{\alpha,\Phi}(E)}\right)\right]^{-1}\le \mathcal{B}_{\alpha,\Phi}(E).$$

As a consequence of Theorem 8.1, we prove in Corollary 8.2 that the capacities $\mathcal{P}_{\alpha,\Phi}$ and $\mathcal{B}_{\alpha,\Phi}$ have the same null sets in $B^n(0,R_0)$.

In order to prove the inequality $\mathcal{B}_{\alpha,\Phi}(E)\le C\varphi(\mathcal{P}_{\alpha,\Phi}(E))\mathcal{P}_{\alpha,\Phi}(E)^p$ of Theorem 8.1, we are going to show in Theorem 7.6 that for every function f from the Orlicz space $L^\Phi(\Omega)$ there exists a positive constant C, which does not depend on f, such that

$$C^{-1}\|f\|_{L^\Phi(\Omega)}\le\left(\int_0^1 f^*(t)^p\varphi\left(\frac{1}{t}\right)dt\right)^{1/p}\le C\|f\|_{L^\Phi(\Omega)}. \tag{1.1}$$

For the definition of f^* and Orlicz spaces, we refer to Definition 3.2 and Section 5, respectively. One essential part of the proof for inequalities (1.1) is the following Hardy's inequality: if $p\in(1,\infty)$ and $t^p\varphi(t)$ satisfies the conditions in 6.1, then

$$\left[\int_0^1\left(\frac{\varphi(\frac{1}{t})^{1/p}}{t}\int_0^t f^*(s)\right)^p dt\right]^{1/p}\le\frac{p}{p-1}\left[\int_0^1 f^*(t)^p\varphi\left(\frac{1}{t}\right)dt\right]^{1/p}.$$

We prove this Hardy's inequality in Lemma 7.2.

This paper is organized as follows. In Section 2, we give notation. The definitions and basic properties of the decreasing rearrangement and the maximal function are recalled in Section 3. Sections 4 and 5 are devoted to Banach function spaces and Orlicz spaces, respectively. The definitions of Sobolev–Orlicz capacities are given in Section 6. We prove inequalities (1.1) in Section 7. The proofs for Theorem 8.1 and Corollary 8.2 are given in Section 8, along with some examples, which show that our results are more general than the ones of Adams and Hurri-Syrjänen [2].

2. Notation

In this paper, the letter C will denote a positive constant, not necessarily the same in different occurrences. If, for two quantities Q_1 and Q_2, there exists a positive constant C such that $C^{-1}Q_2 \le Q_1 \le CQ_2$, then we write $Q_1 \sim Q_2$. If Q_1 and Q_2 are two norms such that $Q_1 \sim Q_2$, then we say that Q_1 and Q_2 are equivalent. We will work in $\mathbb{R}^n$ with the Euclidean metric. If $x \in \mathbb{R}^n$ and $r > 0$,

$$B^n(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$$

is the ball with center x and radius r.

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ and let $\mathcal{M}_0(\Omega)$ denote the real valued Lebesgue measurable functions on Ω that are finite almost everywhere with respect to the Lebesgue measure. We say that a property holds almost everywhere if it holds almost everywhere with respect to the Lebesgue measure. Let m denote Lebesgue n-measure. We shall say that a set or a function is measurable if it is Lebesgue measurable. Let χ_Ω be the characteristic function of Ω defined by

$$\chi_\Omega(x) = \begin{cases} 1, & \text{when } x \in \Omega, \\ 0, & \text{when } x \notin \Omega. \end{cases}$$

When we integrate a Lebesgue measurable function, we denote Lebesgue n-measure by dx. Recall that if $p \in [1,\infty)$, then a measurable function $f : \Omega \to \mathbb{R}$ belongs to the Lebesgue space $L^p(\Omega)$ if and only if

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} < \infty.$$

We define the Bessel kernel G_α for $x \in \mathbb{R}^n$ and $\alpha > 0$ by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha-n}{2}} e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \frac{dt}{t},$$

where Γ is the function $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt$. It follows that G_α is positive, G_α belongs to $L^1(\mathbb{R}^n)$, and if $0 < \alpha < n$, then $G_\alpha(x) \sim |x|^{\alpha-n}$ in a fixed ball $B^n(0,R_0)$. For these and some other properties of the Bessel kernel, we refer to [1, pp. 9–11].

3. Decreasing rearrangement and maximal function

In the following, we are going to recall the definition of the decreasing rearrangement of a measurable, almost everywhere finite function.

Definition 3.1. Suppose that a function f belongs to $\mathcal{M}_0(\Omega)$. Distribution function of f is the function $m_f : [0,\infty) \to [0,\infty]$ defined by

$$m_f(\lambda) = m\big(\{x \in \Omega : |f(x)| > \lambda\}\big).$$

Definition 3.2. The decreasing rearrangement of $f \in \mathcal{M}_0(\Omega)$ is the function $f^* : [0,\infty) \to [0,\infty]$ defined by

$$f^*(t) = \inf\{\lambda : m_f(\lambda) \le t\}.$$

We use the convention $\inf \emptyset = \infty$. Suppose that f and g belong to $\mathcal{M}_0(\Omega)$. Then f^* is nonnegative, decreasing, and rightcontinuous on $[0,\infty)$. If $|g| \le |f|$ almost everywhere on Ω, then $g^* \le f^*$. We have $(af)^* = |a| f^*$ for all $a \in \mathbb{R}$ and $(|f|^p)^* = (f^*)^p$ for all $p \in (0,\infty)$. Moreover, if $(f_n)_{n=1}^\infty$ is such that every f_n belongs to $\mathcal{M}_0(\Omega)$ and $|f_n| \nearrow |f|$, then $f_n^* \nearrow f^*$. The proofs of these and some other properties of the decreasing rearrangement may be found in [5, Proposition 1.7, p. 41].

Definition 3.3. Suppose that f belongs to $\mathcal{M}_0(\Omega)$. Then $f^{**} : (0,\infty) \to [0,\infty]$ is the maximal function of f^* defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds.$$

We have collected some properties of the maximal function in the following proposition.

