

## ALMOST HOMOMORPHISMS OF COMPACT GROUPS

JÁN ŠPAKULA AND PAVOL ZLATOŠ

ABSTRACT. A continuous mapping between compact topological groups which is “almost” a homomorphism need not be uniformly close to a homomorphism. The distance of the almost homomorphism from a strict homomorphism can be shown to depend not just on the degree of its nonhomomorphy but also on some “continuity scale.”

### Introduction

Let  $G, H$  be groups, the latter endowed with a (left) invariant metric  $\varrho$ , and  $\varepsilon$  be a positive real number. A mapping  $f : G \rightarrow H$  is called an  $\varepsilon$ -homomorphism if  $\varrho(f(x)f(y), f(xy)) \leq \varepsilon$  for all  $x, y \in G$ . Two mappings  $f, g : G \rightarrow H$  are said to be  $\varepsilon$ -close if  $\varrho(f(x), g(x)) \leq \varepsilon$  for each  $x \in G$ . More generally, if  $H$  is a topological group and  $V$  is a neighborhood of the unit element in  $H$ , then  $f : G \rightarrow H$  is called a  $V$ -homomorphism if  $f(x)f(y)f(xy)^{-1} \in V$  for all  $x, y \in G$ . Two mappings  $f, g : G \rightarrow H$  are said to be  $V$ -close if  $f(x)g(x)^{-1} \in V$  for each  $x \in G$ .

In this paper we examine some conditions under which a continuous  $\delta$ -homomorphism  $f : G \rightarrow H$  of compact metrizable groups is  $\varepsilon$ -close to a continuous homomorphism  $\varphi : G \rightarrow H$ , as well as their topological generalization. Let us remark that we include the separation axiom  $T_1$  into the definition of a topological group.

The origins of this problem can be traced back to Ulam: it is discussed in the Scottish Book [12, p. 11], in terms of “epsilon stability” of functions  $\mathbb{R} \rightarrow \mathbb{R}$ , as well as of functions on metrizable groups.

A theorem by Pólya and Szegő, occurring as an exercise in [13, Problem 99], can easily be restated as follows: If  $\varepsilon > 0$  and  $f : \mathbb{Z} \rightarrow \mathbb{R}$  is an  $\varepsilon$ -homomorphism, then the limit  $a = \lim_{n \rightarrow \infty} f(n)/n$  exists and  $f$  is  $\varepsilon$ -close to the homomorphism  $\varphi : \mathbb{Z} \rightarrow \mathbb{R}$  given by  $\varphi(n) = an$ . Using methods of nonstandard analysis, this result was generalized by Luxemburg [11], altering the  $\varepsilon$ -homomorphy and  $\varepsilon$ -closeness conditions.

---

Received December 15, 2003; received in final form December 9, 2004.  
2000 *Mathematics Subject Classification*. Primary 22C05. Secondary 39B52, 41A65.  
Research supported by the VEGA grant 1/0261/03.

Partly as a byproduct and partly as one of the tools of their studies in differential geometry, Grove, Karcher and Ruh established that, for  $0 < \delta \leq \pi/6$ , every continuous  $\delta$ -homomorphism of compact Lie groups is  $\varepsilon$ -close to a continuous homomorphism, where  $\varepsilon = (1 + \delta/2 + 0.64\delta^3)\delta < 2\delta$ . This is proved for an appropriately chosen biinvariant Riemannian metric in [5] and modified to Finsler metric in [6].

A fairly general result, covering all the above mentioned cases, was proved by Kazhdan in [9]: If  $G$  is an amenable group and  $H = U(X)$  is the group of all unitary operators on some Hilbert space  $X$  with the usual operator norm, then any continuous  $(\varepsilon/2)$ -homomorphism  $f : G \rightarrow H$ , where  $0 < \varepsilon < 1/100$ , is  $\varepsilon$ -close to some continuous homomorphism  $\varphi : G \rightarrow H$ . A more elementary proof of this result, working for any amenable group  $G$  and finite dimensional compact Lie group  $H$ , was given by Alekseev, Glebskii and Gordon in [1].

