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GEOMETRIC CHARACTERIZATIONS OF EXISTENTIALLY CLOSED FIELDS WITH OPERATORS

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Abstract. This paper concerns the basic model-theory of fields of arbitrary characteristic with operators. Simplified geometric axioms are given for the model-companion of the theory of fields with a derivation. These axioms generalize to the case of several commuting derivations. Let a *D-field* be a field with a derivation *or* a difference-operator, called D. The theory of D-fields is companionable. The existentially closed D-fields can be characterized geometrically without distinguishing the two cases in which D can fall. The class of existentially closed fields with a derivation *and* a difference-operator is elementary only in characteristic 0.

0. Introduction

On a field, a *jet-operator* is, roughly, a function whose behavior at sums and products is determined by polynomials, *and* whose value at 0 and 1 is 0. The term is from Alexandru Buium [4], who shows that on a field of characteristic 0, every jet-operator is equivalent to a *derivation* or a *difference-operator*. Piotr Kowalski [10] shows that this remains true in positive characteristic, provided that one generalizes the notion of a derivation.

The present paper is concerned with a *uniform* and *geometric* treatment of fields with derivations and difference-operators.

Thomas Scanlon [20] provides a way to begin, defining a **D-field** as a structure (K, e, D), where K is a field, $e \in K$, and D is an endomorphism of the additive group of K satisfying

$$(*) \qquad D(x \cdot y) = Dx \cdot y + (x + e \cdot Dx) \cdot Dy.$$

If $e = 0$, then D is a **derivation**, and (K, D) is a **differential field**. In any case, $e \cdot D$ is the map $x \mapsto x^\sigma - x$ for some endomorphism σ of K, so $e \cdot D$ is the **difference-operator** associated with σ, and (K, σ) is a **difference-field**. As Scanlon notes, 'this formal connection [between differential and difference-fields] supports the view that differential and difference-algebra are instances of the same theory.'

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By piecing together what is known about differential and difference-fields, one can show that the theory of D-fields is *companionable*. (The definition is reviewed at the end of this section.) Then the *model-companion* of this theory is a mathematically motivated model-complete theory whose completions are, respectively, (0) ω-stable; (1) stable, but not super-stable; and (2) simple, but not stable.

For the model-companion DCF_0 of the theory DF_0 of differential fields of characteristic 0, geometric axioms are given in a paper with Anand Pillay [18]. Here, 'geometric' means that the axioms refer to **varieties**, which for us are just zero-sets of polynomials; they are irreducible when this matters. Say (K, D) is a differential field. If V is a variety over K, then the **prolongation** $\tau(V)$ is the variety obtained by applying D to the polynomials over K that are 0 on V. If $(K, D) \models \mathrm{DCF}_0$, then K is algebraically closed, and every subvariety of $\tau(V)$ that projects generically onto V contains a K-rational point $(\mathbf{a}, D\mathbf{a})$; and these observations *characterize* the models of DCF_0 among the models of DF_0.

A derivation on a field of characteristic 0 extends *uniquely* to the algebraic closure of the field. Because of this, in Section 1 below, we can streamline the geometric approach of [18], giving axioms of DCF_0 that refer to varieties alone, and not to their prolongations. These re-formulated axioms can be seen as a special case of the axioms in [17] for DCF_0^m, the model-companion for the theory of fields of characteristic 0 with m commuting derivations. Rather, the new axioms for DCF_0 suggest a neater way to express the axioms for DCF_0^m in general, given in Section 2.

In the case of positive characteristic p, Carol Wood [22] shows how to come to terms with the fact that a non-trivial differential field cannot be perfect. She gives axioms for DCF_p using Seidenberg's elimination-theory for differential equations (as Abraham Robinson did for DCF_0; Wood gives simpler axioms for DCF_p in [23], parallel to those of Blum for DCF_0). *Geometric* axioms for DCF_p are a special case in Kowalski's analysis [11] of **derivations of powers of Frobenius**. These are additive maps δ satisfying

$$\delta(x \cdot y) = \delta x \cdot y^\sigma + x^\sigma \cdot \delta y, \tag{†}$$

where σ is a power of the Frobenius map $x \mapsto x^p$, so that $\sigma^{-1} \circ \delta$ and $\delta \circ \sigma^{-1}$ are derivations in the usual sense (albeit not on the same field). In case σ is the identity, Kowalski's axioms correspond to those of [18]; in particular, they involve prolongations.

As in the characteristic-zero case, we can write geometric axioms for DCF_p without reference to prolongations. We can also write the axioms independently of characteristic, getting the theory DCF of existentially closed differential fields of arbitrary characteristic. Likewise, we shall axiomatize DCF^m, the model-companion of the theory of fields of arbitrary characteristic with m commuting derivations.

We can approach the theory of fields with distinguished automorphism σ in the same spirit. This theory has the model-companion ACFA, for which Angus Macintyre [13] and Zoé Chatzidakis and Ehud Hrushovski [5] have published geometric axioms. These axioms inspired the original geometric axioms for DCF_0. Where the latter axioms refer to $\tau(V)$, the former refer to $V \times V^\sigma$. In the present paper, as we re-formulate the axioms for DCF_0, so too, in Section 3, for ACFA. In contrast to the case of a derivation, we cannot avoid applying σ to a variety. Still, we need not form the Cartesian product. (Thus, logically, we can strengthen the axioms for ACFA. The main point is that we can simplify them, at least slightly; the corresponding simplification in the case of derivations is much greater.)

In Section 4, we shall also adjust the definition of D-field so that there are two additional named operators present. There will be a derivation δ and an endomorphism σ, of which, however, at least one is trivial. Then D is δ if this is non-trivial; otherwise D is $x \mapsto x^\sigma - x$. In the larger language, we shall be able to axiomatize the existentially closed D-fields *without* distinguishing the cases in which D can fall.

Finally, in Section 5, of the class of fields with a derivation and an endomorphism that have no required interaction, we can say enough about the sub-class of existentially closed members to see that it is not elementary. For example, if (K, δ, σ) is in this class, let K^σ be the image of σ. Then K/K^σ is purely inseparable; but if $\operatorname{char} K = p$, then there need be no n such that $K^{p^n} \subseteq K^\sigma$. In characteristic 0, such problems disappear, so there is a model-companion.

The notational conventions of the present paper are as in [17]; in particular, tuples are bold-face, indices on their entries may be superscripts, and indices start with 0.

Words being defined (perhaps implicitly) are in **bold**; technical terms being emphasized, but not defined, are *slanted*; other emphasized words are in the usual *italic*.

Functions are generally written to the left of their arguments, although the field-endomorphism σ is written as a superscript (as above), by analogy with the Frobenius endomorphism $x \mapsto x^p$.

If V is a variety over K, and $\mathbf{x}$ is an n-tuple of elements of the function-field $K(V)$, then $\mathbf{x}$ is the generic point over K of a sub-variety W of affine n-space $\mathbb{A}^n$. Also, $\mathbf{x}$ can be understood as a rational map from V to $\mathbb{A}^n$, and as a *dominant* rational map into W. Finally, $\mathbf{x}$ determines an embedding $f \mapsto f(\mathbf{x}) : K(W) \to K(V)$, which can be considered as an inclusion; then the rational map $\mathbf{x}$ is **separable** if $K(V)$ is separable over $K(W)$. (All field-extensions in characteristic 0 are separable; in characteristic p, the extension L/K is separable if and only if L^p and K are linearly disjoint over K^p.) If $K(V)$ is separable over K, then V itself may be called separable.

Over a theory T, a model $\mathfrak{A}$ is **existentially closed** if $\mathfrak{A} \preccurlyeq_1 \mathfrak{B}$ whenever $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B} \models T$. (This definition can be found in [21, § 2]. Here $\mathfrak{A} \preccurlyeq_1 \mathfrak{B}$ means that quantifier-free formulas with parameters from A have solutions in $\mathfrak{A}$, provided they have solutions in $\mathfrak{B}$; equivalently, all *primitive* sentences over A that are true in $\mathfrak{B}$ are true in $\mathfrak{A}$.) A structure can be called existentially closed if it is an existentially closed model of its own universal theory (by [21, Theorem 2.4]). If the class of existentially closed models of an $\forall\exists$ theory T is elementary, then the theory of the class is the **model-companion** of T. More generally, a theory T has model-companion T^* if $T_\forall = T^*_\forall$ and T^* is **model-complete** ($T^* \cup \operatorname{diag} \mathfrak{M}$ is complete whenever $\mathfrak{M} \models T^*$); model-complete theories are always $\forall\exists$.