Proposition 3.4. *Suppose that functions f and g belong to $\mathcal{M}_0(\Omega)$. Let $(f_n)_{n=1}^\infty$ be such that each f_n belongs to $\mathcal{M}_0(\Omega)$. Then f^{**} is nonnegative, decreasing, and continuous on $(0,\infty)$. Moreover:*

(1) $f^{**} \equiv 0$ *if and only if* $f = 0$ *almost everywhere;*
(2) $f^* \le f^{**}$;
(3) *if* $|g| \le |f|$ *almost everywhere then* $g^{**} \le f^{**}$;
(4) $(cf)^{**} = |c| f^{**}$ *for all* $c \in \mathbb{R}$;
(5) $(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t)$ *for all* $t > 0$;
(6) *if* $|f_n| \nearrow |f|$, *then* $f_n^{**} \nearrow f^{**}$.

These properties are shown in [5, Proposition 3.2, p. 52], except for the property (5), which is shown in [5, Theorem 3.4, p. 55].

4. Banach function spaces

Although we are interested in Orlicz spaces, it turns out to be useful to study a more general setting, namely Banach function spaces. The following definition is from [6, Definition 3.1.1, p. 64].

Definition 4.1. A mapping $\rho : \mathcal{M}_0(\Omega) \to [0,\infty]$ is a Banach function norm if all functions f and g in $\mathcal{M}_0(\Omega)$ have the following properties:

(1) $\rho(f) = 0$ if and only if $f = 0$ almost everywhere; $\rho(af) = a\rho(f)$ for all $a \ge 0$, and $\rho(f+g) \le \rho(f) + \rho(g)$;

(2) if $0 \le g \le f$ almost everywhere, then $\rho(g) \le \rho(f)$;

(3) if $(f_n)_{n=1}^\infty$ is a sequence of nonnegative functions such that each function f_n belongs to $\mathcal{M}_0(\Omega)$ and $f_n \nearrow f$ almost everywhere, then $\rho(f_n) \nearrow \rho(f)$;

(4) if E is a measurable subset of Ω, and $m(E) < \infty$, then $\rho(\chi_E) < \infty$;

(5) if E is a measurable subset of Ω, and $m(E) < \infty$, and f is nonnegative, then there exists a positive constant C, possibly depending on E, such that

$$\int_E f(x)\,dx \le C\rho(f).$$

DEFINITION 4.2. Let ρ be a Banach function norm. The collection $X = X(\rho)$ of all functions f in $\mathcal{M}_0(\Omega)$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, we define

$$\|f\|_X = \rho(|f|).$$

A Banach function space X is a linear space and it is shown in [6, Theorem 3.1.3, p. 65] that $(X, \|\cdot\|_X)$ is a Banach space.

5. Orlicz spaces

The following definition is from [6, Definition 3.4.9, p. 96].

DEFINITION 5.1. A function $\Phi : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, strictly increasing, and convex, and it satisfies

$$\lim_{t\to 0+} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0.$$

It can be shown that

$$\Phi(t) = \int_0^t \phi(s)\,ds$$

for some nondecreasing, rightcontinuous function $\phi : [0,\infty) \to [0,\infty)$.

DEFINITION 5.2. Let Φ be a Young function. Its complementary Young function $\Psi : [0,\infty) \to [0,\infty)$ is defined by

$$\Psi(t) = \int_0^t \psi(s)\,ds,$$

where $\psi(s) = \sup\{r | \phi(r) \le s\}$ for $s \ge 0$.

Let Ω be a measurable subset of $\mathbb{R}^n$ and let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. The Orlicz class is defined by

$$\tilde{L}^\Phi(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, f \text{ measurable}, \int_\Omega \Phi(|f(x)|)\,dx < \infty \right\}.$$

Note that in general $\tilde{L}^\Phi(\Omega)$ is not a linear space. For example, let $\Phi(t) = e^t$ and $\Omega =]0,1[$; then the function

$$u(t) = -\frac{1}{2}\log t \in \tilde{L}^\Phi(\Omega),$$

but

$$2u(t) = -\log t \notin \tilde{L}^{\Phi}(\Omega).$$

DEFINITION 5.3. Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. Suppose that Ω is a measurable subset of $\mathbb{R}^n$, and f belongs to $\mathcal{M}_0(\Omega)$. Then the Luxemburg norm is defined by

$$\|f\|_{L^{\Phi}(\Omega)} = \inf\left\{k > 0 : \int_{\Omega} \Phi\left(\frac{|f(x)|}{k}\right) dx \leq 1\right\}.$$

It is shown in [6, Theorem 3.4.16, p. 99] that the Luxemburg norm is a Banach function norm. Hence, the collection of functions which belong to $\mathcal{M}_0(\Omega)$ and satisfy $\|f\|_{L^{\Phi}(\Omega)} < \infty$ is a Banach function space. We are now able to define Orlicz spaces.

DEFINITION 5.4. Let Φ be a Young function and let Ω be a measurable subset of $\mathbb{R}^n$. The Orlicz space $L^{\Phi}(\Omega)$ is the Banach function space of all functions $f \in \mathcal{M}_0(\Omega)$ that satisfy $\|f\|_{L^{\Phi}(\Omega)} < \infty$.

We say that a Young function $\Phi : [0,\infty) \to [0,\infty)$ satisfies Δ_2-condition, if there exists a positive constant C such that

$$\Phi(2t) \leq C\Phi(t) \quad \text{for all } t \geq 0.$$

Note that a Young function Φ satisfies Δ_2-condition if and only if for each $l > 1$ there is a constant $C(l)$ greater than one such that $\Phi(lt) \leq C(l)\Phi(t)$ for all $t \geq 0$.

REMARK 5.5. If a Young function Φ satisfies Δ_2-condition, then $\tilde{L}^{\Phi}(\Omega)$ is a linear space and $\tilde{L}^{\Phi}(\Omega) = L^{\Phi}(\Omega)$.

