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THE BOUNDARY PROBLEM FOR L_1-PREDUALS

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ABSTRACT. Let E be an L_1-predual and $B \subset B_{E^*}$ be a boundary. We show that any bounded $\sigma(E,B)$-compact subset of E is weakly compact. We also present an example of an L_1-predual E that is not angelic in the $\sigma(E, \operatorname{ext} B_{E^*})$-topology.

1. Introduction

If E is a Banach space, let B_E stand for its closed unit ball. A subset B of the closed dual unit ball B_{E^*} is called a *boundary*, if for each $x \in E$ there exists $b \in B$ such that $\|x\| = b(x)$. We consider on E the locally convex topology $\sigma(E,B)$ generated by all functionals from B. The following open *boundary problem* was formulated by Godefroy in [13, Question V.2]:

Let $K \subset E$ be a bounded $\sigma(E,B)$-compact set. Is K weakly compact?

Despite serious effort of many mathematicians, only partial results are known. The answer is known to be positive, if

- $B = \operatorname{ext} B_{E^*}$, i.e., B consists of all extreme points of B_{E^*} (see [3, Theorem 1]),
- K is convex (see [10, Sections 8, 8.1, Corollary 1]),
- E does not contain $\ell_1[0,1]$ (see [5, Theorem D]),
- $E = \mathcal{C}(L)$ for some compact Hausdorff topological space L (see [4, Proposition 3]),
- B is relatively sequentially compact in (B_{E^*}, weak^*) (see [7, Corollary C]), or
- $E = \ell_1(\Gamma)$ (see [6, Theorem 4.9]).

Among goals of our paper is to provide the positive answer for the boundary problem in case E is an L_1*-predual*, i.e., E^* is isometric to $L_1(\mu)$ for a

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suitable measure μ. The most important examples of L_1-preduals are $\mathcal{C}(L)$ spaces and spaces of affine continuous functions on Choquet simplices (see [11, Proposition 3.23]). We refer the reader to [16] for a classification of L_1-preduals.

A more general version of the boundary problem is the following question on *angelicity* of a boundary topology:

Let E be a Banach space and $B \subset B_{E^*}$ be a boundary. Is $(B_E, \sigma(E,B))$ angelic?

(We recall that a regular topological space X is angelic, if every relatively countably compact subset A of X is relatively compact and its closure $\overline{A}$ is made up of the limits of sequences from A.) The point of this question is that its affirmative answer would provide a positive solution for the boundary problem via the Simons lemma [18, Theorem 8]. Moreover, all known examples of boundary topologies are angelic on bounded sets. We recall a classical example of an angelic space, namely the space $\mathcal{C}(L)$ endowed with the topology of pointwise convergence (see [12, Theorem 462B]). Hence, any Banach space is angelic in its weak topology (see [12, Theorem 462D]).

An ingenious construction of Moors and Reznichenko in [17, Section 4] provides an example of a Banach space E and its $\sigma(E, \operatorname{ext} B_{E^*})$-compact subset K that is not angelic in the $\sigma(E, \operatorname{ext} B_{E^*})$-topology (of course, K is not bounded in E). This answers a question asked by Cascales and Shvydkoy in [6, Problem 4.11]. In the second part of our paper, we show that the Banach space E in their construction is even an L_1-predual. Hence, the assumption of boundedness is essential in the question of angelicity of boundary topologies even for L_1-preduals. This might be of some interest since a particular example of an L_1-predual, namely a $\mathcal{C}(L)$ space, has the property that any boundary topology is angelic on $\mathcal{C}(L)$ (see [4, Theorem 5]).

We summarize the results of our paper in the following theorem.

THEOREM 1.1.

(a) *Let E be an L_1-predual and $B \subset B_{E^*}$ be a boundary. Then any bounded $\sigma(E,B)$-compact subset K of E is weakly compact. Moreover, the space $(B_E, \sigma(E,B))$ is angelic.*

(b) *There exists a subset K of an L_1-predual E such that K is a compact nonangelic space in the $\sigma(E, \operatorname{ext} B_{E^*})$-topology.*

If E is a Banach space, we write weak (respectively weak*) for the weak (respectively weak*) topology. If A is a subset of E, $\operatorname{co} A$ (respectively $\operatorname{span} A$) is the convex (respectively linear) hull of A. Throughout the paper, we consider the space E to be canonically embedded in its double dual E^{**}.