The existentially closed models of any theory are just those models that omit certain types. Indeed, a model $\mathfrak{M}$ of T is an existentially closed model just in case, for all primitive formulas $\phi(\mathbf{x})$ in the language of T, for all tuples $\mathbf{a}$ from M, if $T \cup \operatorname{diag} \mathfrak{M} \cup \{\phi(\mathbf{a})\}$ is consistent, then $\mathfrak{M} \models \phi(\mathbf{a})$. Now, the following conditions are equivalent:

(0) $T \cup \operatorname{diag} \mathfrak{M} \cup \{\phi(\mathbf{a})\}$ is inconsistent.
(1) $T \models \theta(\mathbf{a}, \mathbf{b}) \to \neg\phi(\mathbf{a})$ for some open formula θ and some tuple $\mathbf{b}$ from M such that $\mathfrak{M} \models \theta(\mathbf{a}, \mathbf{b})$.
(2) $T \models \forall \mathbf{x}\ (\phi(\mathbf{x}) \to \forall \mathbf{y}\ \neg\theta(\mathbf{x}, \mathbf{y}))$ for some open θ such that $\mathfrak{M} \models \exists \mathbf{y}\ \theta(\mathbf{a}, \mathbf{y})$.

For any primitive ϕ, let Θ_ϕ be the set of universal consequences of $T \cup \{\phi\}$. Condition (2) is that $\mathfrak{M}$ omits Θ_ϕ. So a model $\mathfrak{M}$ of T is an existentially closed model if and only if $\mathfrak{M}$ omits each type $\Theta_\phi \cup \{\neg\phi\}$.

I thank the anonymous referee for reading carefully and for insisting on the spelling out of some details; this led to some important corrections and improvements, as for example in the development of Lemma 1.2.

1. Differential fields

The co-domain of a derivation **on** a field need only be a vector-space *over* that field. Let an **extension** of a derivation be a derivation of which the first is a restriction. On any field K, the zero-derivation has the extension $f \mapsto f'$ to $K(X)$ and, more generally, has the n extensions $\partial/\partial X^j$ or ∂_j to $K(X^0, \dots, X^{n-1})$. Moreover, any derivation δ on K has the unique extension $f \mapsto f^\delta$ to $K(X^0, \dots, X^{n-1})$ that takes each X^j to 0.

FACT 1.1. *Suppose δ is a derivation on a field K.*

(0) *If $f \in K(X^0, \dots, X^{n-1})$, and $\mathbf{a} \in K^n$, then*

$$\delta(f(\mathbf{a})) = \sum_{j<n} \partial_j f(\mathbf{a}) \cdot \delta a^j + f^\delta(\mathbf{a})$$

if $f(\mathbf{a})$ is defined. In case $n = 1$, this is

$$\delta(f(a)) = f'(a) \cdot \delta a + f^{\delta}(a). \tag{‡}$$

(1) *If a is transcendental over K, or if $\operatorname{char} K = p$ and $a \in K^{1/p} \smallsetminus K$ and $\delta(a^p) = 0$, then the formula (‡) uniquely determines an extension of δ to $K(a)$, once the derivative δa is chosen arbitrarily.*

(2) *If $a \in K^{\text{sep}}$, then δ extends uniquely to $K(a)$; and if f is the minimal polynomial of a over K, then $\delta a = -f^{\delta}(a)/f'(a)$.*

Proof. See for example [12, ch. VIII, § 5, p. 369]. □

Fact 1.1 (1) suggests an analogy between differential fields of null and positive characteristic; the analogy can be described in terms of *closure-operators* (as defined for example in [1, Definition 3.1.4, p. 53]). If L is a field with subfield K, then L becomes a *pre-geometry* when equipped with the closure-operator

$$\operatorname{cl}_K^{\text{alg}} : A \longmapsto K(A)^{\text{alg}} \cap L : \mathcal{P}(L) \longrightarrow \mathcal{P}(L).$$

Therefore L has a *basis*—a maximal *independent* subset—with respect to this closure-operator; such a basis is precisely a transcendence-basis of L/K. (See also [14] for an early account of transcendence-bases along these lines.) If $\operatorname{char} K = p$, then another closure-operator that makes L/K a pre-geometry is

$$\operatorname{cl}_K^{p} : A \longmapsto L^p K(A) : \mathcal{P}(L) \longrightarrow \mathcal{P}(L);$$

a basis of *this* pre-geometry can be called a p**-basis** of L/K (or of L/L^pK). An **(absolute)** p-basis of L is then a p-basis of L/L^p. (See also [15, § 4].) That B is a p-basis of L/K means that L, as a vector-space over L^pK, has a basis consisting of the monomials

$$\prod_{x \in B} x^{s(x)},$$

where s is a map from B to p whose support $B \smallsetminus s^{-1}(0)$ is finite. That L/K is separable means that any (absolute) p-basis of K is p**-independent** in L—is included in a p-basis of L.

Every separating transcendence-basis in characteristic p is a p-basis, by [15, Lemma 3, p. 382]. The converse holds if L/K has a *finite* separating transcendence-basis, but not generally [15, p. 385], since the field $\mathbb{F}_p(X_n : n \in \omega)$ has the p-basis $(X_n - X_{n+1}^p : n \in \omega)$, over which the field is not algebraic.

It may be worth noting that, in the sense of Kolchin [9, ch. 0, § 2, pp. 3–4], an **inseparability-basis** of L/K is a minimal generating set of L with respect to the closure-operator $A \mapsto K(A)^{\text{sep}} \cap L$. This operator fails generally to have the *exchange property*, since $\mathbb{F}_p(X)^{\text{sep}} \smallsetminus \mathbb{F}_p^{\text{sep}}$ contains X^p, but $\mathbb{F}_p(X^p)^{\text{sep}}$ does not contain X. So the operator does not make L a pre-geometry, and inseparability-bases are not guaranteed to exist. Indeed, there is a standard counterexample: The extension $\mathbb{F}_p(X)^{p^{-\infty}}/\mathbb{F}_p$ has no inseparability-basis.

For any field L that includes K, on $\mathcal{P}(L)$ define

$$\mathrm{cl}_K = \begin{cases} \mathrm{cl}^{\mathrm{alg}}_K, & \text{if } \operatorname{char} K = 0; \\ \mathrm{cl}^p_K, & \text{if } \operatorname{char} K = p. \end{cases}$$

Henceforth, let **independence** in L/K and **bases** of L/K be understood with respect to cl_K. In characteristic p, the following is a generalization of [8, ch. IV, § 7, Theorem 17, p. 181]:

LEMMA 1.2. *Suppose L/K is a field-extension, B is a basis of L/K, and δ is a derivation from K to L.*

(0) *If δ extends to L and sends B into L, then L becomes a differential field.*
(1) *If* $\operatorname{char} K = 0$*, then δ extends to L.*
(2) *If δ extends to L, then δ extends uniquely to L after arbitrary choice of those δx such that $x \in B$.*
(3) *Hence, if δ extends to L, or if* $\operatorname{char} K = 0$*, then δ extends so as to make L a differential field.*

Proof. Claim (0) follows from Fact 1.1 (0) and (2).

Suppose now $\operatorname{char} K = 0$. By induction on a well-ordering of B and by Fact 1.1 (1), we can extend δ to $K(B)$ after arbitrary choice of δx when $x \in B$; and the extension of δ is then unique. Then δ extends further, and uniquely, to L by Fact 1.1 (2). This proves Claim (2) when $\operatorname{char} K = 0$, and also Claim (1).

For the other case of Claim (2), suppose $\operatorname{char} K = p$ and δ extends to L. Then δ is zero on L^p, and this determines the extension to L^pK by Fact 1.1 (0); this extension still has co-domain L; so we can replace K with L^pK. Now we can use induction and Fact 1.1 (1) as before to extend δ uniquely to $K(B)$ after arbitrary choice of the δx with x in B. But now $K(B) = L$ (since we assumed $K = L^pK$); so Claim (2) is established in all cases.