A different direction in examining this topic was taken by Kanovei and Reeken in [8]. Here  $G$  is assumed to be a Borel group endowed with a  $\sigma$ -additive probability measure  $\mu$  and  $H = \prod_{n \in \mathbb{N}} H_n$  is the direct product of a family of Borel groups  $H_n$ . An invariant Hamming-like pseudometric on  $H$  stems from a submeasure  $\nu$  on  $\mathbb{N}$ :

$$d_\nu(x, y) = \nu\{i \in \mathbb{N}; x_i \neq y_i\},$$

for  $x, y \in H$ . It is shown that, for any  $\varepsilon > 0$  and  $\nu$  satisfying certain version of Fubini theorem, every  $\mu$ -measurable (Baire measurable) mapping  $f : G \rightarrow H$  which is almost everywhere an  $\varepsilon$ -homomorphism already is  $6\varepsilon$ -close ( $63\varepsilon$ -close) to a Borel homomorphism  $\varphi : G \rightarrow H$ .

The above mentioned results are remarkable for the explicit assessment of the bound  $\varepsilon$  on the distance of the  $\delta$ -homomorphism to the nearest homomorphism, expressed in terms of  $\delta$ . This was enabled by some additional structure, entering the picture case by case. However, one cannot hope to establish any explicit relation between  $\delta$  and  $\varepsilon$  in general, not even for compact groups  $G, H$ . Nevertheless, it is still tempting to formulate the following

**CONJECTURE.** *Let  $G$  and  $H$  be compact groups, the topology of  $H$  being induced by a (left) invariant metric. Then to each  $\varepsilon > 0$  there is a  $\delta > 0$ , such that every continuous  $\delta$ -homomorphism  $f : G \rightarrow H$  is  $\varepsilon$ -close to some continuous homomorphism  $\varphi : G \rightarrow H$ .*

Assuming additionally that  $G$  is finite, the Conjecture can easily be proved using, e.g., methods of nonstandard analysis. Anderson in [2] developed a general nonstandard machinery enabling him to prove, under some compactness assumptions, that objects almost having some property are close to objects with that property. The special case of the Conjecture can also be obtained as one of the immediate corollaries. It directly follows from the Corollary to our Theorem 2, as well.

According to a remark in Kazhdan [9], “by similar methods the result of (his) Theorem 1 for compact groups was obtained in [6].” As we understand it, he seems to have ascribed to Grove *et al.* (at least) the proof of the above Conjecture. However, they proved it just for compact *Lie* groups and not for all compact groups. Moreover, the Conjecture, as it stands, is not true. This is rather surprising, especially in view of the fact that the “material” of which we will put together the counterexample is taken from [9], again.

To get some positive results, we introduce the notion of “continuity scale.” Then the Conjecture can be retained, by showing that  $\delta$  can be chosen depending not just on  $\varepsilon$  but also on some continuity scale  $\Gamma$ , given in advance.

### 1. The counterexample

A counterexample to the Conjecture is provided by the following

**THEOREM 1.** *There exist compact metric abelian groups  $G$ ,  $H$ , the latter endowed with an invariant metric  $\varrho$ , such that for each  $\delta > 0$  there is a continuous  $\delta$ -homomorphism  $f : G \rightarrow H$ , satisfying*

$$\sup_{x \in G} \varrho(f(x), \varphi(x)) \geq 1$$

for every homomorphism  $\varphi : G \rightarrow H$ .

The proof is based on the following result due to Kazhdan [9]. We quote it slightly reformulated and with complete proof here.

**LEMMA 1.** *There is a compact metric abelian group  $H$ , endowed with an invariant metric  $\varrho$ , such that for each  $\delta > 0$  there is a compact metric abelian group  $G$  and a continuous  $\delta$ -homomorphism  $g : G \rightarrow H$ , satisfying*

$$\sup_{x \in G} \varrho(g(x), \varphi(x)) \geq 1$$

for every homomorphism  $\varphi : G \rightarrow H$ .

