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OBATA'S THEOREM FOR KÄHLER MANIFOLDS

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Abstract. It is known that, in a complete Riemannian manifold (M, g), if the Hessian of a real valued function satisfies some suitable conditions, then it restricts the geometry of (M, g). In this paper we give a characterization of a certain class of Kähler manifolds admitting a real valued function u such that the Hessian has two eigenvalues u and $\frac{1+u}{2}$.

1. Introduction

It is known that, in a complete Riemannian manifold (M, g), if the Hessian of a real valued function satisfies some suitable conditions, then we get information about the geometry of the manifold (M, g). In fact, Obata [5] gave a characterization showing that a complete Riemannian manifold of dimension $n \geq 2$ is isometric to the round sphere (S^n, ds^2) of constant sectional curvature 1 if and only if there is a real valued function $u \in C^2(M)$ such that the Hessian of u, $\nabla^2 u$, satisfies the equation $\nabla^2 u = -u \,\mathrm{Id}$. Also there are other works characterizing some classes of Riemannian manifolds under suitable conditions on the Hessian:

For Kähler manifolds, an analogue of Obata's theorem characterizing the complex projective space $\mathbb{CP}^n$ with constant holomorphic sectional curvature is proved in [6]. In [2], it is shown that compact rank-1 symmetric spaces are those complete Riemannian manifolds (M, g) admitting a real valued function u such that the Hessian of u has at most two eigenvalues $-u$ and $-\frac{1+u}{2}$, under some mild hypothesis on (M, g). See [2], [3] and [6] for details.

In this paper, we give a characterization of a certain class of Kähler manifolds. More precisely, we prove:

Theorem 1. *Let (M, g, J) be a Kähler manifold of dimension $2n$. Let $u \in C^2(M)$ be a real valued function with critical points such that*

(1) *the Hessian of u, $\nabla^2 u$, has two eigenvalues u and $\frac{u+1}{2}$ and the eigenvalue u is of multiplicity 2, and*
(2) *∇u and $J\nabla u$ are eigenvectors of $\nabla^2 u$ with eigenvalue u.*

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Then the following holds.

(1) *If the function u has a minimum, then (M, g) is isometric to the complex hyperbolic space $(\mathbb{CH}^n, ds^2)$ of constant holomorphic sectional curvature -1.*

(2) *If the function u has a maximum, then there exists a totally geodesic submanifold M_0 of co-dimension 2 such that (M, g) is diffeomorphic to the normal bundle of M_0. Furthermore, the fibre over each point in M_0 is isometric to the simply connected surface $(\mathbb{H}^2, ds^2)$ of constant curvature -1.*

2. Preliminaries

We refer to [7] for basic definitions and tools used in this paper.

Let (M, g) be a complete Riemannian manifold and $u \in C^2(M)$. We let $X := \frac{\nabla u}{\|\nabla u\|}$ on $\{q \in M : \nabla u(q) \neq 0\}$.

The following two propositions are proved in [2]. For the sake of completeness, we sketch the proof of these results here.

Proposition 2. *Let (M, g) be a complete Riemannian manifold and $u \in C^2(M)$. Then the integral curves of X are geodesics if and only if ∇u is an eigenvector of $\nabla^2 u$.*

Proof. Let γ be an integral curve of X. Then γ is a geodesic if and only if $\nabla_X X = 0$ along γ. We will now prove that $\nabla_X X = 0$ along γ is equivalent to ∇u being an eigenvector of $\nabla^2 u$. On $\{q \in M : \nabla u(q) \neq 0\}$,

$$\begin{aligned}\nabla_X X &= \frac{1}{\|\nabla u\|}\nabla_X \nabla u + X\left(\frac{1}{\|\nabla u\|}\right)\nabla u\\ &= \frac{1}{\|\nabla u\|}\nabla_X \nabla u - \frac{X(\|\nabla u\|)}{\|\nabla u\|^2}\nabla u\\ &= \frac{1}{\|\nabla u\|}\nabla_X \nabla u - \frac{\langle \nabla_X \nabla u, \nabla u\rangle}{\|\nabla u\|^3}\nabla u\\ &= \frac{1}{\|\nabla u\|}\nabla_X \nabla u - \frac{1}{\|\nabla u\|}\langle \nabla_X \nabla u, X\rangle X.\end{aligned}$$

Hence $\nabla_X X = 0$ if and only if

$$\frac{1}{\|\nabla u\|}\nabla_X \nabla u = \frac{1}{\|\nabla u\|}\langle \nabla_X \nabla u, X\rangle X.$$

This completes the proof. □

Proposition 3. *Let (M, g) be a complete Riemannian manifold and $u \in C^2(M)$ be such that the integral curves of X are geodesics. Then u does not have saddle points.*

Proof. Let us assume the contrary and arrive at a contradiction.

Let $p \in M$ be a saddle point of the function u. Then $\nabla^2 u(p)$ has both positive and negative eigenvalues. Hence there is an open neighbourhood W of $p \in M$ such that the flow lines of X have the form of hyperbolas near the point p and in this open set they form a saddle. We may assume that $W = \exp_p(W_1)$, where W_1 is an open neighbourhood of $0 \in T_pM$. We also assume that W is geodesically convex. (See [1] and [4].) Let $E^{us} \subseteq T_pM$ denote the eigensubspace of $\nabla^2 u(p)$ on which $\nabla^2 u(p)$ is negative definite and let $E^s \subseteq T_pM$ denote the eigensubspace of $\nabla^2 u(p)$ on which $\nabla^2 u(p)$ is positive definite. Let $W^{us} := \exp_p(W_1 \cap E^{us})$ and $W^s := \exp_p(W_1 \cap E^s)$. Then the integral curves of X through any point in W^{us} will start from p and diverge near p and the integral curves of X through any point in W^s will converge to p. (See [1].)