So now, by Claim (1), if $\operatorname{char} K = 0$, then δ extends to L. In any case, if δ extends to L, then by Claim (2), it extends so as to send B into L; then by Claim (0), the extension makes L a differential field. This establishes Claim (3) and the theorem. □

For any field-extension L/K, let $\mathrm{Der}(L/K)$ be the vector-space over L consisting of derivations from L to itself that are 0 on K. The **universal K-linear derivation** on L (as defined in [6, § 16, p. 386]) can be understood as the map $\mathrm{d}_K : L \to \mathrm{Der}(L/K)^*$ given by

$$D(\mathrm{d}_K\, x) = Dx.$$

If $S \subseteq \mathrm{Der}(L/K)^*$, let $\langle S\rangle^L$ be the L-linear span of S. Then there is a uniform definition of cl_K:

LEMMA 1.3. *Let L/K be a field-extension, and let d_K be the universal K-linear derivation on L. Then cl_K is the map*

$$A \longmapsto \{x \in L : \mathrm{d}_K\, x \in \langle \mathrm{d}_K\, a : a \in A\rangle^L\} : \mathcal{P}(L) \longrightarrow \mathcal{P}(L).$$

Proof. Being a derivation on L that is 0 on K, the map d_K takes dependent sets to L-linearly dependent sets, by Fact 1.1 (0); it takes independent sets to L-linearly independent sets, by Lemma 1.2 (2). □

The subspace $\langle \mathrm{d}\, x : x \in L\rangle^L$ of $\mathrm{Der}(L/K)^*$ can be denoted

$$\Omega^1_{L/K}.$$

This can be understood as the space of *Kähler differentials* of L over K, and its dual is naturally isomorphic to $\mathrm{Der}(L/K)$.

In the following, the **kernel** of a derivation is its constant-field, that is, its kernel as a homomorphism of abelian groups.

LEMMA 1.4. *Suppose (K, δ) is a differential field, and $K \subseteq L$.*

(0) *If δ extends to $\tilde{\delta}$ on L, then $\ker\tilde{\delta}$ is linearly disjoint from K over $\ker\delta$.*
(1) *If $\mathrm{char}\, K = p$, and $L^p(\ker\delta)$ is linearly disjoint from K over $\ker\delta$, then δ extends to L.*

Proof. That (0) is true is a special case of [9, ch. II, § 1, Corollary 1, p. 87]. For an alternative proof, suppose δ does extend to $\tilde{\delta}$ on L. Let $\mathbf{a}$ be an n-tuple of elements of $\ker\tilde{\delta}$ that are linearly dependent over K. Shortening the tuple as necessary, we may assume that its null-space

$$\{\mathbf{x} \in K^n : \mathbf{a}\cdot\mathbf{x} = 0\}$$

has dimension 1. Then we may assume that this space is spanned by a single element $\mathbf{b}$ whose first entry b^0 is 1. But $\tilde{\delta}\mathbf{a} = \mathbf{0}$, so

$$0 = \tilde{\delta}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\delta\mathbf{b}.$$

Thus $\delta\mathbf{b}$ is in the null-space of $\mathbf{a}$ and is therefore a multiple of $\mathbf{b}$. But $\delta b^0 = 0$, so $\delta\mathbf{b} = \mathbf{0}$, which means $\mathbf{b} \in (\ker\delta)^n$. Thus $\mathbf{a}$ is linearly dependent over $\ker\delta$. This proves (0).

Suppose now that the hypotheses of (1) hold. Let B be a p-basis of $K/\ker\delta$. Then δ on K is determined by $\delta|_B$, by Lemma 1.2. Also, B is included in a p-basis of $L/\ker\delta$; so the zero-derivation on $\ker\delta$ extends to L to agree with δ on K. □

In the terminology of [22], the differential field (K, δ) is **(differentially) perfect** if $\mathrm{char}\, K = 0$, or else $\mathrm{char}\, K = p$ and $\ker\delta \subseteq K^p$. The terminology is chosen because of the following lemma (which is equivalent to a slight generalization of [9, ch. II, § 3, Proposition 5(a), p. 92]).

LEMMA 1.5. *Suppose (K, δ) is a differential field. The following are equivalent:*

(0) *If δ extends to L, then L/K is separable.*
(1) *If δ extends to L, and L/K is algebraic, then L/K is separable.*
(2) *(K, δ) is differentially perfect.*

Proof. We may assume that $\operatorname{char} K = p$. Trivially, (0) implies (1).

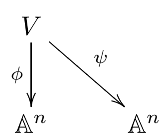
Suppose (2) fails. Then there is β in $\ker\delta \smallsetminus K^p$. Let $L = K(\beta^{1/p})$. Then L/K is algebraic; also $L^p \subseteq \ker\delta$, so δ extends to L by Lemma 1.4 (1). Thus (1) fails.

Finally, if (2) holds, and δ extends to L, then L^p is linearly disjoint from K over K^p by Lemma 1.4 (0), so L/K is separable; thus (0) holds. □

The following theorem will turn out to be a special case of Corollary 2.7 below. (Rather, it is *almost* a special case; the weakening of Condition (2) in the theorem uses Lemma 1.2, which doesn't generalize to several commuting derivations.)

THEOREM 1.6. *A differential field (K, δ) is existentially closed just in case it satisfies the following conditions:*

(0) *K is separably closed.*
(1) *(K, δ) is differentially perfect.*
(2) *For every variety V over K, if there are rational maps*



for some n, where ϕ is dominant and separable, then V has a K-rational point P such that ϕ and ψ are regular at P, and

$$\delta \circ \phi(P) = \psi(P).$$

In Condition (2), it is sufficient to assume $n = \dim V$.

Proof. Existentially closed differential fields meet Condition (0) by Fact 1.1 (2) and Lemma 1.2; they meet Condition (1) by Lemmas 1.2 and 1.5.

Condition (2) is that if an n-tuple $\mathbf{x}$ of elements of $K(V)$ extends to a separating transcendence-basis of this field over K, and $\mathbf{y}$ is an arbitrary n-tuple of elements of $K(V)$, then V has a K-rational point $\mathbf{a}$ such that each member of each equation

$$(\S) \qquad \delta(x^i(\mathbf{a})) = y^i(\mathbf{a})$$

is well-defined, and the equations hold.

Suppose for the moment that $\mathbf{a}$ is a *generic* point of V. Then the set of elements $x^i(\mathbf{a})$ of $K(\mathbf{a})$ extends to a separating transcendence-basis B of this field over K. By Fact 1.1, we can extend δ to $K(\mathbf{a})$. By Lemma 1.2 (2) then, since the $y^i(\mathbf{a})$ are in $K(\mathbf{a})$, we can extend δ so that the equations (§) hold and δ maps all of B into $K(\mathbf{a})$. This extension makes $K(\mathbf{a})$ a differential field by Lemma 1.2 (0).

Moreover, each of the x^i or y^i is an equivalence-class of quotients $f_{\mathrm{n}}/f_{\mathrm{d}}$ or $g_{\mathrm{n}}/g_{\mathrm{d}}$ of polynomials over K; so the equations (§) are implied by the satisfaction, by $\mathbf{a}$, of some quantifier-free formulas of the form

$$(\delta f_{\mathrm{n}} \cdot f_{\mathrm{d}} - f_{\mathrm{n}} \cdot \delta f_{\mathrm{d}}) \cdot g_{\mathrm{d}} = f_{\mathrm{d}}^2 \cdot g_{\mathrm{n}} \wedge f_{\mathrm{d}} \neq 0 \wedge g_{\mathrm{d}} \neq 0$$

in the signature of rings with constants from K. Hence existentially closed differential fields meet Condition (2) as well.

Suppose conversely that (K, δ) meets the given conditions. We have to look at primitive sentences over (K, δ). We can simplify such a sentence as in [17, Lemma 5.5]: we can replace the inequations by equations, using the Rabinowitsch-trick, and we can replace each derivative with a new variable. The result is the statement that a system

$$(\P) \qquad \bigwedge_f f = 0 \wedge \bigwedge_{i<k} \delta X^i = g^i$$

has a solution, where the (finitely numerous) f and the g^i are in the polynomial-ring $K[X^0, \dots, X^{r-1}]$ for some r, and $k \leqslant r$. Suppose the system (¶) has a solution $\mathbf{b}$ from an extension of (K, δ); we have to find a K-rational solution.

Now, we are assuming that δ extends to $K(\mathbf{b})$ so as to map $K(b^0, \dots, b^{k-1})$ into $K(\mathbf{b})$. By Lemma 1.2 (3), we may assume that δ has been extended so as to map all of $K(\mathbf{b})$ into itself.

By Lemma 1.5, the extension $K(\mathbf{b})/K$ is separable. Let $(h^j(\mathbf{b}) : j < n)$ be a basis of $K(\mathbf{b})/K$, and say $\delta h^j(\mathbf{b}) = q^j(\mathbf{b})$ when $j < n$, for some rational functions h^j and q^j over K. These equations determine the extension of δ from K to $K(\mathbf{b})$. In particular, they determine the equations $\delta b^i = g^i(\mathbf{b})$ where $i < k$. Indeed, let F^i be irreducible polynomials over K such that $F^i(h^0(\mathbf{b}), \dots, h^{n-1}(\mathbf{b}), b^i) = 0$. By Fact 1.1, we have

$$\sum_{j<n} \partial_j F^i(h^0(\mathbf{b}), \dots, h^{n-1}(\mathbf{b}), b^i) \cdot q^j(\mathbf{b}) + \\ + \partial_n F^i(h^0(\mathbf{b}), \dots, h^{n-1}(\mathbf{b}), b^i) \cdot g^i(\mathbf{b}) + \\ + (F^i)^\delta(h^0(\mathbf{b}), \dots, h^{n-1}(\mathbf{b}), b^i) = 0,$$

and these equations can be solved for $g^i(\mathbf{b})$.

