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ESTIMATES OF GREEN FUNCTIONS FOR SOME PERTURBATIONS OF FRACTIONAL LAPLACIAN

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Abstract. Suppose that Y_t is a d-dimensional symmetric Lévy process such that its Lévy measure differs from the Lévy measure of the isotropic α-stable process ($0 < \alpha < 2$) by a finite signed measure. For a bounded Lipschitz open set D we compare the Green functions of the process Y with those of its stable counterpart, and we prove several comparability results, both one-sided and two-sided. In particular, assuming an additional condition about the difference between the densities of the Lévy measures, namely that it is of the order of $|x|^{-d+\varrho}$ as $|x| \to 0$, where $\varrho > 0$, we prove that the Green functions are comparable, provided D is connected.

These results apply, for example, to the relativistic α-stable process. The bounds for its Green functions were previously known for $d > \alpha$ and smooth sets. Here we consider also the one-dimensional case for $\alpha \geq 1$, and we prove that the Green functions for a bounded open interval are comparable, a case that, to the best of our knowledge, had not been treated in the literature.

1. Introduction

The purpose of the paper is to study estimates of the Green functions of bounded open sets of a symmetric Lévy process Y_t in $\mathbb{R}^d$. We assume that its Lévy measure is close in some sense, which we specify later, to the Lévy measure of the isotropic α-stable process. From the point of view of infinitesimal generators, the generator of the semigroup corresponding to Y_t can be considered as a perturbation of the fractional Laplacian by a bounded linear operator. The potential theory of stable processes has been extensively investigated in recent years (see [2], [4], [6], [15]), and there are several results providing estimates of the Green functions of bounded $C^{1,1}$ sets (see [14] and [7]) or even bounded Lipschitz sets ([12], [3]). We intend to make a comparison of the Green functions of the process Y_t with those of its stable counterpart.

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One of the first results in this direction was given in [16], where the relativistic α-stable process was considered. This is a process whose characteristic function is of the form

$$E^0 e^{iz\cdot Y_t} = e^{-t((|z|^2+m^{2/\alpha})^{\alpha/2}-m)}, \quad z \in \mathbb{R}^d,$$

where $0 < \alpha < 2$ and $m > 0$ is a parameter. Observe that for $m = 0$ this process reduces to the isotropic α-stable process. The main result of [16] says that the Green function of a bounded $C^{1,1}$ set is comparable to the Green function of the isotropic α-stable process if $d > \alpha$. Later, in [8], this result was derived by a different method. In the present paper, we develop the method from [16] to derive several extensions of the results proved therein. The main results are contained in the following two theorems.

THEOREM 1.1. *Let $D \subset \mathbb{R}^d$ be a bounded connected Lipschitz open set. Suppose that Y_t is a symmetric pure jump Lévy process in $\mathbb{R}^d$ with $d \geq 1$ and $\nu^Y(x)$ is the density of its Lévy measure. By $\widetilde{\nu}(x)$ we denote the density of the Lévy measure of the isotropic stable process and by $\widetilde{G}_D$ its Green function of D. Assume that $\sigma(x) = \widetilde{\nu}(x) - \nu^Y(x) \geq 0,\ x \in \mathbb{R}^d$, and $\sigma(x) \leqslant c|x|^{\varrho-d}$ for $|x| \leqslant 1$, where $c, \varrho > 0$. Then there exists a constant $C = C(d, \alpha, D, \varrho, c)$, such that*

$$C^{-1}\widetilde{G}_D(x,y) \leqslant G^Y_D(x,y) \leqslant C\widetilde{G}_D(x,y),$$

for all $x, y \in D$.

In the next theorem we remove the assumption about the positivity of the function σ at the cost of some mild assumption about the behaviour of the density of the Lévy measure.

THEOREM 1.2. *Let $d > \alpha$. With the same notation as in the previous theorem assume that there are positive constants c and ϱ such that $|\sigma(x)| \leqslant c|x|^{-d+\varrho}$ for $|x| \leqslant 1$, and that $\nu^Y(x)$ is bounded on $B^c(0,1)$. Then there is a constant $C = C(d, \alpha, D, \varrho, \sigma)$ such that for any $x, y \in D$,*

$$C^{-1}\widetilde{G}_D(x,y) \leqslant G^Y_D(x,y) \leqslant C\widetilde{G}_D(x,y).$$

Observe that in the first theorem the assumption about the positivity of σ enables us to not require any assumptions about the behaviour of $\nu^Y(x)$ away from the origin except that it has to be dominated by $\widetilde{\nu}$. For example, $\nu^Y(x)$ can vanish outside some neighborhood of the origin. Of course, the assumptions are readily checked for the relativistic process (see [16] for the description of the Lévy measure), so the theorem extends to bounded Lipschitz domains the main result of [16] (see also [8]). In addition, note that it covers the one-dimensional case for $\alpha \geq 1$, which was not treated in either of the two papers cited above. Actually, both papers assumed $d \geq 2$, but the proofs remain valid for $d > \alpha$. To the best of our knowledge, the one-dimensional result is new and fills a gap in the potential theory of the relativistic process.

The methods we apply are elementary and are based on the fact that for any two pure jump processes such that the difference between their Lévy measures is a *positive* and *finite* measure one can represent one of the processes as a sum of the other and an independent compound Poisson process. A different approach in taken in [8], where the problem in the $C^{1,1}$ case was tackled by the so-called drift transform technique. After obtaining the main results of the present paper, the authors found on the website of Panki Kim a paper of Kim and Lee [13] with results similar to ours, but for even more general sets (so called κ-fat sets). The method they use is essentially designed in [8], so our methods and results can be viewed as an alternative approach to the problem of comparing the Green functions. On the other hand, there is a difference between the results in [13] and in the present paper which is worthwhile to mention. Namely, one of their core assumptions is a certain condition (see Theorem 2.2 in [8]), which in our setting is equivalent to

$$\inf_{x,y\in D} \frac{\nu^Y(x-y)}{\widetilde{\nu}(x-y)} > 0.$$

Essentially this means that the Lévy measure ν^Y cannot vanish anywhere if we want to consider a domain D of a large diameter. Our method can handle the situation when the Lévy measure ν^Y vanishes outside some neighborhood of the origin, which seems not to be possible with the other method used in [8] or [13].

The paper is organized in the following way. In Section 2 we set up the notation and state the definitions and basic facts needed in the sequel. At first, we do not assume that Y_t is compared with the stable process, but instead work in a slightly more general setting, where Y_t is compared with another Lévy process X_t under appropriate assumptions about their Lévy measures. In Section 3 we prove the main estimates along with some other related results. To prove Theorem 1.2 we first prove the estimates for sets of small diameter, and then use this to prove the Boundary Harnack Principle (BHP) for the process Y_t in the case when its Lévy measure dominates the Lévy measure of the isotropic α-stable process.

2. Preliminaries

In $\mathbb{R}^d$, $d \geqslant 1$, we consider a symmetric Lévy processes X_t such that its characteristic triplet is equal to $(0, \nu, 0)$, where ν is its (nonzero) Lévy measure. That is, its characteristic function is given by

$$E^0 e^{iz\cdot X_t} = e^{-t\int_{\mathbb{R}^d}(1-\cos(z\cdot w))\nu(dw)}, \quad z \in \mathbb{R}^d.$$

If the measure ν is absolutely continuous with respect to the Lebesgue measure, then we denote its density by $\nu(x)$. We assume that the transition densities of X_t exist and we denote them by $p(t, x, y)$. Moreover, they are assumed to be bounded and defined for every $x, y \in \mathbb{R}^d$. The potential kernel for X_t is given by

$$U(x,y) = U(x-y) = \int_0^\infty p(t, x-y)dt,$$

if the integral is finite, that is, the process is transient.

We use the notation $C = C(\alpha, \beta, \gamma, \dots)$ to mean that the constant C depends on $\alpha, \beta, \gamma, \dots$. The values of constants may change from line to line, but they are always strictly positive and finite. The dependence on usual quantities (e.g., d, α) is sometimes not explicitly indicated in the notation.

We write $f \approx g$ on D to denote that the functions f and g are comparable, that is, there exists a constant C such that

$$C^{-1}f(x) \le g(x) \le Cf(x), \quad x \in D.$$

Let $D \subset \mathbb{R}^d$ be an open set. By τ_D we denote the first exit time from D, that is,

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

Next, we investigate the boundedness of the first moment of τ_D.

Lemma 2.1. *For any bounded open set D there exists a constant $C = C(D)$ such that*

$$\sup_{x \in \mathbb{R}^d} E^x \tau_D \leqslant C.$$

Proof. The proof of this lemma uses the same arguments as in the classical case of Brownian motion (see [9]). The argument therein requires the existence of $t_0 > 0$ such that $\sup_{x\in\mathbb{R}^d} P^x(X_{t_0} \in D) < 1$.

The process is nonzero. Hence one can find $y \in R^d$, $y \neq 0$, such that the real-valued process $\langle y, X_t\rangle$ is a nonzero Lévy process. Since D is bounded, we can find r such that $D \subset \{z \in R^d : |\langle y, z\rangle| \le r\}$. Hence by Lemma 48.3 in [17] we obtain

$$\sup_{x\in\mathbb{R}^d} P^x(X_t \in D) \le \sup_{x\in\mathbb{R}^d} P^x(|\langle y, X_t\rangle| \le r) = O(t^{-1/2}), \quad t \to \infty. \qquad \square$$

In order to study the killed process on exiting D we construct its transition densities by the classical formula

$$p_D(t,x,y) = p(t,x,y) - r_D(t,x,y),$$

where

$$r_D(t,x,y) = E^x[t \geqslant \tau_D; p(t-\tau_D, X_{\tau_D}, y)].$$

The arguments used for Brownian motion (see, e.g., [9]) will prevail in our case and one can easily show that the transition density $p_D(t,x,y)$, $t > 0$, satisfies the Chapman-Kolmogorov equation (semigroup property). Moreover, the transition density $p_D(t,x,y)$ is a symmetric function (x,y) a.s. Assuming some other mild conditions on the transition densities of the (free) process, one can actually show that $p_D(t,x,y)$ can be chosen as continuous functions of (x,y).

Next, we define the Green function of the set D,

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)dt.$$

Let us see that the integral is well defined. We have

$$\int_D G_D(x,y)dy = \int_D \int_0^\infty p_D(t,x,y)dtdy = \int_0^\infty P^x(\tau_D > t)dt = E^x\tau_D < \infty.$$

Hence for every $x \in \mathbb{R}^d$ the Green function $G_D(x,y)$ is well defined (y) a.s. Again, under assumptions which make the function $p_D(t,x,y),\ t > 0$, continuous in the arguments x, y, one can show that the Green function is a continuous function (in the extended sense) on $D \times D$.

It is well known that if the Lévy measure is absolutely continuous with respect to the Lebesgue measure, then the distribution of X_{τ_D} restricted to $\overline{D}^c$ is absolutely continuous as well (see [11]) and the density is given by the so-called Ikeda-Watanabe formula:

$$P_D(x,z) = \int_D G_D(x,y)\nu(y-z)dy, \quad (x,z) \in D \times \overline{D}^c.$$

We call $P_D(x,z)$ the Poisson kernel. Under some other mild conditions X_{τ_D} has zero probability of belonging to the boundary of D, so in this case the Poisson kernel fully describes the distribution of the exiting point.

We say that a measurable function u is *harmonic* with respect to X_t in an open set D if for every bounded open set U satisfying $\overline{U} \subset D$,

$$u(x) = E^x u(X_{\tau_U}), \quad x \in U.$$

If

$$u(x) = E^x u(X_{\tau_D}), \quad x \in D,$$

then we say that u is *regular harmonic* with respect to X_t in an open set D.