It is usually difficult to calculate a complementary Young function. The following result, shown in [7, Theorem 4.3, p. 26], makes it sometimes easier to see whether a complementary Young function satisfies Δ_2-condition.

PROPOSITION 5.6. *Suppose that a Young function Φ has a strictly increasing continuous derivative. Then the complementary Young function Ψ satisfies Δ_2-condition if and only if there exists a real number β such that*

$$\frac{t\Phi'(t)}{\Phi(t)} > \beta > 1 \quad \textit{for all } t > 0.$$

We need the next lemma in the proof of Theorem 8.1.

LEMMA 5.7. *Let Φ be a Young function, which satisfies Δ_2-condition. Suppose that f belongs to $L^{\Phi}(\Omega)$, and $\|f\|_{L^{\Phi}(\Omega)} > 0$. Then*

$$\int_{\Omega} \Phi\left(\frac{|f(x)|}{\|f\|_{L^{\Phi}(\Omega)}}\right) dx = 1.$$

Proof. Let us show that $\int_\Omega \Phi(\frac{|f(x)|}{\|f\|_{L^\Phi}})\,dx \le 1$. Let $(k_n)_{n=1}^\infty$ be a sequence of positive numbers such that $k_n \searrow \|f\|_{L^\Phi}$. By Fatou's lemma

$$\begin{aligned}\int_\Omega \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi}}\right) dx &= \int_\Omega \liminf_{n\to\infty} \Phi\left(\frac{|f(x)|}{k_n}\right) dx \\ &\le \liminf_{n\to\infty} \int_\Omega \Phi\left(\frac{|f(x)|}{k_n}\right) dx \le 1.\end{aligned}$$

Let us show that $\int_\Omega \Phi(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}})\,dx \ge 1$. Since Φ satisfies Δ_2-condition, for each number l greater than one there is a constant $C(l)$ depending on l such that $\Phi(lt) \le C(l)\Phi(t)$ for all $t \ge 0$. Note that since Φ is continuous, $C(l) \searrow 1$ as $l \searrow 1$. If there existed $l > 1$ such that

$$\int_\Omega \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\right) dx < \frac{1}{C(l)},$$

then

$$\begin{aligned}1 &> C(l) \int_\Omega \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\right) dx \ge \int_\Omega \Phi\left(l\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\right) dx \\ &= \int_\Omega \Phi\left(\frac{|f(x)|}{\frac{1}{l}\|f\|_{L^\Phi(\Omega)}}\right) dx.\end{aligned}$$

Since $\frac{1}{l}\|f\|_{L^\Phi(\Omega)} < \|f\|_{L^\Phi(\Omega)}$, the above inequality contradicts with the fact that $\|f\|_{L^\Phi(\Omega)} = \inf\{k > 0 : \int_\Omega \Phi(\frac{|f(x)|}{k})\,dx \le 1\}$. Thus,

$$\int_\Omega \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\right) dx \ge \frac{1}{C(l)} \quad \text{for all } l > 1.$$

Therefore,

$$\int_\Omega \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\right) dx \ge 1. \qquad \square$$

The following result is useful. The proof may be found in [6, Corollary 3.2.8, p. 75].

Proposition 5.8. *Let $\Phi : [0,\infty) \to [0,\infty)$ be continuous, strictly increasing, and such that $\Phi(0) = 0$. If $f \in \mathcal{M}_0(\Omega)$, then*

$$\int_\Omega \Phi(|f(x)|)\,dx = \int_0^{m(\Omega)} \Phi(f^*(t))\,dt.$$

6. Capacities

In this section, we introduce the capacities we are going to study.

CONDITIONS FOR Φ 6.1. Let $\varphi : [0,\infty) \to [1,\infty)$ be an increasing function such that for some positive constant C

$$\varphi(t^2) \le C\varphi(t) \quad \text{for all } t \in [0,\infty), \tag{6.1}$$

and there exists a positive number ε less than one such that

$$\int_0^1 t^{-\varepsilon}\varphi\left(\frac{1}{t}\right) dt < \infty. \tag{6.2}$$

Further, suppose that $p \in (1,\infty)$ and let $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(t) = t^p\varphi(t)$ be a Young function such that its complementary Young function satisfies Δ_2-condition.

Note that if Φ is of the above type, then Φ satisfies Δ_2-condition. Moreover, it follows from (6.1) that there is a positive constant C such that

$$\varphi(st) \le C\varphi(s)\varphi(t) \quad \text{for all } s,t \in [0,\infty). \tag{6.3}$$

DEFINITION 6.2. Suppose that a function Φ satisfies the conditions in 6.1. Fix positive R_0 and let E be any subset of $B^n(0,R_0)$. Suppose that $\alpha = n/p$. We define

$$\mathcal{B}_{\alpha,\Phi}(E) = \inf\left\{\int_{B^n(0,R_0)} \Phi(f(x))\,dx \,\middle|\, f \text{ nonnegative, } (G_\alpha * f)(x) \ge 1 \text{ on } E\right\}$$

and

$$\mathcal{P}_{\alpha,\Phi}(E) = \inf\{\|f\|_{L^\Phi(B^n(0,R_0))} | f \text{ nonnegative, } (G_\alpha * f)(x) \ge 1 \text{ on } E\}.$$

Here, $\inf\emptyset = \infty$.

Since Φ satisfies Δ_2-condition, Remark 5.5 implies that $\|f\|_{L^\Phi(B^n(0,R_0))}$ and $\int_{B^n(0,R_0)} \Phi(f(x))\,dx$ are finite if and only if $f \in L^\Phi(B^n(0,R_0))$. Hence, we may take the infimum in Definition 6.2 over all functions belonging to the set

$$\{f \in L^\Phi(B^n(0,R_0)) | f \text{ is nonnegative, } (G_\alpha * f)(x) \ge 1 \text{ on } E\}.$$

We have collected the basic properties of the capacities $\mathcal{P}_{\alpha,\Phi}$ and $\mathcal{B}_{\alpha,\Phi}$ in the next proposition. Since these properties are shown in [1, pp. 25–26] for L^p-capacities and the proofs are similar, we omit the proof.