Let U be the algebraic set over K consisting of those specializations of $\mathbf{b}$ where the h^j and the q^j are well-defined and the $\partial_n F^i(h^0, \dots, h^{n-1}, X^i)$ are not 0. Suppose $\mathbf{x} \in U$. Then the x^i are separable over $K(h^0(\mathbf{x}), \dots, h^{n-1}(\mathbf{x}))$ when $i < k$; and if δ extends to this field so that $\delta h^j(\mathbf{x}) = q^j(\mathbf{x})$, then δ extends further to the x^i, and $\delta x^i = g^i(\mathbf{x})$.

Now, there is a variety V over K consisting of precisely one tuple $(\mathbf{x}, \mathbf{y})$ for each $\mathbf{x}$ in U. By the weak form of Condition (2), with $(h^0, \dots, h^{n-1})$ as ϕ, and with $(q^0, \dots, q^{n-1})$ as ψ, we can conclude that (¶) has a K-rational solution. □

In the weak form of Condition (2), if ϕ is written as a tuple $\mathbf{c}$, then there is a variety W with a generic point $(\mathbf{c}, d)$ such that $d \in K(\mathbf{c})^{\text{sep}}$, and there is a birational map $\chi : W \to V$ such that $\phi \circ \chi$ is $(\mathbf{x}, y) \mapsto \mathbf{x}$. In Condition (2) then, we can replace ϕ with $\phi \circ \chi$, and V with an open subset of W (namely, the set of regular points of χ). So, we can write the conditions of Theorem 1.6 in a more explicitly first-order way:

(0) $\forall x\ \exists y\ (f'(x) = 0 \lor f(y) = 0)$ for all polynomials f in one variable (over the universe).
(1) $\forall x\ \exists y\ (p \cdot 1 = 0 \land \delta x = 0 \to y^p = x)$ for all primes p.
(2) $\exists \mathbf{x}\ (f(\mathbf{x}) = 0 \land \bigwedge_{i \leqslant n} g^i(\mathbf{x}) \neq 0 \land \bigwedge_{i<n} g^i(\mathbf{x}) \cdot \delta x^i = h^i(\mathbf{x}))$ for all polynomials f, g^i and h^i in $n+1$ variables such that $\partial_n f \neq 0$ and $f \nmid g^i$, for all n in ω.

So the theory DF of differential fields has a model-companion, DCF, which is the theory of **differentially closed fields** of arbitrary characteristic.

We shall generalize to several derivations in the next section. Meanwhile, for the sake of an analogy with difference-fields, we give an alternative axiomatization of DCF.

Let (K, δ) be an arbitrary differential field. Suppose the variety V over K is the zero-set of the prime ideal I of the ring $K[\mathbf{X}]$. If f is in this ring, then $\delta f \in K[\mathbf{X}, \delta\mathbf{X}]$. The zero-set of all f and δf such that $f \in I$ has been denoted $\tau(V)$, presumably by analogy with the tangent-bundle $T(V)$; but here I shall just write $\delta(V)$. I shall also write π_0 for the map $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} : \delta(V) \to V$.

The next lemma (in case of positive characteristic) is related to [11, Fact 1.6].

Lemma 1.7. *Let (K, δ) be a differential field. Suppose V is a variety over K containing a tuple $\mathbf{a}$, and $\mathbf{b}$ is a tuple of the same length. If δ extends to $K(\mathbf{a})$ so that $\delta\mathbf{a} = \mathbf{b}$, then $(\mathbf{a}, \mathbf{b}) \in \delta(V)$. The converse holds, provided $\mathbf{a}$ is a generic point of V.*

Proof. See [12, ch. VIII, § 5], or use [8, ch. IV, § 6, Theorem 14, p. 172]. □

Certain generalizations of the lemma are not possible:

- Though δ extend to $K(\mathbf{a})$ so that $\delta\mathbf{a} = \mathbf{b}$, if $\operatorname{char} K = p$, it may be that δ does not extend further to $K(\mathbf{a}, \mathbf{b})$. For example, if α is transcendental over K, we can define $\delta(\alpha^p) = \alpha$, but then $\delta\alpha$ cannot be defined [11, Remark after Fact 1.6].
- The map $\pi_0 : \delta(V) \to V$ need not be dominant. Let $K = \mathbb{F}_p(\alpha, \beta)$, where $\{\alpha, \beta\}$ is algebraically independent. Define $\delta\alpha = \delta\beta = 1$. Let

$f = \alpha \cdot X^p + \beta \cdot Y^p$, and let V be the zero-set of f. Since $\delta f = X^p + Y^p$, the image of $\delta(V)$ under π_0 is $\{(0,0)\}$.

In characteristic 0, the following can be seen as a corollary of [18, Theorem 2.1].

Theorem 1.8. *In Theorem* 1.6*, we can replace Condition* (2) *with:*

(3) *For every variety V over K, if $s : V \to \delta(V)$ is a rational section of π_0, then V has a K-rational point P such that $s(P) = (P, \delta P)$.*

Proof. The necessity of Condition (3) is by Lemma 1.7. For its sufficiency, consider the system (¶)—with the attendant notation—of the proof of Theorem 1.6. In that proof, once the solution $\mathbf{b}$ is chosen, it is noted that, by Lemma 1.2 (3), we may assume that δ maps $\mathbf{b}$ into $K(\mathbf{b})$. This means we may assume that g^i in $K(\mathbf{X})$ exist also when $k \leqslant i < r$ so that $\delta b^i = g^i(\mathbf{b})$ for all i less than r. Let s be $\mathbf{x} \mapsto (\mathbf{x}, \mathbf{g}(\mathbf{x}))$ on V. Then $s(\mathbf{b}) \in \delta(V)$ by Lemma 1.7, so s is a section of π_0. Condition (3) now yields a K-rational solution of (¶). □

Following [2, (1.1), p. 4] and [3, **(0.6)**, pp. 3f.], we can refer to the pair (V, s) in Condition 3 as a **δ-variety**, and to the point P as a K-rational **δ-point**. So the condition is that δ-varieties have K-rational δ-points. Also, if $(V, s)^\sharp = \{P \in V : s(P) = (P, \delta P)\}$ as in [19, p. 3], then the condition is that $(V, s)^\sharp$ contains a K-rational point.

A corollary will be needed for the uniform treatment in Section 4. First, note that, if f is a rational map on V, then δf can be understood as a rational map on $\delta(V)$. Moreover, by Lemma 1.7, if ϕ is a rational map from V onto W, then $(\phi, \delta\phi)$ is a rational map from $\delta(V)$ into $\delta(W)$ making the following diagram commute:

$$\begin{array}{ccc} V & \xleftarrow{\ \pi_0\ } & \delta(V) \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle (\phi,\delta\phi)} \\ W & \xleftarrow[\ \pi_0\ ]{} & \delta(W) \end{array}$$

Corollary 1.9. *In Theorem* 1.6*, we can replace Condition* (2) *with:*

(4) *If $\phi : V \to W$ is a rational map of varieties over K, and if $s : V \to \delta(V)$ is a rational section of π_0, then V has a K-rational point P such that $\delta\phi \circ s(P) = \delta \circ \phi(P)$.*

It is sufficient to require ϕ to be dominant.

Proof. Condition (4) is sufficient, since Condition (3) is the special case when ϕ is the identity. For the necessity, suppose $\mathbf{b}$ is a generic point of V. Then δ extends to $K(\mathbf{b})$ so that $(\mathbf{b}, \delta\mathbf{b}) = s(\mathbf{b})$. Hence $\delta \circ \phi(\mathbf{b}) = \delta\phi(\mathbf{b}, \delta\mathbf{b}) = \delta\phi \circ s(\mathbf{b})$. □

2. Fields with several derivations

We can generalize Theorem 1.6 to several derivations, because we can generalize the relevant arguments of [17] to arbitrary characteristic. Indeed, let us remove from [17] the blanket assumption that rings and fields have characteristic 0. In particular, let us allow models of DF^m to have *any* characteristic. (We can specify characteristic with a subscript, as in DF_0^m or DF_p^m.) In characteristic p, all transcendence-bases should be replaced with p-bases, and 'transcendence-degree' should be read as **p-dimension**—the size of a p-basis. Also, the following additional changes should be made:

In [17, Fact 3.1], by Lemmas 1.2 and 1.4 above, if $\operatorname{char} K = p$, then the extension $f \mapsto f^\delta$ exists just in case L^p is linearly disjoint from K over $\ker\delta$; such a condition is also required for the conclusion about more general extensions of δ.

