Illinois Journal of Mathematics
Volume 46, Number 4, Winter 2002, Pages 1233–1258
S 0019-2082

ON ANALYTIC AND MEROMORPHIC FUNCTIONS AND SPACES OF Q_K-TYPE

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ABSTRACT. Starting from a nondecreasing function $K : [0, \infty) \to [0, \infty)$, we introduce a Möbius-invariant Banach space Q_K of functions analytic in the unit disk in the plane. We develop a general theory of these spaces, which yields new results and also, for special choices of K, gives most basic properties of Q_p-spaces. We have found a general criterion on the kernels K_1 and K_2, $K_1 \le K_2$, such that $Q_{K_2} \subsetneqq Q_{K_1}$, as well as necessary and sufficient conditions on K so that $Q_K = \mathcal{B}$ or $Q_K = \mathcal{D}$, where the Bloch space $\mathcal{B}$ and the Dirichlet space $\mathcal{D}$ are the largest, respectively smallest, spaces of Q_K-type. We also consider the meromorphic counterpart $Q_K^\#$ of Q_K and discuss the differences between Q_K-spaces and $Q_K^\#$-classes.

1. Introduction

Let Δ be the unit disk in the complex plane, and let $dA(z)$ be the Euclidean area element on Δ. Let $H(\Delta)$ (resp. $M(\Delta)$) denote the class of functions that are analytic (resp. meromorphic) in Δ. The Green's function in Δ with singularity at $a \in \Delta$ is given by $g(z, a) = \log 1/|\varphi_a(z)|$, where $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is a Möbius transformation of Δ. For $0 < r < 1$, let $\Delta(a, r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudohyperbolic disk with center $a \in \Delta$ and radius r.

For $0 < p < \infty$, we define spaces Q_p and M_p by

$$Q_p = \left\{ f \in H(\Delta) : \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^2 (g(z, a))^p \, dA(z) < \infty \right\},$$

$$M_p = \left\{ f \in H(\Delta) : \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^2 (1 - |\varphi_a(z)|)^p \, dA(z) < \infty \right\}.$$

Received March 26, 2002; received in final form June 14, 2002.
2000 *Mathematics Subject Classification.* 30D45, 30D50.

We know that $Q_1 = BMOA$, the space of all analytic functions of bounded mean oscillation (cf. [11] and Theorem 5 in [7]). For each $p \in (1, \infty)$ the space Q_p is the *Bloch space* $\mathcal{B}$ (cf. [2] and [15]), defined as

$$\mathcal{B} = \left\{ f \in H(\Delta) : \|f\|_{\mathcal{B}} = \sup_{z \in \Delta} (1 - |z|^2)|f'(z)| < \infty \right\}.$$

When we study meromorphic functions in Δ, it is natural to replace $|f'(z)|$ in these expressions by the spherical derivative $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ and obtain the classes $Q_p^{\#}$, $M_p^{\#}$ and $\mathcal{N}$, the class of normal functions in Δ, respectively (see, for example, Aulaskari, Xiao and Zhao [5] and Wulan [19]). The meromorphic counterpart of $BMOA$ is the set UBC of meromorphic functions of uniformly bounded characteristic introduced by Yamashita [22].

It turns out that we have

$Q_p = M_p$ (Aulaskari, Stegenga and Xiao [3]),

$Q_p^{\#} \subsetneqq M_p^{\#}$ (Aulaskari, Wulan and Zhao [4] and Wulan [19]).

Is there a more general structure behind these facts? To consider a more general case, we let $K : [0, \infty) \to [0, \infty)$ be a right-continuous and nondecreasing function and define Q_K and $Q_K^{\#}$ as follows.

Definition 1. $f \in H(\Delta)$ belongs to the space Q_K if

$$\|f\|_K^2 = \|f\|_{Q_K}^2 = \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^2 K(g(z, a))\, dA(z) < \infty. \tag{1.1}$$

Definition 2. $f \in M(\Delta)$ belongs to the class $Q_K^{\#}$ if

$$\sup_{a \in \Delta} \iint_{\Delta} (f^{\#}(z))^2 K(g(z, a))\, dA(z) < \infty. \tag{1.2}$$

Modulo constants, Q_K is a Banach space under the norm defined in (1.1). $Q_K^{\#}$ is not a linear space. It is clear that Q_K and $Q_K^{\#}$ are Möbius invariant.

Remark 1. For $0 < p < \infty$, $K(t) = t^p$ gives the space Q_p and the class $Q_p^{\#}$. Choosing $K(t) = (1 - e^{-2t})^p$, we obtain M_p and $M_p^{\#}$.

Remark 2. Choosing $K(t) = 1$, we get the Dirichlet space $\mathcal{D}$ and the spherical Dirichlet class $\mathcal{D}^{\#}$. For a fixed r, $0 < r < 1$, we choose

$$K_0(t) = \begin{cases} 1, & t \geq \log(1/r), \\ 0, & 0 < t < \log(1/r). \end{cases}$$

Then we obtain

$$\iint_{\Delta} |f'(z)|^2 K_0(g(z, a))\, dA(z) = \iint_{\Delta(a,r)} |f'(z)|^2\, dA(z),$$

and

$$\iint_{\Delta} {f^{\#}}^2(z) K_0(g(z,a))\, dA(z) = \iint_{\Delta(a,r)} {f^{\#}}^2(z)\, dA(z).$$

We conclude that $Q_{K_0} = \mathcal{B}$ (cf. Axler [6]) and $Q^{\#}_{K_0} = \mathcal{B}^{\#}$, where $\mathcal{B}^{\#}$ is the class of spherical Bloch functions (cf. Section 3). It is easy to see that $\mathcal{N} \subsetneqq \mathcal{B}^{\#}$ (cf. Lappan [13] and the discussion after Definition 2.1 in Wulan [19]).

Which properties of K_1 and K_2 imply that $Q_{K_1} = Q_{K_2}$ or $Q^{\#}_{K_1} = Q^{\#}_{K_2}$? We shall develop a general theory for Q_K and $Q^{\#}_K$ spaces which answers these questions and which gives most basic properties of Q_p, $Q^{\#}_p$, and $M^{\#}_p$. Examples of functions in Q_K for different kernels K are given in Theorems 2.2–2.9. A preliminary version of our results can be found in [9]. We note that problems of this type are also discussed in [18].

Let us introduce the following notation. By writing $f(r) \approx g(r)$ as $r \to r_0$, we mean that there exist positive constants c_1 and c_2 such that

$$c_1 \le \frac{f(r)}{g(r)} \le c_2$$

in a neighbourhood of r_0.

2. Q_K-spaces

For a nondecreasing function $K : [0,\infty) \to [0,\infty)$ we say that the space Q_K is trivial if Q_K contains only constant functions. Whether our space Q_K is trivial or not depends on the integral

$$\int_0^{1/e} K(\log(1/\rho))\rho\, d\rho = \int_1^{\infty} K(t) e^{-2t}\, dt. \tag{2.1}$$

PROPOSITION 2.1.

(i) *If the integral* (2.1) *is divergent, then the space* Q_K *is trivial.*

(ii) *If the integral* (2.1) *is convergent, then* $Q_K \subset \mathcal{B}$.

Proof. (i) Let $f_a = f \circ \varphi_a$. Then $|f_a'(0)| = |f'(a)|(1-|a|^2)$. Assume that there exists $f \in Q_K$ such that $f_a'(0) \ne 0$ for some $a \in \Delta$. By subharmonicity, we have

$$\begin{aligned}\iint_{\Delta} |f'(z)|^2 K(g(z,a))\, dA(z) &= \iint_{\Delta} |f_a'(w)|^2 K(\log|1/w|)\, dA(w)\\ &\ge 2\pi\, |f_a'(0)|^2 \int_0^{1/e} K(\log(1/\rho))\rho\, d\rho.\end{aligned}$$

Thus the integral (2.1) must be convergent and we have proved (i).