Let $\varepsilon > 0$ be such that the closed ball $\overline{B(p, 2\varepsilon)}$ of radius ε and center p is contained in W.

Let $x \in S(p, \varepsilon) \setminus W^s$ and γ_x be the integral curve of the vector field X such that $\gamma_x(0) = x$. Then the geodesic γ_x passes through $B(p, 2\varepsilon)$ and $d(\gamma_x(t), \gamma_x(s)) \leq 4\varepsilon$ for $\gamma_x(t),\ \gamma_x(s) \in B(p, 2\varepsilon)$. Therefore, for the proof of this proposition, we restrict such geodesics to the interval $[0, 4\varepsilon]$. If $d(x, W^s)$ is small, then the exit point of the geodesic γ_x from $B(p, 2\varepsilon)$ is close to W^{us}.

Now we fix a point $q \in W^s \cap S(p, \varepsilon)$. Let $q_n \in S(p, \varepsilon) \setminus W^s$ be a sequence of points converging to the point q. Let $\gamma_n : [0, 4\varepsilon] \to W$ be the integral curve of X such that $\gamma_n(0) = q_n$. By the local compactness of the unit tangent bundle UM, the sequence $(\gamma_n(0), \gamma_n'(0))$ has a convergent subsequence converging to a point (q, w) in UM. Without loss of generality we assume that the original sequence itself is convergent. Let $\gamma \colon [0, 4\varepsilon] \to W$ be the limiting geodesic with $\gamma(0) = q$ and $\gamma'(0) = w$. Since the sequence of points q_n converge to the point q in W^s, the exit point of the sequence of geodesics γ_n in $B(p, 2\varepsilon)$ will converge to a point in W^{us}. Hence the limiting geodesic will pass through the point p and it will be broken at p. Since the geodesics γ_n are all minimizing, the geodesic γ is also minimizing. This is a contradiction. Hence the function u cannot have saddle points. □

In the following lemma, we describe the function u along the integral curves of X.

LEMMA 4. *Let (M, g) be a complete Riemannian manifold and $u \in C^2(M)$ be such that ∇u is an eigenvector of $\nabla^2 u$ with eigenvalue u. Let γ be an integral curve of X. Then there exist constants A_γ and B_γ such that $u(\gamma(t)) = A_\gamma e^t + B_\gamma e^{-t}$ for all t in $\mathbb{R}$.*

Proof. Let γ be an integral curve of X. We have seen in Proposition 2 that γ is a geodesic. Since (M, g) is a complete Riemannian manifold, the geodesic γ is defined on all of $\mathbb{R}$ and $\gamma'(t) = X(\gamma(t))$ whenever $\nabla u(\gamma(t)) \neq 0$.

We will show that the function u has at most one critical point along the geodesic γ and there exist constants A_γ and B_γ such that $u(\gamma(t)) = A_\gamma e^t + B_\gamma e^{-t}$ for all t in $\mathbb{R}$.

Let $U_\gamma := \{t \in \mathbb{R} : \nabla u(\gamma(t)) \neq 0\}$. Then U_γ is the largest open subset of $\mathbb{R}$ on which the geodesic γ is defined as an integral curve of the vector field X.

If the function u does not have critical points along the geodesic γ, then $U_\gamma = \mathbb{R}$ and

$$\begin{aligned}(u \circ \gamma)''(t) &= \left\langle \nabla_{\gamma'(t)} \nabla u, \gamma'(t) \right\rangle \\ &= u(\gamma(t))\end{aligned}$$

for every t in $\mathbb{R}$. Therefore there exist constants A_γ and B_γ such that $u(\gamma(t)) = A_\gamma e^t + B_\gamma e^{-t}$ for all $t \in \mathbb{R}$.

Let us now assume that u has critical points along γ and prove the result.

In this case $U_\gamma \neq \mathbb{R}$. Let U_1 be a connected component of U_γ.

Suppose $U_1 = (a, b)$ for some a, $b \in \mathbb{R}$. First we observe that the points $\gamma(a)$ and $\gamma(b)$ are critical points of the function u. We can show as above that

$$(u \circ \gamma)''(t) = u(\gamma(t))$$

for all $t \in (a, b)$. Therefore there exist constants A_γ and B_γ such that $u(\gamma(t)) = A_\gamma e^t + B_\gamma e^{-t}$ for all $t \in (a, b)$. Further,

$$\begin{aligned}(u \circ \gamma)'(t) &= \langle \nabla u(\gamma(t)), \gamma'(t) \rangle \\ &= \left\langle \nabla u(\gamma(t)), \frac{\nabla u(\gamma(t))}{\| \nabla u(\gamma(t)) \|} \right\rangle \\ &= \| \nabla u(\gamma(t)) \|\end{aligned}$$

for every $t \in (a, b)$. Since the points $\gamma(a)$ and $\gamma(b)$ are critical points of the function u, it follows that

$$\begin{aligned}0 &= \| \nabla u(\gamma(a)) \| \\ &= \lim_{t \to a} \| \nabla u(\gamma(t)) \| \\ &= \lim_{t \to a} (u \circ \gamma)'(t) \\ &= \lim_{t \to a} A_\gamma e^t - B_\gamma e^{-t} \\ &= A_\gamma e^a - B_\gamma e^{-a}\end{aligned}$$

and by similar arguments $A_\gamma e^b - B_\gamma e^{-b} = 0$. This is possible only if $A_\gamma = B_\gamma = 0$, a contradiction. This proves that every connected component of U_γ is an infinite interval. Hence $U_1 = (-\infty, a)$ or (b, ∞) for some real numbers a, b in $\mathbb{R}$.