If X is a locally compact space, we write $\mathcal{M}^+(X)$ (respectively $\mathcal{M}^1(X)$) for the set of all positive (respectively probability) Radon measures on X. If X is compact, we consider $\mathcal{M}^+(X)$ endowed with the weak* topology given by all continuous functions on X. We write ε_x for the Dirac measure at a point $x \in X$.

If X is a compact convex subset of a locally convex space, a convex set $F \subset X$ is a *face*, if $x, y \in F$, whenever $x, y \in X$ and some $\alpha \in (0,1)$ satisfy $\alpha x + (1-\alpha)y \in F$. We write $\mathfrak{A}(X)$ for the Banach space of all affine continuous functions on X endowed with the sup-norm.

If μ is a probability measure on X, let $r(\mu)$ stand for the *barycenter* of μ (see [1, p. 12]). The convex cone of all convex continuous functions determines a partial ordering on $\mathcal{M}^+(X)$, namely $\mu \preceq \nu$ if and only if $\mu(f) \leq \nu(f)$ for any continuous convex function f on X. The set X is called a *Choquet simplex* (briefly a *simplex*) if for every point $x \in X$ there exists a unique probability measure μ maximal with respect to $\preceq$ such that $r(\mu) = x$ (see [2, Theorem 7.3]).

If X is a set and B a subset of X, we write τ_B for the topology of pointwise convergence on B for the space $\mathbb{R}^X$ of all functions from X to $\mathbb{R}$.

2. Boundaries of compact convex sets

A different point of view on the boundary problem is the following. Let X be a compact convex subset of a locally convex space. A set $B \subset X$ is a boundary of X, if every function from $\mathfrak{A}(X)$ attains its maximum on B (cf. [17, Section 2, p. 7]). Then the boundary problem can be reformulated as follows:

Let $K \subset \mathfrak{A}(X)$ be a bounded τ_B-compact set. Is it τ_X-compact?

To see this, we notice that a boundary $B \subset B_{E^*}$ of a Banach space E is also a boundary of the compact convex set (B_{E^*}, weak^*) in the sense mentioned above. Moreover, the topology τ_B on E is nothing else than the topology $\sigma(E, B)$ and $\tau_{B_{E^*}}$ coincides on E with the weak topology.

Conversely, if $B \subset X$ is a boundary of a compact convex set X, the dual unit ball $B_{\mathfrak{A}(X)^*}$ can be identified with $\operatorname{co}(X \cup -X)$. (We refer the reader to [1, Chapter 2, Section 2] and [2, Theorem 4.7] for proofs of this representation.) Then $B \cup -B$ is a boundary of the Banach space $\mathfrak{A}(X)$ and the topologies τ_B and $\sigma(\mathfrak{A}(X), B \cup -B)$ may be identified as well as the topology τ_X with the weak topology on $\mathfrak{A}(X)$.

In this setting, Khurana proved in [14, Theorem 1] that any bounded $\tau_{\operatorname{ext} X}$-compact subset of $\mathfrak{A}(X)$ is τ_X-compact. It also follows from his proof, or from the method of [3], that the space $(B_{\mathfrak{A}(X)}, \tau_{\operatorname{ext} X})$ is angelic.

As was already mentioned in the Introduction, the space $\mathfrak{A}(X)$ is an L_1-predual for any Choquet simplex X (see [11, Proposition 3.23]). Hence, Theorem 1.1(a) yields the following corollary.

Corollary 2.1. *Let $B \subset X$ be a boundary of a Choquet simplex X. Then*

(a) *any bounded τ_B-compact set $K \subset \mathfrak{A}(X)$ is τ_X-compact, and*

(b) *$(B_{\mathfrak{A}(X)}, \tau_B)$ is an angelic space.*

3. L_1-preduals and boundary topologies

Our solution of the boundary problem in L_1-preduals starts with Lemma 3.1. It enables to use geometrical properties of separable L_1-preduals (see Lemmas 3.2 and 3.3) and employ the technique of [3]. The key Lemma 3.4 relies upon the fact that any extreme point of B_{E^*} for a separable L_1-predual space E is even weak* exposed (see Lemma 3.3(b)). Since any weak* exposed point of B_{E^*} is contained in any boundary $B \subset B_{E^*}$, we get that the set of extreme points is contained in an arbitrary boundary. As was pointed out in [4, Theorem 6], this property already implies the positive answer for the boundary problem.

