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THE OLLP AND $\mathcal{T}$-LOCAL REFLEXIVITY OF OPERATOR SPACES

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Abstract. In this paper, we study two 'dual' problems in the operator space theory. We first show that if L is a finite-dimensional operator space, then L has the OLLP if and only if for any indexed family of operator spaces $(W_i)_{i\in I}$ and a free ultrafilter $\mathcal{U}$ on I, we have a complete isometry

$$\prod(L\hat{\otimes}W_i)/\mathcal{U} = L\hat{\otimes}\prod W_i/\mathcal{U}.$$

Next, we show that if W is an operator space, then $(T_n\check{\otimes}W)^{**} = T_n\check{\otimes}W^{**}$ holds if and only if W is $\mathcal{T}$-locally reflexive, if and only if for any finitely representable operator spaces V, we have an isometry $\mathcal{I}(V,W^*) = (V\check{\otimes}W)^*$.

1. Introduction

Many problems in operator spaces are naturally motivated by both Banach space theory and C^*-algebraic theory. The exactness for C^*-algebras was first introduced by Kirchberg [10], and extended to operator spaces by Pisier [14]. In [14], Pisier showed that the notion of exactness for operator spaces is closely connected to a commutation property involving ultraproducts.

Theorem 1.1 ([14]). *Suppose that L is a finite-dimensional operator space. Then L is exact if and only if for any indexed family of operator spaces $(W_i)_{i\in I}$ and a free ultrafilter $\mathcal{U}$ on I, we have a complete isometry*

$$\prod(L\check{\otimes}W_i)/\mathcal{U} = L\check{\otimes}\prod W_i/\mathcal{U}.$$

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In this paper, we consider a natural 'dual' problem of Theorem 1.1. That is, which condition is the following completely isometry

$$\prod(L\hat{\otimes}W_i)/\mathcal{U} = L\hat{\otimes}\prod W_i/\mathcal{U} \tag{1}$$

equivalent to? We show in Section 2 that (1) holds for any indexed family of operator spaces $(W_i)_{i\in I}$ if and only if L has the OLLP.

For any operator space W, we always have

$$(M_n\check{\otimes}W)^{**} = M_n\check{\otimes}W^{**}. \tag{2}$$

The second problem which we are interested in this paper is sufficient and necessary conditions for which the following equations hold:

$$(M_n\hat{\otimes}W)^{**} = M_n\hat{\otimes}W^{**}$$

and

$$(T_n\check{\otimes}W)^{**} = T_n\check{\otimes}W^{**}.$$

This can be considered as the 'dual' problem of (2). The 'dual' problem is closely related to the notion of '$\mathcal{T}$-local reflexivity'. The analysis of $\mathcal{T}$-local reflexivity rests upon a careful study of finitely representable integrals and completely ∞-summing mappings. These results are presented in Section 3 and Section 4, respectively. The main result on the second 'dual' problem is proved in Section 5 (Theorem 5.2).

For the convenience of the readers, we recall some of the basic notations and terminologies in operator spaces; the details can be found in [5], [15]. Given a Hilbert space $\mathcal{H}$, we let $\mathcal{B}(\mathcal{H})$ denote the space of all bounded linear operators on $\mathcal{H}$. For each natural number $n \in \mathbf{N}$, there is a canonical norm $\|\cdot\|_n$ on the $n\times n$ matrix space $M_n(\mathcal{B}(\mathcal{H}))$ given by identifying $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$. We call this family of norms $\{\|\cdot\|_n\}$ an operator space matrix norm on $\mathcal{B}(\mathcal{H})$. An operator space V is a norm closed subspace of some $\mathcal{B}(\mathcal{H})$ equipped with the distinguished operator space matrix norm inherited from $\mathcal{B}(\mathcal{H})$. An abstract matrix norm characterization of operator spaces was given in [17]. The morphisms in the category of operator spaces are the completely bounded linear maps. Given operator spaces V and W, a linear map $\varphi : V \to W$ is completely bounded if the corresponding linear mappings $\varphi_n : M_n(V) \to M_n(W)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ are uniformly bounded, i.e.,

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbf{N}\} < \infty.$$

A map φ is completely contractive (respectively, completely isometric, a complete quotient) if $\|\varphi\|_{cb} \leq 1$ (respectively, for each $n \in \mathbf{N}$, φ_n is an isometry, a quotient map). We denote by $CB(V,W)$ the space of all completely bounded maps from V into W. It is known that $CB(V,W)$ is an operator space with the operator space matrix norm given by identifying $M_n(CB(V,W)) = CB(V,M_n(W))$. In particular, if V is an operator space, then its dual space

V^* is an operator space with operator space matrix norm given by the identification $M_n(V^*) = CB(V, M_n)$. Given operator spaces V and W, and a completely bounded mapping $\varphi : V \to W$, the corresponding adjoint mapping $\varphi^* : W^* \to V^*$ is completely bounded with $\|\varphi^*\|_{cb} = \|\varphi\|_{cb}$. Furthermore, $\varphi : V \to W$ is a completely isometric injection if and only if φ^* is a completely quotient mapping. On the other hand, if $\varphi : V \to W$ is a surjection, then φ is a completely quotient mapping if and only if φ^* is a completely isometric injection. We use the notations $V\check{\otimes}W$ and $V\hat{\otimes}W$ for the injective and projective operator space tensor products (see [1], [2]). The operator space tensor products share many of the properties of the Banach space analogues. For example, we have the natural complete isometries

$$(V\hat{\otimes}W)^* = CB(V, W^*),\ (V\hat{\otimes}W)^* = CB(W, V^*)$$

and the completely isometric injection

$$V^*\check{\otimes}W \hookrightarrow CB(V, W).$$

The tensor product $\check{\otimes}$ is injective in the sense that if $\varphi : W \to Y$ is a completely isometric injection, then so is

$$\mathrm{id}_V \otimes \varphi : V\check{\otimes}W \to V\check{\otimes}Y.$$

On the other hand, the tensor product $\hat{\otimes}$ is projective in the sense that if $\varphi : W \to Y$ is a completely quotient mapping, then so is

$$\mathrm{id}_V \otimes \varphi : V\hat{\otimes}W \to V\hat{\otimes}Y.$$

In the following, we give some definitions of local properties for operator spaces. Given an operator space V, we define:

1. *Exactness (see Pisier* [14]*).* For any finite dimensional subspace L of V and every $\epsilon > 0$, there exist an integer n and a subspace $S \subseteq M_n$ such that $d_{cb}(L, S) < 1 + \epsilon$.
2. *Local reflexivity (see Effros, Junge and Ruan* [6]*).* For any finite dimensional operator space L, every complete contraction $\varphi : L \to V^{**}$ is the point-weak* limit of a net of complete contractions $\varphi_\alpha : L \to V$.
3. *OLLP (see Ozawa* [13]*).* Given any unital C^*-algebra $\mathcal{A}$ with ideal $\mathcal{J} \subseteq \mathcal{A}$ and a complete contraction $\varphi : V \to \mathcal{A}/\mathcal{J}$, for every finite dimensional subspace E of V, there exists a complete contraction $\tilde{\varphi} : E \to \mathcal{A}$ such that $\pi \circ \tilde{\varphi} = \varphi|_E$, where $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ is the canonical quotient mapping. We say that V has the OLP if we can always take $E = V$ in the preceding definition.
4. *Local lifting property (LLP) (see Kye and Ruan* [12]*).* Given any operator spaces $W \subseteq Y$ and a complete contraction $\varphi : V \to Y/W$, for every finite dimensional subspace E of V and $\epsilon > 0$, there exists a completely bounded linear map $\tilde{\varphi} : E \to Y$ such that $\|\tilde{\varphi}\|_{cb} < 1 + \epsilon$

and $\pi \circ \tilde{\varphi} = \varphi|_E$, where $\pi : Y \to Y/W$ is the canonical quotient mapping.