Now [17, Fact 3.3] is incorrect as it stands, by [22, Theorem 2]. But replace DF^1 with the theory of *perfect* differential fields (with one derivation); then the claim holds by [22, Lemma 5].

In [17, Lemma 3.4], the field K^{a} (that is, K^{alg}) should be K^{sep}.

To generalize [17, Lemma 3.6], we generalize the definition of *perfect*: A model $(K, D_0, \dots, D_{m-1})$ of DF^m can be called **(differentially) perfect** if $\operatorname{char} K = 0$, or if $\operatorname{char} K = p$ and $K^p = \bigcap_{i<m} \ker D_i$. So being perfect here means satisfying the sentence

$$\forall x\ \exists y\ (p \cdot 1 = 0 \wedge \bigwedge_{i<m} D_i x = 0 \to y^p = x)$$

whenever p is prime. Let us refer to the theory of differentially perfect models of DF^m as PDF^m. In [17, Lemma 3.6], the theories $\mathrm{PDF}_p^m \cup \{\alpha\}$ are consistent, having the models $(\mathbb{F}_p(X^0, \dots, X^{m-1}), \partial_0, \dots, \partial_{m-1})$. Also, it is now $\mathrm{PDF}^m \cup \{\alpha\}$ that has the amalgamation property. The proof in characteristic 0 should have noted that the fields L_i can and must be assumed free over K. The same is true in positive characteristic, but only by Lemma 2.3 below.

In [17, § 4], the first two sub-sections require no change. In particular, if $(K, D_0, \dots, D_{m-1}) \models \mathrm{DF}^m$, and the D_i span E over K, then (K, E) is also called a differential field and is equipped with the derivation $\mathrm{d} : K \to E^*$ given by $D(\mathrm{d}\, x) = Dx$; if $\operatorname{char} K = p$, then (K, E) is perfect if and only if $\ker \mathrm{d} \subseteq K^p$.

In [17, Lemma 4.7], Condition (1) could have been given more simply as: There is also an additive map $\mathrm{d} : E^* \to \mathrm{A}^2(E)$ such that $\mathrm{d}(\mathrm{d}\, y \cdot x) = \mathrm{d}\, x \wedge \mathrm{d}\, y$ when $x, y \in K$. It should be noted then that, if this condition holds, so that (0) also holds, then d is given by the equation near the bottom of [17, p. 933], so that

$$(D_0, D_1)\, \mathrm{d}\, \theta = D_0(D_1\theta) - D_1(D_0\theta) - [D_0, D_1]\theta$$

when $\theta \in E^*$. This observation is needed in proving [17, Lemma 4.9].

In the sub-section called 'Extensions' [17, p. 935], the discussion leading up to the 'Frobenius Theorem' [17, 4.11] needs some modification. It is assumed here that (K, E) is a differential field, and L/K is a field-extension. If $\operatorname{char} K = p$, then possibly the restriction-map $D \mapsto D|_K : \operatorname{Der}(L) \to L \otimes_K \operatorname{Der}(K)$ is not surjective. The definition of $\operatorname{Der}(L/E)$ stands in any case; but the ensuing [17, Lemma 4.10] for characteristic 0 should be supplemented with the following, where $p\text{-}\dim(L/K)$ denotes the p-dimension of L/K if $\operatorname{char} K = p$:

Lemma 2.1. *Suppose* (K, E) *is a differential field,* $\operatorname{char} K = p$, *and* L/K *is a field-extension.*

(0) *If* L/K *is separable, then the map*

$$\Psi : D \longmapsto D|_K : \operatorname{Der}(L/E) \longrightarrow L \otimes_K E$$

is surjective;

(1) *if* Ψ *is surjective, then* $\dim_L \operatorname{Der}(L/E) = \dim_K E + p\text{-}\dim(L/K)$.

Proof. By Lemma 1.4 (1), the map is surjective; by [17, Fact 3.1] for characteristic p, or Lemma 1.3, the dimension of its kernel is $p\text{-}\dim(L/K)$. □

In the remainder of [17, § 4], if $\operatorname{char} K = p$, then it should be assumed that the map Ψ in Lemma 2.1 *is* surjective. It should be noted that $\operatorname{Der}(L/E)$ is naturally isomorphic to the dual of $\Omega^1_{L/E}$. If $(x^j : j < \mu)$ is a basis of L/K, then $(\mathrm{d}\, x^j : j < \mu)$ is a basis of $\Omega^1_{L/E}$—not simply, (as wrongly suggested six lines before [17, Lemma 4.11],) but *modulo* $E^* \otimes_K L$, by [17, Fact 3.1]. In fact,

$$\Omega^1_{L/E} \cong \Omega^1_{L/K} \oplus (E^* \otimes_K L),$$

though not canonically; $E^* \otimes_K L$ is the kernel of the restriction-map from $\Omega^1_{L/E}$ to $\Omega^1_{L/K}$ (whose dual is the embedding of $\operatorname{Der}(L/K)$ in $\operatorname{Der}(L/E)$).

Again, [17, Lemma 4.12] can be taken as a definition of *integrable*. (A minor correction in the proof is that [17, p. 937, l. 3] should read '…its further restriction to K is in $L' \otimes_K E$…'.) In particular, if $\Omega^1_{L/E}$ *has* an integrable subspace, then the map Ψ in Lemma 2.1 is surjective.

We now have the following generalization of Lemma 1.4 above.

Lemma 2.2. *Suppose* $(K, D_0, \dots, D_{m-1})$ *is a differential field, the* D_i *span* E, *and* $K \subseteq L$.

(0) *If each* D_i *extends to* $\tilde{D}_i$ *on* L, *then* $\bigcap_{i<m} \ker \tilde{D}_i$ *is linearly disjoint from* K *over* $\ker \mathrm{d}$.

(1) *If* $\operatorname{char} K = p$, *and* $L^p(\ker \mathrm{d})$ *is linearly disjoint from* K *over* $\ker \mathrm{d}$, *then* (K, E) *has an extension* $(L, \tilde{E})$.

Proof. Claim (0) is [9, ch. II, § 1, Corollary 1, p. 87]. Alternatively, under the assumptions, each $\ker \tilde{D}_i$ is linearly disjoint from K over $\ker D_i$, by

Lemma 1.4 (0). Suppose then that $(\mathbf{a}, b)$ from $\bigcap_{i<m} \ker \tilde{D}_i$ is minimally linearly dependent over K. Then $\mathbf{a}$ is independent over K, but b is a $(\ker D_i)$-linear combination of $\mathbf{a}$ for each i. It should be the *same* combination in each case (otherwise subtraction yields a dependence for $\mathbf{a}$); so the combination is over $\bigcap_{i<m} \ker D_i$, which is $\ker \mathrm{d}$.

Suppose now that the hypotheses of (1) hold. Then $L^p(\ker D_i)$ is linearly disjoint from K over $\ker D_i$ (by [7, Lemma VI.2.3, p. 319]) for each i less than n; so each D_i extends to an element of $\mathrm{Der}(L)$, and $E^* \otimes_K L$ embeds in $\Omega^1_{L/E}$. Let B be a p-basis of L/K. Let W be the span of the $\mathrm{d}\,x$ such that $x \in B$. Then $\Omega^1_{L/E} = W \oplus (E^* \otimes_K L)$, and $\mathrm{d}\,W = \Omega^1_{L/E} \wedge W$, so $(L, \ker W)$ extends (K, E) by [17, Theorem 4.11]. □

Then Lemma 1.5 also generalizes:

Lemma 2.3. *Suppose (K, E) is a differential field. The following are equivalent:*

(0) *If $(K, E) \subseteq (L, \tilde{E})$, then L/K is separable.*
(1) *If $(K, E) \subseteq (L, \tilde{E})$, and L/K is algebraic, then L/K is separable.*
(2) *(K, E) is differentially perfect.*

Proof. As for Lemma 1.5. □

The generalization of [17, Lemma 5.2] is the following:

Lemma 2.4. *In any existentially closed model of DF^m, the D_i are linearly independent, the model itself is differentially perfect, and the underlying field is separably closed.*

Proof. The original proof of [17, Lemma 5.2] yields the first and last points; the middle point is by Lemma 2.3. □

In the ensuing discussion in [17], we may therefore assume that $(K, D_0, \dots, D_{m-1})$ is a model of $\mathrm{PDF}^m \cup \{\alpha\} \cup \mathrm{SCF}$ (where SCF is the theory of separably closed fields). We can drop [17, Theorem 5.3] for now. We can generalize [17, Lemma 5.5] as Theorem 2.5 below.