LEMMA 3.1. *Let Y be a separable subspace of an L_1-predual E. Then there exists a separable L_1-predual Z such that $Y \subset Z \subset E$.*

Proof. See [15, Chapter 7, Section 23, Lemma 1]. □

If E is a Banach space and Y a locally convex space, a multivalued mapping $\varphi : B_{E^*} \longrightarrow Y$ is called *convex* if for any $x^*, y^* \in B_{E^*}$ and $\alpha \in [0,1]$,

$$\alpha\varphi(x^*) + (1-\alpha)\varphi(y^*) \subset \varphi\big(\alpha x^* + (1-\alpha)y^*\big).$$

The mapping φ is *weak* lower semicontinuous* if

$$\varphi^{-1}(U) = \{x^* \in B_{E^*} : \varphi(x^*) \cap U \neq \emptyset\}$$

is weak* open in B_{E^*} for each $U \subset Y$ open. We say that φ is *odd* if $\varphi(-x^*) = -\varphi(x^*)$ for each $x^* \in B_{E^*}$.

A *selection for* φ is a mapping $f : B_{E^*} \longrightarrow Y$ such that $f(x^*) \in \varphi(x^*)$, $x^* \in B_{E^*}$.

THEOREM 3.2. *Let E be an L_1-predual and Y be a Fréchet space. Let $\varphi : B_{E^*} \longrightarrow Y$ be a convex odd weak* lower semicontinuous mapping with nonempty closed convex values. Let $F \subset B_{E^*}$ be a face of B_{E^*} such that $H = \mathrm{co}(F \cup -F)$ is weak* closed and $h : H \longrightarrow Y$ is a weak* continuous odd affine selection of $\varphi\restriction_H$.*

Then φ admits an affine odd weak continuous selection f such that $f\restriction_H = h$.*

Proof. See [16, Theorem 2.2] or [15, Chapter 7, Section 22, Theorem 2]. □

LEMMA 3.3. *Let E be a separable L_1-predual and $x^* \in \mathrm{ext}\, B_{E^*}$.*

(a) *If $y^* \in \mathrm{ext}\, B_{E^*} \setminus \{x^*, -x^*\}$, then there exists an element $x \in B_E$ such that $x^*(x) = 1$ and $y^*(x) = 0$.*

(b) *There exists $x \in B_E$ such that $x^*(x) = 1$ and*

$$|y^*(x)| < 1 \quad \textit{for each } y^* \in B_{E^*} \setminus \mathrm{co}(\{x^*\} \cup \{-x^*\}).$$

Proof. Assume that E is a separable L_1-predual space and $x^* \in \operatorname{ext} B_{E^*}$ is given. We denote $H = \operatorname{co}(\{x^*\} \cup \{-x^*\})$. For the proof of (a), let y^* be a point of $\operatorname{ext} B_{E^*}$ that is distinct from x^* and $-x^*$.

We define a multivalued mapping $\varphi : B_{E^*} \longrightarrow [-1,1]$ as

$$\varphi(z^*) = \begin{cases} 0, & z^* \in \{y^*, -y^*\}, \\ [-1,1], & \text{otherwise.} \end{cases}$$

Then φ is weak* lower semicontinuous odd convex mapping with nonempty closed convex values in $\mathbb{R}$. Further,

$$h : H \longrightarrow [-1,1],$$
$$\lambda x^* + (1-\lambda)(-x^*) \mapsto 2\lambda - 1, \quad \lambda \in [0,1],$$

is a weak* continuous odd affine selection of $\varphi\restriction_H$.

According to Theorem 3.2, φ admits a weak* continuous odd affine selection $f : B_{E^*} \longrightarrow [-1,1]$ such that $f = h$ on H. Hence, there exists $x \in B_E$ such that $f(z^*) = z^*(x)$, $z^* \in B_{E^*}$. Then $x^*(x) = 1$ and $y^*(x) = 0$. This concludes the proof of (a).