(5) *Finitely representable in* $\{T_n\}_{n\in\mathbf{N}}$ *(or simply, finitely representable) (see Effros, Junge and Ruan* [6]*).* For every finite-dimensional subspace E of V and $\epsilon > 0$, there exists a subspace F of some T_n such that $d_{cb}(E, F) < 1 + \epsilon$.

In Section 3 and Section 4, we discuss operator space mapping ideals. An operator space mapping ideal $\mathcal{O}$ is an assignment to each pair of operator spaces V, W of a linear space $\mathcal{O}(V, W)$ of completely bounded mappings $\varphi : V \to W$, together with an operator space matrix norm $\|\cdot\|_{\mathcal{O}}$, such that for each $\varphi \in M_n(\mathcal{O}(V, W))$,

(a) $\|\varphi\|_{cb} \leq \|\varphi\|_{\mathcal{O}}$, and

(b) for any linear mapping $r : U \to V$ and $s : W \to X$

$$\|s_n \circ \varphi \circ r\|_{\mathcal{O}} \leq \|s\|_{cb} \cdot \|\varphi\|_{\mathcal{O}} \cdot \|r\|_{cb}.$$

We define the completely nuclear mappings $\mathcal{N}(V, W)$ to be the image of the canonical mapping $\Phi : V^*\hat{\otimes}W \to V^*\check{\otimes}W \subseteq CB(V, W)$ with the quotient operator space structure determined by the identification

$$\mathcal{N}(V, W) \cong \frac{V^*\hat{\otimes}W}{\ker\Phi}.$$

We let ν be the corresponding norm on $\mathcal{N}(V, W)$.

Given operator spaces V and W, we define a mapping $\varphi : V \to W$ to be completely integral if

$$\iota(\varphi) = \sup\{\nu(\varphi|_S) : S \subseteq V \text{ finite dimensional}\} < \infty.$$

We let $\mathcal{I}(V, W)$ denote the set of all completely integral mappings.

If $\varphi : V \to W$ is a linear mapping of operator spaces, then we define $\pi_1(\varphi)$ in $[0, \infty]$ by

$$\begin{aligned}\pi_1(\varphi) &= \|\operatorname{id}_{T_\infty} \otimes \varphi : T_\infty\check{\otimes}V \to T_\infty\hat{\otimes}W\| \\ &= \sup\{\|\operatorname{id}_{T_r} \otimes \varphi : T_r\check{\otimes}V \to T_r\hat{\otimes}W\| : r \in \mathbf{N}\}.\end{aligned}$$

If $\pi_1(\varphi) < \infty$, we say that φ is a completely 1-summing mapping from V into W and let $\Pi_1(V, W)$ denote the space of all completely 1-summing mappings from V into W.

$\mathcal{N}(\cdot,\cdot)$, $\mathcal{I}(\cdot,\cdot)$ and $\Pi_1(\cdot,\cdot)$ are all operator space mapping ideals. The details may be found in [5].

2. The OLLP

The following lemma is a corollary of Theorem 2.5 in [13], but we can prove it directly.

LEMMA 2.1. *Suppose that L is a finite-dimensional operator space. Then L has the OLLP if and only if L^* is exact.*

Proof. From Theorem 14.4.1 in [5], L^* is exact if and only if for any C^*-algebra $\mathcal{A}$ with closed ideal $\mathcal{J} \subseteq \mathcal{A}$, the natural mapping

$$\mathcal{A}\check{\otimes}L^* \to (\mathcal{A}/\mathcal{J})\check{\otimes}L^*$$

is a completely quotient mapping. Thus the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}\check{\otimes}L^* \arrow[r] \arrow[d, equal] & (\mathcal{A}/\mathcal{J})\check{\otimes}L^* \arrow[d, equal] \\
CB(L,\mathcal{A}) \arrow[r] & CB(L,\mathcal{A}/\mathcal{J})
\end{tikzcd}$$

implies that L has the OLLP if and only if L^* is exact. □

The following result can be considered as the 'dual' result of Theorem 1.1.

THEOREM 2.2. *Suppose that L is a finite-dimensional operator space. Then L has the OLLP if and only if for any indexed family of operator spaces $(W_i)_{i\in I}$ and a free ultrafilter $\mathcal{U}$ on I, we have a complete isometry*

$$\prod(L\hat{\otimes}W_i)/\mathcal{U} = L\hat{\otimes}\prod W_i/\mathcal{U}.$$

Proof. Suppose that we have $\prod(L\hat{\otimes}W_i)/\mathcal{U} = L\hat{\otimes}\prod W_i/\mathcal{U}$ for any indexed family of operator spaces $(W_i)_{i\in I}$. We can identify L^* with a subspace of M_∞, and we write $P^n : M_\infty \to M_n$ for the truncation mapping and we let $\rho_n = (P^n)|_{L^*}$ and

$$S_n = \rho_n(L^*) \subseteq M_n.$$

If $m \le n$, then $P^m \circ \rho_n = \rho_m$ and thus for each $v \in M_p(L^*)$,

$$\begin{aligned}\|(\mathrm{id}_{M_p} \otimes \rho_m)(v)\| &\le \|P^m\|_{cb} \cdot \|(\mathrm{id}_{M_p} \otimes \rho_n)(v)\| \\ &\le \|(\mathrm{id}_{M_p} \otimes \rho_n)(v)\|.\end{aligned}$$

As in the second proof of Theorem 14.1.1 in [5], we may select $n_0 \in \mathbf{N}$ such that $n \ge n_0$ implies that $P^n : L^* \to M_n$ is one-to-one, and therefore $\rho_n : L^* \to S_n$ is a linear isomorphism. We let $\sigma_n = \rho_n^{-1}$ for $n \ge n_0$, and $\sigma_n = 0$ for $n < n_0$, and we fix a ultrafilter $\mathcal{U}$ on the set $\mathbf{N}$. The mapping

$$\rho = (\rho_n)_{\mathcal{U}} : L^* \to \prod S_n/\mathcal{U}$$

is a completely isometric surjection. In fact, for any $x \in M_p(L^*)$,

$$\begin{aligned}\|(\mathrm{id}_{M_p} \otimes \rho)(x)\| &= \|(\pi_{\mathcal{U}})_p((\mathrm{id}_{M_p} \otimes \rho_n)(x))\| \\ &= \lim_{\mathcal{U}} \|(\mathrm{id}_{M_p} \otimes \rho_n)(x)\| = \|x\|.\end{aligned}$$

From Corollary 10.3.7 in [5], $\prod S_n/\mathcal{U}$ and L^* have the same finite dimension and so ρ is also a surjection. The inverse mapping of ρ is

$$\sigma = (\sigma_n)_{\mathcal{U}} : \prod S_n/\mathcal{U} \to \prod L^*/\mathcal{U} = L^*.$$

It follows that σ is a complete isometry. From the hypothesis, we have

$$\prod (L\hat{\otimes} S_n)/\mathcal{U} = L\hat{\otimes} \prod S_n/\mathcal{U}.$$

From Corollary 10.3.4 in [5], the natural mapping

$$\prod (L\hat{\otimes} S_n)^*/\mathcal{U} \to \left(\prod (L\hat{\otimes} S_n)/\mathcal{U}\right)^*$$

is a completely isometric injection. It follows from Corollary 10.3.7 in [5] that the dimensions of $\prod (L\hat{\otimes} S_n)^*/\mathcal{U}$ and $(\prod (L\hat{\otimes} S_n)/\mathcal{U})^*$ are the same as the finite dimension of $L\hat{\otimes} S_n$. So

$$\prod (L\hat{\otimes} S_n)^*/\mathcal{U} \cong \left(\prod (L\hat{\otimes} S_n)/\mathcal{U}\right)^*.$$