PROPOSITION 6.3. *The capacities $\mathcal{P}_{\alpha,\Phi}$ and $\mathcal{B}_{\alpha,\Phi}$ have the following properties*:

(1) $\mathcal{P}_{\alpha,\Phi}(\emptyset) = 0$ *and* $\mathcal{B}_{\alpha,\Phi}(\emptyset) = 0$;

(2) *if E_1 and E_2 are subsets of $B^n(0,R_0)$, and $E_1 \subset E_2$, then* $\mathcal{P}_{\alpha,\Phi}(E_1) \le \mathcal{P}_{\alpha,\Phi}(E_2)$ *and* $\mathcal{B}_{\alpha,\Phi}(E_1) \le \mathcal{B}_{\alpha,\Phi}(E_2)$;

(3) *if* $(E_i)_{i=1}^{\infty}$ *is a sequence of sets such that* $\bigcup_{i=1}^{\infty} E_i$ *is a subset of* $B^n(0, R_0)$, *then*

$$\mathcal{P}_{\alpha,\Phi}\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mathcal{P}_{\alpha,\Phi}(E_i) \quad \textit{and} \quad \mathcal{B}_{\alpha,\Phi}\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mathcal{B}_{\alpha,\Phi}(E_i);$$

(4) *if* E *is a subset of* $B^n(0, R_0)$, *then*

$$\mathcal{P}_{\alpha,\Phi}(E) = \inf\{\mathcal{P}_{\alpha,\Phi}(U) | U \text{ is open, } E \subset U\}$$

and

$$\mathcal{B}_{\alpha,\Phi}(E) = \inf\{\mathcal{B}_{\alpha,\Phi}(U) | U \text{ is open, } E \subset U\}.$$

7. A new norm in $L^{\Phi}(\Omega)$

In the rest of this paper, we suppose that the function $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(t) = t^p \varphi(t)$ satisfies the conditions in 6.1. In this section, we assume that Ω has finite measure. For simplicity, let $m(\Omega) = 1$. We shall show that if $f \in L^{\Phi}(\Omega)$, then

$$\left(\int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} \sim \|f\|_{L^{\Phi}(\Omega)}.$$

PROPOSITION 7.1. *Suppose that a function* f *belongs to* $\mathcal{M}_0(\Omega)$. *Then the function* f *belongs to* $L^{\Phi}(\Omega)$ *if and only if*

$$\left(\int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} < \infty. \tag{7.1}$$

Proof. Since Φ is a Young function and Φ satisfies Δ_2-condition, it follows from Remark 5.5 that $f \in L^{\Phi}(\Omega)$ if and only if $\int_\Omega \Phi(|f(x)|)\, dx < \infty$.

Assume that (7.1) holds. Let us first show that f belongs to $L^1(\Omega)$. Since $\int_\Omega |f(x)|^p\, dx = \int_0^1 f^*(t)^p\, dt$ by Proposition 5.8, and $\varphi(\frac{1}{t}) \ge 1$ for all $t \in (0,1)$, we have

$$\int_\Omega |f(x)|^p\, dx = \int_0^1 f^*(t)^p\, dt \le \int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt < \infty.$$

It follows from Hölder's inequality that

$$\int_\Omega |f(x)|\, dx \le m(\Omega)^{\frac{p-1}{p}} \left(\int_\Omega |f(x)|^p\, dx\right)^{1/p}.$$

Thus, f belongs to $L^1(\Omega)$.

If $t \in (0,1)$, then

$$f^*(t) \le f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds \le \frac{1}{t}\int_0^1 f^*(s)\, ds = \frac{\|f\|_{L^1(\Omega)}}{t}.$$

Further, φ is increasing, $\varphi(st) \leq C\varphi(s)\varphi(t)$ for all $s, t \in [0, \infty)$, and, by Proposition 5.8, $\int_\Omega \Phi(|f(x)|)\,dx = \int_0^1 \Phi(f^*(t))\,dt$. Hence, we obtain

$$\begin{aligned}\int_\Omega |f(x)|^p \varphi(|f(x)|)\,dx &= \int_0^1 f^*(t)^p \varphi(f^*(t))\,dt \\ &\leq \int_0^1 f^*(t)^p \varphi\left(\frac{\|f\|_{L^1(\Omega)}}{t}\right) dt \\ &\leq C\varphi(\|f\|_{L^1(\Omega)}) \int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt < \infty.\end{aligned}$$

Therefore, f belongs to $L^\Phi(\Omega)$.

Suppose then that f belongs to $L^\Phi(\Omega)$. It follows from Proposition 5.8 that

$$\int_0^1 f^*(t)^p \varphi(f^*(t))\,dt < \infty.$$

Since Φ satisfies the conditions in 6.1, there is a positive number ε less than one such that $\int_0^1 t^{-\varepsilon}\varphi(\frac{1}{t})\,dt < \infty$. Let us set

$$G = \{t \in [0,1] | f^*(t)^p > t^{-\varepsilon}\}$$

and $F = [0,1] \backslash G$. Note that either G or F may be empty.