For the proof of (b), let $y^* \in \operatorname{ext} B_{E^*} \setminus H$ be given. Using (a), we find a point $x_{y^*} \in B_E$ such that $x^*(x_{y^*}) = 1$ and $y^*(x_{y^*}) = 0$, and a weak* open neighbourhood U_{y^*} of y^* such that $|u^*(x_{y^*})| < 1$ for each $u^* \in U_{y^*}$. Since $\operatorname{ext} B_{E^*} \setminus H$ is a weak* separable metrizable space, we can select countably many points $\{y_n^* : n \in \mathbb{N}\} \subset \operatorname{ext} B_{E^*} \setminus H$ such that

$$\operatorname{ext} B_{E^*} \Big\backslash H \subset \bigcup_{n=1}^{\infty} U_{y_n^*}.$$

We set

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n} x_{y_n^*}.$$

Then $x \in B_E$, $x^*(x) = 1$ and $|u^*(x)| < 1$ for each $u^* \in \operatorname{ext} B_{E^*} \setminus H$.

Let $z^* \in B_{E^*} \setminus H$ be given. Using the Choquet representation theorem [9, Theorem 4.43], we find a measure $\mu \in \mathcal{M}^1(B_{E^*})$ carried by $\operatorname{ext} B_{E^*}$ that represents z^*, i.e.,

$$\int_{\operatorname{ext} B_{E^*}} u^*(y)\, d\mu(u^*) = z^*(y) \quad \text{for each } y \in E.$$

Since $z^* \notin H$, $\mu(\operatorname{ext} B_{E^*} \setminus H) > 0$. Thus,

$$z^*(x) = \int_{\operatorname{ext} B_{E^*} \cap H} u^*(x)\, d\mu(u^*) + \int_{\operatorname{ext} B_{E^*} \setminus H} u^*(x)\, d\mu(u^*) < 1.$$

By symmetry, $(-z^*)(x) < 1$. This finishes the proof. □

LEMMA 3.4. *Let E be an L_1-predual and $B \subset B_{E^*}$ be a boundary. If $K \subset B_E$ is $\sigma(E,B)$-relatively countably compact, then it is weakly relatively sequentially compact.*

Proof. If E is an L_1-predual, it follows from Lemma 3.3 that E satisfies property $(\mathscr{S})$ from [4, Definition 2]. Thus, the proof of [4, Theorem 6] yields that any sequence in K has a weakly convergent subsequence, which is the required conclusion. □

Proof of Theorem 1.1(a). Let E be an L_1-predual, $B \subset E$ be a boundary and $K \subset E$ a bounded $\sigma(E,B)$-compact set. Without loss of generality, we may assume that B is symmetric and $K \subset B_E$.

Since K is $\sigma(E,B)$-countably compact, K is relatively weakly sequentially compact by Lemma 3.4. As K is $\sigma(E,B)$-closed, it is a weakly closed set as well. Hence, K is weakly compact by the Eberlein–Šmulyan theorem [9, Theorem 4.47].

For the proof of the second assertion in (a), let $A \subset B_E$ be $\sigma(E,B)$-relatively countably compact. According to Lemma 3.4, A is weakly relatively countably compact. Since the weak topology is angelic, $\overline{A}^{\text{weak}}$ is a weakly compact set. Since $\sigma(E,B)$-topology is weaker than the weak topology, $\overline{A}^{\text{weak}}$ is also $\sigma(E,B)$-compact, and hence $\sigma(E,B)$-closed. Thus,

$$\overline{A}^{\sigma(E,B)} \subset \overline{A}^{\text{weak}} \subset \overline{A}^{\sigma(E,B)}.$$

Since the identity mapping

$$\mathrm{id} : (\overline{A}^{\text{weak}}, \text{weak}) \longrightarrow (\overline{A}^{\text{weak}}, \sigma(E,B))$$

is continuous, it is a homeomorphism and both topologies coincide on $\overline{A}^{\text{weak}}$. In particular, $\sigma(E,B)$ is angelic on $\overline{A}^{\text{weak}}$, which concludes the proof. □

4. An example of an L_1-predual

The aim of this section is a proof of Theorem 1.1(b), i.e., the proof of the assertion that *there exist an L_1-predual E and its subset K such that the topological space $(K, \sigma(E, \operatorname{ext} B_{E^*}))$ is compact and nonangelic.*

For its proof, we recall an ingenious construction by Moors and Reznichenko in [17, Section 4]. They presented a general construction that produces compact convex sets with various interesting properties. In [17, Example 4.8], they found a compact convex set X and a $\tau_{\operatorname{ext} X}$-compact set $K \subset \mathfrak{A}(X)$ that is not angelic.

We briefly remind their construction and show that the set X is moreover a simplex. Thus, $\mathfrak{A}(X)$ is an L_1-predual and K is its $\tau_{\operatorname{ext} X}$-compact nonangelic set. According to Section 2, Theorem 1.1(b) follows.