(ii) Conversely, if the integral (2.1) is convergent and $f \in Q_K$, it follows from the inequality above that $\|f\|_{\mathcal{B}} < \infty$, i.e., we have $Q_K \subset \mathcal{B}$. □

We note that a necessary condition for the space Q_K to be nontrivial is that

$$\lim_{t\to\infty} K(t)e^{-2t} = 0. \tag{2.2}$$

It is easy to see that (2.2) is not a sufficient condition for the space Q_K to be nontrivial.

It is also easy to see that the condition $K(t)e^{-2t} = O(t^{-s})$ for $s > 1$ is sufficient for the integral (2.1) to be convergent. The convergence of (2.1) is related to the growth order of K. The log-order of the function $K(r)$ is defined as

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ K(r)}{\log r}, \tag{2.3}$$

where $\log^+ x = \max\{\log x, 0\}$. If $0 < \rho < \infty$, the log-type of the function $K(r)$ is defined as

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log^+ K(r)}{r^\rho}. \tag{2.4}$$

Proposition 2.2. *Let ρ and σ be the log-order and the log-type of a nondecreasing function K.*

(i) *If $\rho > 1$, then the space Q_K is trivial.*
(ii) *If $\rho = 1$ and $\sigma > 2$, then the space Q_K is trivial.*

A proof of this result can be found in [18].

Remark 3. In the critical case $\rho = 1$ and $\sigma = 2$, Q_K may be trivial or nontrivial.

From now on and through the remainder of Sections 2 and 3 we assume that the function $K : [0,\infty) \to [0,\infty)$ is right-continuous and nondecreasing and that the integral (2.1) is convergent.

Theorem 2.1. *Assume that $K(1) > 0$ and set $K_1(r) = \inf(K(r), K(1))$. Then $Q_K = Q_{K_1}$.*

Proof. Since $K_1 \le K$ and K_1 is nondecreasing, it is clear that $Q_K \subset Q_{K_1}$. It remains to prove that $Q_{K_1} \subset Q_K$. We note that

$$g(z,a) > 1, \quad z \in \Delta(a, 1/e),$$
$$g(z,a) \le 1, \quad z \in \Delta \setminus \Delta(a, 1/e).$$

Thus $K(g(z,a)) = K_1(g(z,a))$ in $\Delta \setminus \Delta(a,1/e)$. It suffices to deal with integrals over $\Delta(a,1/e)$. If $f \in Q_{K_1}$, f is a Bloch function by Proposition 2.1(ii). It follows that

$$\begin{aligned}\iint\limits_{\Delta(a,1/e)} &|f'(z)|^2 K(g(z,a))\,dA(z) \\ &\le \|f\|_{\mathcal{B}}^2 \iint\limits_{\Delta(a,1/e)} (1-|z|^2)^{-2} K(g(z,a))\,dA(z) \\ &= \|f\|_{\mathcal{B}}^2 \iint\limits_{\Delta(0,1/e)} (1-|w|^2)^{-2} K(\log|1/w|)\,dA(w) \\ &= 2\pi\|f\|_{\mathcal{B}}^2 \int_0^{1/e} r(1-r^2)^{-2} K(\log(1/r))\,dr.\end{aligned}$$

The right hand member gives a bound for the supremum over $a \in \Delta$ of the first term in this chain of inequalities. Hence $f \in Q_K$ and Theorem 2.1 is proved. □

Corollary 2.1. *An analytic function f belongs to $\mathcal{B}$ if and only if there exists an r, $0 < r < 1$, such that $K(\log(1/r)) > 0$ and*

$$\sup_{a\in\Delta} \iint\limits_{\Delta(a,r)} |f'(z)|^2 K(g(z,a))\,dA(z) < \infty.$$

Proof. If $f \in \mathcal{B}$, this supremum is finite for any $r \in (0,1)$, by the argument in the proof of Theorem 2.1. Conversely, if the supremum is finite, then

$$\sup_{a\in\Delta} \iint\limits_{\Delta(a,r)} |f'(z)|^2\,dA(z) \le \sup_{a\in\Delta} \frac{1}{K(\log(1/r))} \iint\limits_{\Delta(a,r)} |f'(z)|^2 K(g(z,a))\,dA(z).$$

Since $|f'(z)|^2$ is subharmonic, we conclude that $f \in \mathcal{B}$. □

Theorem 2.2. *Let $0 < p < \infty$. Assume that $K(r) \approx r^p$ as $r \to 0$. Then $Q_K = Q_p$.*

Proof. By Theorem 2.1, it suffices to compare the functions $K(r)$ and r^p in a neighbourhood of the origin. □

Corollary 2.2 (Aulaskari, Stegenga and Xiao [3]). *We have $Q_p = M_p$ for $0 < p < \infty$.*

Proof. We note that

$$1 - \exp(-2g(z,a)) = 1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

We choose $K(t) = (1 - e^{-2t})^p$. It is easy to see that (2.1) is convergent and $K(r) \approx r^p$ as $r \to 0$. By Theorem 2.2, $Q_K = Q_p$. We conclude that $Q_p = M_p$. $\square$

THEOREM 2.3. *$Q_K = \mathcal{B}$ if and only if*

$$\int_0^1 K(\log(1/r))(1-r^2)^{-2}\, r\, dr < \infty. \tag{2.5}$$

Choosing $K(r) = r^p$, we obtain that (2.5) holds for $p > 1$ and that (2.5) fails for $0 < p \le 1$; that is, we have:

COROLLARY 2.3 (Aulaskari and Lappan [2] and Aulaskari, Xiao and Zhao [5]). *We have $Q_p = \mathcal{B}$ when $p > 1$ and $Q_p \subsetneqq \mathcal{B}$ when $0 < p \le 1$.*

Proof of Theorem 2.3. Let us first assume that (2.5) holds. From Proposition 2.1(ii), we know that $Q_K \subset \mathcal{B}$. To prove that $\mathcal{B} \subset Q_K$, we assume that $f \in \mathcal{B}$ and observe that

$$\iint_\Delta |f'(z)|^2 K(g(z,a))\, dA(z) \le \|f\|_{\mathcal{B}}^2 \iint_\Delta (1-|z|^2)^{-2} K(g(z,a))\, dA(z)$$
$$= 2\pi \|f\|_{\mathcal{B}}^2 \int_0^1 (1-r^2)^{-2} K(\log(1/r)) r\, dr < \infty.$$

Hence $f \in Q_K$ and we have proved that (2.5) is a sufficient condition for $Q_K = \mathcal{B}$.