Thus, we have

$$\begin{aligned} CB\left(\prod S_n/\mathcal{U}, L^*\right) &\cong (L\hat{\otimes} \prod S_n/\mathcal{U})^* \cong \left(\prod (L\hat{\otimes} S_n)/\mathcal{U}\right)^* \\ &\cong \prod (L\hat{\otimes} S_n)^*/\mathcal{U} \cong \prod CB(S_n, L^*)/\mathcal{U}. \end{aligned}$$

This implies that

$$\lim_{\mathcal{U}} \|\sigma_n\|_{cb} = \|\pi_{\mathcal{U}}((\sigma_n))\|_{cb} = \|(\sigma_n)_{\mathcal{U}}\| = \|\sigma\|_{cb} = 1.$$

Given $\epsilon > 0$, there exists an integer $n(\epsilon) \geq n_0$ such that $\|\sigma_{n(\epsilon)}\|_{cb} < 1 + \epsilon$, and hence $d_{cb}(L^*, S_{n(\epsilon)}) < 1 + \epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that L^* is exact. Lemma 2.1 shows that L has the OLLP.

Conversely, suppose that L has the OLLP. For any $\epsilon > 0$, it follows from Theorem 2.5 in [13] that we may find a completely bounded isomorphism $r : L \to Q$, where Q is a quotient of some T_n, such that

$$\|r\|_{cb} \cdot \|r^{-1}\|_{cb} < 1 + \epsilon.$$

Since Q is a quotient of some T_n, Corollary 10.3.9 in [5] shows that

$$\theta : Q\hat{\otimes} \prod W_i/\mathcal{U} \cong \prod (Q\hat{\otimes} W_i)/\mathcal{U}.$$

Thus, from the following diagram

$$\begin{tikzcd} Q\hat{\otimes} \prod W_i/\mathcal{U} \arrow[r, "\theta"] & \prod (Q\hat{\otimes} W_i)/\mathcal{U} \arrow[d, "(r^{-1} \otimes \mathrm{id}_{W_i})_{\mathcal{U}}"] \\ L\hat{\otimes} \prod W_i/\mathcal{U} \arrow[u, "r \otimes \mathrm{id}"] & \prod (L\hat{\otimes} W_i)/\mathcal{U} \end{tikzcd},$$

we define

$$\Phi = (r^{-1} \otimes \mathrm{id}_{W_i})_{\mathcal{U}} \circ \theta \circ (r \otimes \mathrm{id}),$$

its inverse

$$\Phi^{-1} = (r^{-1} \otimes \mathrm{id}) \circ \theta^{-1} \circ (r \otimes \mathrm{id}_{W_i})_{\mathcal{U}}$$

and

$$\begin{aligned}\|\Phi\|_{cb} &\le \|(r^{-1} \otimes \mathrm{id}_{W_i})_{\mathcal{U}}\|_{cb} \cdot \|\theta\|_{cb} \cdot \|r \otimes \mathrm{id}\,\|_{cb} \\ &\le \lim_{\mathcal{U}} \|r^{-1} \otimes \mathrm{id}_{W_i}\,\|_{cb} \cdot \|r \otimes \mathrm{id}\,\|_{cb} \\ &= \|r^{-1}\|_{cb} \cdot \|r\|_{cb} < 1 + \epsilon,\end{aligned}$$

where the second inequality follows from Proposition 10.3.2 in [5]. Similarly, $\|\Phi^{-1}\|_{cb} < 1 + \epsilon$. Since $\epsilon > 0$ is arbitrary, Φ and Φ^{-1} are completely contractive. Therefore Φ is a completely isometric surjection and

$$L \hat{\otimes} \prod W_i/\mathcal{U} \cong \prod (L \hat{\otimes} W_i)/\mathcal{U}. \qquad \square$$

We can give another ultraproduct characterization of exactness.

Corollary 2.3. *Suppose that L is a finite-dimensional operator space. Then L is exact if and only if for any indexed family of operator spaces $(W_i)_{i \in I}$ and a free ultrafilter $\mathcal{U}$ on I, we have a complete isometry*

$$\prod (L^* \hat{\otimes} W_i)/\mathcal{U} = L^* \hat{\otimes} \prod W_i/\mathcal{U}.$$

3. Finite-representably integral mappings

First we recall some equivalent conditions of completely integral mappings and exactly integral mappings. Given operator spaces V and W, and a linear mapping $\varphi : V \to W$, it was shown in [6] that φ is a completely integral mapping if and only if

$$\iota(\varphi) = \sup\big\{\|\,\mathrm{id}_L \otimes \varphi : L \check{\otimes} V \to L \check{\otimes} W\| : \\ \forall \text{ finite- dimensional operator space } L\big\} < \infty;$$

φ is an exactly integral mapping if and only if

$$\iota^{ex}(\varphi) = \sup\{\|\,\mathrm{id}_{L^*} \otimes \varphi : L^* \check{\otimes} V \to L^* \check{\otimes} W\| : \forall L \subseteq M_n, n \in \mathbf{N}\} < \infty.$$

Similarly, we can give the following definition.

Definition 3.1. If $\varphi : V \to W$ is a linear mapping of operator spaces, then we define $\iota^{fr}(\varphi)$ in $[0, \infty]$ by

$$\iota^{fr}(\varphi) = \sup\{\|\,\mathrm{id}_{L^*} \otimes \varphi : L^* \check{\otimes} V \to L^* \hat{\otimes} W\| : L \subseteq T_n, n \in \mathbf{N}\}.$$

This definition is 'stable' in the sense that we may replace the bounded norms with completely bounded norms. To see this, let us suppose that $\iota^{fr}(\varphi) \leq 1$. Let us fix $L \subseteq T_n$. We have

$$\| \mathrm{id}_{L^*} \otimes \varphi \|_{cb} = \sup_{p \in \mathbf{N}} \{ \| \mathrm{id}_{M_p} \otimes \mathrm{id}_{L^*} \otimes \varphi : M_p(L^* \check{\otimes} V) \to M_p(L^* \hat{\otimes} W) \| \}.$$

Since the inclusion $T_p(L) \hookrightarrow T_p(T_n) = T_{pn}$ is completely isometric, it follows from Theorem 8.1.10 in [5] and the definition of ι^{fr} that the two mappings in the diagram

$$\begin{aligned} M_p(L^* \check{\otimes} V) = M_p(L^*) \check{\otimes} V = T_p(L)^* \check{\otimes} V \to T_p(L)^* \hat{\otimes} W = (M_p \check{\otimes} L^*) \hat{\otimes} W \\ \to M_p \check{\otimes} (L^* \hat{\otimes} W) = M_p(L^* \hat{\otimes} W) \end{aligned}$$

are contractions, and thus $\| \mathrm{id}_{L^*} \otimes \varphi \|_{cb} \leq 1$. If we let $L = \mathbf{C}$, then $\|\varphi\|_{cb} \leq 1$, and thus $\|\varphi\|_{cb} \leq \iota^{fr}(\varphi)$.