*Proof.* Let  $p$  be any prime number and  $H$  be the (compact abelian) group  $\mathbb{Z}_p$  of  $p$ -adic integers with its usual norm. I.e.,  $\mathbb{Z}_p$  is the completion of the ring  $\mathbb{Z}$  of all integers with respect to the norm

$$|x|_p = p^{-o_p(x)},$$

where  $p^{o_p(x)}$  is the highest (integer) power of  $p$  dividing  $x \in \mathbb{Z}$ ; for  $x = 0$  we put  $o_p(0) = \infty$  and  $|0|_p = 0$ .

Given  $\delta > 0$ , choose an  $n$  such that  $p^{-n} < \delta$  and denote by  $G$  the finite discrete (hence compact) cyclic group  $\mathbb{Z}/p^n\mathbb{Z}$ . Then the (continuous) mapping  $g : G \rightarrow H$ , where  $g(x) \in \mathbb{Z} \subseteq \mathbb{Z}_p$  is the representative of  $x \in \{0, 1, \dots, p^n - 1\}$  under the obvious identification of the integers  $\mathbb{Z}$  with a subgroup of  $\mathbb{Z}_p$ , is

a  $p^{-n}$ -homomorphism. Indeed, for any  $x, y \in G$  the expression  $g(x) + g(y) - g(x + y)$  equals either 0 or  $p^n$ , hence

$$|g(x) + g(y) - g(x + y)|_p \leq |p^n|_p = p^{-n} \leq \delta.$$

However, as  $\mathbb{Z}_p$  has no torsion, the only homomorphism  $\mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}_p$  is the trivial one  $\theta \equiv 0$ . Hence

$$\sup_{x \in G} |g(x) - \theta(x)|_p = |1 - 0|_p = 1. \quad \square$$

*Proof of Theorem 1.* Let  $H$  be guaranteed by Lemma 1 and  $(\delta_n)_{n=1}^\infty$  be any sequence of positive reals, converging to 0. By Lemma 1, one can find a sequence  $(G_n)$  of compact metric abelian groups and a sequence of mappings  $(g_n)$ , such that each  $g_n : G_n \rightarrow H$  is a continuous  $\delta_n$ -homomorphism; nonetheless, every homomorphism  $\varphi : G_n \rightarrow H$  satisfies  $\sup_{x \in G_n} \varrho(g_n(x), \varphi(x)) \geq 1$ . Then  $G = \prod_{n=1}^\infty G_n$ , endowed with the product topology, is a compact metrizable abelian group. Denote by  $\pi_n : G \rightarrow G_n$  the canonical projection onto the  $n$ th factor and by  $\iota_n : G_n \rightarrow G$  its canonical embedding into the product. One can readily see that the composition  $f_n = g_n \circ \pi_n : G \rightarrow H$  is a continuous  $\delta_n$ -homomorphism. Now, given any homomorphism  $\psi : G \rightarrow H$  we have

$$\sup_{u \in G} \varrho(f_n(u), \psi(u)) \geq \sup_{u \in \iota_n G_n} \varrho(g_n(\pi_n u), \psi(u)) = \sup_{x \in G_n} \varrho(g_n(x), \psi(\iota_n x)) \geq 1,$$

as  $\psi \circ \iota_n : G_n \rightarrow H$  obviously is a homomorphism. □

### 2. The “almost-near” theorem

Thus we have seen that the  $\delta > 0$ , called for in the Conjecture, cannot be chosen depending just on given compact groups  $G, H$ , and  $\varepsilon > 0$ . To get some positive results, we introduce an additional parameter to control the continuity of the  $\delta$ -homomorphisms considered.

For a topological group  $G$  we denote by  $\mathcal{O}_G$  the filter of all neighborhoods of its unit element. Given two topological groups  $G, H$ , and a neighborhood basis  $\mathcal{B} \subseteq \mathcal{O}_H$ , any mapping  $\Gamma : \mathcal{B} \rightarrow \mathcal{O}_G$  will be called a  $(G, H)$  continuity scale. Then a mapping  $f : G \rightarrow H$  is said to be  $\Gamma$ -continuous if  $xy^{-1} \in \Gamma(V)$  implies  $f(x)f(y)^{-1} \in V$  for all  $V \in \mathcal{B}$  and  $x, y \in G$ .