Conversely, assume that $Q_K = \mathcal{B}$. To prove that (2.5) is a necessary condition, we study the Bloch function

$$f(z) = \sum_{k=1}^\infty z^{2^k},$$

(cf. Proposition 8.12 in [15]). We introduce

$$L(r) = \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta = 2\pi \sum_{k=1}^\infty 2^{2k} r^{2^{k+1}-2}.$$

In the interval $[1/2, 1)$, we have $\log r \ge 2(r-1)$, and it follows that

$$r^{2^{k+1}-2} \ge \exp\{2^{k+2}(r-1)\}, \qquad r \in [1/2, 1),$$

and thus that there is a positive constant C (which will be specified below) such that

$$L(r) \ge C(1-r)^{-2}, \qquad r \in [3/4, 1).$$

Since $Q_K = \mathcal{B}$, we have

$$\infty > \iint_\Delta |f'(z)|^2 K(\log(1/|z|))dA(z) = \int_0^1 K(\log(1/r))L(r)rdr$$
$$\geq C \int_{3/4}^1 K(\log(1/r))(1-r)^{-2}rdr.$$

It remains to prove the estimate of $L(r)$ used above. The general term in the series defining $L(r)$ can be estimated from below by $(1-r)^{-2}t^2\exp\{-4t\}$, where $t = 2^k(1-r)$. It is easy to see that $\sup_{t>0} t^2e^{-4t} = (4e^2)^{-1}$ is assumed for $t = 1/2$. For $r \in [3/4, 1)$, we find k so that $1/2 \leq 2^k(1-r) < 1$. For this k we have

$$2^{2k}(1-r)^2\exp\{-4(2^k)(1-r)\} \geq e^{-4},$$

and

$$L(r) \geq 2\pi e^{-4}(1-r)^{-2} = C(1-r)^{-2}, \qquad r \in [3/4, 1).$$

Hence (2.5) holds and we have proved Theorem 2.3. □

Let us now consider the space $Q_{K,0}$, defined as

$$\left\{ f \in H(\Delta) : \iint\limits_\Delta |f'(z)|^2 K(g(z,a))\, dA(z) \to 0,\ |a| \to 1 \right\},$$

and the relation between this space and the space $\mathcal{B}_0$, defined as

$$\mathcal{B}_0 = \left\{ f \in H(\Delta) : \lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0 \right\}.$$

Theorem 2.4.

(i) $Q_{K,0} \subset \mathcal{B}_0$.

(ii) *An analytic function f belongs to $\mathcal{B}_0$ if and only if there exists an r, $0 < r < 1$, such that*

$$\lim_{|a|\to 1} \iint\limits_{\Delta(a,r)} |f'(z)|^2 K(g(z,a))\, dA(z) = 0.$$

Proof. To prove (i), we note that if $f \in Q_{K,0}$ and $K(1) > 0$, then

$$K(1) \iint\limits_{\Delta(a,1/e)} |f'(z)|^2\, dA(z) \leq \iint\limits_\Delta |f'(z)|^2 K(g(z,a))\, dA(z) \to 0, \quad |a| \to 1.$$

By subharmonicity,

$$(1-|z|^2)^2|f'(z)|^2 \leq \mathrm{Const} \cdot \iint\limits_{\Delta(z,1/e)} |f'(\zeta)|^2\, dA(\zeta) \to 0, \quad |z| \to 1,$$

which proves the first part of Theorem 2.4.

To prove the second part, we note that

$$\iint\limits_{\Delta(a,r)} |f'(z)|^2 K(g(z,a)) dA(z)$$
$$\leq \sup_{\Delta(a,r)} \left\{(1-|z|^2)^2 |f'(z)|^2\right\} 2\pi \int_0^r \rho(1-\rho^2)^{-2} K(\log(1/\rho)) d\rho,$$

(see the computation in the last part of the proof of Theorem 2.1). If $f \in \mathcal{B}_0$, the supremum term tends to zero as $|a| \to 1$ and the integral in (ii) will tend to zero. Conversely, if the limit of the integral in (ii) is zero, it follows that

$$\lim_{|a|\to 1} \iint\limits_{\Delta(a,r_1)} |f'(z)|^2 dA(z) = 0, \qquad |a| \to 1,$$

where $r_1 \in (0, r]$ is chosen so that $K(\log(1/r_1)) > 0$. Since $|f'(z)|^2$ is subharmonic, we must have $f \in \mathcal{B}_0$. □

THEOREM 2.5. *$Q_{K,0} = \mathcal{B}_0$ if and only if (2.5) holds.*

Proof. Let us first assume that (2.5) holds. By Theorem 2.4, it suffices to prove that $\mathcal{B}_0 \subset Q_{K,0}$. Suppose that $f \in \mathcal{B}_0$. Since (2.5) holds, for given $\varepsilon > 0$ there exists an r, $0 < r < 1$, such that

$$\int_r^1 K(\log(1/\rho))/(1-\rho^2)^{-2} \rho \, d\rho < \varepsilon.$$

Thus,

$$\iint\limits_{\Delta\setminus\Delta(a,r)} |f'(z)|^2 K(g(z,a))\, dA(z) \tag{2.6}$$
$$\leq \|f\|_{\mathcal{B}}^2 \iint\limits_{\Delta\setminus\Delta(a,r)} (1-|z|^2)^{-2} K(g(z,a))\, dA(z)$$
$$= \|f\|_{\mathcal{B}}^2 \iint\limits_{r\leq|w|<1} (1-|w|^2)^{-2} K(\log(1/|w|)) dA(w)$$
$$= 2\pi \|f\|_{\mathcal{B}}^2 \int_r^1 K(\log(1/\rho))(1-\rho^2)^{-2} \rho \, d\rho$$
$$< 2\pi \|f\|_{\mathcal{B}}^2 \, \varepsilon.$$

Since $f \in \mathcal{B}_0$, $|f'(\varphi_a(w))|(1-|\varphi_a(w)|^2) \to 0$ as $|a| \to 1$ uniformly in $|w| < r$. Making the change of variables $z = \varphi_a(w)$, we see that

(2.7)

$$\begin{aligned}
&\iint\limits_{\Delta(a,r)} |f'(z)|^2 K(g(z,a))\, dA(z) \\
&\quad = \iint\limits_{|w|<r} |f'(\varphi_a(w))|^2 (1-|\varphi_a(w)|^2)^2 (1-|w|^2)^{-2} K(\log(1/|w|))\, dA(w) \\
&\quad \le 2\pi \sup_{|w|<r} |f'(\varphi_a(w))|^2 (1-|\varphi_a(w)|^2)^2 \int\limits_0^1 K(\log(1/\rho))(1-\rho^2)^{-2} \rho\, d\rho \\
&\quad \le \text{Const} \cdot \sup_{|w|<r} |f'(\varphi_a(w))|^2 (1-|\varphi_a(w)|^2)^2 .
\end{aligned}$$

Combining (2.6) and (2.7), we get

$$\lim_{|a|\to 1} \iint\limits_{\Delta} |f'(z)|^2 K(g(z,a))\, dA(z) = 0,$$

which shows that $f \in Q_{K,0}$. Thus (2.5) is a sufficient condition for $Q_{K,0} = \mathcal{B}_0$ to hold.

Conversely, we shall assume that (2.5) does not hold and prove that there exists a function in $\mathcal{B}_0 \setminus Q_{K,0}$.

If (2.5) does not hold, we can find a continuous, strictly decreasing function $h : [0,1) \to (0,1]$ tending to zero at 1 such that

$$\int_0^1 h(r) K(\log(1/r))(1-r^2)^{-2} r dr = \infty. \tag{2.8}$$

For a given $r \in [3/4, 1)$ we find an integer k such that $1/2 \le 2^k(1-r) < 1$. Then $1 - 2^{-k} < r$. We define $a_k = h(1-2^{-k})^{1/2}$ and consider the gap series

$$f_0(z) = \sum_1^\infty a_k z^{2^k}.$$

Since $a_k \to 0$ as $k \to \infty$, it is clear that $f_0 \in \mathcal{B}_0$ (cf. Proposition 8.12 in [15]). If

$$L_0(r) = \int_0^{2\pi} |f_0'(re^{i\theta})|^2 d\theta = 2\pi \sum_1^\infty a_k^2 2^{2k} r^{2^{k+1}-2},$$

we argue in the same way as in the second half of the proof of Theorem 2.3 to deduce that

$$L_0(r) \ge 2\pi h(r)(1-r)^{-2} e^{-4} = Ch(r)(1-r)^{-2}, \qquad r \in [3/4, 1).$$

It follows that

$$\iint_\Delta |f_0'(z)|^2 K(\log(1/|z|))dA(z) \geq C\int_{3/4}^1 h(r)K(\log(1/r))(1-r^2)^{-2}rdr = \infty.$$

Hence $f_0 \in \mathcal{B}_0 \setminus Q_{K,0}$, which shows that $Q_{K,0} \subsetneqq \mathcal{B}_0$ and finishes the proof of Theorem 2.5. □

In a natural way, the Bloch space $\mathcal{B}$ is the largest Möbius-invariant subspace in the linear space of analytic functions on Δ equipped with the topology of uniform convergence on compact sets (cf. Rubel and Timoney [17] and Arazy, Fisher and Peetre [1]). In our work, this is clear, since $Q_K \subset \mathcal{B}$ holds for all nondecreasing functions K. The smallest Möbius-invariant subspace in our scale is the Dirichlet space $\mathcal{D}$ (modulo constants) under the norm $\|f\|_{\mathcal{D}} = \{\iint_\Delta |f'(z)|^2 dA(z)\}^{1/2}$ (cf. Remark 2).