If $\iota^{fr}(\varphi) < \infty$, we say that φ is a finitely representable integral (or simply, f.r. integral) and we refer to $\iota^{fr}(\varphi)$ as the f.r. integral norm of φ. We let $\mathcal{I}^{fr}(V, W)$ denote the space of all f.r. integral mappings from V into W. For any matrix $\varphi = [\varphi_{ij}] \in M_m(\mathcal{I}^{fr}(V, W))$,

$$\begin{aligned} \iota^{fr}_m(\varphi) = \sup\big\{ \| \mathrm{id}_{L^*} \otimes \varphi = [\mathrm{id}_{L^*} \otimes \varphi_{ij}] : L^* \check{\otimes} V \to M_m(L^* \hat{\otimes} W) \| : \\ \forall L \subseteq T_n, n \in \mathbf{N} \big\}. \end{aligned}$$

It is routine to check that $\mathcal{I}^{fr}(V, W)$ is a linear space, and ι^{fr} is an operator space matrix norm on $\mathcal{I}^{fr}(V, W)$. Let us suppose that we are given mappings $r : U \to V, s : W \to X$, and $\varphi : V \to M_m(W)$. Then it is apparent from the diagram

$$L^* \check{\otimes} U \xrightarrow{\mathrm{id}_{L^*} \otimes r} L^* \check{\otimes} V \xrightarrow{\mathrm{id}_{L^*} \otimes \varphi} M_m(L^* \hat{\otimes} W) \xrightarrow{(\mathrm{id}_{L^*} \otimes s)_m} M_m(L^* \hat{\otimes} X)$$

that $\iota^{fr}_m(s_m \circ \varphi \circ r) \leq \|s\|_{cb} \cdot \iota^{fr}_m(\varphi) \cdot \|r\|_{cb}$. Therefore, $\mathcal{I}^{fr}(\cdot, \cdot)$ is an operator space mapping ideal.

Proposition 3.2. *For any operator spaces V and W, a linear mapping $\varphi : V \to W$ satisfies $\iota^{fr}(\varphi) \leq 1$ if and only for each $n \in \mathbf{N}$, $L \subseteq T_n$ and complete contraction $\psi : L \to V$, $\nu(\varphi \circ \psi) \leq 1$.*

Proof. This is apparent from the commutative diagram

$$\begin{tikzcd} L^* \check{\otimes} V \arrow[r, "\mathrm{id}_{L^*} \otimes \varphi"] \arrow[d, equal] & L^* \hat{\otimes} W \arrow[d, equal] \\ CB(L, V) \arrow[r] & \mathcal{N}(L, W) \end{tikzcd}$$

□

From Proposition 3.2, we have

$$\iota^{fr}(\varphi) = \sup\{ \nu(\varphi \circ \psi) : \forall \psi \in CB(L, V)_{\|\cdot\|_{cb} \leq 1}, L \subseteq T_n, n \in \mathbf{N} \}.$$

Proposition 3.3. *$\mathcal{I}^{fr}(\cdot,\cdot)$ is a local operator space mapping ideal and $\iota^{fr}(\varphi) \leq \iota(\varphi)$ for any linear mapping $\varphi : V \to W$.*

Proof. Since $\mathcal{I}^{fr}(\cdot,\cdot)$ is an operator space mapping ideal, it is clear that for every finite-dimensional subspace $S \subseteq V$,

$$\iota^{fr}(\varphi|_S) \leq \iota^{fr}(\varphi).$$

On the other hand, for any $L \subseteq T_n$ and any complete contraction $\psi : L \to V$, it follows from Proposition 3.2 that

$$\nu(\varphi \circ \psi) = \nu(\varphi|_S \circ \psi) \leq \iota^{fr}(\varphi|_S),$$

where we let $S = \psi(L)$. Proposition 3.2 shows that

$$\begin{aligned}\iota^{fr}(\varphi) &= \sup\{\nu(\varphi \circ \psi) : \forall \psi \in CB(L,V)_{\|\cdot\|_{cb} \leq 1}, L \subseteq T_n, n \in \mathbf{N}\}\\ &\leq \sup\{\iota^{fr}(\varphi|_S) : \forall \text{ finite-dimensional subspace } S \subseteq V\}.\end{aligned}$$

This implies that $\mathcal{I}^{fr}$ is a local operator space mapping ideal.

If $\nu(\varphi) \leq 1$, then for any $n \in \mathbf{N}$, $L \subseteq T_n$, and each complete contraction $\psi : L \to V$, we have

$$\nu(\varphi \circ \psi) \leq \nu(\varphi) \cdot \|\psi\|_{cb} \leq 1.$$

From Proposition 3.2, $\iota^{fr}(\varphi) \leq 1$ and $\iota^{fr}(\varphi) \leq \nu(\varphi)$ in general. Since ι^{fr} is local,

$$\begin{aligned}\iota^{fr}(\varphi) &= \sup\{\iota^{fr}(\varphi|_S) : \forall \text{ finite-dimensional subspace } S \subseteq V\}\\ &\leq \sup\{\nu(\varphi|_S) : \forall \text{ finite-dimensional subspace } S \subseteq V\}\\ &= \iota(\varphi).\end{aligned}$$

□

Given a finite-rank mapping $\psi : W \to V$, we define

$$\gamma_{ST}(\psi) = \inf\{\|a\|_{cb} \cdot \|b\|_{cb}\},$$

where the infimum is taken over all factorizations

$$\begin{tikzcd} & L \arrow[dr, "a"] & \\ W \arrow[ur, "b"] \arrow[rr, "\psi"] & & V \end{tikzcd}$$

with $L \subseteq T_n$ and $n \in \mathbf{N}$. It is easy to see that this determines a norm on $\mathcal{F}(W,V)$, and we let $\gamma^o_{ST}(W,V)$ denote the corresponding normed space.

Lemma 3.4. *If W is finite-dimensional, we have an isometric isomorphism $\mathcal{I}^{fr}(V,W) = \gamma^o_{ST}(W,V)^*$.*

Proof. From the definition of ι^{fr}, we have

$$\begin{aligned}\iota^{fr}(\varphi) &= \sup\{\|\operatorname{id}_{L^*}\otimes\varphi : L^*\check{\otimes}V \to L^*\hat{\otimes}W\| : \quad \forall L \subseteq T_n, n \in \mathbf{N}\}\\ &= \sup\{\|(\operatorname{id}_{L^*}\otimes\varphi)(u)\|_{L^*\hat{\otimes}W} :\\ &\qquad \|u\|_{L^*\check{\otimes}V} \le 1, L \subseteq T_n, n \in \mathbf{N}\}\\ &= \sup\{\|(\operatorname{id}_{L^*}\otimes\iota_W\circ\varphi)(u)\|_{L^*\hat{\otimes}W^{**}} :\\ &\qquad \|u\|_{L^*\check{\otimes}V} \le 1, L \subseteq T_n, n \in \mathbf{N}\}\\ &= \sup\{| < (\operatorname{id}_{L^*}\otimes\varphi)(u), v > | : \quad \|u\|_{L^*\check{\otimes}V} \le 1,\\ &\qquad \|v\|_{L\check{\otimes}W^*} \le 1, L \subseteq T_n, n \in \mathbf{N}\},\end{aligned}$$

where the third equation follows from the complete isometric injection $L^*\hat{\otimes}W \hookrightarrow L^*\hat{\otimes}W^{**}$ and since L, W are finite-dimensional, the fourth equation follows from $(L\check{\otimes}W^*)^* \cong L^*\hat{\otimes}W^{**}$. Thus, if we let u and v correspond to the functions $a \in CB(L, V)$ and $b \in CB(W, L)$, then a simple calculation with elementary matrices leads to the formula

$$\iota^{fr}(\varphi) = \sup\{|\operatorname{trace}(\varphi\circ\psi)| : \psi = a\circ b, \|a\|_{cb}, \|b\|_{cb} \le 1\}.$$

We conclude from the definition of γ^o_{ST} that we have an isometric injection

$$\mathcal{I}^{fr}(V, W) \hookrightarrow \gamma^o_{ST}(W, V)^*,$$

and since W is finite-dimensional,

$$\mathcal{I}^{fr}(V, W) = \gamma^o_{ST}(W, V)^*. \qquad \square$$

The following result provides some motivation for our terminology 'finitely representable integral mapping'. The identities (2) and (3) in Theorem 3.5 are (complete) isometries.