The following lemma is a simple consequence of Lemma 2.1 and the boundedness of $p(t,x)$.

LEMMA 2.2. *For any $x \in D$ and $t \geq 1$ we have*

$$p_D(t,x,y) \leqslant C(X)\frac{E^x\tau_D E^y\tau_D}{t^2} \quad (y) \text{ a.s. } .$$

Proof. Observe that for $s \geqslant 0$,

$$\begin{aligned}\sup_{x,y\in D} p_D(s+1/4,x,y) &\leqslant \sup_{x,y\in\mathbb{R}^d} p(s+1/4,x-y) = \sup_{x\in\mathbb{R}^d} p(1/4,\cdot)*p(s,x)\\ &\leqslant \sup_{x\in\mathbb{R}^d} p(1/4,x) = C_1.\end{aligned}$$

Hence, by the Chapman-Kolmogorov equation we obtain for $t \geqslant 1/2$ and (y) a.s.

$$p_D(t,x,y) = \int_D p_D(t/2,x,z)p_D(t/2,z,y)dz \leqslant C_1 P^x(\tau_D > t/2).$$

Applying again the Chapman-Kolmogorov equation together with the above inequality we get for $t \geqslant 1$,

$$\begin{aligned} p_D(t,x,y) &\leqslant C_1 P^x(\tau_D > t/4) \int_D p_D(t/2,z,y)dz \\ &= C_1 P^x(\tau_D > t/4) P^y(\widehat{\tau}_D > t/2), \end{aligned}$$

where $\widehat{\tau}_D = \inf\{t > 0 : -X_t \in D\}$. But the process X_t is symmetric, so $\{X_t\} \overset{D}{=} \{-X_t\}$. Hence

$$P^y(\widehat{\tau}_D > t/2) = P^y(\tau_D > t/2).$$

Therefore, we have

$$p_D(t,x,y) \leqslant C_1 P^x(\tau_D > t/4) P^y(\tau_D > t/2).$$

An application of Chebyshev's inequality completes the proof. □

Remark 2.3. If X_t is an isotropic stable process, then by similar arguments we have for $t > 0$ and $x, y \in D$,

$$p_D(t,x,y) \leqslant C(\alpha,d) \frac{E^x \tau_D E^y \tau_D}{t^{2+d/\alpha}}.$$

In one of our general results (Theorem 3.1) we require the following property which exhibits a relation between the moments of the exit times and the Green function.

Property A. *There is a constant $c = c(D)$ such that*

$$E^x \tau_D E^y \tau_D \leq c G_D(x,y), \quad x, y \in D.$$

At first glance the above condition looks a bit restrictive, but actually it holds in the stable case ([15], [6], [1]) and it is usually derived as a consequence of the intrinsic ultracontractivity of the killed process. In a recent paper of the first author (see [10]) the intrinsic ultracontractivity is studied under much broader assumptions. For example, the above property holds if $p_D(t,\cdot,\cdot)$ is continuous in x, y and the Lebesgue measure is absolutely continuous with respect to the Lévy measure.

From now on we consider two symmetric Lévy processes Y_t and X_t such that the signed measure $\sigma = \nu^X - \nu^Y$ is finite, where ν^Y, ν^X are the Lévy measures of Y_t and X_t, respectively. We use this notational convention throughout the whole paper; e.g., we denote the transition density of X_t by $p^X(t,x)$ and the transition density of Y_t by $p^Y(t,x)$ (the density must exist since the measure $\sigma = \nu^X - \nu^Y$ is finite). Later on we will require one of the processes, say X_t, to be the isotropic stable process. The aim of this paper is to make comparisons between the two processes in various aspects of which the relationship of the Green functions is our main target. Some of the results are general, but our typical situation is a comparison between the isotropic stable process and another process such that their Lévy measures are sufficiently close to each other.

With the assumption that $\sigma = \nu^X - \nu^Y$ is finite we have the following formula comparing infinitesimal generators on $L^1(\mathbb{R}^d)$ of these processes:

$$\mathcal{A}^Y = \mathcal{A}^X - P, \qquad \text{where } P\varphi(x) = \sigma * \varphi(x) - \sigma(\mathbb{R}^d)\varphi(x).$$

The fact that P is a bounded operator implies that the domains of these generators coincide.

As mentioned above, very often the process X_t is taken to be the isotropic α-stable process, $0 < \alpha < 2$. To emphasize its role, we denote it by $\widetilde{X}_t$. This process has the following characteristic function:

$$E^0 e^{iz\cdot\widetilde{X}_t} = e^{-t|z|^\alpha}, \quad z \in \mathbb{R}^d.$$

From now on, we will use the tilde sign to denote functions, measures, etc., corresponding to $\widetilde{X}_t$. For example, its Lévy measure is given by the formula

$$\widetilde{\nu}(B) = \int_B \mathscr{A}(-\alpha, d)|x|^{-d-\alpha}dx,$$

where

$$\mathscr{A}(\rho, d) = \frac{\Gamma((d-\rho)/2)}{\pi^{d/2}2^\rho|\Gamma(\rho/2)|}.$$

The potential kernel, which is well defined for $\alpha < d$, is given by

$$\widetilde{U}(x) = \mathscr{A}(\alpha, d)|x|^{\alpha-d}, \quad x \in \mathbb{R}^d.$$

The next two lemmas provide the basic tools for examining the relationship between the Green functions. In the first lemma we compare the moments of exit times assuming only that $\sigma = \nu^X - \nu^Y$ is a finite signed measure, while in the second lemma we require that σ is nonnegative. This assumption gives a nice inequality involving the transitions densities. Although both lemmas have already appeared in the literature under some additional assumptions (see [16]), for the reader's convenience we provide the proofs.

LEMMA 2.4. *Let D be a bounded open set and let $\sigma = \nu^X - \nu^Y$ be finite. Then we have on D,*

$$E^x\tau_D^X \approx E^x\tau_D^Y.$$

Proof. Suppose that $\sigma = \sigma_+ - \sigma_-$ is the Jordan decomposition of σ. Let V_t be a compound Poisson process independent of X_t with Lévy measure σ_- and let $V_t^{'}$ be a compound Poisson process independent of Y_t with Lévy measure σ_+. We put $Z_t = X_t + V_t$. Then, of course, we have $\{Z_t\} \overset{D}{=} \{Y_t + V_t^{'}\}$, where $\stackrel{D}{=}$ means equality in distribution. Hence it is enough to show that $E^x\tau_D^Z \approx E^x\tau_D^X$.

Let us define a stopping time T by $T = \inf\{t > 0 : V_t \neq 0\}$. The processes X_t and V_t are mutually independent. Therefore X_t and T are independent as well. Besides, $Z_t = X_t$ for $0 \leqslant t < T$. We set $m = \sigma_-(\mathbb{R}^d)$.

First, we claim that $E^x(\tau_D^X) \leqslant 2E^x(\tau_D^X \wedge t)$ for t large enough. Indeed, by the Markov property and Lemma 2.1 we have

$$\begin{aligned} E^x\tau_D^X &= E^x(\tau_D^X \wedge t) + E^x(\tau_D^X > t; \tau_D^X - t) \\ &= E^x(\tau_D^X \wedge t) + E^x(\tau_D^X > t; E^{X_t}\tau_D^X) \\ &\leqslant E^x(\tau_D^X \wedge t) + CP^x(\tau_D^X > t) \\ &\leqslant E^x(\tau_D^X \wedge t) + C\frac{E^x\tau_D^X}{t}, \end{aligned}$$

which proves our claim for $t \geqslant 2C$.

Because $\tau_D^Z \wedge T = \tau_D^X \wedge T$, by the independence T and X_t we get

$$\begin{aligned} E^x\tau_D^Z \geqslant E^x(\tau_D^Z \wedge T) = E^x(\tau_D^X \wedge T) = \int_0^\infty E^x(\tau_D^X \wedge t) me^{-mt} dt \\ \geqslant \int_{2C}^\infty E^x(\tau_D^X \wedge t) me^{-mt} dt \geqslant \frac{1}{2} e^{-2Cm} E^x\tau_D^X. \end{aligned}$$

Now, we prove the upper bound. Again, by the strong Markov property and Lemma 2.1 we arrive at

$$\begin{aligned} E^x\tau_D^Z &= E^x(\tau_D^Z \wedge T) + E^x(\tau_D^Z > T; \tau_D^Z - T) \\ &\leqslant E^x\tau_D^X + E^x(\tau_D^Z > T; E^{Z_T}\tau_D^Z) \\ &\leqslant E^x\tau_D^X + CP^x(\tau_D^Z > T), \end{aligned}$$

but

$$P^x(\tau_D^Z > T) \leqslant P^x(\tau_D^X \geqslant T) = m\int_0^\infty P^x(\tau_D^X \geqslant t)e^{-mt} dt \leqslant mE^x\tau_D^X,$$

which completes the proof. □

LEMMA 2.5. *Suppose that* $\sigma = \nu^X - \nu^Y$ *is a nonnegative finite measure and* D *is an open set. Then for any* $x \in D$ *and* $t > 0$,

$$p_D^Y(t, x, \cdot) \leqslant e^{mt} p_D^X(t, x, \cdot) \quad a.s. \ .$$

If, in addition, we assume that $p^Y(t, \cdot)$ *and* $p^X(t, \cdot)$ *are continuous, then we have for* $x, y \in D$,

$$r_D^Y(t, x, y) \leqslant e^{2mt} r_D^X(t, x, y).$$

Proof. We put $m = \sigma(\mathbb{R}^d) < \infty$, and define a compound Poisson process V_t with the Lévy measure σ independent of Y_t. Note that the process $Y_t + V_t$

is a copy of the process X_t. Hence we may assume that $X_t = Y_t + V_t$. The random variable

$$T = \inf\{t \geqslant 0 : V_t \neq 0\} \tag{2.1}$$

has exponential distribution with intensity m. Then Y_t and T are independent and for $0 \leqslant t < T$ we have $X_t = Y_t$.