Since $f^*(t) > t^{-\frac{\varepsilon}{p}} \geq 1$ on G, we have $f^*(t)^{\frac{p}{\varepsilon}} \leq f^*(t)^{2^k}$ on G for some positive integer k. It follows from (6.1) that $\varphi(f^*(t)^{\frac{p}{\varepsilon}}) \leq \varphi(f^*(t)^{2^k}) \leq C\varphi(f^*(t))$ on G. Moreover, $f^*(t)^{\frac{p}{\varepsilon}} > t^{-1}$ on G, and $f^*(t)^p \leq t^{-\varepsilon}$ on F. Hence,

$$\begin{aligned}\int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt &\leq \int_G f^*(t)^p \varphi(f^*(t)^{\frac{p}{\varepsilon}})\,dt + \int_F t^{-\varepsilon}\varphi\left(\frac{1}{t}\right) dt \\ &\leq C\int_G f^*(t)^p \varphi(f^*(t))\,dt + \int_F t^{-\varepsilon}\varphi\left(\frac{1}{t}\right) dt \\ &\leq C\int_0^1 f^*(t)^p \varphi(f^*(t))\,dt + \int_0^1 t^{-\varepsilon}\varphi\left(\frac{1}{t}\right) dt \\ &< \infty.\end{aligned}$$

Thus, the proof is complete. □

The difficulty with (7.1) is that it defines only a quasinorm in $L^\Phi(\Omega)$. We shall show that we obtain a norm in $L^\Phi(\Omega)$, if we replace f^* by f^{**}. Moreover, we shall prove that this norm is equivalent to the quasinorm in (7.1). The following Hardy's inequality is essential to us.

LEMMA 7.2. *Suppose that a function f belongs to $\mathcal{M}_0(\Omega)$. Then for all $p \in (1, \infty)$*

$$\left(\int_0^1 f^{**}(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p}. \tag{7.2}$$

Proof. It follows from [8, Theorem 1, p. 32] that if there is a positive constant B such that

$$\left(\int_t^1 s^{-p}\varphi\left(\frac{1}{s}\right)ds\right)^{\frac{1}{p}}\left(\int_0^t \varphi\left(\frac{1}{s}\right)^{-\frac{1}{p-1}}ds\right)^{\frac{p-1}{p}} \le B \tag{7.3}$$

for all $t\in(0,1)$, then

$$\left(\int_0^1 f^{**}(t)^p\varphi\left(\frac{1}{t}\right)dt\right)^{1/p} \le C\left(\int_0^1 f^{*}(t)^p\varphi\left(\frac{1}{t}\right)dt\right)^{1/p},$$

where C is a positive constant such that

$$C \le p^{\frac{1}{p}}\left(\frac{p}{p-1}\right)^{\frac{p-1}{p}}B.$$

Since

$$p^{\frac{1}{p}}\left(\frac{p}{p-1}\right)^{\frac{p-1}{p}}\frac{1}{(p-1)^{\frac{1}{p}}}=\frac{p}{p-1},$$

we need only show that inequality (7.3) holds with $B=(p-1)^{-\frac{1}{p}}$.

Since $s\mapsto\varphi(\frac{1}{s})$ is decreasing,

$$\left(\int_t^1 s^{-p}\varphi\left(\frac{1}{s}\right)ds\right)^{\frac{1}{p}} \le \left(\varphi\left(\frac{1}{t}\right)\int_t^1 s^{-p}\,ds\right)^{\frac{1}{p}} \le \varphi\left(\frac{1}{t}\right)^{\frac{1}{p}}\frac{t^{\frac{1-p}{p}}}{(p-1)^{\frac{1}{p}}}.$$

Since $s\mapsto\varphi(\frac{1}{s})^{-\frac{1}{p-1}}$ is increasing,

$$\left(\int_0^t \varphi\left(\frac{1}{s}\right)^{-\frac{1}{p-1}}ds\right)^{\frac{p-1}{p}} \le \left(\varphi\left(\frac{1}{t}\right)^{-\frac{1}{p-1}}\int_0^t ds\right)^{\frac{p-1}{p}} = \varphi\left(\frac{1}{t}\right)^{-\frac{1}{p}}t^{\frac{p-1}{p}}.$$

Hence, by combining the above estimates, we have

$$\left(\int_t^1 s^{-p}\varphi\left(\frac{1}{s}\right)ds\right)^{\frac{1}{p}}\left(\int_0^t \varphi\left(\frac{1}{s}\right)^{-\frac{1}{p-1}}ds\right)^{\frac{p-1}{p}} \le \frac{1}{(p-1)^{\frac{1}{p}}},$$

and we obtain inequality (7.2). □

PROPOSITION 7.3. *Suppose that a function f belongs to $L^\Phi(\Omega)$. Then both $f^*(\cdot)\varphi(\frac{1}{\cdot})^{1/p}$ and $f^{**}(\cdot)\varphi(\frac{1}{\cdot})^{1/p}$ belong to $L^p(0,1)$. Further,*

$$\left(\int_0^1 f^{*}(t)^p\varphi\left(\frac{1}{t}\right)dt\right)^{1/p} \sim \left(\int_0^1 f^{**}(t)^p\varphi\left(\frac{1}{t}\right)dt\right)^{1/p}. \tag{7.4}$$

Proof. By Proposition 7.1, we have $(\int_0^1 f^*(t)^p\varphi(\frac{1}{t})\,dt)^{1/p}<\infty$, that is $f^*(\cdot)\varphi(\frac{1}{\cdot})^{1/p}$ belongs to $L^p(0,1)$. It follows from Lemma 7.2 that

$$\left(\int_0^1 f^{**}(t)^p\varphi\left(\frac{1}{t}\right)dt\right)^{1/p} \le \frac{p}{p-1}\left(\int_0^1 f^{*}(t)^p\varphi\left(\frac{1}{t}\right)dt\right)^{1/p}.$$

In particular, $f^{**}(\cdot)\varphi(\frac{1}{\cdot})^{1/p}$ belongs to $L^p(0,1)$.

Since $f^*(t) \le f^{**}(t)$ for all $t \in (0,1)$, we have

$$\left(\int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} \le \left(\int_0^1 f^{**}(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p}. \tag{7.5}$$

Hence, the proof is complete. □

The following lemma is shown in [5, Corollary 1.1.9, p. 7].

Lemma 7.4. *If two Banach function spaces consist of the same set of functions, then their norms are equivalent.*

We shall show that if f^* is replaced by f^{**} in (7.1), then we obtain a Banach function norm, which turns out to be a new norm in $L^\Phi(\Omega)$.

Lemma 7.5. *The mapping* $\rho : \mathcal{M}_0(\Omega) \to [0,\infty]$ *defined by*

$$\rho(f) = \left(\int_0^1 f^{**}(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p}$$

is a Banach function norm.