For a metrizable  $H$ , a neighborhood basis  $\mathcal{B} \subseteq \mathcal{O}_H$  can be represented by a sequence  $(\varepsilon_n)$  of positive reals converging to 0. Then a  $(G, H)$  continuity scale  $\Gamma$  is described by a sequence  $(\Gamma_n)$  of neighborhoods in  $G$ , and  $\Gamma$ -continuity of a mapping  $f : G \rightarrow H$  simply means that  $xy^{-1} \in \Gamma_n$  implies  $\varrho(f(x), f(y)) < \varepsilon_n$  for  $x, y \in G$  and each  $n$ .

Obviously, if  $f : G \rightarrow H$  is  $\Gamma$ -continuous with respect to some  $(G, H)$  continuity scale  $\Gamma$  then it is (uniformly) continuous. Conversely, assuming the axiom of choice (in case of metrizable  $H$ , countable choice would suffice) one can readily show that any uniformly continuous mapping  $f : G \rightarrow H$  is  $\Gamma$ -continuous with respect to some  $(G, H)$  continuity scale  $\Gamma$ . The point is

that a set of mappings  $G \rightarrow H$ ,  $\Gamma$ -continuous with respect to some  $(G, H)$  continuity scale  $\Gamma$ , already is *equicontinuous*. Again, the reversed implication can be shown using the axiom of choice.

**THEOREM 2.** *Let  $G, H$  be compact topological groups. Then for each  $V \in \mathcal{O}_H$  and any  $(G, H)$  continuity scale  $\Gamma$  there is a  $U \in \mathcal{O}_H$  such that every  $\Gamma$ -continuous  $U$ -homomorphism  $f : G \rightarrow H$  is  $V$ -close to a continuous homomorphism  $\varphi : G \rightarrow H$ .*

*Proof.* We will give two proofs of the theorem: first based on the Ascoli lemma, second making use of methods of nonstandard analysis—see, e.g., Davis [4], or Arkeryd, Cutland and Henson [3], mainly the parts [7] by Henson and [10] by Loeb. Both proofs start in the same way.

Let  $V \in \mathcal{O}_H$ ,  $\mathcal{B} \subseteq \mathcal{O}_H$  be a neighborhood basis of the unit in  $H$ , and  $\Gamma : \mathcal{B} \rightarrow \mathcal{O}_G$  be a  $(G, H)$  continuity scale. We can assume, without loss of generality, that  $\mathcal{B}$  consists of symmetric sets, only. Admit, to get a contradiction, that for each  $U \in \mathcal{B}$  there is a  $\Gamma$ -continuous  $U$ -homomorphism  $f_U : G \rightarrow H$ , such that each continuous homomorphism  $\varphi : G \rightarrow H$  satisfies  $f_U(x)\varphi(x)^{-1} \notin V$  for some  $x \in G$ .

*Standard continuation.* Let us denote by  $C_\Gamma$  the system of all  $\Gamma$ -continuous mappings  $f : G \rightarrow H$ . They obviously form an equicontinuous family. As  $H$  is compact, the set  $C_\Gamma(x) = \{f(x); f \in C_\Gamma\} \subseteq H$  is precompact for each  $x \in G$ . As  $G$  is compact, the compact-open topology on the space  $C(G, H)$  of all continuous functions  $G \rightarrow H$  coincides with the topology of uniform convergence on  $C(G, H)$ . By the Ascoli lemma,  $C_\Gamma$  is a precompact set in  $C(G, H)$  endowed with any of the mentioned topologies.