GENERAL CONSTRUCTION 4.1. Let X, Y be a couple of compact convex sets such that $\operatorname{ext} Y$ is closed. Let $y_\infty \in \operatorname{ext} Y$ be fixed and $\varphi\colon \operatorname{ext} Y \setminus \{y_\infty\} \longrightarrow X \setminus \operatorname{ext} X$ be continuous and injective. We define the following subsets of $X \times Y$ as

$$A = \operatorname{ext} X \times \{y_\infty\} \quad \text{and} \quad B = \big\{(\varphi(y), y) \in X \times Y : y \in \operatorname{ext} Y \setminus \{y_\infty\}\big\}.$$

Let

$$Z = \overline{\operatorname{co}}(A \cup B).$$

LEMMA 4.2. *Let Z be constructed as above. Then the following assertions hold.*

(a) $\operatorname{ext} Z = A \cup B$.
(b) $B = \overline{B} \setminus (X \times \{y_\infty\})$, *in particular, B is a locally compact space and a Borel subset of Z.*
(c) *If X, Y are simplices, then Z is a simplex as well.*

Proof. For the proof of (a), we refer the reader to [17, Theorem 4.1].

To verify (b), we notice that this easily follows from the compactness of $\operatorname{ext} Y$.

Thus, we have to prove (c). First, we notice that X may be identified with $X \times \{y_\infty\}$.

CLAIM 4.2.1. *If λ is a maximal measure on Z, then λ is carried by $(X \times \{y_\infty\}) \cup B$.*

Proof. Given a maximal measure λ, [2, Theorem 6.8] yields that λ is carried by

$$\overline{\operatorname{ext} Z} = \overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Hence, the assertion follows from (b). $\square$

CLAIM 4.2.2. *If λ is a maximal measure on Z such that $\lambda\restriction_{X\times\{y_\infty\}}$ is nonzero, then $\lambda\restriction_{X\times\{y_\infty\}}$ is maximal on $X \times \{y_\infty\}$.*

Proof. Let $\lambda \in \mathcal{M}^1(Z)$ be a maximal measure. We write $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1 \in \mathcal{M}^+(X \times \{y_\infty\})$ and λ_2 is carried by $Z \setminus (X \times \{y_\infty\})$. Using [1, Lemma I.4.7] we find a measure $\omega \in \mathcal{M}^+(X \times \{y_\infty\})$ such that $\lambda_1 \preceq \omega$ and ω is maximal with respect to $\preceq$ (here the ordering $\preceq$ is considered on the set $X \times \{y_\infty\}$).

Given any convex continuous function f on Z, we have

$$\begin{aligned}\lambda(f) &= \int_{X\times\{y_\infty\}} f(s,t)\, d\lambda_1(s,t) + \int_B f(s,t)\, d\lambda_2 \\ &\leq \int_{X\times\{y_\infty\}} f(s,t)\, d\omega(s,t) + \int_B f(s,t)\, d\lambda_2.\end{aligned}$$

Since λ is maximal, $\lambda = \omega + \lambda_2$ and $\lambda_1 = \omega$ is maximal on $X \times \{y_\infty\}$. $\square$

Let $(x,y) \in Z$ be given. We are going to show that there exists a unique maximal measure on Z whose barycenter is (x,y).

Before stating the next claim, we recall that given a continuous mapping $\phi : W_1 \longrightarrow W_2$ of a locally compact space W_1 onto a locally compact space W_2, $\phi(\omega) \in \mathcal{M}^+(W_2)$ denotes the image of a measure $\omega \in \mathcal{M}^+(W_1)$ (we refer the reader to [12, Theorem 418I] for more information on images of Radon measures).

Claim 4.2.3. *Let* $\pi : Z \longrightarrow Y$ *denote the restriction of the projection of* $X \times Y$ *onto* Y *and let* $\psi : \operatorname{ext} Y \setminus \{y_\infty\} \longrightarrow B$ *be defined as* $\psi(y) = (\varphi(y), y)$. *Then for any measure* $\lambda \in \mathcal{M}^+(Z)$ *carried by* B, *it holds* $\psi(\pi(\lambda)) = \lambda$.