Proof. The proof consists of straightforward calculations based on the properties of the maximal function. Here, we prove the triangle inequality $\rho(f + g) \le \rho(f) + \rho(g)$. We may assume that $\rho(f) + \rho(g) < \infty$. Since functions $f^{**}(\cdot)\varphi(\frac{1}{\cdot})^{1/p}$ and $g^{**}(\cdot)\varphi(\frac{1}{\cdot})^{1/p}$ belong to $L^p(0,1)$, we obtain by using property (5) of Proposition 3.4 and Minkowski's inequality,

$$\begin{aligned}
\rho(f+g) &= \left(\int_0^1 [(f+g)^{**}(t)]^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} \\
&\le \left(\int_0^1 [f^{**}(t) + g^{**}(t)]^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} \\
&= \left(\int_0^1 \left[f^{**}(t)\varphi\left(\frac{1}{t}\right)^{1/p} + g^{**}(t)\varphi\left(\frac{1}{t}\right)^{1/p}\right]^p dt\right)^{1/p} \\
&\le \left(\int_0^1 f^{**}(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} + \left(\int_0^1 g^{**}(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} \\
&= \rho(f) + \rho(g).
\end{aligned}$$

□

Theorem 7.6. *Suppose that a function* f *belongs to* $\mathcal{M}_0(\Omega)$. *Then* $f \in L^\Phi(\Omega)$ *if and only if* $(\int_0^1 f^{**}(t)^p \varphi(\frac{1}{t})\, dt)^{1/p} < \infty$. *Further, for all* $f \in L^\Phi(\Omega)$

$$\left(\int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt\right)^{1/p} \sim \|f\|_{L^\Phi(\Omega)}. \tag{7.6}$$

Proof. If $f \in L^\Phi(\Omega)$, then by Proposition 7.3, $(\int_0^1 f^{**}(t)^p \varphi(\frac{1}{t})\,dt)^{1/p} < \infty$. On the other hand, if we suppose that $(\int_0^1 f^{**}(t)^p \varphi(\frac{1}{t})\,dt)^{1/p} < \infty$, then inequality (7.5) yields $(\int_0^1 f^{*}(t)^p \varphi(\frac{1}{t})\,dt)^{1/p} < \infty$. Proposition 7.1 implies $f \in L^\Phi(\Omega)$. Thus, we have shown that $f \in L^\Phi(\Omega)$ if and only if $(\int_0^1 f^{**}(t)^p \times \varphi(\frac{1}{t})\,dt)^{1/p} < \infty$.

Let us prove (7.6). By Lemma 7.5, the collection of functions $f \in \mathcal{M}_0(\Omega)$ which satisfy $(\int_0^1 f^{**}(t)^p \varphi(\frac{1}{t})\,dt)^{1/p} < \infty$ is a Banach function space. Since we have shown that $f \in L^\Phi(\Omega)$ if and only if $(\int_0^1 f^{**}(t)^p \varphi(\frac{1}{t})\,dt)^{1/p} < \infty$, Lemma 7.4 yields

$$\left(\int_0^1 f^{**}(t)^p \varphi\left(\frac{1}{t}\right)dt\right)^{1/p} \sim \|f\|_{L^\Phi(\Omega)}$$

for all $f \in L^\Phi(\Omega)$. Hence, (7.6) follows from Proposition 7.3. □

8. Null sets for capacities $\mathcal{P}_{\alpha,\Phi}$ and $\mathcal{B}_{\alpha,\Phi}$

Recall that the function $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(t) = t^p\varphi(t)$, with $p \in (1,\infty)$, satisfies the conditions in 6.1.

THEOREM 8.1. *Let E be a subset of $B^n(0,R_0)$. Suppose that α is a positive real number such that $\alpha p = n$. Then there is a positive constant C, depending on n, p, R_0 and φ, such that*

$$\mathcal{B}_{\alpha,\Phi}(E) \le C\mathcal{P}_{\alpha,\Phi}(E)^p \varphi(\mathcal{P}_{\alpha,\Phi}(E)). \tag{8.1}$$

Further, if $\mathcal{P}_{\alpha,\Phi}(E) > 0$, then

$$C^{-1}\mathcal{P}_{\alpha,\Phi}(E)^p \left[\varphi\left(\frac{1}{\mathcal{P}_{\alpha,\Phi}(E)}\right)\right]^{-1} \le \mathcal{B}_{\alpha,\Phi}(E). \tag{8.2}$$

We set $\mathcal{P}_{\alpha,\Phi}(E)^{-1} = 0$ in (8.2), if $\mathcal{P}_{\alpha,\Phi}(E) = \infty$.

Proof. We may assume $m(B^n(0,R_0)) = 1$. Since $\mathcal{P}_{\alpha,\Phi}(E)$ is finite if and only if $\mathcal{B}_{\alpha,\Phi}(E)$ is finite, we may assume that these capacities are finite. Then the set

$$\mathcal{F}_E = \{f \in L^\Phi(B^n(0,R_0)) | f \text{ is nonnegative, } (G_\alpha * f)(x) \ge 1 \text{ on } E\}$$

is not empty.

Let us prove inequality (8.1). Suppose that a function f belongs to $\mathcal{F}_E$. Let us write

$$N^p := \int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right)dt$$

and

$$M^p := \int_0^1 f^*(t)^p \varphi(f^*(t))\,dt.$$

Then, by Theorem 7.6, $N^p \sim \|f\|^p_{L^\Phi(B^n(0,R_0))}$, and by Proposition 5.8, $M^p = \int_{B^n(0,R_0)} \Phi(f(x))\,dx$.