Since every connected component of U_γ is an infinite interval, it follows that either U_γ is connected or U_γ has two connected components and $U_\gamma = (-\infty, a) \cup (b, \infty)$.

Let $U_\gamma = (-\infty, a) \cup (b, \infty)$. We claim that $a = b$. Suppose $a < b$. This means that $\gamma(t)$ is a critical point of the function u for every point $t \in [a, b]$. Hence $\nabla u(\gamma(t)) = 0$ for all $t \in [a, b]$ and

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(\gamma(t)) &= \frac{\partial}{\partial t} \langle \nabla u(\gamma(t)), \gamma'(t) \rangle \\ &= 0 \end{aligned}$$

for all $t \in [a, b]$. In particular, $(u \circ \gamma)''(a) = 0 = (u \circ \gamma)''(b)$. Since $\nabla u(\gamma(t)) \neq 0$, for $t < a$, we have that $u(\gamma(t)) = (u \circ \gamma)''(t)$ for $t < a$. Therefore

$$\begin{aligned} u(\gamma(a)) &= \lim_{t \to a} u(\gamma(t)) \\ &= \lim_{t \to a} (u \circ \gamma)''(t) \\ &= (u \circ \gamma)''(a) \\ &= 0. \end{aligned}$$

Further, $(u \circ \gamma)'(a) = 0$. Therefore, if $u(\gamma(t)) = A_\gamma e^t + B_\gamma e^{-t}$ for $t \in U_1$, we get that $A_\gamma e^a + B_\gamma e^{-a} = 0$ and $A_\gamma e^a - B_\gamma e^{-a} = 0$. This implies that $A_\gamma = 0 = B_\gamma$, a contradiction.

Hence $U_\gamma = (-\infty, a) \cup (a, \infty)$ and $u(\gamma(t)) = A_\gamma e^t + B_\gamma e^{-t}$ for all $t \in \mathbb{R}$.

If U_γ is connected, then $U_\gamma = (-\infty, a)$ or (b, ∞). Using the same arguments as above we can show that this is not possible. This completes the proof. □

We will now describe the minimum and maximum of the function u.

Proposition 5. *Let (M, g) be a complete Riemannian manifold of dimension n and $u \in C^2(M)$ be such that the Hessian of u, $\nabla^2 u$, has at most two eigenvalues u and $\frac{1+u}{2}$, and ∇u is an eigenvector of $\nabla^2 u$ with eigenvalue u. Let $p \in M$ be a critical point of u. Then the following holds.*

(1) *If the multiplicity of the eigenvalue u is n, then the Hessian of u at the point p, $\nabla^2 u(p)$, is non-degenerate.*
(2) *If the multiplicity of the eigenvalue u is not equal to n, then the Hessian $\nabla^2 u(p)$ is non-degenerate iff the point p is a minimum for the function u.*

Proof. Let $p \in M$ be a critical point of u.

If the multiplicity of the eigenvalue u is n, then $\nabla^2 u = u \,\mathrm{Id}$. In this case, if u has a critical point, it has been proved in [5] and [3] that $\nabla^2 u(p)$ is non-degenerate. Further, it has also been shown that p is the only critical point of the function u and $u(q) = u(p) \cosh d(p, q)$ for all $q \in M$. Hence we omit the proof here.

We will now prove the second part of the proposition.

Let p be a critical point of the function u such that $\nabla^2 u(p)$ is non-degenerate. We will show that $\nabla^2 u(p)$ is positive definite.

Since $\nabla^2 u(p)$ is non-degenerate, there exists an open neighbourhood W of p such that p is the only critical point of the function u in W. We may assume that the open neighbourhood W is geodesically convex.

Since u does not have saddle points, the point p must either be a local maximum or a local minimum. Hence all the integral curves γ of X passing through the points in $W \setminus \{p\}$ must either start from p and diverge near p- if p is a maximum or converge to p- if p is a minimum in W.

Since W is geodesically convex, given a point $q \neq p \in W$, there exists a unique geodesic γ_{pq} passing through p and q. On the other hand, given a point $q \neq p$ in W, there is a unique integral curve of X passing through q which must either converge to the point p or start from the point p. Therefore the geodesic γ_{pq} must be tangential to the vector field X at q. This means that every vector $E \in T_pM$ is an eigenvector of $\nabla^2 u(p)$. This proves that $u(p) = \frac{1+u(p)}{2}$. Hence $u(p) = 1$ and $\nabla^2 u(p)$ is positive definite. Thus we have shown that the point p is a local minimum for the function u.

Conversely assume that the point p is a local minimum for the function u. Hence the Hessian of u at p, $\nabla^2 u(p)$, is positive semi-definite. Since the eigenvalues of $\nabla^2 u(p)$ are $u(p)$ and $\frac{1+u(p)}{2}$, it is enough to show that $u(p) > 0$.