Let A be a Borel subset of D. Since $Y_t = X_t$, for $t < T$ we infer that $\{\tau_D^Y > t\} \cap \{T > t\} = \{\tau_D^X > t\} \cap \{T > t\}$. By the independence of Y_t and T we have

$$\begin{aligned} P^x(t < \tau_D^Y; Y_t \in A) P^x(T > t) &= P^x(t < \tau_D^Y; Y_t \in A; T > t) \\ &= P^x(t < \tau_D^X; X_t \in A; T > t) \\ &\leqslant P^x(t < \tau_D^X; X_t \in A). \end{aligned}$$

So we obtain that (y) a.s.,

$$p_D^Y(t,x,y) P^x(T > t) \leqslant p_D^X(t,x,y).$$

But T has exponential distribution with intensity m, that is,

$$P^x(T > t) = e^{-mt}.$$

The second inequality is proved analogously, using the first with $D = \mathbb{R}^d$ in the intermediate step. Moreover, the continuity of $p^Y(t,\cdot)$ and $p^X(t,\cdot)$ is required to justify the last step:

$$\begin{aligned} r_D^Y(t,x,y)e^{-mt} &= E^x[t \geqslant \tau_D^Y; p^Y(t - \tau_D^Y, Y_{\tau_D^Y}, y)] P^x(T > t) \\ &= E^x[\tau_D^Y \leqslant t < T; p^Y(t - \tau_D^Y, Y_{\tau_D^Y}, y)] \\ &= E^x[\tau_D^X \leqslant t < T; p^Y(t - \tau_D^X, X_{\tau_D^X}, y)] \\ &\leqslant e^{mt} E^x[\tau_D^X \leqslant t; p^X(t - \tau_D^X, X_{\tau_D^X}, y)] \\ &= e^{mt} r_D^X(t,x,y). \end{aligned}$$

□

The next lemma is a sort of comparison between transition densities in the sense that a "nice" behaviour of these for one process implies that the transition densities of the second are uniformly bounded away from zero. The "nice" behaviour is present, for example, if the first process is the isotropic stable process. We use this result in the sequel to ensure that the transition densities of the killed process are continuous and to obtain Property A. We define the exponential of a signed finite measure σ by

$$\exp\{\sigma\}(A) = e^{-\sigma(\mathbb{R}^d)} \sum_{n=0}^{\infty} \frac{\sigma^{*n}(A)}{n!}, \quad \text{where } A \subset \mathbb{R}^d \text{ is a Borel set.}$$

LEMMA 2.6. *Suppose that ν^X and ν^Y are absolutely continuous and $\sigma(x) = \nu^X(x) - \nu^Y(x)$ is an integrable function such that $|p^X(t,\cdot) * \sigma(x)| + |\sigma(x)| \leq c_1$*

for $|x| \geq \delta$ *and* $t \leq 1$. *If* $p^X(t,x) \leqslant c_2 t^{-\zeta}$ *for* $t \leq 1$, *where* $\zeta > 0$, *and* $p^X(t,x) \leqslant c_3(\delta)$ *for* $|x| \geq \delta$, *then there is a constant* C *such that*

$$p^Y(t,x) \leq C, \quad |x| \geq ([\zeta] \vee 1)\,\delta \text{ and } t > 0.$$

Proof. Suppose that $\int_{\mathbb{R}^d} |\sigma(x)|dx = M < \infty$. We put $\int_{\mathbb{R}^d} \sigma(x)dx = m$. We can write

$$\begin{aligned} p^Y(t,x) &= p^X(t,\cdot) * \exp\{-t\sigma\} \\ &= p^X(t,x)e^{tm} + \sum_{n=1}^{\infty} \frac{(-t)^n p^X(t,\cdot) * \sigma^{*n}(x)}{n!} e^{tm}. \end{aligned}$$

Observe that

$$\left|p^X(t,\cdot) * \sigma^{*n}(x)\right| \leq \sup_{y \in R^d} p^X(t,y) M^n \leqslant c_2 \frac{M^n}{t^\zeta},$$

so for $t \leq 1$ we have

$$\left|\sum_{n \geq \zeta}^{\infty} \frac{(-t)^n p^X(t,\cdot) * \sigma^{*n}(x)}{n!} e^{tm}\right| \leq C \sum_{n \geq \zeta}^{\infty} \frac{t^{n-\zeta} M^n}{n!} = Ce^M < \infty. \tag{2.2}$$

Now, we show that if $|p^X(t,\cdot) * \sigma(x)| + |\sigma(x)| \leq c(1)$ for $|x| \geq \delta$ and $t \leq 1$, then

$$\left|p^X(t,\cdot) * \sigma^{*n}(x)\right| \leq c(n), \quad |x| \geq n\delta. \tag{2.3}$$

We assume (2.3) for n and we prove it for $n+1$. Observe that

$$\begin{aligned} \left|p^X(t,\cdot) * \sigma^{*(n+1)}(x)\right| &\leqslant \int_{B^c(x,n\delta)} \left|p^X(t,\cdot) * \sigma^{*n}(x-y)\right| |\sigma(y)|dy \\ &\quad + \int_{B(x,n\delta)} \left|p^X(t,\cdot) * \sigma^{*n}(x-y)\right| |\sigma(y)|dy \\ &\leq c(n)M + c_1 M^n, \end{aligned}$$

because if $y \in B(x, n\delta)$, then $|y| \geq |x| - |x-y| \geq \delta$. Combining (2.2) and (2.3) and using that $p^X(t,x) \leq c(\delta)$ for $|x| \geq \delta$, we complete the proof for $t \leqslant 1$.

Next, for $t > 1$ we have

$$\sup_{x \in \mathbb{R}^d} p^Y(t,x) = \sup_{x \in \mathbb{R}^d} p^Y(1,\cdot) * p^Y(t-1,x) \leqslant \sup_{x \in \mathbb{R}^d} p^Y(1,x) = C,$$

which proves the conclusion for $t > 1$. □

The following lemma is an attempt to find a condition under which the potential kernel of a process is comparable at the vicinity of the origin with the stable potential kernel. It will play an important role in proving the upper bound for the Green function G_D^Y by its stable counterpart (see Theorem 3.23).

LEMMA 2.7. *Let $d > \alpha$. Let $-\sigma = \nu^Y - \tilde{\nu}$ be a nonnegative finite measure such that $\tilde{U} * (-\sigma)(x) \le C\tilde{U}(x)$ for $|x| \le 1$. Then for some constant $C > 1$,*

$$C^{-1}\tilde{U}(x) \le U^Y(x) \le C\tilde{U}(x), \quad |x| \le 1.$$

Proof. Suppose that $-\sigma = \nu^Y - \tilde{\nu} \ge 0$. Let $-\sigma(\mathbb{R}^d) = m > 0$. We can write

$$p^Y(t,x) = \widetilde{p}(t,\cdot) * \exp\{-t\sigma\} = \widetilde{p}(t,x)e^{-tm} + \sum_{n=1}^{\infty} \frac{t^n \widetilde{p}(t,\cdot) * (-\sigma)^{*n}(x)}{n!} e^{-tm}.$$

Observe that

$$\widetilde{p}(t,\cdot) * (-\sigma)^{*n}(x) \le \sup_{y \in \mathbb{R}^d} \widetilde{p}(t,y) m^n = C \frac{m^n}{t^{d/\alpha}},$$

so for $n > d/\alpha - 1$ we have

$$\begin{aligned}\int_0^\infty \frac{t^n \widetilde{p}(t,\cdot) * (-\sigma)^{*n}(x)}{n!} e^{-tm} dt &\le C \int_0^\infty \frac{t^{n-d/\alpha} m^n}{n!} e^{-tm} dt \\ &\le C \frac{\Gamma(n+1-d/\alpha)}{n!} m^{d/\alpha+1} \le C \frac{m^{d/\alpha+1}}{n^{d/\alpha}}.\end{aligned}$$

This implies that

$$\int_0^\infty \sum_{n > d/\alpha - 1}^{\infty} \frac{t^n \widetilde{p}(t,\cdot) * (-\sigma)^{*n}(x)}{n!} e^{-tm} \le C \sum_{n > d/\alpha - 1}^{\infty} \frac{m^{d/\alpha+1}}{n^{d/\alpha}} < \infty. \tag{2.4}$$

Next, estimating $t^n e^{-tm} \le C(n,m) < \infty$, we have

$$\int_0^\infty \frac{t^n \widetilde{p}(t,\cdot) * (-\sigma)^{*n}(x)}{n!} e^{-tm} dt \le C(n,m)\, \widetilde{U} * (-\sigma)^{*n}(x).$$

Let

$$\tilde{U}(x) = \frac{\mathscr{A}}{|x|^{d-\alpha}}.$$

If we assume that $\tilde{U} * (-\sigma)(x) \le C\tilde{U}(x)$ for $|x| \le 1$, then we claim that

$$\tilde{U} * (-\sigma)^{*n}(x) \le C(n)\tilde{U}(x), \quad |x| \le 1. \tag{2.5}$$

We check this for $n = 2$, since the general case will follow by induction.

$$\begin{aligned}\tilde{U} * \sigma^{*2}(x) &= \int_{B(x,1)} \tilde{U} * (-\sigma)(x-y)(-\sigma)(dy) + \int_{B^c(x,1)} \tilde{U}(x-y)\sigma^{*2}(dy) \\ &\le C \int_{B(x,1)} \tilde{U}(x-y)(-\sigma)(dy) + \mathscr{A}\, m^2 \\ &\le C^2\tilde{U}(x) + \mathscr{A}\, m^2 \le C(2)\tilde{U}(x),\end{aligned}$$

because $\lim_{|x|\to 0} \tilde{U}(x) = \infty$. By (2.4) and (2.5) we conclude that $U^Y(x) \le C\tilde{U}(x)$, $|x| \le 1$.

Getting the reverse inequality is almost immediate, since

$$\widetilde{p}(t,x) \le e^{tm} p^Y(t,x)$$

by Lemma 2.5 together with the fact that $\widetilde{p}(t,\cdot)$ and $p^Y(t,\cdot)$ are continuous. The following estimate is well known:

$$\widetilde{p}(t,x) \leqslant C(d,\alpha)\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right). \tag{2.6}$$

Hence for $|x| \leqslant 1$,

$$\tilde{U}(x) \leqslant C \int_0^1 \widetilde{p}(t,x)dt,$$

for some constant $C = C(d,\alpha)$. Therefore

$$\tilde{U}(x) \leqslant \int_0^1 \widetilde{p}(t,x)dt \le e^m \int_0^1 p^Y(t,x)dt \le e^m U^Y(x),$$

for $|x| \le 1$. □

REMARK 2.8. If $-\sigma(x)$ is a nonnegative density of a finite measure and

$$-\sigma(x) \le C|x|^{-d+\varrho}, \quad |x| \le 1,$$

where $\varrho > 0$, then the condition $\tilde{U} * (-\sigma)(x) \le C\tilde{U}(x)$ for $|x| \le 1$ is satisfied.

The last lemma in this section is intended to treat the one-dimensional situation while comparing two processes of which one is a symmetric α-stable process with $\alpha \ge 1$ (the recurrent case). This case is different from the case $\alpha < 1$ (the transient case) and requires somewhat different arguments.

LEMMA 2.9. *Let $d = 1$, $\alpha \ge 1$ and $0 < t_0 \le 1$. Suppose that $\sigma = \widetilde{\nu} - \nu^Y$ is a finite measure. Then there exists a constant $C = C(m, M)$ such that*

$$\int_0^{t_0} \left|\widetilde{p}(t,x) - e^{-2mt} p^Y(t,x)\right| dt \leqslant C\, t_0^{2-1/\alpha},$$

where $m = \sigma(\mathbb{R})$ and $M = |\sigma|(\mathbb{R})$.

Proof. Let $\sigma(\mathbb{R}) = m$ and $|\sigma|(\mathbb{R}) = M > 0$. We can write

$$p^Y(t,x) = \widetilde{p}(t,\cdot) * \exp\{-t\sigma\} = \widetilde{p}(t,x)e^{tm} + \sum_{n=1}^{\infty} \frac{(-t)^n \widetilde{p}(t,\cdot) * \sigma^{*n}(x)}{n!} e^{tm}.$$

Next,

$$|\widetilde{p}(t,\cdot) * \sigma^{*n}(x)| \le \sup_{y\in\mathbb{R}} \widetilde{p}(t,y) M^n = C\frac{M^n}{t^{1/\alpha}}.$$

Using this estimate we obtain

$$\begin{aligned}
&\left|\widetilde{p}(t,x) - e^{-2mt}p^Y(t,x)\right| \\
&\quad= \left|\widetilde{p}(t,x)(1-e^{-mt}) - \sum_{n=1}^{\infty} \frac{(-t)^n \widetilde{p}(t,\cdot) * \sigma^{*n}(x)}{n!} e^{-mt}\right| \\
&\quad\le \widetilde{p}(t,x)(1-e^{-mt}) + \frac{C}{t^{1/\alpha}} \sum_{n=1}^{\infty} \frac{(tM)^n}{n!} e^{-mt}.
\end{aligned}$$

From this it easily follows that there is a constant $C = C(m, M)$ such that

$$\left|\widetilde{p}(t,x) - e^{-2mt}p^Y(t,x)\right| \le Ct^{1-1/\alpha}, \quad t \le 1.$$

Now the conclusion follows by integration. □

3. Comparability of the Green functions

In this section we prove our main results. We start with a general one-sided estimate of Green functions.