Theorem 3.5. *For any operator space V, the following are equivalent.*

(1) *V is finitely representable.*
(2) *$\mathcal{I}(V, S) = \mathcal{I}^{fr}(V, S)$ for any finite-dimensional subspace $S \subseteq V$.*
(3) *$\mathcal{I}(V, W) = \mathcal{I}^{fr}(V, W)$ for any operator space W.*

Proof. (1)⇒(3): Let $\varphi : V \to W$ be a f.r. integral mapping. Since V is finitely representable, for any finite-dimensional subspace $S \subseteq V$ and $\epsilon > 0$, there exists a linear isomorphism ψ from S onto an operator subspace L of some T_n such that $\|\psi\|_{cb} < 1 + \epsilon$ and $\|\psi^{-1}\|_{cb} < 1$. It follows from Proposition 3.2 that

$$\iota(\varphi|_S) = \nu(\varphi|_S\circ\psi^{-1}\circ\psi) \le \nu(\varphi|_S\circ\psi^{-1})\cdot\|\psi\|_{cb} \le \iota^{fr}(\varphi|_S)(1+\epsilon).$$

If we let $\epsilon \to 0$, we have

$$\iota(\varphi|_S) \le \iota^{fr}(\varphi|_S),$$

and thus by the local property of ι and ι^{fr},

$$\iota(\varphi) \le \iota^{fr}(\varphi).$$

From Proposition 3.3, we have $\iota(\varphi) = \iota^{fr}(\varphi)$, and this shows that (1)⇒(3).

(3)⇒(2): This is obvious.

(2)⇒(1): For any fixed finite-dimensional subspace $S \subseteq V$, it follows from the definition of γ^o_{ST} that we have a norm-decreasing linear isomorphism (both sides coincide with the linear space $S^* \otimes V$)

$$\theta : \gamma^o_{ST}(S, V) \to CB(S, V).$$

Let us consider the adjoint of this mapping θ,

$$\theta^* : CB(S, V)^* \to \gamma^o_{ST}(S, V)^*.$$

Since

$$CB(S, V)^* = (S^* \check{\otimes} V)^* = \mathcal{I}(V, S)$$

and from Lemma 3.4,

$$\gamma^o_{ST}(S, V)^* = \mathcal{I}^{fr}(V, S),$$

it follows from the hypothesis of (2) that θ^* is an isometry, and thus θ must itself be an isometry. If $\iota : S \to V$ is the inclusion mapping, then it follows that for any $\epsilon > 0$, we have a commutative diagram

$$\begin{tikzcd}
 & L \arrow[dr, "a"] & \\
S \arrow[ur, "b"] \arrow[rr, "\iota"] & & V
\end{tikzcd}$$

where L is an operator subspace of some T_n, and $\|a\|_{cb} \cdot \|b\|_{cb} < 1 + \epsilon$. Thus, by definition, S is finitely representable, and the same follows for V. □

4. Completely ∞-summing mappings

Completely 1-summing mappings have been studied by Effros and Ruan [4] and completely p-summing mappings ($1 < p < +\infty$) have been considered by Pisier [16]. In this section, we will define and study (completely) ∞-summing mappings. Although the results for Banach spaces and operator spaces outlined in this section are largely parallel to each other, we shall find that some of novel aspects of operator space theory arise when one considers the behavior under duality.

Definition 4.1. Given Banach spaces X and Y and a linear mapping $\varphi : X \to Y$, we define the ∞-summing norm of φ by

$$\begin{aligned}
\pi^B_\infty(\varphi) &= \| \operatorname{id}_{c_0} \otimes \varphi : c_0 \overset{\lambda}{\otimes} X \to c_0 \overset{\gamma}{\otimes} Y \| \\
&= \sup_{n \in \mathbf{N}} \{ \| \operatorname{id}_{l^n_\infty} \otimes \varphi : l^n_\infty \overset{\lambda}{\otimes} X \to l^n_\infty \overset{\gamma}{\otimes} Y \| \}.
\end{aligned}$$

We say that φ is ∞-summing if $\pi^B_\infty(\varphi) < \infty$. It is evident that π^B_∞ is a norm on the space $\Pi^B_\infty(X,Y)$ of all ∞-summing mappings, and in fact the isometric embedding

$$\Pi^B_\infty(X,Y) \hookrightarrow B(c_0 \overset{\lambda}{\otimes} X, c_0 \overset{\gamma}{\otimes} Y) : \varphi \longmapsto \mathrm{id}_{c_0} \otimes \varphi$$

may be used to see that it is a Banach space.

We note that if we are given a diagram

$$D \xrightarrow{r} X \xrightarrow{\varphi} Y \xrightarrow{s} G,$$

then we have a corresponding diagram

$$c_0 \overset{\lambda}{\otimes} D \xrightarrow{\mathrm{id}_{c_0} \otimes r} c_0 \overset{\lambda}{\otimes} X \xrightarrow{\mathrm{id}_{c_0} \otimes \varphi} c_0 \overset{\gamma}{\otimes} Y \xrightarrow{\mathrm{id}_{c_0} \otimes s} c_0 \overset{\gamma}{\otimes} G,$$

from which it follows that

$$\pi^B_\infty(s \circ \varphi \circ r) \leq \|s\| \cdot \pi^B_\infty(\varphi) \cdot \|r\|,$$

and thus

$$\Pi^B_\infty : (X,Y) \longmapsto (\Pi^B_\infty(X,Y), \pi^B_\infty)$$

is a Banach space mapping ideal.

Proposition 4.2. *For any Banach spaces X, Y, $\varphi : X \to Y$ satisfies $\pi^B_\infty(\varphi) \leq 1$ if and only if for each $n \in \mathbf{N}$ and contraction $\theta : l^n_1 \to X$, $\nu^B(\varphi \circ \theta) \leq 1$.*

Proof. This is apparent from the commutative diagram

$$\begin{tikzcd}
l^n_\infty \overset{\lambda}{\otimes} X \arrow[r] \arrow[d, equal] & l^n_\infty \overset{\gamma}{\otimes} Y \arrow[d, equal] \\
B(l^n_1, X) \arrow[r] & \mathcal{N}^B(l^n_1, Y).
\end{tikzcd}$$

□

Corollary 4.3. *Π^B_∞ is a local Banach space mapping ideal, and for any linear mapping $\varphi : X \to Y$, $\pi^B_\infty(\varphi) \leq \iota^B(\varphi)$.*

Proof. Since Π^B_∞ is a Banach space mapping ideal, it is clear that for every finite-dimensional subspace $S \subseteq X$

$$\pi^B_\infty(\varphi|_S) \leq \pi^B_\infty(\varphi).$$

On the other hand, suppose that for any finite-dimensional subspace $S \subseteq X$, $\pi^B_\infty(\varphi|_S) \leq 1$. For every $n \in \mathbf{N}$ and contraction $\theta : l^n_1 \to X$, we set $S = \theta(l^n_1)$. Since $\pi^B_\infty(\varphi|_S) \leq 1$, it follows from Proposition 4.2 that

$$\nu^B(\varphi \circ \theta) = \nu^B(\varphi|_S \circ \theta) \leq 1.$$

Proposition 4.2 shows that $\pi^B_\infty(\varphi) \leq 1$ and therefore Π^B_∞ is a local Banach space mapping ideal.