Let

$$E_{\frac{1+u(p)}{2}} := \{E \in T_pM \colon \nabla^2 u(p)(E) = \frac{1+u(p)}{2} E\}.$$

Since $u(p) \geq 0$, the Hessian $\nabla^2 u(p)$ is positive definite on $E_{\frac{1+u(p)}{2}}$. Therefore there exists an open neighbourhood W_1 of 0 in T_pM such that on the set $W^s := \exp_p(W_1 \cap E_{\frac{1+u(p)}{2}})$, the point p is the only critical point of the function u and the integral curves of the vector field X passing through W^s will all converge to the point p. Therefore, for every unit vector $v \in E_{\frac{1+u(p)}{2}}$, there exists an $\varepsilon > 0$ such that the geodesic $\gamma_v(t) := \exp_p(tv)$ is in W^s and it is an integral curve of X in $W^s \setminus \{p\}$. Since $\gamma(0) = p$ is the only critical point of the function u along γ, we can write the function u along γ as $u(\gamma(t)) = A_\gamma e^t + B_\gamma e^{-t}$ for $0 < |t| < \varepsilon$. Since $\nabla u(\gamma(0)) = 0$, we see that $0 = \|\nabla u(\gamma(0))\| = A_\gamma - B_\gamma$, i.e., $A_\gamma = B_\gamma$. Therefore $u(\gamma(t)) = 2A_\gamma \cosh t$. Now we use the fact that $\nabla u \neq 0$ in $W^s \setminus \{p\}$ to conclude that $A_\gamma \neq 0$. This shows that $u(p) > 0$ and hence $\nabla^2 u(p)$ is non-degenerate. □

COROLLARY 6. *Let (M, g) and u be as in Proposition 5. Then a critical point p of the function u is a local maximum iff $\nabla^2 u(p)$ is degenerate. Furthermore, in this case the value of the function at the point p is -1, i.e., $u(p) = -1$.*

Proof. It follows from Proposition 5 that p is a maximum for the function iff $\nabla^2 u(p)$ is degenerate and negative semi-definite. Therefore $u(p) \leq 0$ and $\frac{1+u(p)}{2} \leq 0$. But, if $u(p) = 0$, then we get that $\frac{1+u(p)}{2} > 0$, a contradiction.

On the other hand, if $u(p) < 0$ and $\frac{1+u(p)}{2} < 0$, then we get that $\nabla^2 u(p)$ is non-degenerate, a contradiction. Hence $u(p) = -1$. □

Our proof of the main result depends on the following two theorems.

THEOREM 7. *Let (M, g) and u be as in Proposition 5. Let p be a minimum for the function u. Then*

(1) *$u(q) = u(p) \cosh d(p, q)$ for every point $q \in M$, and*
(2) *$\exp_p : T_pM \to M$ is a diffeomorphism.*

THEOREM 8. *Let (M, g) and u be as in Proposition 5. Assume that $M_0 := \{q \in M : u(q) = \max_{p\in M} u(p)\}$ is non-empty. Then M_0 is a totally geodesic submanifold of M.*

3. Proof of Theorems 7 and 8

Proof of Theorem 7. Let p be a point of minimum for the function u. Then it follows from Proposition 5 that $\nabla^2 u(p)$ is non-degenerate and $u(p)$ is the only eigenvalue of $\nabla^2 u(p)$. If $\nabla^2 u$ has two eigenvalues u and $\frac{1+u}{2}$, then $u(p) = 1$. If the Hessian has only one eigenvalue, we may assume that $u(p) = 1$, by dividing the function u by a suitable constant.

Let γ be a geodesic starting at the point p. We have shown, in Proposition 5, that γ is an integral curve of X on $U_\gamma = \mathbb{R} \setminus \{0\}$ and $u(\gamma(t)) = u(p) \cosh t$ for all $t \in \mathbb{R}$.

Let $q \neq p \in M$. Then there is a length minimizing geodesic joining p and q. We have shown in Proposition 5 that such a geodesic must be an integral curve of the vector field X and further $u(q) = u(p) \cosh d(p, q)$. Therefore $\nabla u(q) = u(p) \sinh d(p, q) \nabla d(p, q) \neq 0$, where $\nabla d(p, .)$ denotes the radial vector field starting at p. This means that the point q is an ordinary point for the function u and hence there is a unique integral curve γ of the vector field X passing through the point q. This proves that given a point $q \neq p$, there is a unique geodesic γ_q such that $\gamma_q(0) = p$ and $d(p, \gamma_q(t)) = |t|$ for all $t \in \mathbb{R}$. Thus we have shown that the geodesics starting at p are rays. Hence the map $\exp_p : T_pM \to M$ is a diffeomorphism. □

Using Theorem 7, we prove the following theorem.

THEOREM 9. *Let (M, g), u and $p \in M$ be as in Theorem 7. Then the following holds.*

(1) *If the multiplicity of the eigenvalue u is 1, then (M, g) is isometric to the simply connected hyperbolic space $(\mathbb{H}^n, ds^2)$ of constant curvature $-1/4$.*
(2) *If the multiplicity of the eigenvalue u is n, then (M, g) is isometric to the simply connected hyperbolic space $(\mathbb{H}^n, ds^2)$ of curvature -1.*

Proof. We give a proof for the first claim. The proof is similar to the proof of Theorem 1(2) of [2].

Since the multiplicity of the eigenvalue u is 1, every vector $E \perp \nabla u$ is an eigenvector of $\nabla^2 u$ with eigenvalue $\frac{1+u}{2}$. Therefore the vector subbundle $E_{\frac{1+u}{2}} := \{E \in TM : \nabla^2 u(E) = \frac{1+u}{2} E\}$ is parallel along the integral curves of X.