THEOREM 3.1. *Let D be a bounded open set and $\sigma = \nu^X - \nu^Y$ be a nonnegative finite measure. Suppose that for one of the processes X_t or Y_t its Green function satisfies Property A. Then there exists a constant $C = C(\sigma, D, \alpha, d)$ such that for $x \in D$,*

$$G_D^Y(x,y) \leqslant C G_D^X(x,y) \quad (y) \text{ a.s.}.$$

Proof. Denote $\sigma(\mathbb{R}^d) = m$. From Lemmas 2.2 and 2.5 we get that (y) almost surely

$$\begin{aligned}
G_D^Y(x,y) &= \int_0^{t_0} p_D^Y(t,x,y)dt + \int_{t_0}^{\infty} p_D^Y(t,x,y)dt \\
&\leqslant e^{mt_0} \int_0^{t_0} p_D^X(t,x,y)dt + C_1 \int_{t_0}^{\infty} t^{-2} E^x \tau_D^Y E^y \tau_D^Y dt,
\end{aligned}$$

for $t_0 \ge 1$. Hence

$$G_D^Y(x,y) \leqslant c G_D^X(x,y) + \frac{C_1}{t_0} E^x \tau_D^Y E^y \tau_D^Y.$$

If Y_t satisfies

$$E^x \tau_D^Y E^y \tau_D^Y \le C_2 G_D^Y(x,y), \tag{3.1}$$

then for $t_0 = \max\{1, 2C_1C_2\}$ we get

$$G_D^Y(x,y) \leqslant 2c G_D^X(x,y).$$

Now, suppose that (3.1) holds for X_t. Then by Lemma 2.4 we have

$$G_D^Y(x,y) \leqslant c G_D^X(x,y) + C_3 E^x \tau_D^X E^y \tau_D^X \leqslant C G_D^X(x,y),$$

which completes the proof. □

Kulczycki in [15] showed that for the isotropic α-stable process Property A is satisfied for any bounded open set D, so we obtain the following corollary.

COROLLARY 3.2. *Let D be a bounded open set. If $\sigma = \widetilde{\nu} - \nu^Y$ is a nonnegative and finite measure, then there is a constant C such that*

$$G_D^Y(x,y) \leqslant C\widetilde{G}_D(x,y).$$

If $\nu^Y - \widetilde{\nu}$ is a nonnegative and finite measure, then

$$\widetilde{G}_D(x,y) \leqslant CG_D^Y(x,y).$$

Suppose that $p_D^X(t,x,\cdot)$ and $p_D^X(t,\cdot,x)$ are continuous for any $x \in D$. If the Lebesgue measure is absolutely continuous with respect to the Lévy measure of X_t, then the following theorem is true for any bounded open set D. On the other hand, if there exists a radius $r > 0$ such that the density ν_{ac}^X of the absolutely continuous part of the Lévy measure satisfies

$$\inf_{x\in B(0,r)} \nu_{ac}^X(x) > 0,$$

then the following theorem holds for any connected Lipschitz bounded set D (see [10]).

THEOREM 3.3. *For every $t > 0$ there is a constant $c = c(t, D, \alpha)$ such that*

$$cE^x\tau_D^X E^y\tau_D^X \leqslant p_D^X(t,x,y), \qquad x,y \in D.$$

If we integrate the above inequality with respect to dt, we get Property A for X_t:

$$CE^x\tau_D^X E^y\tau_D^X \leqslant G_D^X(x,y).$$

Therefore from Theorem 3.1 we obtain the following corollary.

COROLLARY 3.4. *Let $p_D^X(t,\cdot,\cdot)$ be continuous for every $t > 0$, and let the finite measure $\sigma = \nu^X - \nu^Y$ be nonnegative. Suppose that the Lebesgue measure is absolutely continuous with respect to ν^X. Then for any bounded open set D there exists a constant $C = C(\sigma, D, \alpha, d)$ such that for $x \in D$,*

$$G_D^Y(x,y) \leqslant CG_D^X(x,y), \quad (y) \text{ a.s..}$$

Our next goal is to reverse the above estimate. We are not able to do this under the above assumptions alone, but instead need some additional assumptions. We proceed in several steps. In the first step, we take advantage of the following lemma, which can be proved in the same way as Lemma 7 in [16].

LEMMA 3.5. *Let $\sigma = \nu^X - \nu^Y$ be a nonnegative finite measure. Suppose that $G_D^X(x,\cdot)$ and $G_D^Y(x,\cdot)$ are continuous. Then*

$$G_D^X(x,y) \leqslant G_D^Y(x,y) + E^x[\tau_D^X > T; G_D^X(X_T,y)],$$

where T is defined by (2.1).

This lemma can be rewritten in the following way, which is more useful for further analysis.

Corollary 3.6. *Suppose that $\sigma = \nu^X - \nu^Y$ is a nonnegative finite measure, and $G_D^X(x, \cdot)$ and $G_D^Y(x, \cdot)$ are continuous. Then*

$$G_D^X(x, y) \leqslant G_D^Y(x, y) + \int_D \int_{D-w} G^Y(x, w) G^X(w + z, y) \sigma(dz) dw.$$

Proof. See the proof of Lemma 9 in [16]. □

From now on we assume that $X_t = \tilde{X}_t$ and that the measure $\sigma = \widetilde{\nu} - \nu^Y$ is finite and absolutely continuous. We will use the following notational convention: in the case when a measure μ is absolutely continuous we denote its density by $\mu(x)$. Thus, $\sigma(x)$ is the density of $\widetilde{\nu} - \nu^Y$. Moreover, we assume a particular behavior of $\sigma(x)$ near 0, namely, we suppose that there exist $\varrho > 0$ and C such that

$$|\sigma(x)| \leqslant C|x|^{\varrho - d}, \quad |x| \leqslant 1. \tag{3.2}$$

In addition, we assume that $\sigma(x)$ is bounded on $B^c(0, 1)$, which obviously is equivalent to the boundedness of $\nu^Y(x)$ on $B^c(0, 1)$.

For example, the above conditions are satisfied by the Lévy measure of the relativistic process (see [16]) and the Lévy measure of the α-stable process truncated to $B(0, 1)$ ($\nu^Y(x) = \mathbf{1}_{B(0,1)}(x)\widetilde{\nu}(x)$).

With these assumptions we have that the characteristic function of Y_t is integrable, so $p^Y(t, \cdot)$ is bounded and continuous. Moreover, by (2.6) we get that for any $\delta > 0$,

$$\widetilde{p}(t, x) \leqslant C(\delta), \quad |x| \geqslant \delta.$$

Therefore from Lemma 2.6 we obtain that the transition density of Y_t also satisfies

$$p^Y(t, x) \leqslant C(\delta), \quad |x| \geqslant \delta.$$

This property enables us to prove, similarly as for the Brownian motion in [9], that $p_D^Y(t, x, \cdot)$ and $p^Y(t, \cdot, y)$ are continuous, and that $G_D^Y(x, \cdot)$ and $G_D^Y(\cdot, y)$ are continuous, too. Hence, under the present assumptions, in the statements of all results proved so far, the estimates hold for every y, and not just for almost all y.

Furthermore, by (3.2) there exist a radius r and a constant c such that $\widetilde{\nu}(x) \leq c\nu^Y(x)$ on $B(0, r)$. So, $\inf_{x \in B(0,r)} \nu^Y(x) > 0$. Therefore from Theorem 3.3 we have that for any bounded connected Lipschitz open set the process Y_t satisfies property A. That is, we have the following corollary.

Corollary 3.7. *Let $\sigma(x) = \widetilde{\nu}(x) - \nu^Y(x)$ be an integrable function satisfying* (3.2). *Moreover, let σ be bounded on $B^c(0, 1)$. Then Property A holds* *for Y_t and any bounded connected Lipschitz open set. On the other hand, if we assume that $\nu^Y \geqslant \widetilde{\nu}$, then Property A holds for Y_t and any bounded open set.*

Let $D \subset \mathbb{R}^d$ be a bounded connected Lipschitz open set with Lipschitz character (r_0, λ) (see [12], [2] for the definitions). We need to introduce some additional notations related to D. We assume that D is a nonempty bounded open set. We put $\mathbf{r}_0 = r_0/\operatorname{diam}(D)$ and $\kappa = 1/(2\sqrt{1+\lambda^2})$. The set $\{x \in D : \delta_D(x) \geqslant r_0/2\}$ is nonempty. We choose one of its elements and denote it by $x_0 = x_0(D)$. We also fix a point x_1 such that $|x_0 - x_1| = r_0/4$. For any $x, y \in D$ let $r = r(x,y) = \delta_D(x) \vee \delta_D(y) \vee |x-y|$. If $r \leqslant r_0/32$, we let $A_{x,y}$ be an element of the set

$$\mathcal{B}(x,y) = \{A \in D : B(A, \kappa r) \subset D \cap B(x, 3r) \cap B(y, 3r)\},$$

and if $r > r_0/32$, we set $A_{x,y} = x_1$.

For bounded Lipschitz open sets Jakubowski [12] proved the following theorem, which gives estimates of the Green function for the isotropic α-stable process in the case $d \geq 2$. If $d = 1$, then an analogous theorem is true also for $\alpha < 1$ (see, e.g., [5]).

THEOREM 3.8. *Let D be a bounded open Lipschitz set and $d > \alpha$. There is a constant $C_1 = C_1(d, \lambda, r_0, \operatorname{diam}(D), \alpha)$ such that for every $x, y \in D$ we have*

$$C_1^{-1} \frac{\widetilde{\phi}_D(x)\widetilde{\phi}_D(y)}{\widetilde{\phi}_D^2(A_{x,y})} |x-y|^{\alpha-d} \leqslant \widetilde{G}_D(x,y) \leqslant C_1 \frac{\widetilde{\phi}_D(x)\widetilde{\phi}_D(y)}{\widetilde{\phi}_D^2(A_{x,y})} |x-y|^{\alpha-d},$$

where $\widetilde{\phi}_D(x) = \widetilde{G}_D(x, x_0) \wedge \mathscr{A}(d,\alpha) r_0^{\alpha-d} \approx E^x \tau_D$.

From the scaling property of the Green function for the isotropic α-stable process we have the following remark.

REMARK 3.9. The constant C_1 depends on the constants r_0 and $\operatorname{diam}(D)$ only via their ratio $\mathbf{r}_0$.

Now, we recall estimates for the Green function of the isotropic α-stable process if $1 = d \leqslant \alpha$. Their proof can be found, e.g., in [5].

THEOREM 3.10. *Let $d = 1$ and D be an open interval. Then we have on $D \times D$,*

$$\widetilde{G}_D(x,y) \approx \begin{cases} \log\left(\dfrac{(\delta_D(x)\delta_D(y))^{1/2}}{|x-y|} + 1\right), & \alpha = 1, \\ \min\left\{(\delta_D(x)\delta_D(y))^{(\alpha-1)/2}, \dfrac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|}\right\}, & 1 < \alpha. \end{cases}$$

Theorem 3.8 and Theorem 3.10 imply the following corollary.