First, I correct a flaw in the definition of 'eliminable' on [17, p. 939]. There (and everywhere else in the paper), the word 'place' should be understood more generally than usual. If $W \subseteq \Omega^1_{L/E}$, and L is $K(\mathbf{a})$, and $\mathbf{b}$ is a specialization of $\mathbf{a}$ over K, then W is **eliminable** if it vanishes under the substitution-map $f(\mathbf{a}) \mapsto f(\mathbf{b})$. This map is well-defined on the localization $\mathfrak{O}$ of $K[\mathbf{a}]$ at the ideal $\{f(\mathbf{a}) : f(\mathbf{b}) = 0\}$. This ideal generates in $\mathfrak{O}$ its unique maximal ideal $\mathfrak{m}$, the field $\mathfrak{O}/\mathfrak{m}$ being isomorphic to $K(\mathbf{b})$ over K. Now, $\mathfrak{O}$ is *not* generally a valuation-ring; but nothing in [17] requires it to be. So, for 'valuation-', read 'local' everywhere. In particular, in [17, Lemma 5.4], the ring $\mathfrak{O}$ need only be a local ring such that $K \subseteq \mathfrak{O} \subseteq L$. (Strictly, $\mathfrak{O}$ need not even be local; $\mathfrak{m}$ should just be *some* maximal ideal.) For the additive map of that lemma to be surjective, it is enough that the extension L/K be separable; but this case is all that is needed for the following.

THEOREM 2.5. *The existentially closed models of* DF^m *are just the differential fields* $(K, D_0, \dots, D_{m-1})$ *such that:*

(0) *K is separably closed;*
(1) *$(K, D_0, \dots, D_{m-1})$ is differentially perfect;*
(2) *the span E over K of the derivations D_i has dimension m;*
(3) *for any finitely generated extension L of K, every integrable subspace W of $\Omega^1_{L/E}$ is eliminable.*

The last condition can be weakened by requiring W to have, modulo $E^* \otimes_K L$*, a basis of the form $(\mathrm{d}\, X^k : k < r)$, where $(X^k : k < r)$ is independent in L/K.*

Proof. Except for the weakening of Condition (3), the argument of [17] remains correct in arbitrary characteristic, provided that we make the terminological corrections just noted. (Also, on [17, p. 940, l. −1], the word 'integrable' should be 'eliminable'.)

In the original argument that, with the other conditions, the weak form of (3) is sufficient, an integrable subspace W of some $\Omega^1_{L/E}$ is found. Here $L = K(\mathbf{a}, \mathbf{b})$;—rather, $L = K(A \cup B)$ for some finite sets A and B, and $W = \langle \mathrm{d}\, a - \theta^a : a \in A \rangle^L$ for some θ^a in $E^* \otimes_K L$. Then L/K is separable, by [17, Lemma 4.12] and Lemma 2.3. Suppose $\operatorname{char} K = p$. If $a \in L^p \cap A$, then $\theta^a = 0$. Let $W' = \langle \mathrm{d}\, a - \theta^a : a \in A \smallsetminus L^p \rangle^L$. Since W is integrable, so is W'; also, if W' is eliminable, then so is W. Now, A has a p-independent subset C such that $W' = \langle \mathrm{d}\, a - \theta^a : a \in C \rangle^L$, by Lemma 1.3. So the weak form of Condition (3) is enough in general. □

Let (K, E) be a differential field such that $\dim E = m$, and let V be a variety over K. A rational map from V to $\mathbb{A}^m$ over K is an element of $\mathbb{A}^m(K(V))$. This space can also be written $\mathbb{A}^m(K) \otimes_K K(V)$. Now, $\mathbb{A}^m(K) \cong_K E^*$ as vector-spaces. Let us say that the elements of $E^* \otimes_K K(V)$ are the **rational maps** from V to E^*. If V is separable, then this space embeds in $\Omega^1_{K(V)/E}$. Then [17, Theorem 5.7] becomes the following.

THEOREM 2.6. *Let (K, E) be a differential field, and let V be a variety over K. Suppose $\mathbf{x}$ is a dominant separable rational map from V to $\mathbb{A}^r$, and $\mathbf{y}$ is an r-tuple of rational maps from V to E^*. Let W be the subspace of $\Omega^1_{K(V)/E}$ spanned by the forms $y^i - \mathrm{d}\, x^i$. Then the following are equivalent:*

(0) *W is integrable, that is, the differential field (K, E) has an extension in which $\mathrm{d}\, x^i = y^i$ in each case.*
(1) *The subspace $\mathrm{d}\, W$ of $\Omega^2_{K(V)/E}$ is linearly disjoint from $\mathrm{A}^2(E) \otimes_K K(V)$.*

Proof. The tuple $\mathbf{x}$ is an initial segment of a basis $(x^k : k < n)$ of $K(V)/K$. With the x^k in place of the X^k, the argument of [17, Theorem 5.7] now goes through. Condition (3) there is equivalent to Condition (1) above, by [17, Lemma 4.8]. □

In the lemma, let us denote $\mathrm{d}\,W$ by $\mathrm{d}\,\mathbf{y}/\,\mathrm{d}\,\mathbf{x}$. Then Condition (1) is that $\mathrm{d}\,\mathbf{y}/\,\mathrm{d}\,\mathbf{x}$ contains no non-trivial rational map from V to $\mathrm{A}^2(E)$.

To Theorem 2.5, we now have:

Corollary 2.7. *The existentially closed models of* DF^m *are just the differential fields* $(K, D_0, \dots, D_{m-1})$ *such that:*

(0) *K is separably closed;*
(1) *$(K, D_0, \dots, D_{m-1})$ is differentially perfect;*
(2) *the span E over K of the derivations D_i has dimension m;*
(3) *for every variety V over K, if there are rational maps*

$$\begin{array}{ccc} V & & \\ {\scriptstyle\phi}\downarrow & \searrow^{\psi} & \\ \mathbb{A}^n & & (E^*)^n \end{array}$$

where ϕ is dominant and separable, then V has a K-rational point P such that $\mathrm{d}\circ\phi(P) = \psi(P)$, provided *that $\mathrm{d}\,\psi/\,\mathrm{d}\,\phi$ does not contain a non-trivial rational map from V to $\mathrm{A}^2(E)$.*

As the conditions are first-order, DF^m *is companionable.*

Proof. The maps ϕ and ψ correspond to $\mathbf{x}$ and $\mathbf{y}$ in Theorem 2.6. □

Remark 2.8. An element of $\mathrm{A}^q(E) \otimes_K K(V)$ induces, for each field L that includes K, a partial map from $V(L)$ to $\mathrm{A}^q(E) \otimes_K L$. More generally, an element of $\Omega^q_{K(V)/E}$ can be written as $\theta(\mathbf{b})$, where $\mathbf{b}$ is a generic point of V; by [17, Lemma 5.4], if $\mathbf{a} \in V(L)$, then we have a partial map

$$\theta(\mathbf{b}) \longmapsto \theta(\mathbf{a}) : \Omega^q_{K(V)/E} \dashrightarrow \Omega^q_{L/E}.$$

So a particular form $\theta(\mathbf{b})$ induces

$$\mathbf{a} \longmapsto \theta(\mathbf{a}) : V(L) \dashrightarrow \Omega^q_{L/E}.$$

As we can consider $\mathbb{A}^m$ as a functor $L \mapsto \mathbb{A}^m(L)$ from the category of fields that include K (with inclusions) to the category of vector-spaces (with inclusions), so we have a functor $L \mapsto \Omega^q_{L/E}$, which we might denote Ω^q_E. But this is not a variety, and the map $\mathbf{a} \mapsto \theta(\mathbf{a})$ is not generally a rational map from V to Ω^q_E. Indeed, the map $\mathbf{a} \mapsto \theta(\mathbf{a})$ generally involves differentiation, as when θ is $\mathrm{d}\,f$ for some non-constant f in $K(V)$: then $\theta(\mathbf{a}) = \mathrm{d}(f(\mathbf{a}))$.