Let γ be an integral curve of X. It follows from Theorem 7 that the geodesic γ passes through the point p. Hence we may assume that $\gamma(0) = p$. Therefore $U_\gamma = (-\infty, 0) \cup (0, \infty)$.

Let W denote the Jacobi field describing the variation of the geodesic γ such that $W(0) = 0$ and $W'(0) = E \in \{E \in TM : \nabla^2 u(E) = \frac{1+u}{2} E\}$ of unit norm. Since $[W, \gamma'] = 0$ along γ, it follows that $\nabla_X W = \nabla_W X$ whenever $\nabla u(\gamma(t)) \neq 0$. Using the fact $u(\gamma(t)) = \cosh t$ along the geodesic γ, we see that $\nabla_X W = W'$ along the geodesic γ. Therefore, for every $t \in U_\gamma$,

$$\begin{aligned} W'(t) &= \frac{1}{\| \nabla u(\gamma(t)) \|} \nabla_W \nabla u \\ &= \frac{1 + u(\gamma(t))}{2} \frac{1}{\| \nabla u(\gamma(t)) \|} W(t) \\ &= \frac{1}{2} \frac{\cosh \frac{t}{2}}{\sinh \frac{t}{2}} W(t) \end{aligned}$$

and

$$\frac{\langle W'(t), W(t) \rangle}{\| W(t) \|^2} = \frac{1}{2} \frac{\cosh \frac{t}{2}}{\sinh \frac{t}{2}}.$$

Therefore

$$\frac{d}{dt} \log \left(\frac{\| W \|}{\sinh \frac{t}{2}} \right) = 0$$

for all $t \in \mathbb{R}$. Hence $\frac{\| W \|}{\sinh \frac{t}{2}} = \frac{\| W \|}{\sinh \frac{t}{2}} |_{t=0} = 2$. Thus $\| W(t) \| = 2 \sinh \frac{t}{2}$ along the geodesic γ. Since $E_{\frac{1+u}{2}}$ is parallel along the integral curves of the vector field X, we can write $W(t) = 2 \sinh \frac{t}{2} E(t)$, where E is a unit vector field parallel along γ. Therefore

$$\begin{aligned} R(W, \gamma')\gamma' &= -W'' \\ &= -\frac{1}{4} W \end{aligned}$$

along the geodesic γ. Hence the sectional curvature $\langle R(E, X)X, E \rangle = -1/4$ for every unit vector E in $E_{\frac{1+u}{2}}$ on $M \setminus \{p\}$.

We are now ready to prove that (M, g) is isometric to $(\mathbb{H}^n, ds^2)$ of constant curvature $-1/4$.

We choose a point o in $\mathbb{H}^n$ and fix an isometry $i : T_pM \to T_o\mathbb{H}^n$. We define a map $\Phi : M \to \mathbb{H}^n$ by $\Phi(q) := \exp_o \circ i \circ \exp_p^{-1}(q)$. Then Φ maps the geodesics

γ starting at p onto the geodesics $\overline{\gamma}$ starting at o in $\mathbb{H}^n$ and it also maps the geodesic spheres of radius r around the point p bijectively onto the geodesic spheres of radius r around o in $\mathbb{H}^n$ for all $r > 0$. To complete the proof, we need only to show that the derivative $d\Phi$ of the map Φ is norm preserving. But this follows very easily from the observation that any Jacobi field W describing the variation of any geodesic γ starting at p such that $W(0) = 0$ and $W'(0)$ a unit vector in $E_{\frac{1+u}{2}}$ is of the form $W(t) = 2\sinh\frac{t}{2}E(t)$, where $E(t)$ is a vector field parallel along the geodesic γ and its image $d\Phi(W(t))$ is the normal Jacobi field describing the variation of the geodesic $\overline{\gamma} = \Phi(\gamma)$ starting at o in $\mathbb{H}^n$.

The proof of the second part of the theorem is similar. We give a brief sketch of the proof. In this case, we first observe that $\nabla^2 u = u\,\mathrm{Id}$. Let γ be a geodesic starting at the point p. By a similar computation as above, we conclude that a Jacobi field W describing the variation of the geodesic γ such that $W(0)$ and $W'(0) \perp \gamma'(0)$ is of the form $W(t) = \sinh t\, E(t)$, where E is a parallel vector field along γ. Now the rest of the proof is same as above. (See also [3].) □

Lemma 10. *Let (M, g) and u be as in Theorem 8. Assume that $M_0 \neq \emptyset$. Let γ be an integral curve of the vector field X. Then*

(1) *$U_\gamma = (-\infty, c) \cup (c, \infty)$ for some $c \in \mathbb{R}$ and*
(2) *$u(\gamma(c)) = -1$.*

Proof. We have shown in Corollary 6 that $M_0 := \{q \in M : u(q) = -1\}$. Therefore $u(p) \leq -1$ for every point p in M.

Let γ be an integral curve of X. If we show that $U_\gamma \neq \mathbb{R}$, then we are through. Assume on the contrary that $U_\gamma = \mathbb{R}$ and let A_γ and B_γ be two constants such that $u(\gamma(t)) = A_\gamma e^t + B_\gamma e^{-t}$ for all $t \in \mathbb{R}$.