COROLLARY 3.11. *Let D be a bounded Lipschitz open set if $d \geq 2$ or a bounded open interval if $d = 1$. For $|x-y| \geqslant \theta > 0$ there exists a constant $C(\theta)$ such that*

$$\widetilde{G}_D(x,y) \leqslant C(\theta) E^x \widetilde{\tau}_D E^y \widetilde{\tau}_D.$$

Proof. We prove only the case $d \geq 2$, since the other case follows immediately from Theorem 3.10 and the fact that $E^x \tau_D \approx (\delta_D(x))^{\alpha/2}$.

By Theorem 3.8

$$\widetilde{G}_D(x,y) \leqslant C_1 \frac{\widetilde{\phi}_D(x)\widetilde{\phi}_D(y)}{\widetilde{\phi}_D^2(A_{x,y})} |x-y|^{\alpha-d}.$$

Next, note that, by the definition of the point $A = A_{x,y}$, we have

$$\delta_D(A) \geq \kappa|x-y| \wedge r_0/4 \geq \kappa\theta \wedge r_0/4 = r_\theta,$$

which shows that

$$\widetilde{\phi}_D(A) \approx E^A \widetilde{\tau}_D \geq E^A \widetilde{\tau}_{B(A,r_\theta)} = C r_\theta^\alpha.$$

This implies the conclusion. □

A consequence of Lemmas 13 and 15 from [12] is the following lemma.

LEMMA 3.12. *There are constants $C = C(d,\lambda,\alpha,\mathbf{r}_0)$ and $\gamma = \gamma(d,\lambda,\alpha) < \alpha < d$ such that for every $x,y,z,w \in D$ we have*

$$\frac{\widetilde{\phi}_D^2(A_{x,y})}{\widetilde{\phi}_D(A_{x,w})\widetilde{\phi}_D(A_{z,y})} \leqslant C \max\left\{1, \frac{|x-y|^\gamma}{|x-w|^\gamma}, \frac{|x-y|^\gamma}{|z-y|^\gamma}, \frac{|x-y|^{2\gamma}}{|x-w|^\gamma |z-y|^\gamma}\right\}.$$

Proof. First, we assume that $|x-y| \leqslant |x-w|$. Then it can be proved using similar methods as in Lemma 13 of [12] that

$$\widetilde{\phi}_D(A_{x,y}) \leqslant C(d,\lambda,\alpha,\mathbf{r}_0)\widetilde{\phi}_D(A_{x,w}). \tag{3.3}$$

Now, let $|x-w| \leqslant |x-y|$. Then from the proof of Lemma 15 in [12] we infer that

$$\widetilde{\phi}_D(A_{x,y}) \leqslant C(d,\lambda,\alpha,\mathbf{r}_0) \frac{|x-y|^\gamma}{|x-w|^\gamma} \widetilde{\phi}_D(A_{x,w}), \tag{3.4}$$

for some $0 < \gamma < \alpha$. Combining (3.3) and (3.4) completes the proof. □

LEMMA 3.13. *Let $x \neq y \in D$, $-d < \varrho$ and $0 < a, b$. We set*

$$Q(x,y) = \int_D \int_D |y-z|^{a-d} |z-w|^\varrho |w-x|^{b-d} dz dw.$$

Then there exists a constant $C = C(d,a,b,\varrho)$ *such that*

$$Q(x,y) \leqslant C \begin{cases} |x-y|^{a+\varrho+b}, & a+\varrho+b<0, \\ 1+\log\left(\frac{\operatorname{diam}(D)}{|x-y|}\right), & a+\varrho+b=0, \\ (\operatorname{diam}(D))^a \left(1+\log\left(\frac{\operatorname{diam}(D)}{|x-y|}\right)\right), & a=b=-\varrho, \\ (\operatorname{diam}(D))^{a+\varrho+b}, & \textit{otherwise}. \end{cases}$$

Proof. By changing variables to $u = \frac{z-y}{|x-y|}$ and $v = \frac{w-x}{|x-y|}$ we get

$$Q(x,y) = |x-y|^{a+b+\varrho} \int_{\frac{D-y}{|x-y|}} \int_{\frac{D-x}{|x-y|}} |u|^{a-d}|v|^{b-d}|u-v-\mathbf{q}|^{\varrho} du dv,$$

where $\mathbf{q} = \frac{x-y}{|x-y|}$.

For $\varrho + a < 0$ we have

$$\int_{\mathbb{R}^d} |u|^{a-d}|u-v-\mathbf{q}|^{\varrho} du = C_{d,a,\varrho}|v+\mathbf{q}|^{a+\varrho},$$

and for $\varrho + a + b < 0$ we have

$$\int_{\mathbb{R}^d} |v|^{b-d}|v+\mathbf{q}|^{a+\varrho} dv = C_{d,a,b,\varrho},$$

which proves the first case. If $\varrho + a + b = 0$, then we have

$$\begin{aligned} \int_{\frac{D-x}{|x-y|}} |v|^{b-d}|v+\mathbf{q}|^{a+\varrho} dv &\leqslant \int_{B(0,2)} |v|^{b-d}|v+\mathbf{q}|^{a+\varrho} dv + \\ &\quad + 2^{-\varrho-a} \int_{B(0,\operatorname{diam}(D)/|x-y|)\setminus B(0,2)} |v|^{-d} dv \\ &= C(d,a,b,\varrho) + \\ &\quad + C(d,a,\varrho)\left(\log\left(\frac{\operatorname{diam}(D)}{|x-y|}\right) - \log(2)\right) \vee 0 \\ &\leqslant C(d,a,b,\varrho)\left\{1+\log\left(\frac{\operatorname{diam}(D)}{|x-y|}\right)\right\}. \end{aligned}$$

If $0 < \varrho + a + b < b$, then

$$\begin{aligned} \int_{\frac{D-x}{|x-y|}} |v|^{b-d}|v+\mathbf{q}|^{a+\varrho} dv &\leqslant \int_{B(0,2)} |v|^{b-d}|v+\mathbf{q}|^{a+\varrho} dv + \\ &\quad + 2^{-\varrho-a} \int_{B(0,\frac{\operatorname{diam}(D)}{|x-y|})\setminus B(0,2)} |v|^{\varrho+a+b-d} dv \\ &\leqslant C(d,a,b,\varrho)\left\{1+\left(\frac{\operatorname{diam}(D)}{|x-y|}\right)^{\varrho+a+b}\right\}. \end{aligned}$$

The remaining cases can be proved in the same way. □

LEMMA 3.14. *Let $d > \alpha$. Suppose that there exist positive constants ϱ and $c_1 = c_1(\mathrm{diam}(D))$ such that $|\sigma(x)| \leqslant c_1|x|^{\varrho-d}$ for $|x| \le \mathrm{diam}(D)$. Then there exists a constant $C = C(d, \lambda, \mathbf{r}_0, \alpha, \varrho)$ such that for all $x, y \in D$,*

$$\int_D \int_D \widetilde{G}_D(y,z)|\sigma(z-w)|\widetilde{G}_D(w,x)dwdz \leqslant c_1 C\left(\mathrm{diam}(D)\right)^{\zeta_1}|x-y|^{\zeta_2}\widetilde{G}_D(x,y),$$

for some $\zeta_1 \geqslant 0$ and $\zeta_2 > 0$.

Proof. From Theorem 3.8 and Lemma 13 in [12] we obtain

$$\begin{aligned}\frac{\widetilde{G}_D(x,w)\widetilde{G}_D(z,y)}{\widetilde{G}_D(x,y)} &\approx \left(\frac{|x-y|}{|x-w||y-z|}\right)^{d-\alpha}\frac{\widetilde{\phi}_D(w)\widetilde{\phi}_D(z)\widetilde{\phi}_D^2(A_{x,y})}{\widetilde{\phi}_D^2(A_{x,w})\widetilde{\phi}_D^2(A_{z,y})}\\ &\leqslant \left(\frac{|x-y|}{|x-w||y-z|}\right)^{d-\alpha}\frac{\widetilde{\phi}_D^2(A_{x,y})}{\widetilde{\phi}_D(A_{x,w})\widetilde{\phi}_D(A_{z,y})}.\end{aligned}$$

Because $|\sigma(x)| \leqslant c_1|x|^{\varrho-d}$, for $|x| \le \mathrm{diam}(D)$, we get $|\sigma(w-z)| \le c_1|w-z|^{\varrho-d}$ on $D \times D$. So, by Lemma 3.12 it is enough to prove that for some $\zeta_1 \geqslant 0$ and $\zeta_2 > 0$,

$$\int_D \int_D \frac{|x-w|^{\alpha-\rho_1-d}|w-z|^{\varrho-d}|z-y|^{\alpha-\rho_2-d}}{|x-y|^{\alpha-d-\rho_1-\rho_2}}dwdz \leqslant C(\mathrm{diam}(D))^{\zeta_1}|x-y|^{\zeta_2},$$

for some $C = C(d, \rho_1, \rho_2, \varrho)$, where $\rho_1, \rho_2 \in \{0, \gamma\}$. Recall that $\gamma < \alpha$. Hence the above inequality is a consequence of Lemma 3.13. □

By inspecting the estimates from Theorem 3.10 one obtains the following remark.

REMARK 3.15. In the case $d = 1 \leqslant \alpha$ the above lemma does not hold. This is why the proof given below of Theorem 1.1 in the one-dimensional case for $\alpha \ge 1$ requires arguments different from those in the general case.

3.1. Proof of Theorem 1.1. Throughout this subsection we assume that $\sigma = \widetilde{\nu} - \nu^Y$ is a finite nonnegative absolutely continuous measure and that its density satisfies

$$\sigma(x) \leqslant C|x|^{\varrho-d}, \quad |x| \le 1,$$

for some positive ϱ. Then there is a constant $c = c(C, d, \alpha, \mathrm{diam}(D))$ such that $\sigma(x) \leqslant c|x|^{\varrho-d}$ for $|x| \le \mathrm{diam}(D)$. Let D be a bounded connected Lipschitz open set. Then Property A holds for Y_t by Theorem 3.3.

By Corollaries 3.2 and 3.6 we have the inequality

$$C_1^{-1}G_D^Y(x,y) \leqslant \widetilde{G}_D(x,y) \leqslant G_D^Y(x,y) + C_1\widetilde{R}_D(x,y), \tag{3.5}$$

where $\widetilde{R}_D(x,y) = \int_D \int_D \widetilde{G}_D(x,w)\sigma(w-z)\widetilde{G}_D(z,y)dwdz$.

By Corollary 3.11 we have, for $|x-y| \geqslant \theta > 0$,

$$\widetilde{G}_D(x,y) \leqslant C(\theta)E^x\widetilde{\tau}_D E^y\widetilde{\tau}_D.$$

Hence, by Property A and Lemma 2.4 we get

$$\widetilde{G}_D(x,y) \leqslant C(\theta) G_D^Y(x,y), \quad |x-y| \geqslant \theta > 0.$$

It remains to show that $\widetilde{R}_D(x,y) \leqslant \frac{1}{2C_1}\widetilde{G}_D(x,y)$ if $|x-y|$ is small enough. But for $d > \alpha$ this is a consequence of Lemma 3.14. This completes the proof for $d > \alpha$.

Next, we deal with the case $1 = d \leqslant \alpha$. We need to show that $\widetilde{G}_D(x,y) \leqslant CG_D^Y(x,y)$ if $|x-y|$ is small enough. Recall that in this case D is a bounded open interval.