Since f^* is decreasing and $t \mapsto \varphi(\frac{1}{t})$ is decreasing, we have

$$f^*(t)^p \varphi\left(\frac{1}{t}\right) \le f^*(s)^p \varphi\left(\frac{1}{s}\right),$$

whenever $0 < s \le t < 1$. It follows that for all $t \in (0,1)$

$$\begin{aligned} f^*(t)^p \varphi\left(\frac{1}{t}\right) t = f^*(t)^p \varphi\left(\frac{1}{t}\right) \int_0^t ds \le \int_0^t f^*(s)^p \varphi\left(\frac{1}{s}\right) ds \\ \le \int_0^1 f^*(s)^p \varphi\left(\frac{1}{s}\right) ds = N^p. \end{aligned}$$

The above estimate yields

$$f^*(t) \le \frac{N}{t^{1/p}} \varphi\left(\frac{1}{t}\right)^{-1/p} \quad \text{for all } t \in (0,1). \tag{8.3}$$

Since $t \mapsto t^{\frac{p-1}{p}} \varphi(\frac{1}{t})^{-1/p}$ is increasing on $(0,1)$ and $\varphi(1) \ge 1$,

$$t^{\frac{p-1}{p}} \varphi\left(\frac{1}{t}\right)^{-1/p} \le \varphi(1)^{-1/p} \le 1 \quad \text{for all } t \in (0,1).$$

By inserting inequality (8.3) into the definition of M^p and using the above inequality and the fact that $\varphi(st) \le C\varphi(s)\varphi(t)$ for all $s,t \in [0,\infty)$, we obtain

$$\begin{aligned} \mathcal{B}_{\alpha,\Phi}(E) \le M^p &= \int_0^1 f^*(t)^p \varphi(f^*(t))\,dt \\ &\le \int_0^1 f^*(t)^p \varphi\left(\frac{N}{t^{1/p}} \varphi\left(\frac{1}{t}\right)^{-1/p}\right) dt \\ &\le C\varphi(N) \int_0^1 f^*(t)^p \varphi\left(t^{-\frac{1}{p}} \varphi\left(\frac{1}{t}\right)^{-1/p}\right) dt \\ &= C\varphi(N) \int_0^1 f^*(t)^p \varphi\left(t^{\frac{p-1}{p}} \varphi\left(\frac{1}{t}\right)^{-1/p} \frac{1}{t}\right) dt \\ &\le C\varphi(N) \int_0^1 f^*(t)^p \varphi\left(\varphi(1)^{-1/p} \frac{1}{t}\right) dt \\ &\le C\varphi(N) \int_0^1 f^*(t)^p \varphi\left(\frac{1}{t}\right) dt = C\varphi(N) N^p \\ &\le C\varphi\big(\|f\|_{L^\Phi(B^n(0,R_0))}\big) \|f\|^p_{L^\Phi(B^n(0,R_0))}. \end{aligned}$$

Hence, by taking infimum over all f in $\mathcal{F}_E$, we obtain

$$\mathcal{B}_{\alpha,\Phi}(E) \le C\mathcal{P}_{\alpha,\Phi}(E)^p \varphi(\mathcal{P}_{\alpha,\Phi}(E)).$$

Next, we shall find a lower bound for $\mathcal{B}_{\alpha,\Phi}(E)$. Suppose that E is a subset of $B^n(0,R_0)$ such that $\mathcal{P}_{\alpha,\Phi}(E)>0$, and f belongs to $\mathcal{F}_E$. Then $\mathcal{P}_{\alpha,\Phi}(E)\leq \|f\|_{L^\Phi(B^n(0,R_0))}$. Let us write $\lambda=\|f\|_{L^\Phi(B^n(0,R_0))}$. It follows from Lemma 5.7, and since $\varphi(st)\leq C\varphi(s)\varphi(t)$ for all $s,t\in[0,\infty)$, that

$$\begin{aligned}1&=\int_{B^n(0,R_0)}\Phi\left(\frac{f(x)}{\lambda}\right)dx=\int_{B^n(0,R_0)}\left(\frac{f(x)}{\lambda}\right)^p\varphi\left(\frac{f(x)}{\lambda}\right)dx\\&=\lambda^{-p}\int_{B^n(0,R_0)}f(x)^p\varphi\left(\frac{f(x)}{\lambda}\right)dx\\&\leq C\lambda^{-p}\varphi\left(\frac{1}{\lambda}\right)\int_{B^n(0,R_0)}f(x)^p\varphi(f(x))\,dx\\&=C\lambda^{-p}\varphi\left(\frac{1}{\lambda}\right)\int_{B^n(0,R_0)}\Phi(f(x))\,dx.\end{aligned}$$

Hence, there exists a positive constant C such that

$$C\leq\lambda^{-p}\varphi\left(\frac{1}{\lambda}\right)\int_{B^n(0,R_0)}\Phi(f(x))\,dx,$$

which yields

$$C\frac{\lambda^p}{\varphi(\frac{1}{\lambda})}\leq\int_{B^n(0,R_0)}\Phi(f(x))\,dx. \tag{8.4}$$

Since $\lambda\geq\mathcal{P}_{\alpha,\Phi}(E)>0$ and φ is increasing, we have

$$\varphi\left(\frac{1}{\lambda}\right)\leq\varphi\left(\frac{1}{\mathcal{P}_{\alpha,\Phi}(E)}\right).$$

Further, by combining the above estimate with inequality (8.4),

$$C\frac{\mathcal{P}_{\alpha,\Phi}(E)^p}{\varphi(\frac{1}{\mathcal{P}_{\alpha,\Phi}(E)})}\leq C\frac{\lambda^p}{\varphi(\frac{1}{\lambda})}\leq\int_{B^n(0,R_0)}\Phi(f(x))\,dx,$$

and by taking infimum over all functions in $\mathcal{F}_E$,

$$C\frac{\mathcal{P}_{\alpha,\Phi}(E)^p}{\varphi(\frac{1}{\mathcal{P}_{\alpha,\Phi}(E)})}\leq\mathcal{B}_{\alpha,\Phi}(E).$$

The proof is complete. □

Corollary 8.2. *The capacities $\mathcal{B}_{\alpha,\Phi}$ and $\mathcal{P}_{\alpha,\Phi}$ have the same null sets.*

Proof. Let E be a subset of $B^n(0,R_0)$. If $\mathcal{P}_{\alpha,\Phi}(E)=0$, then it follows from inequality (8.1) that $\mathcal{B}_{\alpha,\Phi}(E)=0$.