Assume that A_γ and B_γ are of the same sign. Then there exists a unique $t_0 \in \mathbb{R}$ such that $A_\gamma e^{t_0} - B_\gamma e^{-t_0} = 0$. Therefore $\nabla u(\gamma(t_0)) = 0$, a contradiction.

Let us now assume that $A_\gamma > 0$ and $B_\gamma \leq 0$. Then $u(\gamma(t)) \to \infty$ as $t \to +\infty$, a contradiction to the fact that $\max u = -1$. Similarly, if $A_\gamma \leq 0$ and $B_\gamma > 0$, then $u(\gamma(t)) \to \infty$ as $t \to -\infty$, a contradiction. Hence $U_\gamma \neq \mathbb{R}$ and the proof is complete. □

Proof of Theorem 8. If $\nabla^2 u$ has only one eigenvalue u, then, using exactly the same arguments as in the proof of Proposition 5, we can show that every point $q \in M_0$ is a non-degenerate critical point of u and $u(p) = u(q)\cosh d(q, p)$ for every point p in M. Thus q is the unique maximum for the function u. Hence $M_0 = \{q\}$ and it is totally geodesic in M.

Let us now assume that $\nabla^2 u$ has two eigenvalues u and $\frac{1+u}{2}$ and prove the result.

Let $q \in M \setminus M_0$ and γ_q be the integral curve of X passing through the point q. From Lemma 10, it follows that $U_{\gamma_q} = (-\infty, c) \cup (c, \infty)$ for some $c \in \mathbb{R}$ and $\gamma_q(c) \in M_0$. This shows that the map $\Phi : M \to M_0$ defined by

$$\Phi(q) := \begin{cases} \exp_q(\cosh^{-1}(-u(q))X(q)) & \text{if } q \notin M_0, \\ q & \text{if } q \in M_0, \end{cases}$$

is onto. This map is also continuous. Hence M_0, being the continuous image of the connected set M, is connected.

Since $\max u = -1$, the Hessian of u at p, $\nabla^2 u(p)$, is $-\operatorname{Id}$ on the vector subspace

$$E_{u(p)} := \{E \in T_pM \colon \nabla^2 u(p)(E) = u(p)E\}$$

for every point $p \in M_0$ and the vector subspace

$$E_{\frac{1+u(p)}{2}} = \{E \in T_pM \colon \nabla^2 u(p)(E) = \frac{1+u(p)}{2}E\}$$

is the kernel of $\nabla^2 u(p)$.

Since the Hessian of u, $\nabla^2 u$, has at most two eigenvalues -1 and 0 on M_0, the rank of $\nabla^2 u$ is constant on M_0. If k is the rank of $\nabla^2 u$ on M_0, then M_0 is a $(n-k)$-dimensional submanifold of M and the normal bundle of M_0 is spanned by the vector field X as we move towards M_0.

We will now show that M_0 is a totally geodesic submanifold of M.

Let $q \in M_0$ and $v \in T_qM_0$. We extend v to a vector field V in a neighbourhood of $q \in M$. We write $V = V_1 + V_2$, where $V_1 \in E_u$ with $V_1(q) = 0$ and $V_2 \in E_{\frac{1+u}{2}}$ such that $V_2(q) = v$. Then

$$\begin{aligned} \nabla_V X &= \nabla_{V_1+V_2} X \\ &= \nabla_{V_1} X + \nabla_{V_2} X \\ &= \frac{u}{\|\nabla u\|} V_1 + \frac{1+u}{2} \frac{1}{\|\nabla u\|} V_2. \end{aligned}$$

Since $u(q) = -1$ and $V_1(q) = 0$, we see that

$$\begin{aligned} \langle \nabla_V X, V \rangle (q) &= \frac{u(q)}{\|\nabla u\|} \|V_1(q)\|^2 + \frac{1+u(q)}{2} \frac{1}{\|\nabla u\|} \|V_2(q)\|^2 \\ &= 0. \end{aligned}$$

Hence M_0 is a totally geodesic submanifold in M. □

4. Proof of Theorem 1

Let

$$E_u := \{E \in TM : \nabla^2 u(E) = uE\}.$$

Then E_u is a subbundle of TM and it is spanned by the vector fields ∇u and $J\nabla u$ whenever $\nabla u \neq 0$. Similarly, let

$$E_{\frac{1+u}{2}} := \{E \in TM : \nabla^2 u(E) = \frac{1+u}{2} E\}.$$

Then $E_{\frac{1+u}{2}}$ is also a subbundle of TM and it is orthogonal to E_u.

Proof of Theorem 1(i). Let p be a point of minimum for the function u. We have proved in Theorem 7 that this point is unique and $\exp_p : T_pM \to M$ is a diffeomorphism. Therefore $\{q \in M : \nabla u(q) \neq 0\} = M \setminus \{p\}$.

For every $v \in T_pM$, we let $\mathbb{R}v$ denote the one dimensional vector subspace spanned by the vector v. Then $\exp_p : \mathbb{R}v \bigoplus \mathbb{R}w \to M$ is a diffeomorphism onto its image for any two linearly independent vectors v and $w \in T_pM$. We also denote by $\mathbb{H}^2_v$ the image of $\mathbb{R}v \bigoplus \mathbb{R}Jv$ under $\exp_p$ for every non-zero vector $v \in T_pM$.

We will now show that the sectional curvature of $\mathbb{H}^2_v$ is -1.