3. Difference-fields

If σ is an endomorphism of the field K, then: (0) σ extends to the algebraic closure of K; (1) σ extends to a field of which it is an *auto*morphism; (2) if $\{\alpha^0, \dots, \alpha^{d-1}\}$ is algebraically independent over K, then σ extends uniquely to $K(\alpha^0, \dots, \alpha^{d-1})$ after *algebraically independent* choices are made for the $\sigma\alpha^i$. As mentioned in Section 0, these give us the following, a slight simplification of a known result:

Theorem 3.1. *The difference-field (K, σ) is existentially closed just in case the following hold:*

(0) *K is algebraically closed.*
(1) *σ is surjective.*
(2) *If V and W are irreducible varieties over K for which there are dominant rational maps*

$$\begin{array}{ccc} V & & \\ {\scriptstyle\phi}\big\downarrow & \searrow{\scriptstyle\psi} & \\ W & & W^{\sigma} \end{array}$$

then V has a K-rational point P such that $\phi(P)^\sigma = \psi(P)$.

Proof. For the necessity of Condition (2), let $\mathbf{a}$ and $\mathbf{c}$ be generic points of W and W^σ; then as in [13, § 1.5, Lemma 5], we can extend σ to an isomorphism from $K(\mathbf{a})$ to $K(\mathbf{c})$, which extends further to an automorphism of a field that includes $K(V)$.

For the sufficiency of the conditions, follow the pattern of the proof of Theorem 1.6. Every primitive sentence over a difference-field (K, σ) says that a system

$$(\|) \qquad \bigwedge_f f = 0 \wedge \bigwedge_{i<k} (X^i)^\sigma = g^i$$

has a solution, where the f and the g^i are in $K[X^0, \dots, X^{r-1}]$, and $k \leqslant r$. Suppose the system $(\|)$ has a solution $\mathbf{b}$. Let V have generic point $\mathbf{b}$ over K, and let W have generic point $(b^i : i < k)$. By Condition (1), we have that $(g^i(\mathbf{b}) : i < k)$ is a generic point of W^σ; so we can apply Condition (2), letting ϕ be $\mathbf{x} \mapsto (x^i : i < k)$, and letting ψ be $\mathbf{x} \mapsto (g^i(\mathbf{x}) : i < k)$. $\square$

In the original geometric treatment, Condition (2) is weakened by the further hypothesis that ϕ and ψ are the projections from a sub-variety of $W \times W^\sigma$. The weakened condition is still sufficient, since, in the proof, for the system $(\|)$, one may assume that $r = 2k$, and each g^i is X^{k+i}.

The same assumption can be made for the system (¶) in the proof of Theorem 1.6; then one is led to the axioms in [18]. A similar assumption could be made in the presence of several derivations, but this was not fruitful in the search for Corollary 2.7.

We have a map $f \mapsto f^\sigma - f : K[\mathbf{X}] \to K[\mathbf{X}, \mathbf{X}^\sigma]$. We can write the co-domain as a quotient

$$K[\mathbf{X}, \mathbf{X}^\sigma, D\mathbf{X}]/(D\mathbf{X} - \mathbf{X}^\sigma + \mathbf{X}),$$

or as $K[\mathbf{X}, D\mathbf{X}]$; in the latter case, we can write $f^\sigma - f$ as Df. For a variety V over K, we can define $D(V)$ by analogy with $\delta(V)$. Then we have an isomorphism

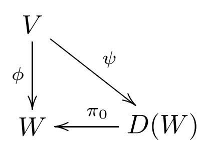
$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{x} + \mathbf{y}) : D(V) \to V \times V^\sigma.$$

Let ρ be the composition of this with the projection onto V^σ, so $\rho(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y}$. As with $\delta(V)$, let π_0 be the projection of $D(V)$ onto V. We can now recast Theorem 3.1:

Corollary 3.2. *In Theorem* 3.1*, we can replace Condition* (2) *with:*

(3) *If $\phi : V \to W$ is a dominant rational map of varieties over K, and $\psi : V \to D(W)$ is a rational map such that $\rho \circ \psi$ is dominant and $\pi_0 \circ \psi = \phi$, then V has a K-rational point P such that $\psi(P) = (\phi(P), D \circ \phi(P))$.*

Thus, both Corollaries 1.9 and 3.2 concern a commutative diagram



where ϕ is dominant. In the former case, where D is a derivation, the map ψ should have a section of the projection of $D(V)$ as a factor. In the latter case, $\rho \circ \psi$ should be dominant.

4. *D*-fields

As in Section 0, we can equip any D-field (K, e, D) with the map $x \mapsto x + e \cdot Dx$, which is an endomorphism σ of K. As mentioned in [20, Remark 2.6], we can now describe D as an additive map satisfying the identity

$$(**) \qquad D(x \cdot y) = Dx \cdot y + x^\sigma \cdot Dy.$$

Let an **operator-field** be a structure (K, σ, D, δ), where σ and δ are respectively an endomorphism and a derivation of K, and both $(**)$ and

$$(\dagger\dagger) \qquad \delta x + x^\sigma = x + Dx$$

are identities. Then for any endomorphism σ and derivation δ of K, the structures $(K, \sigma, \sigma - \mathrm{id}_K, 0)$ and $(K, \mathrm{id}_K, \delta, \delta)$ are operator-fields. In fact, these are the only possibilities:

THEOREM 4.1. *Suppose (K, σ, D, δ) is an operator-field. Then either $\sigma = \mathrm{id}_K$ and $D = \delta$, or $\delta = 0$ and $D = \sigma - \mathrm{id}_K$.*

Proof. From $(**)$, since $xy = yx$, we get

$$Dx \cdot y + x^\sigma \cdot Dy = D(x \cdot y) = Dy \cdot x + y^\sigma \cdot Dx.$$

Therefore

$$(x^\sigma - x) \cdot Dy = (y^\sigma - y) \cdot Dx,$$

that is, D and $\sigma - \mathrm{id}_K$ are linearly dependent. By (††) then, δ and $\sigma - \mathrm{id}_K$ are linearly dependent. So either $\delta = 0$, or $e \cdot \delta = \sigma - \mathrm{id}_K$ for some non-zero e. In the latter case, (K, e, δ) is a D-field, and as we have $(**)$, so we have the identity

$$\delta(x \cdot y) = \delta x \cdot y + x^\sigma \cdot \delta y;$$

this holds trivially if $\delta = 0$. Since δ is also a derivation, we have $\delta(x \cdot y) = \delta x \cdot y + x \cdot \delta y$, so

$$(x^\sigma - x) \cdot \delta y = 0, \tag{‡‡}$$

that is, either $\sigma = \mathrm{id}_K$ or $\delta = 0$. The remainder follows from (††). □

Note then that the name 'operator-field' is not ideal, since it doesn't cover fields with derivations of a power of the Frobenius map.

Let OF be the theory of operator-fields, and let θ be the sentence $\forall x\ x^\sigma = x$. Then OF has a model-companion, OF^*, whose axioms are:

$$\mathrm{OF} \cup \{\neg\theta \to \gamma : \gamma \in \mathrm{ACFA}\} \cup \{\theta \to \gamma : \gamma \in \mathrm{DCF}\}.$$

Towards a more uniform axiomatization, let (K, σ, D, δ) be an operator-field in which $D \neq 0$, and let V be a variety over K. We can define

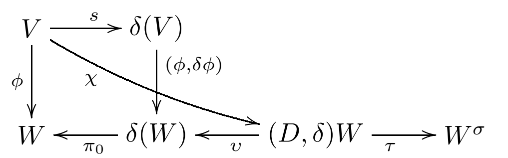
$$(D, \delta)V = \begin{cases} \{(\mathbf{x}, \mathbf{y}, \mathbf{0}) : (\mathbf{x}, \mathbf{y}) \in D(V)\}, & \text{if } \delta = 0 \text{ on } K; \\ \{(\mathbf{x}, \mathbf{y}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in D(V)\}, & \text{if } \delta = D \text{ on } K. \end{cases}$$

This is not uniform either; but we can also define $(D, \delta)V$ as the zero-set of the polynomials f, Df, δf and $\delta g \cdot (Dh - \delta h)$ in $K[\mathbf{X}, D\mathbf{X}, \delta\mathbf{X}]$, where the polynomials f define V, and the g and h are from $K \cup \mathbf{X}$. We have a map τ from $(D, \delta)V$ to V^σ taking $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ to $\mathbf{x} + \mathbf{y} - \mathbf{z}$. We also have a map $\upsilon : (D, \delta)V \to \delta(V)$ taking $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ to $(\mathbf{x}, \mathbf{z})$.