Suppose that $\mathcal{B}_{\alpha,\Phi}(E)=0$. By Proposition 6.3,

$$\mathcal{B}_{\alpha,\Phi}(E)=\inf\{\mathcal{B}_{\alpha,\Phi}(U)|U\text{ is open, }E\subset U\}.$$

Let $\varepsilon > 0$. Then there is an open set U such that $\mathcal{B}_{\alpha,\Phi}(E) < \mathcal{B}_{\alpha,\Phi}(U) < \mathcal{B}_{\alpha,\Phi}(E) + \varepsilon$. Since

$$0 < \mathcal{B}_{\alpha,\Phi}(U) \le C\varphi(\mathcal{P}_{\alpha,\Phi}(U))\mathcal{P}_{\alpha,\Phi}(U)^p,$$

we have $\mathcal{P}_{\alpha,\Phi}(U) > 0$. It follows from inequality (8.2) that

$$C^{-1}\mathcal{P}_{\alpha,\Phi}(U)^p\left[\varphi\left(\frac{1}{\mathcal{P}_{\alpha,\Phi}(U)}\right)\right]^{-1} \le \mathcal{B}_{\alpha,\Phi}(U) < \mathcal{B}_{\alpha,\Phi}(E) + \varepsilon = \varepsilon.$$

Since $\mathcal{P}_{\alpha,\Phi}(E) \le \mathcal{P}_{\alpha,\Phi}(U)$, we obtain $\mathcal{P}_{\alpha,\Phi}(E) = 0$. □

The following example shows that [2, Corollary 2.6, p. 1160] is a special case of our Corollary 8.2.

EXAMPLE 8.3. Let $p \in (1,\infty)$ and $\theta \in [0, p-1]$. Suppose that $\alpha p = n$. Then the Young function $\Phi(t) = t^p(\log(e+t))^\theta$ is an example of a function such that the capacities $\mathcal{P}_{\alpha,\Phi}$ and $\mathcal{B}_{\alpha,\Phi}$ have the same null sets.

We also show that there are other functions Φ, for which $\mathcal{P}_{\alpha,\Phi}$ and $\mathcal{B}_{\alpha,\Phi}$ have the same null sets.

EXAMPLE 8.4. Suppose that $p \in (1,\infty)$, and $\theta \in [0,\infty)$, and $\gamma \in [0,1]$. Let us show that for a Young function $\Phi(t) = t^p(\log(e+t))^\theta e^{(\log\log(e+t))^\gamma}$ the capacities $\mathcal{P}_{\alpha,\Phi}$ and $\mathcal{B}_{\alpha,\Phi}$ have the same null sets, when $\alpha = n/p$. We need only check that the function Φ satisfies the conditions in 6.1. Let us write $\varphi(t) = (\log(e+t))^\theta e^{(\log\log(e+t))^\gamma}$. Let us first show that $\varphi(t^2) \le C\varphi(t)$ for all $t \in [0,\infty)$. Since $(a+b)^\gamma \le a^\gamma + b^\gamma$ for all $\gamma \in [0,1]$ and $a, b \ge 0$, we have

$$\begin{aligned}\varphi(t^2) &= \big(\log(e+t^2)\big)^\theta e^{(\log\log(e+t^2))^\gamma} \le \big(\log\big((e+t)^2\big)\big)^\theta e^{(\log\log((e+t)^2))^\gamma}\\ &= \big(2\log(e+t)\big)^\theta e^{(\log[2\log(e+t)])^\gamma} = 2^\theta\big(\log(e+t)\big)^\theta e^{(\log 2+\log\log(e+t))^\gamma}\\ &\le 2^\theta\big(\log(e+t)\big)^\theta e^{(\log 2)^\gamma+(\log\log(e+t))^\gamma}\\ &= 2^\theta e^{(\log 2)^\gamma}\varphi(t).\end{aligned}$$

Let $\varepsilon \in (0,1)$. We shall show that

$$\int_0^1 t^{-\varepsilon}\varphi\left(\frac{1}{t}\right)dt < \infty.$$

It suffices to show that $\int_0^r t^{-\varepsilon}\varphi(\frac{1}{t})\,dt < \infty$ for some $r \in (0,1)$.

Let us set $\delta = (1-\varepsilon)/2$. Then $\delta \in (0, \frac{1}{2})$. Since

$$\lim_{t\to 0}\frac{t^{-\delta}}{\varphi(\frac{1}{t})} = \infty,$$

there exists $r \in (0,1)$ such that

$$\varphi\left(\frac{1}{t}\right) \le t^{-\delta} \quad \text{for all } t \in (0,r).$$

The above inequality yields

$$\int_0^r t^{-\varepsilon}\varphi\left(\frac{1}{t}\right)dt \le \int_0^r t^{-\varepsilon}\cdot t^{-\delta}\,dt = \int_0^r t^{-\varepsilon}\cdot t^{-\frac{(1-\varepsilon)}{2}} = \int_0^r t^{-\frac{1+\varepsilon}{2}}\,dt < \infty,$$

since $(1+\varepsilon)/2 \in (1/2,1)$.

Let us check that the complementary Young function to Φ satisfies Δ_2-condition. Since

$$\Phi'(t) = pt^{p-1}\varphi(t) + t^p\varphi'(t) \sim t^{p-1}\varphi(t),$$

we obtain that Φ' is continuous and strictly increasing. Further,

$$t\frac{\Phi'(t)}{\Phi(t)} = \frac{pt^p\varphi(t)+t^{p+1}\varphi'(t)}{t^p\varphi(t)} = p + \frac{t\varphi'(t)}{\varphi(t)} \ge p > 1 \quad \text{for all } t>0.$$

Thus, it follows from Proposition 5.6 that the complementary Young function to Φ satisfies Δ_2-condition. Therefore, we have shown that Φ satisfies the conditions in 6.1.

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