Theorem 2.6. *Let $K_1 \leq K_2$ in $(0,1)$ and assume that $K_1(r)/K_2(r) \to 0$ as $r \to 0$ and that the integral in (2.5) is divergent when $K = K_2$. Then $Q_{K_2} \subsetneqq Q_{K_1}$.*

Corollary 2.4.

(i) *(cf. [5]) $Q_s \subsetneqq Q_q, \quad 0 \leq s < q < 1$.*
(ii) *$\mathcal{D} \subset Q_K$. Furthermore, we have $\mathcal{D} = Q_K$ if and only if $K(0) > 0$.*
(iii) *Let*

$$K(t) = \begin{cases} t/|\log t|, & 0 < t \leq 1/e, \\ t, & t > 1/e. \end{cases}$$

Then $BMOA = Q_1 \subsetneqq Q_K \subsetneqq \mathcal{B}$.

Proof of Corollary 2.4. (i) Choose $K_2(r) = r^s$ and $K_1(r) = r^q$ and apply Theorem 2.6.

(ii) If $K(0) > 0$, it is an immediate consequence of Theorem 2.1 that $Q_K = Q_{K(0)} = \mathcal{D}$. Conversely, if $K(0) = 0$, we choose $K_2(r) = K(1)$ and $K_1(r) = K(r)$ and apply Theorem 2.6 to conclude that $\mathcal{D} \subsetneqq Q_K$.

(iii) We choose $K_1(t) = K(t)$ and $K_2(t) = t$ and apply Theorem 2.6 to conclude that $BMOA \subsetneqq Q_K$. From Proposition 2.1(ii), we know that $Q_K \subset \mathcal{B}$. With K as above, the integral (2.5) is divergent. It is now clear from Theorem 2.3 that $Q_K \subsetneqq \mathcal{B}$. □

Proof of Theorem 2.6. We note first that it is clear that $Q_{K_2} \subset Q_{K_1}$. We assume that $Q_{K_2} = Q_{K_1}$ and apply the open mapping theorem (cf. 1.1.4 in [23]), which tells us that the identity map from one of these spaces into the other one is continuous. Thus there exists a constant C such that $\|\cdot\|_{K_2} \leq$

$C\|\cdot\|_{K_1}$. If $K_1(t) \le (2C)^{-1}K_2(t)$, $0 < t \le t_0$, we can choose $r_0 = e^{-t_0}$ and deduce that if $f \in Q_{K_2}$, then

$$\sup_{a\in\Delta} \iint\limits_{\Delta} |f'(z)|^2 K_2(g(z,a))dA(z) \le C \sup_{a\in\Delta} \iint\limits_{\Delta(a,r_0)} |f'(z)|^2 K_1(g(z,a))dA(z) + \frac{1}{2}\sup_{a\in\Delta}\iint_{\Delta} |f'(z)|^2 K_2(g(z,a))dA(z).$$

Consequently,

$$\iint_{\Delta} |f'(z)|^2 K_2(g(z,a))dA(z) \le 2C \sup_{a\in\Delta}\iint_{\Delta(a,r_0)} |f'(z)|^2 K_1(g(z,a))dA(z).$$

Since $Q_{K_2} \subset \mathcal{B}$, we use the fact that we must have $f \in \mathcal{B}$ to deduce that the right hand side of this inequality is majorized by

$$4\pi C\|f\|_{\mathcal{B}}^2 \int_0^{r_0} t(1-t^2)^{-2} K_1(\log(1/t))dt.$$

Hence there exists a constant C' such that

$$\sup_{a\in\Delta}\iint_{\Delta} |f'(z)|^2 K_2(g(z,a))dA(z) \le C'\|f\|_{\mathcal{B}}^2, \quad f \in Q_{K_2}.$$

If $h \in \mathcal{B}$ and $h_r(z) = h(rz)$, $0 < r < 1$, then $\|h_r\|_{\mathcal{B}} \le \|h\|_{\mathcal{B}}$. Since $h_r \in Q_{K_2}$, $0 < r < 1$, we can choose $f = h_r$ in this inequality. Using Fatou's lemma, we deduce that

$$\sup_{a\in\Delta}\iint_{\Delta} |h'(z)|^2 K_2(g(z,a))dA(z) \le C'\|h\|_{\mathcal{B}}^2, \quad h \in \mathcal{B}.$$

We have proved that if $h \in \mathcal{B}$, then $h \in Q_{K_2}$, which means that $Q_{K_2} = \mathcal{B}$. It follows from Theorem 2.3 that the integral (2.5) with $K = K_2$ must be convergent, which contradicts our assumptions. We conclude that we must have $Q_{K_2} \subsetneqq Q_{K_1}$. This finishes the proof of Theorem 2.6. □

Theorem 2.7. *$\mathcal{D} \subset Q_{K,0}$ if and only if $K(0) = 0$.*

An immediate consequence of Theorem 2.7 is that we have $\mathcal{D} \subset Q_{p,0}$ for all p, $0 < p < \infty$.

Proof of Theorem 2.7. Let us first assume that $K(0) = 0$ and that $f \in \mathcal{D}$. For $0 < r < 1$ we have

$$\iint_{\Delta\setminus\Delta(a,r)} |f'(z)|^2 K(g(z,a))\,dA(z)$$
$$\leq K(\log(1/r)) \iint_{\Delta\setminus\Delta(a,r)} |f'(z)|^2\,dA(z) \leq \|f\|^2_{\mathcal{D}} K(\log(1/r)).$$

For given $\varepsilon > 0$ we recall our assumption that K is right-continuous and $K(0) = 0$. We can choose an r_0, $0 < r_0 < 1$, such that

$$\iint_{\Delta\setminus\Delta(a,r_0)} |f'(z)|^2 K(g(z,a))\,dA(z) \leq \|f\|^2_{\mathcal{D}} K(\log(1/r_0)) < \varepsilon$$

for all $a \in \Delta$.

It is easy to see that $f \in \mathcal{B}_0$ if $f \in \mathcal{D}$ (see the proof of Theorem 2.4). Thus, for $\varepsilon > 0$ given, we can find $\delta(\varepsilon)$ such that $|f'(z)|(1-|z|^2) < \varepsilon$ if $1-|z| < \delta(\varepsilon)$. For $|a|$ close to 1, we deduce that

$$\iint_{\Delta(a,r_0)} |f'(z)|^2 K(g(z,a))\,dA(z) \leq \varepsilon^2 \iint_{\Delta(a,r_0)} K(g(z,a))(1-|z|^2)^{-2} dA(z)$$
$$= \varepsilon^2 \iint_{|w|<r_0} K(\log(1/|w|))(1-|w|^2)^{-2} dA(w) = \text{Const}\cdot\varepsilon^2.$$

Hence

$$\lim_{|a|\to 1} \iint_{\Delta} |f'(z)|^2 K(g(z,a))\,dA(z) = 0,$$

which means that $f \in Q_{K,0}$.