As first step we will prove that, for every point $q \in \mathbb{H}^2_v$, the tangent space $T_q\mathbb{H}^2_v = \mathbb{R}\nabla u(q) \bigoplus \mathbb{R}J\nabla u(q) = E_{u(q)}$. We will prove this by showing that, if γ is a unit speed geodesic starting at p and W is the Jacobi field describing the variation of the geodesic γ such that $W(0) = 0$ and $W'(0) = J\gamma'(0)$, then $W(t) = J\nabla u(\gamma(t))$ for $t \in U_\gamma = \mathbb{R} \setminus \{0\}$.

Since the manifold (M, g, J) is Kähler, the complex structure J is parallel. Therefore $\nabla_{\nabla u} J\nabla u = J\nabla_{\nabla u}\nabla u = uJ\nabla u$ on $M \setminus \{p\}$. Further, since $J\nabla u$ is also an eigenvector of $\nabla^2 u$ with eigenvalue u, we see that $\nabla_{J\nabla u}\nabla u = uJ\nabla u = \nabla_{\nabla u}J\nabla u$. Therefore $[J\nabla u, \nabla u] = 0$ on $M \setminus \{p\}$. If $q \neq p$, then

$$\begin{aligned} R(J\nabla u(q), \nabla u(q))\nabla u(q) &= \nabla_{J\nabla u(q)}\nabla_{\nabla u(q)}\nabla u - \nabla_{\nabla u(q)}\nabla_{J\nabla u(q)}\nabla u \\ &= \nabla_{J\nabla u(q)}(u\nabla u) - \nabla_{\nabla u(q)}(uJ\nabla u) \\ &= -\|\nabla u(q)\|^2 J\nabla u(q). \end{aligned}$$

Let γ be a unit speed geodesic starting at the point p. We know that $\gamma'(t) = \frac{\nabla u(\gamma(t))}{\|\nabla u(\gamma(t))\|}$ on $U_\gamma = \mathbb{R} \setminus \{0\}$. Therefore, from what we have shown above $R(J\nabla u(\gamma(t)), \gamma'(t))\gamma'(t) = -J\nabla u(\gamma(t))$ on U_γ. On the other hand, since J is parallel and ∇u is an eigenvector of $\nabla^2 u$ with eigenvalue u, we see that $\frac{D^2}{dt^2}J\nabla u(\gamma(t)) = J\nabla u(\gamma(t))$ on U_γ. Hence $\frac{D^2}{dt^2}J\nabla u + R(J\nabla u, \gamma')\gamma' = 0$ along the geodesic γ. Thus we have shown that the vector field $W(t) := J\nabla u(\gamma(t))$ is the Jacobi field describing the variation of the geodesic γ such that $W(0) = 0$ and $W'(0) = J\gamma'(0)$. Therefore $T_{\gamma(t)}\mathbb{H}^2_v = \mathrm{Span}\{\gamma'(t), J\gamma'(t)\}$ for every $t \neq 0$. This proves that $E_u|_{\mathbb{H}^2_v}$ is the tangent bundle of $\mathbb{H}^2_v \setminus \{p\}$.

Since $\nabla_{\nabla u}J\nabla u = J\nabla_{\nabla u}\nabla u = uJ\nabla u = \nabla_{J\nabla u}\nabla u$ on $M \setminus \{p\}$, it follows that the submanifold $\mathbb{H}^2_v$ is also totally geodesic in M. Therefore the sectional curvature $K_M(\nabla u, J\nabla u)(q) = K_{\mathbb{H}^2_v}(q)$ at all points $q \neq p \in \mathbb{H}^2_v$.

We have already shown that

$$R(J\nabla u, \nabla u)\nabla u = -\|\nabla u\|^2 J\nabla u$$

in $\mathbb{H}^2_v \setminus \{p\}$. Hence the sectional curvature $K_{\mathbb{H}^2_v}(q) = -1$ for all points $q \in \mathbb{H}^2_v \setminus \{p\}$. Since the sectional curvature is a continuous function and equal to -1 on $\mathbb{H}^2_v \setminus \{p\}$, it follows that $K_{\mathbb{H}^2_v} \equiv -1$. This proves that $\mathbb{H}^2_v$ is isometric to the simply connected surface $\mathbb{H}^2$ of constant curvature -1.

Since $\mathbb{H}^2_v$ is totally geodesic for every v in T_pM, the subbundle $E_u\,|_{\mathbb{H}^2_v}$, being the tangent bundle of $\mathbb{H}^2_v$, is parallel along the integral curves γ of the vector field X on $M \setminus \{p\}$. Therefore the subbundle $E_{\frac{1+u}{2}}$, being the orthogonal complement of E_u, is also parallel along the integral curves γ of X on $M \setminus \{p\}$. Now an easy computation shows that $E_{\frac{1+u}{2}}$ is also an eigensubbundle of $R(\cdot\ , X)X$ with eigenvalue $-1/4$ on $M \setminus \{p\}$. This shows that, if W is a Jacobi field along γ describing the variation of γ such that $W(0) = 0$ and $W'(0) \in E_{\frac{1+u}{2}}$, then $W(t) = 2\sinh\frac{t}{2}E(t)$, where $E(t)$ is a vector field parallel along γ such that $E(t) \in E_{\frac{1+u}{2}}$.

Let w, $v \in T_pM$ and $w \perp v, Jv$. Then the map $\exp_p : \mathbb{R}\,v \bigoplus \mathbb{R}\,w \to M$ is a diffeomorphism onto its image. We will also denote this image by $\mathbb{H}^2_{v,w}$. Then it follows from what we have done in the paragraph above that the sectional curvature of $\mathbb{H}^2_{v,w}$ is $-1/4$.