Lemma 3.16. *Let $d = 1$. Then there is a constant $C = C(\alpha, D, m)$ such that for any $x, y \in D$,*

$$\widetilde{R}_D(x,y) \leqslant C\frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|^{1-\varrho\wedge 1}}.$$

Proof. From Theorem 3.10 it is easy to see that

$$\widetilde{G}_D(x,y) \leqslant C\frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|}. \tag{3.6}$$

Hence, for $\varrho < 1$ we can prove in the same way as in Lemma 8 in [16] that

$$\int_D \widetilde{G}_D(x,w)\frac{dw}{|w-y|^{1-\rho}} \leqslant C\frac{(\delta_D(x))^{\alpha/2}}{|x-y|^{1-\rho}}. \tag{3.7}$$

From the above,

$$\int_D \widetilde{G}_D(x,w)\sigma(z-w)dw \leqslant C\int_D \widetilde{G}_D(x,w)\frac{dw}{|w-z|^{1-\varrho}} \leqslant C\frac{\delta_D(x)^{\alpha/2}}{|x-z|^{1-\varrho}}.$$

If $\varrho \geq 1$, then σ is bounded and since $E^x\widetilde{\tau}_D \approx (\delta_D(x))^{\alpha/2}$, we have

$$\int_D \widetilde{G}_D(x,w)\sigma(z-w)dw \leq CE^x\widetilde{\tau}_D \leq c\delta_D(x)^{\alpha/2}.$$

Now, we use the symmetry of the Green function and the inequality (3.7) again to get

$$\widetilde{R}_D(x,y) \leqslant C\frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|^{1-\varrho\wedge 1}}. \qquad \square$$

We are now able to prove the desired lower bound of the Green function for $1 = d \leq \alpha$.

Proposition 3.17. *Let D be a bounded open interval. Let $\alpha \geq 1$. Then there exists a constant $C = C(m, d, \alpha, D)$ such that for any $x, y \in D$,*

$$\widetilde{G}_D(x,y) \leqslant CG_D^Y(x,y).$$

Proof. Note that we only need to consider the case $|x-y| \le \theta$ for some sufficiently small $\theta > 0$. First, we assume that $\delta_D(x)\delta_D(y) \leqslant |x-y|^2$. By Theorem 3.10 this implies that

$$\frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|} \leqslant C\widetilde{G}_D(x,y).$$

Next, we apply Lemma 3.16 to obtain

$$\widetilde{R}_D(x,y) \leqslant C|x-y|^{\varrho\wedge 1}\widetilde{G}_D(x,y),$$

for some constant C. Thus, from (3.5) it follows that

$$\widetilde{G}_D(x,y) \leqslant G_D^Y(x,y) + \widetilde{C}|x-y|^{\varrho\wedge 1}\widetilde{G}_D(x,y). \tag{3.8}$$

By the estimates of $\widetilde{p}_D(t,x,y)$ (Remark 2.3) we have

$$\int_{t_0}^{\infty} \widetilde{p}_D(t,x,y)dt \leqslant Ct_0^{-1-1/\alpha}(\delta_D(x)\delta_D(y))^{\alpha/2}. \tag{3.9}$$

Next, from Lemma 2.5 with $X = \widetilde{X}$ we have

$$\widetilde{p}_D(t,x,y) \leqslant p_D^Y(t,x,y) + \widetilde{p}(t,x,y) - e^{-2mt}p^Y(t,x,y), \tag{3.10}$$

so integrating over $[0,t_0]$, where $t_0 = (\delta_D(x)\delta_D(y))^{\alpha/6} \le 1$, and using Lemma 2.9 together with (3.9) we obtain

$$\begin{aligned} \widetilde{G}_D(x,y) &= \int_0^{t_0} \widetilde{p}_D(t,x,y)dt + \int_{t_0}^{\infty} \widetilde{p}_D(t,x,y)dt \\ &\leqslant G_D^Y(x,y) + \int_0^{t_0} (\widetilde{p}(t,x,y) - e^{-2mt}p^Y(t,x,y))dt + \\ &\quad + Ct_0^{-1-1/\alpha}(\delta_D(x)\delta_D(y))^{\alpha/2} \\ &\leqslant G_D^Y(x,y) + ct_0^{2-1/\alpha} + Ct_0^{-1-1/\alpha}(\delta_D(x)\delta_D(y))^{\alpha/2} \\ &= G_D^Y(x,y) + c(\delta_D(x)\delta_D(y))^{\frac{2\alpha-1}{6}}. \end{aligned} \tag{3.11}$$

Now assume that $|x-y|^2 \leqslant \delta_D(x)\delta_D(y)$ and take into account that in this case $\widetilde{G}_D(x,y) \ge C(\delta_D(x)\delta_D(y))^{(\alpha-1)/2}$, so that we can rewrite (3.11) as

$$\widetilde{G}_D(x,y) \leqslant G_D^Y(x,y) + c(\delta_D(x)\delta_D(y))^{\rho}\widetilde{G}_D(x,y), \tag{3.12}$$

where $\rho = \frac{2-\alpha}{6} > 0$. Observe that (3.12) in the case $|x-y|^2 \leqslant \delta_D(x)\delta_D(y) \le \theta$, and (3.8) in the case $\delta_D(x)\delta_D(y) \le |x-y|^2 \le \theta$ for θ sufficiently small, provide the conclusion in these cases. Of the remaining cases $\delta_D(x)\delta_D(y) \ge \theta$ or $|x-y|^2 \ge \theta$ only the first needs to be considered and can be handled in a very simple way. Indeed, in this situation,

$$(\delta_D(x)\delta_D(y))^{\frac{2\alpha-1}{6}} \le (\delta_D(x)\delta_D(y))^{\frac{\alpha}{2}}\theta^{-\frac{\alpha+1}{6}} \le C\theta^{-\frac{\alpha+1}{6}}G_D^Y(x,y),$$

where the last step follows from the fact that Y_t has Property A and Lemma 2.4. Hence the conclusion holds by (3.11). This completes the proof. □

3.2. Case $\nu^Y \geqslant \widetilde{\nu}$. Throughout this subsection we assume that $\nu^Y \geqslant \widetilde{\nu}$, and, in addition, that D is a bounded Lipschitz open set. Note that in this case, by a result of Sztonyk [18], the process Y does not hit the boundary on exiting D. In verifying the conditions needed for the result to hold (see [18]), we apply Corollary 3.2 and Lemma 2.4 to reduce the problem to the stable case, which is known to be true. Hence, if u is regular harmonic on D with respect to the process Y, then

$$u(x) = E^x u(Y_{\tau_D}) = \int_{D^c} u(z) P_D^Y(x,z)dz, \quad x \in D. \tag{3.13}$$

The aim of this section is to prove that the Green functions are comparable, first for sets D with small diameter, and then for arbitrary bounded Lipschitz open sets. The result for sets D with small diameter allows us to prove a version of the Boundary Harnack Principle (BHP) under the following assumptions:

(G1) $\nu^Y(x) \geqslant \widetilde{\nu}(x)$ for $x \in \mathbb{R}^d \setminus \{0\}$.

(G2) For some $R > 0$ there are constants $c_1 = c_1(R)$ and ϱ such that

$$|\sigma(x)| = |\widetilde{\nu}(x) - \nu^Y(x)| \leqslant c_1 |x|^{\varrho - d} \quad \text{for } |x| \leqslant R.$$

(G3) There is a constant $c_2 = c_2(R)$ such that

$$\nu^Y(x) \leqslant c_2 \nu^Y(y)$$

for any $x, y \in \mathbb{R}^d$ such that $|x-y| \leqslant R/2$ and $|x|, |y| \geqslant R/2$.

Then, after establishing BHP, we show that we can remove the assumption about the diameter of the set D.

We start by iterating the inequality from Corollary 3.6 to obtain, for $G_D^Y(x,\cdot)$ continuous,

$$G_D^Y(x,y) \leqslant \sum_{k=0}^{n} [(H_D^\sigma)^k \widetilde{G}_D(\cdot,y)](x) + [(H_D^\sigma)^{n+1} G_D^Y(\cdot,y)](x), \tag{3.14}$$

where $H_D^\sigma : L^1(D) \to L^1(D)$ is given by

$$[H_D^\sigma f(\cdot)](x) = \int_D \int_D \widetilde{G}_D(x,w) |\sigma(w-z)| f(z) dw dz.$$

We now prove the comparability of Green functions for sets of small diameter. Note that the constant C in the conclusion of the following proposition depends on D through $\mathbf{r}_0$ and λ. This feature is crucial for our future applications.

Proposition 3.18. *Let $d > \alpha$. Let D be a Lipschitz open set and $G_D^Y(x,\cdot)$ be continuous and let ν^Y satisfy* (G1) *and* (G2). *Then there exist constants*

$R_0 = R_0(d,\alpha,\lambda,\mathbf{r}_0,\sigma) \leqslant R$ and $C = C(R_0)$ satisfying the following property: if $\operatorname{diam}(D) \leqslant R_0$, then

$$C^{-1}\widetilde{G}_D(x,y) \leqslant G_D^Y(x,y) \leqslant C\widetilde{G}_D(x,y), \quad x,y \in D.$$

Proof. If $\operatorname{diam}(D) \leqslant R$, we get by Lemma 3.14 that

$$[H_D^\sigma \widetilde{G}_D(\cdot,y)](x) \leqslant C_1 \operatorname{diam}(D)^\zeta \widetilde{G}_D(x,y),$$

for some constant $C_1 = C_1(d,\alpha,\lambda,\mathbf{r}_0,\sigma)$ and $\zeta > 0$. Iterating the above inequality, we obtain that $[(H_D^\sigma)^k \widetilde{G}_D(\cdot,y)](x)$ is bounded by

$$(C_1 \operatorname{diam}(D)^\zeta)^k \widetilde{G}_D(x,y).$$

Setting

$$R_0 = \frac{1}{2} C_1^{-1/\zeta} \wedge R,$$

we obtain for $\operatorname{diam}(D) \leqslant R_0$ that

$$[H_D^\sigma \widetilde{G}_D(\cdot,y)](x) \leqslant \theta \widetilde{G}_D(x,y), \tag{3.15}$$

for some $\theta \leqslant 1/2$.

Next, we show that for any $x \neq y \in D$

$$\lim_{n\to\infty} [(H_D^\sigma)^n G_D^Y(\cdot,y)](x) = 0.$$

Indeed, let us observe that for a positive $f \in L^1(D)$ we have from (3.15) that

$$\begin{aligned}
[(H_D^\sigma)^2 f](x) &= \int_D \int_D [(H_D^\sigma)\widetilde{G}_D(\cdot,w)](x)|\sigma(w-z)|f(z)dzdw \\
&\leqslant \theta \int_D \int_D \widetilde{G}_D(x,w)|\sigma(w-z)|f(z)dzdw \\
&= \theta [H_D^\sigma f](x).
\end{aligned}$$

Iterating, we obtain

$$[(H_D^\sigma)^{n+1} G_D^Y(\cdot,y)](x) \leqslant \theta^n [(H_D^\sigma) G_D^Y(\cdot,y)](x).$$

So it is enough to prove that $[(H_D^\sigma) G_D^Y(\cdot,y)](x)$ is finite. But from Lemma 2.7 we obtain that there is a constant C such that $G^Y(x,y) \leqslant C\widetilde{U}(x-y)$. Hence by Lemma 3.13 we get

$$[(H_D^\sigma) G_D^Y(\cdot,y)](x) \leqslant C \int_D \int_D \widetilde{U}(x-w)|\sigma(w-z)|\widetilde{U}(z-y)dwdz < \infty.$$

Finally, we infer from (3.14) that if $\operatorname{diam}(D) \leqslant R_0$, then

$$G_D^Y(x,y) \leqslant \frac{\theta}{1-\theta} \widetilde{G}_D(x,y),$$

which together with Corollary 3.2 completes the proof. □

REMARK 3.19. The constant $C(R_0)$ in the above theorem converges to 1 if $\operatorname{diam}(D)$ converges to 0.