Conversely, assuming that $K(0) > 0$ and that $f \in \mathcal{D}$, we see that

$$\iint_{\Delta} |f'(z)|^2 dA(z) \leq K(0)^{-1} \iint_{\Delta} |f'(z)|^2 K(g(z,a)) dA(z).$$

Thus, if $f \in \mathcal{D}$ is nonconstant, we must have $f \notin Q_{K,0}$ because if $f \in Q_{K,0}$ we would have

$$\iint_{\Delta} |f'(z)|^2 K(g(z,a)) dA(z) \to 0,\ |a| \to 1.$$

We have thus proved Theorem 2.7. □

For $\alpha \in [0,\infty)$, we let $\mathcal{B}^\alpha$ denote the space of all functions $f \in H(\Delta)$ satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\Delta} (1-|z|^2)^\alpha |f'(z)| < \infty.$$

It is well known that if $\alpha \in [0,1)$, $\mathcal{B}^\alpha$ will coincide with the classical $(1-\alpha)$-Lipschitz class (cf. Theorem 5.1 in [8]). We mention the following result of Yamashita (cf. [21]).

THEOREM Y1. *Let* $f(z) = \sum_1^\infty a_j z^{n_j}$, *where* $n_{j+1}/n_j \geq \lambda > 1, j = 1, 2, \ldots$. *Then* $f \in \mathcal{B}^\alpha, 0 < \alpha < \infty$, *if and only if*

$$\sup_{j\in\mathbb{N}} |a_j| n_j^{1-\alpha} < \infty. \tag{2.9}$$

The following discussion of the relation between α-Bloch spaces $\mathcal{B}^\alpha$ and Q_K-spaces will also give us several examples of functions in Q_K for different kernels K.

THEOREM 2.8. *Let* $1/2 \leq \alpha < 1$. *The following statements are equivalent:*

(i) *We have*

$$I(K,\alpha) = \int_0^1 K(\log(1/r))(1-r^2)^{-2\alpha} r\,dr < \infty. \tag{2.10}$$

(ii) $\mathcal{B}^\alpha \subset Q_{K,0}$.

(iii) $\mathcal{B}^\alpha \subset Q_K$.

Furthermore, we have:

(iv) *Let* $f(z) = f_\alpha(z) = \sum_{j=1}^\infty 2^{-j(1-\alpha)} z^{2^j}$. *If* (2.10) *holds, then* $f \in Q_{K,0}$. *Conversely, if* (2.10) *does not hold, then* $f \notin Q_K$.

We note that if $0 \leq \alpha < 1/2$, then $\mathcal{B}^\alpha \subset \mathcal{D} \subset Q_K$, where the last inclusion is a consequence of Corollary 2.4(ii). If it is also known that $K(0) = 0$, then by Theorem 2.7 we have $\mathcal{B}^\alpha \subset Q_{K,0}$.

Proof of Theorem 2.8. Assume that (2.10) holds. Changing variables in the integral, we see (with $|w| = r$ and $f \in \mathcal{B}^\alpha$) that

$$\begin{aligned}\iint_\Delta |f'(z)|^2 K(g(z,a))\,dA(z) &\leq \|f\|_{\mathcal{B}^\alpha}^2 \iint_\Delta (1-|z|^2)^{-2\alpha} K(g(z,a))\,dA(z)\\ &= \|f\|_{\mathcal{B}^\alpha}^2 \int_0^1 K(\log(1/r))(1-r^2)^{-2\alpha} I(r,\alpha) r\,dr,\end{aligned}$$

where

$$I(r,\alpha) = \int_0^{2\pi} \frac{(1-|a|^2)^{2-2\alpha}}{|1-\bar{a}re^{i\theta}|^{4-4\alpha}}\,d\theta.$$

Let δ be a small positive number. If $r < 1-\delta$, it is clear that $I(r,\alpha)$ tends to 0 uniformly as $|a| \to 1$. For $\{1-\delta < r < 1\}$ and $|a|$ near 1, we have the following estimates of $I(r,\alpha)$:

(a) $I(r,\alpha) \leq \mathrm{C}(1-|a|)^{2-2\alpha}, \quad 3/4 < \alpha < 1$,

(b) $I(r,\alpha) \leq \mathrm{C}(1-|a|)^{1/2} \log \pi/(1-|a|), \quad \alpha = 3/4$,

(c) $I(r,\alpha) \leq \mathrm{C}(1-|a|)^{2\alpha-1}, \quad 1/2 \leq \alpha < 3/4$.

Thus, if $1/2 \le \alpha < 1$, $I(r,\alpha)$ is bounded and the integral over the interval $\{1-\delta < r < 1\}$ is majorized by

$$\text{Const} \cdot \int_{1-\delta}^{1} K(\log(1/r))(1-r^2)^{-2\alpha} r dr.$$

Since (2.10) holds, this expression can be made as small as we like by choosing δ close to 0. It follows that $f \in Q_{K,0}$. We have thus proved that (ii) follows from (i).

It is clear that (ii) $\Rightarrow$ (iii). To prove that (iii) $\Rightarrow$ (i), let us assume that $\mathcal{B}^\alpha \subset Q_K$. Here, we need the following result of J. Xiao (cf. Theorem 2.1.1 in [20]).

Theorem X. *Let $\alpha \in (0,\infty)$. Then there are $f_1,\ f_2$ in $\mathcal{B}^\alpha$ with*

$$|f_1'(z)| + |f_2'(z)| \approx (1-|z|^2)^{-\alpha}, \quad z \in \Delta. \tag{2.11}$$

For $\alpha = 1$, this is proved in Ramey and Ullrich [16] via a study of gap series. Combining this technique with Theorem Y1, Xiao proved Theorem X.

Let f_1 and f_2 be as in Theorem X. Our assumptions show that these functions will also be in Q_K, and we see that

$$\begin{aligned}
\infty > 2 &\iint_\Delta (|f_1'(z)|^2 + |f_2'(z)|^2) K(\log(1/|z|)) dA(z) \\
&\ge \iint_\Delta (|f_1'(z)| + |f_2'(z)|)^2 K(\log((1/|z|)) dA(z) \\
&\ge C \iint_\Delta (1-|z|^2)^{-2\alpha} K(\log(1/|z|)) dA(z) \\
&= C \int_0^1 (1-r^2)^{-2\alpha} K(\log(1/r)) r dr.
\end{aligned}$$

Hence (2.10) holds for $1/2 \le \alpha < 1$ and we have proved (i).

To prove (iv), we apply Theorem Y1 to deduce that $f_\alpha \in \mathcal{B}^\alpha$. If $1/2 \le \alpha < 1$, it follows from the first part of the theorem that if (2.10) holds, then $f_\alpha \in Q_{K,0}$.