If $\nu^B(\varphi) \leq 1$, then for each contraction $\theta : l_1^n \to X$,

$$\nu^B(\varphi \circ \theta) \leq \nu^B(\varphi) \cdot \|\theta\| \leq 1,$$

and from Proposition 4.2,

$$\pi_\infty^B(\varphi) \leq \nu^B(\varphi).$$

Since the mapping ideal is local,

$$\begin{aligned}\pi_\infty^B(\varphi) &= \sup\{\pi_\infty^B(\varphi|_S) : S \text{ finite-dimensional in } X\} \\ &\leq \sup\{\nu^B(\varphi|_S) : S \text{ finite-dimensional in } X\} = \iota^B(\varphi). \qquad \square\end{aligned}$$

Theorem 4.4. *For any Banach spaces X, Y and a linear map $\varphi : X \to Y$, we have $\pi_\infty^B(\varphi) = \pi_1^B(\varphi^*)$.*

Proof. Suppose that $\pi_\infty^B(\varphi) \leq 1$. Thus, for any $n \in \mathbf{N}$,

$$\| \operatorname{id}_{l_\infty^n} \otimes \varphi : l_\infty^n \overset{\lambda}{\otimes} X \to l_\infty^n \overset{\gamma}{\otimes} Y \| \leq 1.$$

Let us consider the adjoint of this mapping. We have

$$(l_\infty^n \overset{\lambda}{\otimes} X)^* = l_1^n \overset{\gamma}{\otimes} X^*$$

and

$$(l_\infty^n \overset{\gamma}{\otimes} Y)^* = B(l_\infty^n, Y^*) = l_1^n \overset{\lambda}{\otimes} Y^*.$$

It follows that

$$\| \operatorname{id}_{l_1^n} \otimes \varphi^* : l_1^n \overset{\lambda}{\otimes} Y^* \to l_1^n \overset{\gamma}{\otimes} X^* \| \leq 1.$$

Therefore,

$$\pi_1^B(\varphi^*) = \sup_{n \in \mathbf{N}} \{ \| \operatorname{id}_{l_1^n} \otimes \varphi^* : l_1^n \overset{\lambda}{\otimes} Y^* \to l_1^n \overset{\gamma}{\otimes} X^* \| \} \leq 1.$$

Conversely, suppose that $\pi_1^B(\varphi^*) \leq 1$. For any $n \in \mathbf{N}$, we have

$$\| \operatorname{id}_{l_1^n} \otimes \varphi^* : l_1^n \overset{\lambda}{\otimes} Y^* \to l_1^n \overset{\gamma}{\otimes} X^* \| \leq 1.$$

Let us consider the adjoint of this mapping. Since any Banach space is locally reflexive, Corollary 14.1.2 in [5] implies that

$$(l_1^n \overset{\lambda}{\otimes} Y^*)^* = l_\infty^n \overset{\gamma}{\otimes} Y^{**}$$

and

$$(l_1^n \overset{\gamma}{\otimes} X^*)^* = B(l_1^n, X^{**}) = l_\infty^n \overset{\lambda}{\otimes} X^{**}.$$

It follows that

$$\| \operatorname{id}_{l_\infty^n} \otimes \varphi^{**} : l_\infty^n \overset{\lambda}{\otimes} X^{**} \to l_\infty^n \overset{\gamma}{\otimes} Y^{**} \| \leq 1$$

and

$$\| \operatorname{id}_{l_\infty^n} \otimes \varphi \| = \|(\operatorname{id}_{l_\infty^n} \otimes \varphi)^{**}\| = \| \operatorname{id}_{l_\infty^n} \otimes \varphi^{**} \| \leq 1.$$

So

$$\pi_\infty^B(\varphi) = \sup_{n\in\mathbf{N}}\{\|\operatorname{id}_{l_\infty^n}\otimes\varphi : l_\infty^n \overset{\lambda}{\otimes} X \to l_\infty^n \overset{\gamma}{\otimes} Y\|\} \le 1.$$

Therefore $\pi_\infty^B(\varphi) = \pi_1^B(\varphi^*)$. $\square$

Corollary 4.5. $\pi_1^B(\varphi) = \pi_\infty^B(\varphi^*)$.

Proof. From the definition of π_1^B and the local reflexivity of any Banach space, we have $\pi_1^B(\varphi) = \pi_1^B(\varphi^{**})$. Thus Theorem 4.4 implies that $\pi_\infty^B(\varphi^*) = \pi_1^B(\varphi^{**}) = \pi_1^B(\varphi)$. $\square$

We now turn our attention to linear mappings of operator spaces $\varphi : V \to W$. The analogs of Theorem 4.4 and Corollary 4.5 are not always true in the theory of operator spaces. This is closely related to the lack of local reflexivity of $\mathcal{B}(\mathcal{H})$.

Definition 4.6. If $\varphi : V \to W$ is a linear mapping operator spaces, then we define π_∞ in $[0,\infty]$ by

$$\begin{aligned}\pi_\infty(\varphi) &= \|\operatorname{id}_{\mathcal{K}_\infty}\otimes\varphi : \mathcal{K}_\infty\check{\otimes}V \to \mathcal{K}_\infty\hat{\otimes}W\| \\ &= \sup\{\|\operatorname{id}_{M_n}\otimes\varphi : M_n\check{\otimes}V \to M_n\hat{\otimes}W\| : \forall n \in \mathbf{N}\}.\end{aligned}$$

This definition is 'stable' in the sense that we may replace the bounded norms with completely bounded norms. To see this, let us suppose that $\pi_\infty(\varphi) \le 1$. Let us fix n. We have

$$\|\operatorname{id}_{M_n}\otimes\varphi\|_{cb} = \sup_{p\in\mathbf{N}}\{\|\operatorname{id}_{M_p}\otimes\operatorname{id}_{M_n}\otimes\varphi : M_p\check{\otimes}(M_n\check{\otimes}V) \to M_p\check{\otimes}(M_n\hat{\otimes}W)\|\}.$$

From Theorem 8.1.10 in [5] and the definition of π_∞, the two mappings in the diagram

$$\begin{aligned}M_p\check{\otimes}(M_n\check{\otimes}V) = M_{pn}\check{\otimes}V \to M_{pn}\hat{\otimes}W &= (M_p\check{\otimes}M_n)\hat{\otimes}W \\ &\to M_p\check{\otimes}(M_n\hat{\otimes}W)\end{aligned}$$

are contractions, and thus $\|\operatorname{id}_{M_n}\otimes\varphi\|_{cb} \le 1$. If we let $n = 1$, then $\|\varphi\|_{cb} \le 1$, and thus $\|\varphi\|_{cb} \le \pi_\infty(\varphi)$. If $\pi_\infty(\varphi) < \infty$, we say that φ is completely ∞-summing and we refer to $\pi_\infty(\varphi)$ as the completely ∞-summing norm of φ. We let $\Pi_\infty(V,W)$ denote the space of all completely ∞-summing mappings from V into W. For any matrix $\varphi = [\varphi_{ij}] \in M_m(\Pi_\infty(V,W))$

$$\pi_{\infty,m}(\varphi) = \|\operatorname{id}\otimes\varphi = [\operatorname{id}\otimes\varphi_{ij}] : \mathcal{K}_\infty\check{\otimes}V \to M_m(\mathcal{K}_\infty\hat{\otimes}W)\|_{cb}.$$

Suppose $r : U \to V, s : W \to X$, and $\varphi : V \to M_m(W)$. Then it is apparent from the diagram

$$\mathcal{K}_\infty\check{\otimes}U \xrightarrow{\operatorname{id}\otimes r} \mathcal{K}_\infty\check{\otimes}V \xrightarrow{\operatorname{id}\otimes\varphi} M_m(\mathcal{K}_\infty\hat{\otimes}W) \xrightarrow{(\operatorname{id}\otimes s)_m} M_m(\mathcal{K}_\infty\hat{\otimes}X)$$

that

$$\pi_{\infty,m}(s_m \circ \varphi \circ r) \leq \|s\|_{cb} \cdot \pi_{\infty,m}(\varphi) \cdot \|r\|_{cb}.$$

Therefore, $\Pi_\infty(\cdot,\cdot)$ is an operator space mapping ideal.