THEOREM 4.2. *The existentially closed models of* OF *are just those models (K, D, δ, σ) such that the following conditions hold:*

(0) *D is non-trivial.*
(1) *K is separably closed.*
(2) *(K, δ) is perfect.*
(3) *σ is surjective.*

(4) *Suppose* $\phi : V \to W$ *and* $\chi : V \to (D,\delta)W$ *are rational maps of varieties over* K*, and* s *is a section of* $\pi_0 : \delta(V) \to V$*, such that the diagram*



commutes and ϕ *and* $\tau \circ \chi$ *are dominant. Then* V *has a* K*-rational point* P *such that* $\chi(P) = (\phi(P), D \circ \phi(P), \delta \circ \phi(P))$.

Proof. The claim follows from Corollaries 1.9 and 3.2. Consider in particular the diagram in Condition (4).

Suppose first that $\delta = 0$ on K. Then χ is $\mathbf{x} \mapsto (\psi(\mathbf{x}), \mathbf{0})$ for some $\psi : V \to D(W)$, and then $\tau \circ \chi$ is $\rho \circ \psi$. Also, υ is $(\mathbf{x}, \mathbf{y}, \mathbf{0}) \mapsto (\mathbf{x}, \mathbf{0})$. So the condition of Corollary 3.2 is satisfied. Also, if this condition is satisfied, then we can let s be $\mathbf{x} \mapsto (\mathbf{x}, \mathbf{0})$, so that the present Condition (4) is satisfied.

Now suppose instead $\delta = D$. Then χ is $(\psi, \pi_1 \circ \psi)$, where ψ is $\upsilon \circ \chi$. Hence $\tau \circ \chi$ is ϕ.

In each case then, Condition (4) is equivalent to the corresponding condition in the respective Corollary. □

5. Two operators together

From the theory OF, if we remove the connection between δ and σ given by (‡‡), then we lose companionability. Let us say that a structure (K, δ, σ) is a **differential and difference-field** if (K, δ) is a differential field, and (K, σ) is a difference-field. These structures compose an elementary class, say with theory DDF. A required characteristic can be indicated, as usual, by a subscript. If $(K, \delta, \sigma) \models \mathrm{DDF}_p$, then $K^p \subseteq \ker(\delta \circ \sigma^n)$ for each n in ω.

Lemma 5.1. *Suppose* (K, δ, σ) *is an existentially closed model of* DDF*. Then:*

(0) K *is separably closed;*
(1) $\bigcap_{n\in\omega} \ker(\delta \circ \sigma^n) \subseteq K^p$*, if* $\operatorname{char} K = p$*;*
(2) K/K^σ *is purely inseparable.*

Proof. Condition (0) is necessary by Theorems 1.6 and 3.1. For the necessity of (1), suppose $\alpha \in \bigcap_{n\in\omega} \ker(\delta \circ \sigma^n) \smallsetminus K^p$, and let $L = K((\alpha^{\sigma^n})^{p^{-1}} : n \in \omega)$. Then δ extends to L by Lemma 1.4 (1), and σ extends to L so that $((\alpha^{\sigma^n})^{p^{-1}})^\sigma = (\alpha^{\sigma^{n+1}})^{p^{-1}}$. For (2), suppose β in K is separable over K^σ. If β is algebraic over K^σ, with minimal polynomial f^σ, then the roots of f are in K by Condition (0), and β is the image under σ of one of them. If β is transcendental over K^σ, then let α be transcendental over K; we can extend δ and σ to $K(\alpha)$ by defining $\delta\alpha = 0$ and $\alpha^\sigma = \beta$. □

For any prime p, let Γ_p be the type

$$\{p \cdot 1 = 0\} \cup \{\delta(x^{\sigma^n}) = 0 : n \in \omega\} \cup \{\forall y\ y^p \neq x\}.$$

Then Condition (1) in Lemma 5.1 is just that (K, δ, σ) omits each Γ_p. Let Δ_p be the type

$$\{p \cdot 1 = 0\} \cup \{\forall y\ y^\sigma \neq x^{p^n} : n \in \omega\}.$$

Condition (2) in the lemma is that each Δ_p is omitted.

THEOREM 5.2. *No definitional expansion of* DDF_p *is companionable.*

Proof. Let $K = \mathbb{F}_p(X_n : n \in \omega)$, and let σ be the endomorphism $X_n \mapsto X_{n+1}$. For any k in ω, a derivation δ_k of K can be defined by

$$\delta_k X_n = \begin{cases} 1, & \text{if } k \leqslant n; \\ 0, & \text{if } n < k. \end{cases}$$

Suppose if possible that T is a definitional expansion of DDF_p with a model-companion T^*. Each structure (K, δ_k, σ) expands to a model of T; this model has an extension $\mathfrak{M}_k$ that is a model of T^*. Writing X_0 as X, we have $X \notin M_k^p$, since $\delta_k(X^{\sigma^k}) \neq 0$; but for each n, for almost all k, we have $\delta_k(X^{\sigma^n}) = 0$. Hence, in a non-principal ultra-product $\mathfrak{N}$ of the structures $\mathfrak{M}_k$, we have $X \notin N^p$, although $\delta(X^{\sigma^n}) = 0$ for all n; so X realizes Γ_p in $\mathfrak{N}$. But the reduct of $\mathfrak{N}$ to the signature of DDF is an existentially closed model of this theory, contradicting Lemma 5.1. □

We can argue similarly using Δ_p: On $\mathbb{F}_p(X)$, let σ be $x \mapsto x^p$, and let δ be $f \mapsto f'$. Say $(\mathbb{F}_p(X), \delta, \sigma^n) \subseteq (K_n, \delta_n, \sigma_n)$. Then $X^{p^{-1}} \notin K_n$, since $\delta X = 1$, so $\{X^{p^{-k-1}} : k < n\} \cap K_n = \varnothing$; therefore $\{X^{p^k} : k < n\} \cap K_n^{\sigma_n} = \varnothing$. Hence X realizes Δ_p in a non-principal ultra-product of the $(K_n, \delta_n, \sigma_n)$.

No definitional expansion of $\mathrm{DDF}_p \cup \{\forall x\ \exists y\ y^\sigma = x\}$ is companionable either. The changes needed in the argument are that, in Lemma 5.1, in Condition (1), the intersection should be over n in $\mathbb{Z}$; and K in the proof of Theorem 5.2 should be $\mathbb{F}_p(X_n : n \in \mathbb{Z})$.

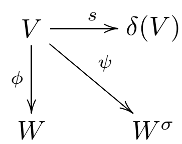
There is no problem in characteristic 0:

THEOREM 5.3. *A model* (K, δ, σ) *of* DDF_0 *is existentially closed just in case the following conditions hold:*

(0) *K is algebraically closed.*

(1) *σ is surjective.*

(2) *For all varieties V and W over K, if there are rational maps*



where ϕ and ψ are dominant, and s is a section of π_0, then V contains a K-rational point P such that $s(P) = (P, \delta P)$ and $\phi(P)^\sigma = \psi(P)$.

Proof. The necessity of the conditions is by Lemma 5.1 and because, in Condition (2), the variety V has a generic point with the desired property. For the sufficiency of (2), note that every primitive sentence over (K, δ, σ) can be written as the statement that a system

$$\bigwedge_f f = 0 \wedge \bigwedge_{i<k} ((X^i)^\sigma = g^i \wedge \delta X^i = h^i)$$

has a solution. Now follow the proofs of Theorems 1.8 and 3.1. □

The theory $\mathrm{OF} \cup \{p \cdot 1 \neq 0 : p \text{ prime}\}$ is the theory of fields of characteristic zero with a jet-operator; its model-companion is $\mathrm{OF}^* \cup \{p \cdot 1 \neq 0 : p \text{ prime}\}$. Because of the derivations of Frobenius, there is no corresponding theory of fields of characteristic p with a jet-operator. However, we can look at the structures (K, δ, σ) where (K, σ) is a difference-field, and δ is an additive map such that Equation (†) of Section 0 is an identity. Then these structures satisfy:

$$\forall x \; \delta(x^{n+1}) = (n+1)(x^\sigma)^n \delta x,$$
$$\forall x \; \forall y \; (x \cdot y = 1 \rightarrow (x^\sigma)^2 \cdot \delta y = -\delta x);$$

In particular, when defined on a domain, δ extends uniquely to the quotient-field. Moreover, the formula of Fact 1.1 (0) becomes:

$$\delta(f(\mathbf{a})) = \sum_{j<n} (\partial_j f(\mathbf{a}))^\sigma \cdot \delta a^j + f^\delta(\mathbf{a}).$$

All of this is noted in [11] in case σ is a power of $x \mapsto x^p$. The arguments of the present section go through to show that the theory of these structures is also not companionable, even if σ is surjective.

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