Let us now assume that $1/2 \le \alpha < 1$ and that $I(K,\alpha) = \infty$. Arguing as in the proofs of Theorems 2.3 and 2.5, we wish to estimate

$$L(r) = \int_0^{2\pi} |f_\alpha'(re^{i\theta})|^2 d\theta = 2\pi \sum_{j=1}^{\infty} 2^{j2\alpha} r^{2^{j+1}-2}.$$

For $r \in [3/4, 1)$, we find k so that $1/2 \le 2^k(1-r) < 1$. Using the inequality $\log r \ge 2(r-1), 1/2 < r < 1$, we see that

$$\begin{aligned}(1-r)^{2\alpha} L(r) &\ge 2\pi \sum_{j=1}^{\infty} (2^j(1-r))^{2\alpha} \exp\left(-2^{j+2}(1-r)\right) \\ &\ge 2^{-2\alpha+1}\pi \sum_{j=1}^{\infty} 2^{(j-k)(2\alpha)} \exp\left(-2^{j-k+2}\right) \\ &\ge 2^{-2\alpha+1}\pi \sum_{0}^{\infty} (2^{j2\alpha} \exp\left(-2^{j+2}\right)) = C.\end{aligned}$$

Hence

$$\begin{aligned}\iint_{\Delta} |f'_\alpha(z)|^2 K(\log(1/|z|)) dA(z) &= \int_0^1 K(\log(1/r)) L(r) r dr \\ &\ge C \int_{3/4}^1 (1-r^2)^{-2\alpha} K(\log(1/r)) r dr.\end{aligned}$$

If the last integral is divergent, we conclude that $f_\alpha \notin Q_K$. This concludes the proof of Theorem 2.8. $\square$

Remark 4. Theorems Y1 and 2.8 and Corollary 2.4(i) should be compared to Theorem 6 in Aulaskari, Xiao and Zhao [5], which says that the gap series

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \qquad n_{k+1}/n_k \ge \lambda > 1, \quad k = 0, 1, \ldots,$$

is in Q_p, $0 < p \le 1$, if and only if

$$\sum_{k=1}^{\infty} 2^{k(1-p)} \sum_{n_j \in I(k)} |a_j|^2 < \infty,$$

where $I(k) = \{n : 2^k \le n < 2^{k+1}, n \in \mathbb{N}\}$. These results are used in [5] to prove that

$$Q_s \subsetneqq Q_q, \quad 0 \le s < q < 1. \tag{2.12}$$

One generalization of (2.12) was given in Theorem 2.6. Theorem 2.8 can be used to give another generalization of this fact. If $K_1 \le K_2$ in the interval $(0, 1)$, it is known that $Q_{K_2} \subset Q_{K_1}$. Under what conditions on the kernels is this inclusion strict?

Corollary 2.5. *Let $K_1 \le K_2$ in $(0, 1)$ and assume that for some $\alpha \in [1/2, 1)$, $I(K_1, \alpha)$ is finite while $I(K_2, \alpha)$ is infinite. Then $Q_{K_2} \subsetneqq Q_{K_1}$.*

Proof. It follows from Theorem 2.8 that $\mathcal{B}^\alpha \not\subset Q_{K_2}$ and that $\mathcal{B}^\alpha \subset Q_{K_1}$. $\square$

To see that Corollary 2.5 is a generalization of (2.12), we note that if $K(t) = K_p(t) = t^p$, $0 \le p < 1$, then $I(K_p, \alpha)$ is finite if $p > 2\alpha - 1$ and infinite if $p \le 2\alpha - 1$. Choosing $\alpha \in [1/2, 1)$ such that $s < 2\alpha - 1 < q$, and applying Corollary 2.5 to $K_q \le K_s$, we obtain (2.12).

Theorem 2.9. *$\log(1-z) \in Q_K$ if and only if*

$$\int_0^1 (1-r^2)^{-1} K(\log(1/r)) r dr < \infty. \tag{2.13}$$

Choosing $K(t) = t^p$, we obtain:

Corollary 2.6. *$\log(1-z) \in Q_p$ for all $p \in (0, \infty)$.*

Proof of Theorem 2.9. A classical rearrangement theorem of Hardy (cf. 10.2 and 10.13 in [12]) tells us that

$$\sup_{a \in \Delta} \iint_\Delta |1-z|^{-2} K\left(\log\left(\frac{1}{|\varphi_a(z)|}\right)\right) dA(z),$$

is assumed for $0 < a < 1$. For a in this range, a change of variables in this integral gives the expression

$$\int_0^1 K(\log(1/r)) r dr \int_0^{2\pi} \frac{(1+a)^2 d\theta}{|1+re^{i\theta}|^2 |1-are^{i\theta}|^2}.$$

A computation shows that the supremum over a of the inner integral is (modulo constants) essentially $(1-r)^{-1}$, and the theorem is proved. □

Corollary 2.7. *If (2.13) holds, then $Q_{K,0} \subsetneqq Q_K$.*

Proof. By Theorems 2.4 and 2.9, we have $\log(1-z) \in Q_K \setminus \mathcal{B}_0 \subset Q_K \setminus Q_{K,0}$. □

We do not know whether we have $Q_{K,0} = Q_K$ when (2.13) does not hold.

In Essén and Xiao [10], some relations between the spaces Q_p and the mean Lipschitz spaces $A(p, \alpha)$ were discussed. Here, $A(p, \alpha)$, $\alpha \in [0, 1]$, consists of functions f analytic in Δ satisfying

$$\|f_r'\|_p = O((1-r)^{\alpha-1}), \ r \to 1,$$

where $f_r'(e^{i\theta}) = f'(re^{i\theta})$ and

$$\|f\|_p = \begin{cases} \left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right)^{1/p}, & 0 < p < \infty, \\ \mathrm{ess}_\theta \sup |f(e^{i\theta})|, & p = \infty. \end{cases}$$

It is easy to see that $\log(1-z) \in A(2p, \alpha)$ for $0 < \alpha < 1/2p$ with $1 \le p < \infty$, and that $\log(1-z) \notin \mathcal{B}_0$. By Theorem 2.4(i) we have $Q_{K,0} \subset \mathcal{B}_0$. It follows that

$$A(2p,\alpha)\not\subset\bigcap_{1-2\alpha<q<1}Q_{q,0},$$

for $\alpha\in(0,1/2p)$ with $p\in[1,\infty)$. This answers in the negative a question asked in the remark on p. 187 of [10].

While the function $\log(1-z)$ is univalent in the unit disk, the gap series of Theorem 2.7(iv) is "∞-valued" there. Comparing (2.5) of Theorem 2.3 with (2.13) of Theorem 2.9, it is natural to conjecture that a much weaker assumption on K is needed to prove that $f\in\mathcal{B}$ is in Q_K if we also assume that f is univalent. We shall study the relation between univalent (or q-valent) functions and Q_K-classes in a separate paper.

3. The meromorphic case: $Q_K^\#$ classes

It is necessary to know for which functions K the classes $Q_K^\#$ will be trivial. Here, the square of the spherical derivative $f^\#(z)^2$ is not necessarily subharmonic, so the proof of Proposition 2.1(ii) does not work. The following theorem also answers a question of Wu and Wulan (cf. [18, Remark 2]).

Theorem 3.1. *If the integral* (2.1) *is divergent, then the space* $Q_K^\#$ *contains only constant functions.*

Proof. Let $f_a=f\circ\varphi_a$. If $Q_K^\#$ is not trivial, we may assume that $f\in Q_K^\#$ is a nonconstant meromorphic function in Δ and we can find $a\in\Delta$ such that $f^\#(a)>0$. Consequently, $f_a^\#(0)=f^\#(a)(1-|a|^2)>0$. By the continuity of $f_a^\#$, we can find a positive number r such that $f_a^\#(w)>f_a^\#(0)/2$ in $\Delta(0,r)$. It follows that

$$\begin{aligned}\iint_\Delta {f^\#}^2(z)K(g(z,a))\,dA(z)&\ge\iint_{\Delta(a,r)}{f^\#}^2(z)K(g(z,a))\,dA(z)\\&=\iint_{|w|<r}{f_a^\#}^2(w)K(\log 1/|w|)\,dA(w)\\&\ge(\pi/2)f_a^\#(0)^2\int_0^r\rho K(\log(1/\rho))d\rho=\infty.\end{aligned}$$