Proposition 4.7. *For any operator spaces V and W, a linear mapping $\varphi : V \to W$ satisfies $\pi_\infty(\varphi) \leq 1$ if and only for each $n \in \mathbf{N}$ and complete contraction $\psi : T_n \to V$, $\nu(\varphi \circ \psi) \leq 1$.*

Proof. This is apparent from the commutative diagram

$$\begin{tikzcd}
M_n \check{\otimes} V \arrow[r, "\mathrm{id} \otimes \varphi"] \arrow[d, equal] & M_n \hat{\otimes} W \arrow[d, equal] \\
CB(T_n, V) \arrow[r] & \mathcal{N}(T_n, W)
\end{tikzcd}$$

□

Corollary 4.8. *The bifunctor $\Pi_\infty : (V, W) \to (\Pi_\infty(V, W), \pi_\infty)$is a local operator space mapping ideal, and for any linear mapping $\varphi : V \to W$, $\pi_\infty(\varphi) \leq \iota^{fr}(\varphi) \leq \iota(\varphi)$.*

Proof. We may use the argument for the Banach ∞-summing norm as Corollary 4.3 to show that $\Pi_\infty(\cdot,\cdot)$ is a local operator space mapping ideal. From Definition 3.1, Definition 4.1 and Proposition 3.3, $\pi_\infty(\varphi) \leq \iota^{fr}(\varphi) \leq \iota(\varphi)$. □

Proposition 4.9. *Given operator spaces V, W and a linear mapping $\varphi : V \to W$, we have $\pi_1(\varphi^*) \leq \pi_\infty(\varphi)$.*

Proof. Suppose that $\pi_\infty(\varphi) \leq 1$. Thus for any $n \in \mathbf{N}$

$$\| \mathrm{id}_{M_n} \otimes \varphi : M_n \check{\otimes} V \to M_n \hat{\otimes} W \| \leq 1.$$

Let us consider the adjoint of this mapping. We have

$$(M_n \check{\otimes} V)^* = T_n \hat{\otimes} V^*$$

and

$$(M_n \hat{\otimes} W)^* = CB(M_n, W^*) = T_n \check{\otimes} W^*.$$

It follows that

$$\| \mathrm{id}_{T_n} \otimes \varphi^* : T_n \check{\otimes} W^* \to T_n \hat{\otimes} V^* \| \leq 1.$$

So

$$\pi_1(\varphi^*) = \sup_{n \in \mathbf{N}} \{ \| \mathrm{id}_{T_n} \otimes \varphi^* : T_n \check{\otimes} W^* \to T_n \hat{\otimes} V^* \| \} \leq 1.$$

□

Corollary 4.10. *If V is an operator space for which the identity mapping $\mathrm{id}_V : V \to V$ satisfies $\pi_\infty(\mathrm{id}_V) < \infty$, then V must be finite dimensional.*

Proof. From Proposition 4.9,

$$\pi_1(\mathrm{id}_{V^*}) = \pi_1(\mathrm{id}_V^*) \leq \pi_\infty(\mathrm{id}_V) < \infty.$$

It follows from the Dvoretzky-Rogers Theorem for operator spaces, V^* must be finite-dimensional and so V is finite-dimensional. □

THEOREM 4.11. *Given operator spaces V, W and a linear mapping $\varphi : V \to W$, we have $\pi_1(\varphi) \leq \pi_\infty(\varphi^*)$. Moreover, we have $\pi_1(\varphi) = \pi_\infty(\varphi^*)$ for any operator space W and linear mapping $\varphi : V \to W$ if and only if $\mathcal{I}(V, M_n) = \mathcal{N}(V, M_n)$ for any $n \in \mathbf{N}$.*

Proof. Since $M_n \hat{\otimes} V^* \to (T_n \check{\otimes} V)^*$ is norm-decreasing, we conclude that

$$\begin{aligned}
\pi_1(\varphi) &= \sup\{\|\mathrm{id} \otimes \varphi : T_n \check{\otimes} V \to T_n \hat{\otimes} W\| : n \in \mathbf{N}\}\\
&= \sup\{\|(\mathrm{id} \otimes \varphi)^* : (T_n \hat{\otimes} W)^* \to (T_n \check{\otimes} V)^*\| : n \in \mathbf{N}\}\\
&\leq \sup\{\|\mathrm{id} \otimes \varphi^* : M_n \check{\otimes} W^* \to M_n \hat{\otimes} V^*\| : n \in \mathbf{N}\}\\
&= \pi_\infty(\varphi^*).
\end{aligned}$$

If $\mathcal{I}(V, M_n) = \mathcal{N}(V, M_n)$, then $M_n \hat{\otimes} V^* \to (T_n \check{\otimes} V)^*$ is isometric, and the above calculation implies that $\pi_1(\varphi) = \pi_\infty(\varphi^*)$.

Conversely, we first prove $\Pi_\infty(T_n, V^*) = \mathcal{N}(T_n, V^*)$. In fact, it follows from Corollary 4.8 that $\pi_\infty(\psi) \leq \nu(\psi)$ for any $\psi : T_n \to V^*$. Suppose that $\pi_\infty(\psi) \leq 1$ for $\psi : T_n \to V^*$. Proposition 4.7 shows that for $\mathrm{id}_{T_n} : T_n \to T_n$,

$$\nu(\psi) = \nu(\psi \circ \mathrm{id}_{T_n}) \leq 1.$$

Therefore, $\nu(\psi) = \pi_\infty(\psi)$ and $\Pi_\infty(T_n, V^*) = \mathcal{N}(T_n, V^*)$.

Thus we have the isometries

$$\mathcal{I}(V, M_n) = \Pi_1(V, M_n) = \Pi_\infty(T_n, V^*) = \mathcal{N}(T_n, V^*) = \mathcal{N}(V, M_n),$$

where the first equation follows from Proposition15.5.1 in [5], the second from the hypothesis and the fourth from Proposition 12.2.5 in [5]. □

COROLLARY 4.12. *If V has the LLP, then we have the isometry*

$$\mathcal{I}(V, W) = \Pi_\infty(V, W)$$

for all operator spaces W.

Proof. Let W be an arbitrary operator space and let $\varphi \in \Pi_\infty(V, W)$ with $\pi_\infty(\varphi) \leq 1$. For any complete contraction $\psi : T_n \to V$, we may identify ψ with a contractive element in $CB(T_n, V) = M_n \check{\otimes} V$. Then $\pi_\infty(\varphi) \leq 1$ implies that $\varphi \circ \psi \in \mathcal{N}(T_n, W) = M_n \hat{\otimes} W$ with $\nu(\varphi \circ \psi) \leq 1$.

Since V has the LLP, it follows from [12] Theorem 3.2 that for any finite-dimensional operator subspace $L \subseteq V$ and $\epsilon > 0$ we have maps $s : L \to T_n$

and $t : T_n \to V$ such that $\|s\|_{cb} \cdot \|t\|_{cb} < 1 + \epsilon$ and $t \circ s = \iota_L$. Certainly, we can suppose that $\|t\|_{cb} < 1, \|s\|_{cb} < 1 + \epsilon$. Then

$$\nu(\varphi|_L) = \nu(\varphi \circ t \circ s) \leq \nu(\varphi \circ t) \cdot \|s\|_{cb} < 1 + \epsilon.$$

Since ϵ is arbitrary, this implies that $\nu(\varphi|_L) \leq 1$. Therefore, $\iota(\varphi) \leq 1$. □

5. $\mathcal{T}$-local reflexivity

Definition 5.1. We say that an operator space W is $\mathcal{T}$-locally reflexive if for any $L \subseteq T_n, n \in \mathbf{N}$, every complete contraction $\varphi : L^* \to W^{**}$ is the point-weak* limit of a net of linear mappings $\varphi_\alpha : L^* \to W$ with $\|\varphi_\alpha\|_{cb} \leq 1$.