Proof. Let $C \subset B$ be a Borel set. Then

$$\psi(\pi(\lambda))(C) = \pi(\lambda)(\psi^{-1}(C)) = \lambda(\pi^{-1}(\psi^{-1}(C))) = \lambda(C).$$

This concludes the proof. □

Claim 4.2.4. *If* $\mu, \nu \in \mathcal{M}^1(Z)$ *are maximal measures with* $r(\mu) = r(\nu) = (x,y)$, *then* $\mu \restriction_B = \nu \restriction_B$.

Proof. Given μ, ν as in the premise, we write

$$\mu = \mu_1 + \mu_2, \qquad \nu = \nu_1 + \nu_2, \tag{4.1}$$

where μ_1, ν_1 are carried by $X \times \{y_\infty\}$ and μ_2, ν_2 are carried by B (here we use Claim 4.2.1). According to Claim 4.2.3, it is enough to show that

$$\pi(\mu_2) = \pi(\nu_2). \tag{4.2}$$

First, we notice that $\pi(\omega)$ is a measure carried by $\operatorname{ext} Y \setminus \{y_\infty\}$ for any $\omega \in \mathcal{M}^+(Z)$ carried by B. Thus, for verification of (4.2), it suffices to check $\pi(\mu_2)(f) = \pi(\nu_2)(f)$ for any $f \in \mathcal{C}(\operatorname{ext} Y)$ with $f(y_\infty) = 0$.

Let f be such a function. Since Y is a simplex, by [1, Theorem II.4.3] there exists a function $h \in \mathfrak{A}(Y)$ such that $h = f$ on $\operatorname{ext} Y$. We set

$$\begin{aligned} 1 \otimes h : X \times Y &\longrightarrow \mathbb{R}, \\ (s,t) &\mapsto h(t). \end{aligned}$$

Then $1 \otimes h$ is an affine continuous function on Z such that $1 \otimes h = 0$ on $X \times \{y_\infty\}$. Hence,

$$\begin{aligned} \pi(\mu_2)(f) &= \mu_2(f \circ \pi) = \mu_2(1 \otimes h) \\ &= \mu_1(1 \otimes h) + \mu_2(1 \otimes h) \\ &= \mu(1 \otimes h) = h(y) = \nu(1 \otimes h) \\ &= \cdots = \pi(\nu_2)(f). \end{aligned}$$

This proves (4.2) and concludes the proof. □

Claim 4.2.5. *If* $\mu, \nu \in \mathcal{M}^1(Z)$ *are maximal measures with* $r(\mu) = r(\nu) = (x,y)$, *then* $\mu \restriction_{X \times \{y_\infty\}} = \nu \restriction_{X \times \{y_\infty\}}$.

Proof. Let μ, ν be decomposed as in (4.1). We show first that

$$\mu_1(h) = \nu_1(h) \tag{4.3}$$

for any continuous affine function h on $X \times \{y_\infty\}$. Given such a function, we define $h \otimes 1 \in \mathfrak{A}(Z)$ similarly as above. Then by Claim 4.2.4,

$$\begin{aligned} \mu_1(h) &= \mu_1(h \otimes 1) = \mu(h \otimes 1) - \mu_2(h \otimes 1) \\ &= (h \otimes 1)(x, y) - \mu_2(h \otimes 1) \\ &= (h \otimes 1)(x, y) - \nu_2(h \otimes 1) \\ &= \cdots = \nu_1(h). \end{aligned}$$

If μ_1, ν_1 are nonzero, Claim 4.2.1yields that both μ_1 and ν_1 are maximal measures on $X \times \{y_\infty\}$. Since $X \times \{y_\infty\}$ is a simplex, equality (4.3) yields $\mu_1 = \nu_1$. This concludes the proof. □

Since Claims 4.2.1, 4.2.4, and 4.2.5 yield assertion (c), the proof is finished. □

Now, we remind Example 4.8 of [17] that provides the desired simplex.

CONSTRUCTION 4.3. We set

$$X = \overline{\text{co}}(\{0\} \cup \{e_n : n \in \mathbb{N}\}) \subset (\mathbb{R}^{\mathbb{N}}, \tau_{\mathbb{N}}),$$

where e_n, $n \in \mathbb{N}$, is the characteristic function of $\{n\}$. Then X is a metrizable simplex with $\operatorname{ext} X = \{0\} \cup \{e_n : n \in \mathbb{N}\}$ being a closed set.