The next result shows that the Poisson kernels for D are comparable under the assumptions of the preceding result. This will provide the necessary tools to establish BHP, which is employed to obtain comparability of Green functions for sets of arbitrary finite diameter.

Proposition 3.20. *Let $d > \alpha$ and D be a bounded Lipschitz open set. Assume that ν^Y satisfies assumptions* (G1) *and* (G2) *and is bounded on $B^c(0,R)$. There exist constants $R_0 = R_0(d,\alpha,\lambda,\mathbf{r}_0,\sigma) \leqslant R/2$ and $C = C(R_0)$ which satisfy, for* $\operatorname{diam}(D) \leqslant R_0$*,*

$$C^{-1}\widetilde{P}_D(x,z) \leqslant P_D^Y(x,z) \leqslant C\widetilde{P}_D(x,z),$$

for any $x \in D$ and $z \in \overline{D}^c : \delta_D(z) \leqslant R_0$. Moreover, if we suppose that ν^Y satisfies assumption (G3) *with $R = 2\,R_0$, then there exists a constant $C(R_0)$ such that*

$$C^{-1}\nu^Y(z-x)E^x\widetilde{\tau}_D \leqslant P_D^Y(x,z) \leqslant C\nu^Y(z-x)E^x\widetilde{\tau}_D,$$

for $x \in D$ and $z \in \overline{D}^c : \delta_D(z) > R_0$.

Proof. By Proposition 3.18 there are constants $\overline{R}_0 \leqslant R/2$ and $C_1(\overline{R}_0)$ such that

$$C_1^{-1}\widetilde{G}_D(x,y) \leqslant G_D^Y(x,y) \leqslant C_1\widetilde{G}_D(x,y),$$

if $\operatorname{diam}(D) \leqslant \overline{R}_0$. Next, by Theorem 1 in [11] we have

$$P_D^Y(x,z) = \int_D \nu^Y(z-y)G_D^Y(x,y)dy.$$

But

$$|\sigma(w)| \leqslant c_1|w|^{-d+\varrho} = c_1\mathscr{A}(-\alpha,d)^{-1}\widetilde{\nu}(w)|w|^{\varrho+\alpha}.$$

So for $z \in \overline{D}^c : \delta_D(z) \leqslant \overline{R}_0$ we have

$$|\sigma(z-y)| \leqslant c_1\mathscr{A}(-\alpha,d)^{-1}(2\overline{R}_0)^{\varrho+\alpha}\widetilde{\nu}(z-y).$$

Hence, we put

$$R_0 = \overline{R}_0 \wedge \frac{1}{2}\left(\frac{\mathscr{A}(-\alpha,d)}{2c_1}\right)^{1/(\alpha+\varrho)}$$

and then

$$|\sigma(z-y)| \leqslant \frac{1}{2}\widetilde{\nu}(x).$$

By the above inequality we obtain

$$\begin{aligned} P_D^Y(x,z) &\leqslant C_1 \int_D \nu^Y(z-y)\widetilde{G}_D(x,y)dy \\ &= C_1\left(\int_D \widetilde{\nu}(z-y)\widetilde{G}_D(x,y)dy + \int_D \sigma(z-y)\widetilde{G}_D(x,y)dy\right) \\ &\leqslant C_1\widetilde{P}_D(x,y) + C_1\int_D |\sigma(z-y)|\widetilde{G}_D(x,y)dy \\ &\leqslant \frac{3}{2}C_1\widetilde{P}_D(x,y), \end{aligned}$$

and

$$\begin{aligned} P_D^Y(x,z) &\geqslant C_1^{-1}\int_D \nu^Y(z-y)\widetilde{G}_D(x,y)dy \\ &\geqslant C_1^{-1}\widetilde{P}_D(x,y) - C_1^{-1}\int_D |\sigma(z-y)|\widetilde{G}_D(x,y)dy \\ &\geqslant \frac{C_1^{-1}}{2}\widetilde{P}_D(x,y), \end{aligned}$$

which completes the proof of the first claim of the theorem.

Now, suppose that there is a constant $c = c(R_0)$ such that $\nu^Y(x) \leqslant c\nu^Y(y)$ for all $|x|, |y| \geqslant R_0$ such that $|x-y| \leq R_0$. Assume that $z \in \overline{D}^c : \delta_D(z) > R_0$. For $x, y \in D$ we have

$$|x-z| \geqslant \delta_D(z) \geqslant R_0 \text{ and, of course, } |x-y| \leqslant \operatorname{diam}(D) \leqslant R_0.$$

Hence, we get

$$\begin{aligned} P_D^Y(x,z) &\leqslant cC_1\nu^Y(x-z)\int_D \widetilde{G}_D(x,y)dy \\ &= cC_1\nu^Y(x-z)E^x\widetilde{\tau}_D. \end{aligned}$$

Similarly, we obtain the lower bound

$$P_D^Y(x,z) \geqslant (cC_1)^{-1}\nu^Y(x-z)E^x\widetilde{\tau}_D. \qquad \square$$

THEOREM 3.21 (Boundary Harnack Principle (BHP)). *Let $d > \alpha$ and D be a bounded Lipschitz open set. Suppose that ν^Y satisfies* (G1)–(G3). *Let $Z \in \partial D$. Then there exists a constant $\rho_0 = \rho_0(D)$ such that for any $\rho \in (0, \rho_0]$ and any two functions u and v which are nonnegative in $\mathbb{R}^d$ and positive, regular harmonic in $D \cap B(Z,\rho)$ the following holds: If u and v vanish on $D^c \cap B(Z,\rho)$, then for $x, y \in D \cap B(Z,\rho\beta)$*

$$\frac{u(x)}{v(x)} \leqslant C\frac{u(y)}{v(y)},$$

for some constant $C = C(D,\alpha,\sigma)$ and $\beta(d,\lambda) \in (0,1)$.

Proof. There is a constant $R_1 = R_1(d, \lambda) \geqslant 1$ (see, e.g., [2]) such that for all $Z \in \partial D$ and $r \in (0, r_0)$, there exists a bounded Lipschitz open set $\Omega(r)$ with Lipschitz constant λR_1 and localization radius $\operatorname{diam}(D)\mathbf{r}_0/R_1$, such that

$$D \cap B(Z, r/R_1) \subset \Omega(r) \subset D \cap B(Z, r).$$

The proof consists of showing that there are constants $C = C(D, \alpha, \sigma)$ and ρ_0 such that for $\rho < \rho_0$ and $z \in \Omega(\rho)^c \cap B^c(Z, \rho/R_1)$,

$$P^Y_{\Omega(\rho)}(x, z) \leqslant C \frac{E^x \widetilde{\tau}_{\Omega(\rho)}}{E^y \widetilde{\tau}_{\Omega(\rho)}} P^Y_{\Omega(\rho)}(y, z), \tag{3.16}$$

where $x, y \in D \cap B(Z, \rho/(R_1 2))$. It is worth mentioning that the constant C is universal for all sets $\Omega(\rho)$, $\rho \leq \rho_0$. This will give the conclusion with $\beta = 1/(2 R_1)$, since by (3.13) we have

$$\begin{aligned} u(x) = E^x u(Y_{\tau_{\Omega(\rho)}}) &= \int_{\Omega(\rho)^c} u(z) P^Y_{\Omega(\rho)}(x, z) dz \\ &= \int_{\Omega(\rho)^c \setminus B(Z, \rho/R_1)} u(z) P^Y_{\Omega(\rho)}(x, z) dz \\ &\leqslant C \frac{E^x \widetilde{\tau}_{\Omega(\rho)}}{E^y \widetilde{\tau}_{\Omega(\rho)}} \int_{\Omega(\rho)^c \setminus B(Z, \rho/R_1)} u(z) P^Y_{\Omega(\rho)}(y, z) dz \\ &= C \frac{E^x \widetilde{\tau}_{\Omega(\rho)}}{E^y \widetilde{\tau}_{\Omega(\rho)}} u(y), \end{aligned}$$

which implies

$$\frac{u(x)}{u(y)} \frac{v(y)}{v(x)} \leqslant C \frac{E^x \widetilde{\tau}_{\Omega(\rho)}}{E^y \widetilde{\tau}_{\Omega(\rho)}} C \frac{E^y \widetilde{\tau}_{\Omega(\rho)}}{E^x \widetilde{\tau}_{\Omega(\rho)}} = C^2.$$

Now we prove (3.16). By Proposition 3.20 there exist constants $\rho_0 < r_0(D)$ and $C_1 = C_1(\rho_0)$ such that for any $\rho \leqslant \rho_0$

$$C_1^{-1} \widetilde{P}_{\Omega(\rho)}(x, z) \leqslant P^Y_{\Omega(\rho)}(x, z) \leqslant C_1 \widetilde{P}_{\Omega(\rho)}(x, z),$$

if $\delta_{\Omega(\rho)}(z) \leqslant \rho_0$. Note that C_1 is universal for all $\Omega(\rho)$.

By Theorem 2 in [12] there is some $C_2 = C_2(\alpha, d, \lambda, \mathbf{r}_0)$ such that for any $x, y \in D$ and $z \in \overline{D}^c$,

$$\widetilde{P}_{\Omega(\rho)}(x, z) \leqslant C_2 \frac{E^x \widetilde{\tau}_{\Omega(\rho)}}{E^y \widetilde{\tau}_{\Omega(\rho)}} \frac{\widetilde{\phi}^2_{\Omega(\rho)}(A_{y,z'})}{\widetilde{\phi}^2_{\Omega(\rho)}(A_{x,z'})} \frac{|y - z|^{d-\alpha}}{|x - z|^{d-\alpha}} \widetilde{P}_{\Omega(\rho)}(y, z),$$

where $z' \in \{A \in D : B(A, \kappa \delta_{\Omega(\rho)}(z)) \subset D \cap B(S, \delta_{\Omega(\rho)}(z))\}$ if $\delta_{\Omega(\rho)}(z) \leqslant r_0/32$, and $z' = x_1$ if $\delta_{\Omega(\rho)}(z) > r_0/32$ for S such that $|z - S| = \delta_{\Omega(\rho)}(z)$. If $x, y \in D \cap B(Z, \rho/(R_1 2))$ and $z \in \Omega(\rho)^c \cap B^c(Z, \rho/R_1)$, then

$$\frac{|y - z|}{|x - z|} \leqslant \frac{|x - z| + |x - y|}{|x - z|} \leqslant \left(1 + \frac{\rho/R_1}{\rho/(2R_1)}\right) = 3.$$

Now, suppose that $\delta_{\Omega(\rho)}(z) \leqslant \rho/32$. Then we obtain

$$|x-z'| \geqslant |x-z|-|z-z'| \geqslant |x-z|-|z-S|-|z'-S| \geqslant \frac{\rho}{2}-2\delta_{\Omega(\rho)}(z) \geqslant \frac{7}{16}\rho > \frac{r_0}{32},$$

while if $\delta_{\Omega(\rho)}(z) > \rho/32$, then $z' = x_1$, so $\delta_{\Omega(\rho)}(z') \geqslant r_0/4$. Therefore, $A_{x,z'} = x_1 = A_{y,z'}$ and, of course, $\widetilde{\phi}_{\Omega(\rho)}(A_{y,z'})/\widetilde{\phi}_{\Omega(\rho)}(A_{x,z'}) = 1$. Hence for $x, y \in D \cap B(Z, \rho/(R_1 2))$ and $z \in \Omega(\rho)^c \cap B^c(Z, \rho/R_1)$ such that $\delta_{\Omega(\rho)}(z) \leqslant \rho_0$ we get

$$P^Y_{\Omega(\rho)}(x,z) \leqslant C_1^2 C_2 3^{d-\alpha} \frac{E^x \widetilde{\tau}_{\Omega(\rho)}}{E^y \widetilde{\tau}_{\Omega(\rho)}} P^Y_{\Omega(\rho)}(y,z).$$

Next, observe that (G1)–(G3) imply that for $r \leq R$ there is a constant $c = c(r)$ such that $\nu^Y(x) \leqslant c\nu^Y(y)$ for all x and y such that $|x-y| \leqslant r$ and $|x|, |y| \geqslant r$. Hence for $\delta_{\Omega(\rho)}(z) \geqslant \rho_0$ we have

$$\begin{aligned} P^Y_{\Omega(\rho)}(x,z) &\leqslant C_3(\rho_0)\nu^Y(z-x)E^x\widetilde{\tau}_{\Omega(\rho)} \leqslant C_3(\rho_0)c(\rho_0)\nu^Y(z-y)E^x\widetilde{\tau}_{\Omega(\rho)} \\ &\leqslant cC_3^2 \frac{E^x\widetilde{\tau}_{\Omega(\rho)}}{E^y\widetilde{\tau}_{\Omega(\rho)}} P^Y_{\Omega(\rho)}(y,z). \end{aligned}$$

This completes the proof of (3.16) and hence that of the theorem. □

For regular harmonic functions which vanish on D^c the following remark holds.