The system  $(f_U)_{U \in \mathcal{B}}$  forms a net over the (downward) directed poset  $(\mathcal{B}, \subseteq)$  in the precompact set  $C_\Gamma$ . Thus there is a neighborhood basis  $\mathcal{B}' \subseteq \mathcal{B}$  and a subnet  $(f_U)_{U \in \mathcal{B}'}$  uniformly converging to a function  $\varphi \in C(G, H)$ . This is to say that for any  $W \in \mathcal{O}_H$  there is a  $U_W \in \mathcal{B}'$  such that  $f_U(x)\varphi(x)^{-1} \in W$  holds for each  $x \in G$  whenever  $U \in \mathcal{B}'$  and  $U \subseteq U_W$ .

In order to check that  $\varphi$  is a group homomorphism, take any  $x, y \in G$  and a symmetric  $W \in \mathcal{O}_H$ . There is a  $U \in \mathcal{B}'$  such that  $U \subseteq U_W \cap W$ . Then the expression  $\varphi(x)\varphi(y)\varphi(xy)^{-1} \in H$  equals the product of the terms

$$\begin{aligned} \varphi(x)(\varphi(y)f_U(y)^{-1})\varphi(x)^{-1} &\in \varphi(x)W\varphi(x)^{-1}, \\ \varphi(x)f_U(x)^{-1} &\in W, \\ f_U(x)f_U(y)f_U(xy)^{-1} &\in U, \\ f_U(xy)\varphi(xy)^{-1} &\in W. \end{aligned}$$

Hence,  $\varphi(x)\varphi(y)\varphi(xy)^{-1} \in \varphi(x)W\varphi(x)^{-1}WUW$ . As such sets form a basis of  $\mathcal{O}_H$  and  $H$  is  $T_1$ , we have  $\varphi(x)\varphi(y) = \varphi(xy)$ .

Finally,  $f_U(x)\varphi(x)^{-1} \in V$  holds for each  $x \in G$  whenever  $U \in \mathcal{B}'$  and  $U \subseteq U_V$ . This contradiction proves the result.

*Nonstandard continuation.* Let us embed the situation into a  $\kappa^+$ -saturated nonstandard universe, where  $\kappa = \max(|\mathcal{B}|, \omega)$ . Then there is a  $*\Gamma$ -continuous internal mapping  $F : *G \rightarrow *H$ , which is a  $*U$ -homomorphism for each  $U \in \mathcal{B}$ , such that for every internal  $*\Gamma$ -continuous homomorphism  $\Phi : *G \rightarrow *H$  there is a  $\xi \in *G$  with  $F(\xi)\Phi(\xi)^{-1} \notin *V$ .

As  $F$  is a  $*U$ -homomorphism, we have  $F(\xi)F(\eta)F(\xi\eta)^{-1} \in *U$ , for each  $U \in \mathcal{B}$ . Hence  $F(\xi\eta) \approx F(\xi)F(\eta)$  for all  $\xi, \eta \in *G$ . Let  $\varphi : G \rightarrow H$  be defined by  $\varphi(x) = {}^\circ F(x)$ , where  $x \in G$  is identified with its image  $*x \in *G$  under the canonical elementary embedding  $*$  :  $G \rightarrow *G$ , and  ${}^\circ : *H \rightarrow H$  is the standard part map, existing by the compactness of  $H$ . Clearly,  $\varphi$  is a group homomorphism — see the diagram

$$\begin{array}{ccc} *G & \xrightarrow{F} & *H \\ * \uparrow & & \downarrow {}^\circ \\ G & \xrightarrow{\varphi} & H \end{array}$$