Further, let $\mathcal{A}$ be a maximal almost disjoint family of infinite subsets of $\mathbb{N}$. Let $\widehat{Y} = \mathcal{A}$ be endowed with the discrete topology, $\alpha(\widehat{Y})$ be its Alexandroff compactification and y_∞ be the point in infinity (see [8, p. 170]). Setting $Y = \mathcal{M}^1(\alpha(\widehat{Y}))$, we get a simplex such that $\alpha(\widehat{Y})$ can be identified with $\operatorname{ext} Y$ via the canonical embedding (see [1, Corollary II.4.2]). We define

$$f : \operatorname{ext} Y \setminus \{y_\infty\} \longrightarrow X \setminus \operatorname{ext} X,$$
$$f(M)(n) = \begin{cases} 2^{-n}, & n \in M, \\ 0, & n \notin M, \end{cases} \qquad M \in \mathcal{A}.$$

Let Z be defined as in Construction 4.1. According to Lemma 4.2(c), Z is a simplex, and thus $\mathfrak{A}(Z)$ is an L_1-predual.

Let $\widehat{K} = \mathcal{A} \cup \mathbb{N}$ with a base of the topology defined as

$$\mathcal{B} = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \big\{\{M\} \cup (M \setminus F) : M \in \mathcal{A}, F \text{ a finite subset of } \mathbb{N}\big\}.$$

Then $\widehat{K}$ is a locally compact space. Let $\alpha(\widehat{K})$ be the Alexandroff compactification of $\widehat{K}$ and $\widehat{k}_\infty$ be the point in infinity.

Let $\pi : \alpha(\widehat{K}) \longrightarrow \mathfrak{A}(Z)$ be defined as

$$\pi(\widehat{k})(x,\mu) = \begin{cases} 2^{\widehat{k}} x(\widehat{k}), & \widehat{k} \in \mathbb{N}, \\ \mu(\{\widehat{k}\}), & \widehat{k} \in \mathcal{A}, \\ 0, & \widehat{k} = \widehat{k}_\infty. \end{cases}$$

Let $K = \pi(\widehat{K})$.

Lemma 4.4. *Let Z and K be as in Construction 4.3. Then the mapping $\pi : \alpha(\widehat{K}) \longrightarrow (K, \tau_{\text{ext}\, Z})$ is a homeomorphism and $(K, \tau_{\text{ext}\, Z})$ is a compact nonangelic space.*

Proof. The fact that π is a homeomorphism is proved in [17, Example 4.8]. It is easy to see that $\widehat{k}_\infty$ is contained in the closure of $\mathbb{N}$ and it cannot be obtained as the limit of a sequence from $\mathbb{N}$. Hence, $\widehat{K}$ is not an angelic space. Thus, K is a $\tau_{\text{ext}\, X}$-compact nonangelic space. □

Question 4.5. The example constructed above shows that there exist a simplex X and a $\tau_{\text{ext}\, X}$-countably compact set $A \subset \mathfrak{A}(X)$ such that $\overline{A}^{\tau_{\text{ext}\, X}}$ is $\tau_{\text{ext}\, X}$-compact but not all the points of the closure can be obtained as the limit of a sequence from A. This violates the second condition required in the definition of angelicity. However, this example does not answer the following question:

Let X be a compact convex set and $A \subset \mathfrak{A}(X)$ be $\tau_{\text{ext}\, X}$-relatively countably compact. Is $\overline{A}^{\tau_{\text{ext}\, X}}$ compact in the topology $\tau_{\text{ext}\, X}$?

The following observation is due to Moors. If A is assumed to be $\tau_{\text{ext}\, X}$-countably compact, A is $\tau_{\text{ext}\, X}$-compact. Indeed, for each $n \in \mathbb{N}$, the set $A \cap nB_{\mathfrak{A}(X)}$ is $\tau_{\text{ext}\, X}$-countably compact. As was mentioned in Section 2, $A \cap nB_{\mathfrak{A}(X)}$ is angelic in the topology $\tau_{\text{ext}\, X}$, and hence $A \cap nB_{\mathfrak{A}(X)}$ is $\tau_{\text{ext}\, X}$-compact. Hence, $A = \bigcup_n A \cap nB_{\mathfrak{A}(X)}$ is Lindelöf in the topology $\tau_{\text{ext}\, X}$. Since A is $\tau_{\text{ext}\, X}$-countably compact, A is $\tau_{\text{ext}\, X}$-compact (see [8, Theorem 3.10.1]).

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