This is a contradiction, and the proof is complete. □

Again, we assume from now on that the function K is right-continuous and nondecreasing and that the integral (2.1) is convergent. We are interested in the class of normal functions

$$\mathcal{N}=\left\{f\in M(\Delta):\|f\|_{\mathcal{N}}=\sup_{z\in\Delta}(1-|z|^2)f^\#(z)<\infty\right\}$$

and the class of spherical Bloch functions

$$\mathcal{B}^{\#} = \left\{ f \in M(\Delta) : \sup_{a\in\Delta} \iint\limits_{\Delta(a,r)} {f^{\#}}^2(z)\, dA(z) < \infty \ \text{ for some } \ r \in (0,1) \right\}.$$

We clearly have $\mathcal{N} \subset \mathcal{B}^{\#}$. In the analytic case, we know that the corresponding definitions with $f^{\#}(z)$ replaced by $|f'(z)|$ both give the space of Bloch functions $\mathcal{B}$. In the meromorphic case, the situation is different. There exists a locally univalent and analytic function $f_0 \in \mathcal{B}^{\#} \setminus \mathcal{N}$ (cf. Lappan [13]). S. Yamashita [22] proved that there is an essential difference between $\mathcal{N}$ and $\mathcal{B}^{\#}$:

Theorem Y2. *A meromorphic function f belongs to $\mathcal{N}$ if and only if*

$$\sup_{a\in\Delta} \iint\limits_{\Delta(a,r)} {f^{\#}}^2(z)\, dA(z) < \pi$$

for some $r \in (0,1)$.

Remark 5. What can we say about the meromorphic analogue of Corollary 2.1? Is the condition that there exists $r \in (0,1)$ such that

$$\sup_{a\in\Delta} \iint_{\Delta(a,r)} f^{\#}(z)^2 K(g(z,a))dA(z) < \infty \tag{3.1}$$

necessary and sufficient for $f \in \mathcal{B}^{\#}$?

If (3.1) holds, we argue as in the proof of Corollary 2.1 to conclude that we must have $f \in \mathcal{B}^{\#}$. In particular, it follows that $Q_K^{\#} \subset \mathcal{B}^{\#}$. (In the analytic case, we have $Q_K \subset \mathcal{B}$.) Conversely, if we assume that $f \in \mathcal{B}^{\#}$ and that K is bounded, it is easy to see that (3.1) will hold. If K is unbounded and $f \in \mathcal{B}^{\#} \setminus \mathcal{N}$, we claim that the supremum in (3.1) will be infinite for all $r \in (0,1)$. To prove the claim, we note that it follows from Theorem Y2 that if $f \in \mathcal{B}^{\#} \setminus \mathcal{N}$, then

$$\sup_{a\in\Delta} \iint_{\Delta(a,r)} f^{\#}(z)^2 dA(z) \geq \pi \quad \text{ for all } r \in (0,1).$$

If $0 < \rho < r$, we see that

$$\iint_{\Delta(a,r)} f^{\#}(z)^2 K(g(z,a))dA(z) \geq K(\log(1/\rho)) \iint_{\Delta(a,\rho)} f^{\#}(z)^2 dA(z).$$

Using the observation above, we deduce that

$$\sup_{a\in\Delta} \iint_{\Delta(a,r)} f^{\#}(z)^2 K(g(z,a))dA(z) \geq \pi K(\log(1/\rho)), \quad 0 < \rho < r.$$

Letting $\rho \to 0$, we conclude that (3.1) cannot hold for any $r \in (0,1)$ which proves our claim.

We conclude that (3.1) is a sufficient condition for $f \in \mathcal{B}^{\#}$. It is also a necessary condition when K is bounded, but not when K is unbounded.

Finally, if we assume that $f \in \mathcal{N}$, it is easy to prove that (3.1) will hold (see the proof of Theorem 3.3(ii) below).

We note that Theorem 2.1 and Theorem 2.2 do not remain valid in the meromorphic case and that it can occur that $Q_K^{\#} \not\subset \mathcal{N}$. We quote a theorem from Wu and Wulan [18], a result which in turn is an immediate consequence of the main result in Aulaskari, Wulan and Zhao [4].

Theorem WW. *Let $0 < p < \infty$. Assume that K is bounded and that $K(r) = O(r^p)$ as $r \to 0$. Then there exists a non-normal function $f_0 \in Q_K^{\#}$.*

Which additional conditions on K are required for the inclusion $Q_K^{\#} \subset \mathcal{N}$? When are the classes $Q_{K_1}^{\#}$ and $Q_{K_2}^{\#}$ identical for $K_1 \neq K_2$? The following relevant result was proved in [18].

Proposition 3.1. *Assume that $K(r) \to \infty$ as $r \to \infty$. Then $Q_K^{\#} \subset \mathcal{N}$.*

If the function K is bounded, Theorem WW tells us that $Q_K^{\#} \subset \mathcal{N}$ does not hold in the general case. However, we have the following result:

Theorem 3.2. *Assume that $K(\infty) = 1$. Then $f \in \mathcal{N}$ if and only if*

$$\sup_{a\in\Delta} \iint\limits_{\Delta(a,r)} f^{\#^2}(z)K(g(z,a))\,dA(z) < \pi, \tag{3.2}$$

for some $r \in (0,1)$.

Proof. Suppose that f is a normal function. Then for $0 < r < 1$,

$$\begin{aligned} \iint\limits_{\Delta(a,r)} f^{\#^2}(z)K(g(z,a))\,dA(z) &\leq \|f\|_N^2 \iint\limits_{\Delta(a,r)} (1-|z|^2)^{-2}K(g(z,a))dA(z) \\ &= \|f\|_N^2 \iint\limits_{|w|<r} (1-|w|^2)^{-2}K(\log\frac{1}{|w|})dA(w) \\ &\leq 2\pi\|f\|_N^2(1-r^2)^{-2}\int_0^r K(\log(1/\rho))\,\rho\,d\rho. \end{aligned} \tag{3.3}$$

Since

$$\int_0^r K(\log(1/\rho))\,\rho\,d\rho \to 0, \;\; r \to 0,$$

we may choose r small enough such that the left hand member in the first inequality in (3.3) is less than $\pi/2$. Thus (3.2) holds.

Conversely, let $C(<\pi)$ be the supremum in (3.2) assumed for some $r_0 \in (0,1)$. Now consider $r \in (0, r_0)$. Since $\Delta(a,r) = \{z \in \Delta : g(z,a) > \log(1/r)\}$,

$$\iint\limits_{\Delta(a,r)} {f^{\#}}^2(z)\, dA(z) \leq \big(K(\log(1/r))\big)^{-1} \iint\limits_{\Delta(a,r_0)} {f^{\#}}^2(z)\, K(g(z,a))\, dA(z)$$
$$\leq C\, \big(K(\log(1/r))\big)^{-1} < \pi$$

if r is small enough. Hence $f \in \mathcal{N}$ according to Theorem Y2. The proof is complete. □

COROLLARY 3.1. *Assume that* $K(\infty) = 1$. *If* $f \in Q_K^{\#}$ *and*

$$\sup_{a\in\Delta} \iint\limits_{\Delta} {f^{\#}}^2(z) K(g(z,a))\, dA(z) < \pi,$$

then $f \in \mathcal{N}$.

The meromorphic analogue of Theorem 2.1 is given by the following result:

THEOREM 3.3. *Assume that* $K(1) > 0$ *and set* $K_1(r) = \inf(K(r), K(1))$.

(i) *If* K *is bounded, then* $Q_K^{\#} = Q_{K_1}^{\#}$.
(ii) *If* K *is unbounded, then* $Q_K^{\#} = \mathcal{N} \cap Q_{K_1}^{\#}$.

Proof. (i) If K is bounded, we have

$$K_1(r) \leq K(r) \leq \frac{K(\infty)}{K(1)} K_1(r),$$

and it is clear that $Q_K^{\#} = Q_{K_1}^{\#}$.