REMARK 3.22. Suppose ν^Y satisfies (G1), (G2) and is bounded on $B^c(0,R)$. Let $Z \in \partial D$. Then there exists a constant $\rho_0 = \rho_0(D)$ such that for any $\rho \in (0, \rho_0]$ and any two functions u and v which are nonnegative in $\mathbb{R}^d$ and positive, regular harmonic in $D \cap B(Z,\rho)$ the following holds: If u and v vanish on D^c, then for $x, y \in D \cap B(Z, \rho\beta)$

$$\frac{u(x)}{v(x)} \leqslant C\frac{u(y)}{v(y)},$$

for some constant $C = C(D, \alpha, \sigma)$ and $\beta(d, \lambda) \in (0,1)$.

THEOREM 3.23. *Let $d > \alpha$ and D be a bounded Lipschitz open set. Assume that ν^Y satisfies assumptions* (G1), (G2) *and is bounded on $B^c(0,R)$. Then for $x, y \in D$ we have*

$$C^{-1}\widetilde{G}_D(x,y) \leqslant G^Y_D(x,y) \leqslant C\widetilde{G}_D(x,y),$$

for some constant $C = C(d, \lambda, \mathbf{r}_0, \sigma)$.

Proof. Observe that for $|x-y| \leqslant N(\delta_D(x) \wedge \delta_D(y))$,

$$G^Y_D(x,y) \geqslant G^Y_{B(x,\delta_D(x)\wedge\delta_D(y)\wedge R_0(D))}(x,y) \geqslant C\widetilde{G}_{B(x,\delta_D(x)\wedge\delta_D(y)\wedge R_0)}(x,y),$$

where R_0 is such that $G^Y_{B(0,R_0)}(x,y) \approx \widetilde{G}_{B(0,R_0)}(x,y)$ (such an R_0 exists by Proposition 3.18). Next, it is easy to see from Theorem 3.4 in [14] that

$$c(N)|x-y|^{\alpha-d} \leqslant \widetilde{G}_{B(x,\delta_D(x)\wedge\delta_D(y)\wedge R_0)}(x,y) \leqslant CG^Y_D(x,y). \tag{3.17}$$

From Lemma 2.7 we have

$$G^Y_D(x,y) \leqslant U^Y(x-y) \leqslant C\widetilde{U}(x-y) = C|x-y|^{\alpha-d}. \tag{3.18}$$

We define similarly as in Theorem 3.8 the truncated Green function for Y_t by

$$\phi^Y_D(x) = G^Y_D(x_1,y) \wedge \mathscr{A}(d,\alpha) r_0^{d+\alpha}.$$

Using Remark 3.22 we can repeat the arguments from Lemma 17 in [12] to show that

$$\phi^Y_D(x) \approx E^x \tau^Y_D.$$

Next, by Lemma 2.4 we get

$$E^x \tau^Y_D \approx E^x \widetilde{\tau}_D.$$

Therefore

$$\phi^Y_D(x) \approx \widetilde{\phi}_D(x). \tag{3.19}$$

From the above and (3.18) we infer that there is a constant r such that $\phi^Y_D(x) = G^Y_D(x,x_0)$ for $x \in D \cap B^c(x_0,r)$. Hence by Harnack's inequality for α-stable harmonic functions we obtain, for all $x,y \in D \cap B^c(x_0,r)$ such that $|x-y| \leqslant N(\delta_D(x)\wedge\delta_D(y))$,

$$G^Y_D(x,x_0) = \phi^Y_D(x) \approx \widetilde{\phi}_D(x) \leqslant C(N)\,\widetilde{\phi}_D(y) \approx \phi^Y_D(y) = G^Y_D(y,x_0). \tag{3.20}$$

Using BHP for Y_t (see Remark 3.22), and taking into account (3.17), (3.18) and (3.20) we can prove a version of Theorem 3.8 with G^Y_D instead of $\widetilde{G}_D$ (see the proof of Theorem 1 in [12]), namely,

$$C_1^{-1}\frac{\phi^Y_D(x)\phi^Y_D(y)}{(\phi^Y_D(A_{x,y}))^2}|x-y|^{\alpha-d} \leqslant G^Y_D(x,y) \leqslant C_1\frac{\phi^Y_D(x)\phi^Y_D(y)}{(\phi^Y_D(A_{x,y}))^2}|x-y|^{\alpha-d}.$$

Applying (3.19) and then comparing the above estimate with the bound from Theorem 3.8 we get the conclusion. □

3.3. Proof of Theorem 1.2. Let $d > \alpha$ and D be a bounded connected Lipschitz open set. Suppose that $|\sigma(x)| \leqslant c_3|x|^{-d+\varrho}$ for $|x| \leqslant 1$, where $\varrho > 0$ and $\nu^Y(x)$ is bounded on $B^c(0,1)$. Then Property A holds for Y_t by Corollary 3.7.

Let $\{Z_t\}$ be a Lévy process with the density of its Lévy measure given by $\nu(x) \vee \widetilde{\nu}(x)$. Then, of course, the process Z_t and the set D satisfy the assumptions of Theorem 3.23. Hence there is a constant C_1 such that

$$C_1^{-1}\widetilde{G}_D(x,y) \leqslant G^Z_D(x,y) \leqslant C_1\widetilde{G}_D(x,y). \tag{3.21}$$

By Corollaries 3.2 and 3.6 we have

$$C_2^{-1}G_D^Y(x,y) \leqslant G_D^Z(x,y) \leqslant G_D^Y(x,y) + C_2 R_D^Z(x,y), \tag{3.22}$$

where $R_D^Z(x,y) = \int_D \int_D G_D^Z(x,w)|\sigma(w-z)|G_D^Z(z,y)dwdz$. From (3.21) and Lemma 3.14 it follows that there exists a constant θ such that for $|x-y| < \theta$,

$$R_D^Z(x,y) \leqslant C_1^2 \widetilde{R}_D(x,y) \leqslant \frac{1}{2C_1}\widetilde{G}_D(x,y) \leqslant \frac{1}{2}G_D^Z(x,y). \tag{3.23}$$

From inequality (3.21) and Corollary 3.11 we obtain the following inequality:

$$G_D^Z(x,y) \leqslant C_1 \widetilde{G}_D(x,y) \leqslant C(\theta)E^x\widetilde{\tau}_D E^y \widetilde{\tau}_D, \quad |x-y| \geqslant \theta.$$

Hence, by Lemma 2.4 and Property A for Y_t we get for $|x-y| \geqslant \theta$,

$$G_D^Z(x,y) \leqslant C G_D^Y(x,y). \tag{3.24}$$

Combining (3.22), (3.23) and (3.24) we arrive at

$$C_3^{-1}G_D^Y(x,y) \leqslant G_D^Z(x,y) \leqslant C_3 G_D^Y(x,y). \tag{3.25}$$

By (3.21) and (3.25) we have

$$C^{-1}\widetilde{G}_D(x,y) \leqslant G_D^Y(x,y) \leqslant C\widetilde{G}_D(x,y),$$

which completes the proof.

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References

[1] R. Bañuelos, *Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators*, J. Funct. Anal. **100** (1991), 181–206. MR 1124298 (92k:35066)

[2] K. Bogdan, *The boundary Harnack principle for the fractional Laplacian*, Studia Math. **123** (1997), 43–80. MR 1438304 (98g:31005)

[3] ———, *Sharp estimates for the Green function in Lipschitz domains*, J. Math. Anal. Appl. **243** (2000), 326–337. MR 1741527 (2001b:31007)

[4] K. Bogdan and T. Byczkowski, *Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains*, Studia Math. **133** (1999), 53–92. MR 1671973 (99m:31010)

[5] H. Byczkowska and T. Byczkowski, *One-dimensional symmetric stable Feynman-Kac semigroups*, Probab. Math. Statist. **21** (2001), 381–404. MR 1911445 (2003d:60092)

[6] Z.-Q. Chen and R. Song, *Intrinsic ultracontractivity and conditional gauge for symmetric stable processes*, J. Funct. Anal. **150** (1997), 204–239. MR 1473631 (98j:60103)

[7] ———, *Estimates on Green functions and Poisson kernels for symmetric stable processes*, Math. Ann. **312** (1998), 465–501. MR 1654824 (2000b:60179)

[8] ———, *Drift transforms and Green function estimates for discontinuous processes*, J. Funct. Anal. **201** (2003), 262–281. MR 1986161 (2004c:60218)

[9] K. L. Chung and Z. X. Zhao, *From Brownian motion to Schrödinger's equation*, Grundlehren der Mathematischen Wissenschaften, vol. 312, Springer-Verlag, Berlin, 1995. MR 1329992 (96f:60140)

[10] T. Grzywny, *Intrinsic ultracontractivity for Lévy processes*, Probab. Math. Statist. **28** (2008), 91–106.

[11] N. Ikeda and S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. **2** (1962), 79–95. MR 0142153 (25 #5546)

[12] T. Jakubowski, *The estimates for the Green function in Lipschitz domains for the symmetric stable processes*, Probab. Math. Statist. **22** (2002), 419–441. MR 1991120 (2004g:60069)

[13] P. Kim and Y.-R. Lee, *Generalized 3G theorem, and application to relativistic stable process on non-smooth open sets*, J. Funct. Anal. **246** (2007), 113–134. MR 2316878

[14] T. Kulczycki, *Properties of Green function of symmetric stable processes*, Probab. Math. Statist. **17** (1997), 339–364. MR 1490808 (98m:60119)

[15] ———, *Intrinsic ultracontractivity for symmetric stable processes*, Bull. Polish Acad. Sci. Math. **46** (1998), 325–334. MR 1643611 (99j:60115)

[16] M. Ryznar, *Estimates of Green function for relativistic α-stable process*, Potential Anal. **17** (2002), 1–23. MR 1906405 (2003f:60087)

[17] K.-I. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999, Translated from the 1990 Japanese original, Revised by the author. MR 1739520 (2003b:60064)

[18] P. Sztonyk, *On harmonic measure for Lévy processes*, Probab. Math. Statist. **20** (2000), 383–390. MR 1825650 (2002c:60126)

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