It is obvious that any locally reflexive operator space is $\mathcal{T}$-locally reflexive.

Theorem 5.2. *Suppose that W is an operator space. Then the following are equivalent.*

(1) *W is $\mathcal{T}$-locally reflexive.*
(2) *For any $L \subseteq T_n, n \in \mathbf{N}$, we have the isometry $L^* \hat{\otimes} W^* = (L \check{\otimes} W)^*$.*
(2)' *For any $L \subseteq T_n, n \in \mathbf{N}$, we have the isometry $\mathcal{N}(W, L^*) = \mathcal{I}(W, L^*)$.*
(3) *For any $n \in \mathbf{N}$, we have the isometry $M_n \hat{\otimes} W^* = (T_n \check{\otimes} W)^*$.*
(3)' *For any $n \in \mathbf{N}$, we have the isometry $\mathcal{N}(W, M_n) = \mathcal{I}(W, M_n)$.*
(4) *For any operator space V which is finitely representable, we have the isometry $\mathcal{I}(V, W^*) = (V \check{\otimes} W)^*$.*
(5) *For any finitely representable operator space V, $V \check{\otimes} : W^{**} = V \check{\otimes} W^{**}$.*
(6) *For any operator space V, we have the isometry $\Pi_1(W, V) = \Pi_\infty(V^*, W^*)$.*

Proof. (2)⇔(2)' and (3)⇔(3)' are immediate from Corollary 12.3.4 in [5], (4)⇔(5) follows from Proposition 14.2.2 in [5], and (3)' ⇔(6) follows from Theorem 4.11.

(2)⇒(3): This is obvious.

(3)⇒(2): For any $L \subseteq T_n$, $n \in \mathbf{N}$, we have the commutative diagram

$$\begin{tikzcd} M_n \hat{\otimes} W^* \arrow[r, phantom, "="] \arrow[d] & (T_n \check{\otimes} W)^* \arrow[d] \\ L^* \hat{\otimes} W^* \arrow[r] & (L \check{\otimes} W)^* \end{tikzcd}$$

where the columns are completely quotient mappings and the top row is isometric from the hypothesis of (3). This implies that we have the isometry

$$L^* \hat{\otimes} W^* = (L \check{\otimes} W)^*.$$

(1)⇔(2): Since for any $L \subseteq T_n, n \in \mathbf{N}$

$$(L^* \hat{\otimes} W^*)^* = CB(L^*, W^{**}) = L \check{\otimes} W^{**},$$

(2) holds if and only if we have the natural isometric isomorphism

$$L\check{\otimes}W^{**} = (L\check{\otimes}W)^{**}.$$

The corresponding mapping is explicitly given by the norm-increasing linear isomorphism

$$\tau : L\check{\otimes}W^{**} \to (L\check{\otimes}W)^{**}.$$

Thus, the relation is isometric if and only if

$$\varphi \in (L\check{\otimes}W^{**})_{\|\cdot\|\leq 1} = CB(L^*, W^{**})_{\|\cdot\|_{cb}\leq 1}$$

implies that

$$\varphi \in (L\check{\otimes}W)^{**}_{\|\cdot\|\leq 1}.$$

From the bipolar theorem, the latter is the case if and only if φ is a weak* limit of elements in

$$(L\check{\otimes}W)_{\|\cdot\|\leq 1} = CB(L^*, W)_{\|\cdot\|_{cb}\leq 1}.$$

Since it is evident that

$$\tau : CB(L^*, W^{**}) \to (L\check{\otimes}W)^{**}$$

is a homeomorphism in the point-weak* and weak* topologies, we are done.

(4)⇒(2): For any $L \subseteq T_n, n \in \mathbf{N}$, we have the isometries

$$L^*\hat{\otimes}W^* = \mathcal{N}(L, W^*) = \mathcal{I}(L, W^*) = (L\check{\otimes}W)^*.$$

(2)⇒(4): From (12.3.9) in [5] we see that

$$S_{int} : \mathcal{I}(V, W^*) \to (V\check{\otimes}W)^*$$

is a contractive injection. Let us suppose that the mapping in (2) is isometric. If we have a contractive functional $F \in (V\check{\otimes}W)^*$, then $F = S(\varphi)$ for some $\varphi : V \to W^*$ (see Chapter 12 in [5]). For any $L \subseteq T_n, n \in \mathbf{N}$, and complete contraction $\psi : L \to V$, we have

$$F \circ (\psi \otimes \mathrm{id}_W) \in (L\check{\otimes}W)^* \text{ and } \varphi \circ \psi : L \to W^*.$$

Since for any $x \in L, y \in W$,

$$(F \circ (\psi \otimes \mathrm{id}_W))(x \otimes y) = F(\psi(x) \otimes y) = \varphi(\psi(x))(y),$$

we have $F \circ (\psi \otimes \mathrm{id}_W) = S(\varphi \circ \psi)$. Thus from (2) and $L^*\hat{\otimes}W^* = \mathcal{N}(L, W^*)$,

$$\nu(\varphi \circ \psi) = \|F \circ (\psi \otimes \mathrm{id}_W)\| \leq \|F\|.$$

Proposition 3.2 shows that $\iota^{fr}(\varphi) \leq \|F\|$. Since V is finitely representable, it follows from Theorem 3.5 that $\iota(\varphi) = \iota^{fr}(\varphi) \leq \|F\|$. Therefore $\iota(\varphi) = \|F\|$, $\varphi \in \mathcal{I}(V, W^*)$ and thus $\mathcal{I}(V, W^*) = (V\check{\otimes}W)^*$. □

Now we can answer the second 'dual' problem raised in Section 1.

Corollary 5.3. *For any $n \in \mathbf{N}$, we have*

$$(T_n \check{\otimes} W)^{**} = T_n \check{\otimes} W^{**} \Leftrightarrow W \text{ is } \mathcal{T}\text{-locally reflexive.}$$

Corollary 5.4. *For any $n \in \mathbf{N}$, we have*

$$(M_n \hat{\otimes} W)^{**} = M_n \hat{\otimes} W^{**} \Leftrightarrow W^* \text{ is } \mathcal{T}\text{-locally reflexive.}$$

Proof. Since $(M_n \hat{\otimes} W)^{**} = (T_n \check{\otimes} W^*)^*$, the result follows from the equivalence of (1) and (3) in Theorem 5.2. ☐

Corollary 5.5. *If W is $\mathcal{T}$-locally reflexive operator space, then any subspace $X \subseteq W$ is $\mathcal{T}$-locally reflexive.*

Proof. For any finitely representable operator space V, this is immediate from Theorem 5.2 (5) and the commutative diagram

$$\begin{tikzcd}
V \otimes_{\vee} X^{**} \arrow[r] \arrow[d] & (V \check{\otimes} X)^{**} \arrow[d] \\
V \otimes_{\vee} W^{**} \arrow[r] & (V \check{\otimes} W)^{**}
\end{tikzcd}$$

in which the columns are automatically isometric. ☐

As in the case of local reflexivity, we can prove the following result.

Proposition 5.6. *An operator space W is $\mathcal{T}$-locally reflexive if and only if that is the case for each separable subspace of W.*

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