From the  $*\Gamma$ -continuity of  $F$  it easily follows that it is S-continuous, hence  $\varphi$  is continuous. Then the mapping  $*\varphi : *G \rightarrow *H$  is an internal S-continuous and  $*\Gamma$ -continuous group homomorphism, extending  $\varphi$ . Moreover, as  $G$  is compact, for each  $\xi \in *G$  we have  $\xi \approx {}^\circ \xi$ , hence

$$F(\xi) \approx F({}^\circ \xi) \approx {}^\circ F({}^\circ \xi) = \varphi({}^\circ \xi) = *\varphi({}^\circ \xi) \approx *\varphi(\xi),$$

contradicting  $F(\xi)*\varphi(\xi)^{-1} \notin *V$  for some  $\xi \in *G$ . □

**COROLLARY.** *Let  $G, H$  be compact topological groups such that the topology of  $H$  is induced by a (left) invariant metric  $\rho$ . Then for each  $\varepsilon > 0$  and any  $(G, H)$  continuity scale  $\Gamma$  there is a  $\delta > 0$  such that every  $\Gamma$ -continuous  $\delta$ -homomorphism  $f : G \rightarrow H$  is  $\varepsilon$ -close to a continuous homomorphism  $\varphi : G \rightarrow H$ .*

**REMARK 1.** It remains an open problem whether the compactness of the groups  $G, H$  can be weakened to local compactness for at least one of them in Theorem 2.

**REMARK 2.** Under a slight and obvious modification of the notions of almost homomorphism and continuity scale, Theorem 2 can readily be generalized from groups to arbitrary compact uniform algebras with finitely many operations, i.e., to universal algebras of finite signature endowed with a compact  $T_1$  uniformity making all the operations (uniformly) continuous.

## REFERENCES

- [1] M. A. Alekseev, L. Y. Glebskiĭ, and E. I. Gordon, *On approximations of groups, group actions and Hopf algebras*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **256** (1999), 224–262, 268. MR 1708567 (2000j:20050)
- [2] R. M. Anderson, “Almost” implies “near”, Trans. Amer. Math. Soc. **296** (1986), 229–237. MR 837809 (87h:03057)
- [3] L.O. Arkeryd, N.J. Cutland, and C.W. Henson (eds.), *Nonstandard analysis*, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, vol. 493, Kluwer Academic Publishers Group, Dordrecht, 1997. MR 1603227 (98i:03006)
- [4] M. Davis, *Applied nonstandard analysis*, Wiley-Interscience [John Wiley & Sons], New York, 1977. MR 0505473 (58 #21590)
- [5] K. Grove, H. Karcher, and E. A. Ruh, *Group actions and curvature*, Invent. Math. **23** (1974), 31–48. MR 0385750 (52 #6609)
- [6] ———, *Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems*, Math. Ann. **211** (1974), 7–21. MR 0355917 (50 #8391)
- [7] C. W. Henson, *Foundations of nonstandard analysis: a gentle introduction to non-standard extensions*, Nonstandard analysis, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 493, Kluwer Acad. Publ., Dordrecht, 1997, pp. 1–49. MR 1603228 (99i:03085)
- [8] V. Kanoveĭ and M. Reeken, *On Ulam’s problem concerning the stability of approximate homomorphisms*, Tr. Mat. Inst. Steklova **231** (2000), 249–283. MR 1841758 (2002d:03082)
- [9] D. Kazhdan, *On  $\varepsilon$ -representations*, Israel J. Math. **43** (1982), 315–323. MR 693352 (84h:22011)
- [10] P. A. Loeb, *Nonstandard analysis and topology*, Nonstandard analysis, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 493, Kluwer Acad. Publ., Dordrecht, 1997, pp. 77–89. MR 1603230
- [11] W. A. J. Luxemburg, *Non-standard analysis*, Mathematics Department, California Institute of Technology, Pasadena, Calif., 1973. MR 0491162 (58 #10428)
- [12] *The Scottish Book*, Birkhäuser Boston, Mass., 1981. MR 666400 (84m:00015)
- [13] G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Dover Publications, New York, N. Y., 1945. MR 0015435 (7,418e)

JÁN ŠPAKULA, DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, 1326 STEVENSON CENTER, NASHVILLE, TN 37240-0001, USA

*E-mail address:* jan.spakula@vanderbilt.edu

PAVOL ZLATOŠ, COMENIUS UNIVERSITY, FACULTY OF MATHEMATICS AND PHYSICS, 84215 BRATISLAVA, SLOVAK REPUBLIC

*E-mail address:* zlatos@fmph.uniba.sk