We will now show that (M, g, J) is isometric to $(\mathbb{CH}^n, ds^2)$, the complex hyperbolic space of constant holomorphic sectional curvature -1.

Let us fix a point $o \in (\mathbb{CH}^n, ds^2)$ and an unitary isometry $I : T_pM \to T_o\mathbb{CH}^n$. Let

$$\Phi : M \to \mathbb{CH}^n$$

be the map defined by

$$\Phi(q) := \exp_o \circ I \circ \exp_p^{-1}(q).$$

Then for any geodesic γ starting at p, the image curve $\overline{\gamma} := \Phi(\gamma)$ is a geodesic staring at the point o in $\mathbb{CH}^n$. To complete the proof of the theorem, we only have to show that $d\Phi$ preserves the lengths of the Jacobi fields along the geodesics γ starting at p.

Before we start with the proof, we recall a few facts about the Jacobi fields on $\mathbb{CH}^n$.

Let us denote by $\overline{R}$ the Riemannian curvature tensor of $\mathbb{CH}^n$. Let σ be a geodesic in $\mathbb{CH}^n$ and $W(t)$ be a Jacobi field along σ such that $W(0) = 0$ and $\|W'(0)\| = 1$. Then

(1) $W(t) = \sinh tE(t)$, where $E(t)$ is a parallel vector field along σ and $E(t) \in E_{-1} := \{w \in T\mathbb{CH}^n : \overline{R}(w, \sigma')\sigma' = -w\}$, if $W'(0) \in E_{-1}$, and
(2) $W(t) = 2\sinh\frac{t}{2}E(t)$, where $E(t)$ is a parallel vector field along σ and $E(t) \in E_{-1/4} := \{w \in T\mathbb{CH}^n : \overline{R}(w, \sigma')\sigma' = -\frac{1}{4}\}$ if $W'(0) \in E_{-1/4}$.

Let γ be a geodesic starting at the point p in M. Let $\gamma'(0) = v$ and $E(t)$ be a vector field parallel along γ such that $E(t) \in E_u$. Since the vector field $W(t) = \sinh t E(t)$ is a Jacobi field along γ, it follows that $E(t) = \frac{1}{\sinh t} d(\exp_p)_{tv}(E(0))$ and $d\Phi_{\gamma(t)}$ maps $d(\exp_p)_{tv}(E(0))$ to $d(\exp_p)_{tI(v)}(I(E(0)))$. Using the fact that the isometry I is unitary, we conclude that the vector $d(\exp_p)_{tI(v)}(I(E(0))) \in E_{-1}$. This proves that $d(\exp_p)_{tI(v)}(I(E(0))) = \frac{\sinh t}{t} I(E(0))$. Hence $d\Phi$ is norm preserving on E_u.

By similar arguments we can show that $d\Phi$ is an isometry on $E_{\frac{1+u}{2}}$. Hence the map $\Phi : M \to \mathbb{CH}^n$ is an isometry. □

Proof of Theorem 1(ii). It follows from Theorem 8 that M_0 is a totally geodesic submanifold of M of dimension $n - k$, where k is the rank of the Hessian $\nabla^2 u$ on M_0.

Using the fact that J is parallel, we see that, if E is an eigenvector of $\nabla^2 u$, then JE is also an eigenvector of $\nabla^2 u$ with the same eigenvalue. Since the multiplicity of the eigenvalue u is 2, it follows that M_0 is a co-dimension two submanifold of M.

Let $q \in M_0$ and $N_q(M_0) := \{w \in T_qM : w \perp T_qM_0\}$ the normal space to M_0 at the point q. Then $N_q(M_0) = \{w \in T_qM_0 : \nabla^2 u(q)(w) = -w\}$ is of dimension 2. It is also a complex vector subspace. For every vector $w \in N_qM_0$, the geodesic γ_w such that $\gamma_w'(0) = w$ is along the direction of ∇u and hence such geodesics are rays starting from q. Therefore $\exp_q : N_q(M_0) \to M$ is a diffeomorphism onto its image. We have shown in Lemma 10 that, if p is a point in M and γ_p an integral curve of X passing through p, then the geodesic γ_p meets M_0 at a unique point q. Hence the normal exponential map $\exp : N(M_0) \to M$ is a diffeomorphism onto M.

For every point $q \in M_0$, we let $\mathbb{H}^2_q := \exp_q(N_q)$. Then an argument exactly same as in the proof of Theorem 1(i) shows that $\mathbb{H}^2_q$ is isometric to $(\mathbb{H}^2, ds^2)$ of constant curvature -1. This completes the proof of Theorem 1. □

5. Concluding Remarks

In the statement of Theorem 1, if we assume that $\frac{u-1}{2}$ as an eigenvalue of $\nabla^2 u$ instead of $\frac{1+u}{2}$, then we can conclude the following:

(1) If the function u has a maximum, then M is isometric to $\mathbb{CH}^n$.
(2) If the function u has a minimum, then there exists a totally geodesic submanifold $M_0 := \{p \in M : u(p) = \min u\}$ of M such that M is diffeomorphic to the normal bundle $N(M_0)$ of M. Further, the fiber over each point is isometric to the simply connected surface $(\mathbb{H}^2, ds^2)$ of constant curvature -1.

The proof is verbatim same as in the proof of Theorem 1 with the words maxima and the minima interchanged.

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