(ii) By Proposition 3.1, we have $Q_K^{\#} \subset \mathcal{N} \cap Q_{K_1}^{\#}$. Now assume that $f \in \mathcal{N} \cap Q_{K_1}^{\#}$. We note that $K(g(z,a)) = K_1(g(z,a))$ in $\Delta \setminus \Delta(a, 1/e)$. (In this domain, we have $g(z,a) \leq 1$.) To compare the two suprema in the integrals defining $Q_K^{\#}$ and $Q_{K_1}^{\#}$, it suffices to deal with integrals over $\Delta(a, 1/e)$. Using our assumption that $f \in \mathcal{N}$, we see that

$$\iint_{\Delta(a,1/e)} f^{\#}(z)^2 K(g(z,a))\,dA(z)$$
$$\le \|f\|^2_{\mathcal{N}} \iint_{\Delta(a,1/e)} (1-|z|^2)^{-2} K(g(z,a))\,dA(z)$$
$$= \|f\|^2_{\mathcal{N}} \iint_{\Delta(0,1/e)} (1-|w|^2)^{-2} K(\log|1/w|)\,dA(w)$$
$$= 2\pi\|f\|^2_{\mathcal{N}} \int_0^{1/e} r(1-r^2)^{-2} K(\log(1/r))\,dr.$$

The right hand member gives a bound for the supremum over $a \in \Delta$ of the first term in this chain of inequalities. Hence $f \in Q_K^{\#}$ and Theorem 3.3 is proved. □

Next, we state conditions on K_1 and K_2 which imply that $Q^{\#}_{K_1} = Q^{\#}_{K_2}$.

THEOREM 3.4. *Assume that K_1 and K_2 are either both bounded or both unbounded and that $K_1(r) \approx K_2(r)$ as $r \to 0$. Then $Q^{\#}_{K_1} = Q^{\#}_{K_2}$.*

COROLLARY 3.3. *Let $0 < p < \infty$ and assume furthermore that K is bounded and that $K(r) \approx r^p$ as $r \to 0$. Then $Q^{\#}_K = M^{\#}_p$.*

Combining Theorems 3.3 and 3.4 and Corollary 3.3, we obtain (cf. Theorem 2.2.2 in Wulan [19]):

COROLLARY 3.4. *Let $0 < p < \infty$ and assume furthermore that K is unbounded and that $K(r) \approx r^p$ as $r \to 0$. Then $Q^{\#}_K = Q^{\#}_p = \mathcal{N} \cap M^{\#}_p$.*

Proof of Theorem 3.4. We define $K_{i,1}(r) = \inf(K_i(r), K_i(1))$, $i = 1, 2$. If K_1 and K_2 are bounded, it follows from our assumptions that $0 < c \le K_1(r)/K_2(r) \le c' < \infty$, $0 < r < \infty$, and it is clear that we have $Q^{\#}_{K_1} = Q^{\#}_{K_2}$. If K_1 and K_2 are unbounded, we use Theorem 3.3 to deduce that

$$Q^{\#}_{K_1} = \mathcal{N} \cap Q^{\#}_{K_{1,1}} = \mathcal{N} \cap Q^{\#}_{K_{2,1}} = Q^{\#}_{K_2}.$$

We thus have proved Theorem 3.4. □

THEOREM 3.5.

(i) *If K is unbounded and* (2.5) *holds, then $Q^{\#}_K = \mathcal{N}$.*
(ii) *If K is bounded and* (2.5) *holds, then $Q^{\#}_K = \mathcal{B}^{\#}$.*
(iii) *In* (i) *(resp.* (ii)*),* (2.5) *is a necessary condition for $Q^{\#}_K = \mathcal{N}$ (resp. $Q^{\#}_K = \mathcal{B}^{\#}$).*

Corollary 3.5 (Aulaskari and Lappan [2] and Wulan [19]).

(i) $Q_p^\# = \mathcal{N}$ *when* $p > 1$ *and* $Q_p^\# \subsetneqq \mathcal{N}$ *when* $0 < p \leq 1$.
(ii) $M_p^\# = \mathcal{B}^\#$ *when* $p > 1$ *and* $M_p^\# \subsetneqq \mathcal{B}^\#$ *when* $0 < p \leq 1$.

Proof of Theorem 3.5. (i) By Proposition 3.1 we have $Q_K^\# \subset \mathcal{N}$. Conversely, if $f \in \mathcal{N}$, we know that $f^\#(z) \leq \text{Const} \cdot (1-|z|^2)^{-1}$ and we can use the argument in the proof of Theorem 2.3 to prove that $f \in Q_K^\#$.

(ii) By Remark 5 we have $Q_K^\# \subset \mathcal{B}^\#$. It suffices to prove that $\mathcal{B}^\# \subset Q_K^\#$. If $f \in \mathcal{B}^\#$, there exists $r \in (0,1)$ such that

$$\iint_{\Delta(a,r)} f^\#(z)^2 dA(z) \leq B < \infty, \qquad \text{for all } a \in \Delta. \tag{3.4}$$

Let us first prove that there exists a constant C_1 depending on r and K (see below) such that

$$\iint_{\Delta} f^\#(z)^2 K(\log(1/|z|)) dA(z) \leq B\|K\|_\infty + C_1. \tag{3.5}$$

Our first observation in the proof of this estimate is that

$$\iint_{|z|<r} f^\#(z)^2 K(\log(1/|z|)) dA(z) \leq B\|K\|_\infty.$$

Let $\Omega_n = \{z : 1-(1-r)^n \leq |z| \leq 1-(1-r)^{n+1}\}$. We wish to cover Ω_n with disks $\Delta(a,r)$ with $|a| = 1-(1-r)^{n+1}$; it suffices to use roughly $C(r(1-r)^{n+1})^{-1}$ such disks, where C is an absolute constant, $n = 1, 2, \ldots$. Hence

$$\begin{aligned}
&\iint_{\Omega_n} f^\#(z)^2 K(\log(1/|z|)) dA(z) \\
&\qquad \leq K(\log \frac{1}{1-(1-r)^n}) BC(r(1-r)^{n+1})^{-1} \\
&\qquad \leq K((1-r)^n \gamma(r)) BC(r(1-r)^{n+1})^{-1},
\end{aligned}$$

where $\gamma(r) = (1-r)^{-1}\log(1/r)$. It follows that

$$\begin{aligned}\iint\limits_{r\le|z|\le 1} & f^{\#}(z)^2K(\log(1/|z|))dA(z)\\ &\le BCr^{-1}\sum_1^\infty(1-r)^{-n-1}K((1-r)^n\gamma(r))\\ &\le BCr^{-2}(1-r)^{-2}\int_0^1 t^{-2}K(t\gamma(r))dt\\ &= BC\gamma(r)r^{-2}(1-r)^{-2}\int_0^{\gamma(r)} s^{-2}K(s)ds = C_1<\infty.\end{aligned}$$

The convergence of the integral follows from (2.5). We have proved that (3.5) holds for all $f\in\mathcal{B}^{\#}$ satisfying (3.4). Since for all $b\in\Delta$,

$$\sup_{a\in\Delta}\iint\limits_{\Delta(a,r)}(f\circ\varphi_b)^{\#}(z)^2dA(z)=\sup_{a\in\Delta}\iint_{\Delta(a,r)}f^{\#}(z)^2dA(z)=B,$$

it follows from (3.4) and (3.5) with $f^{\#}$ replaced by $(f\circ\varphi_b)^{\#}$ that

$$\begin{aligned}\sup_{b\in\Delta}\iint\limits_{\Delta} & f^{\#}(z)^2K(\log\frac{1}{|\varphi_b(z)|})dA(z)\\ &=\sup_{b\in\Delta}\iint\limits_{\Delta}(f\circ\varphi_b)^{\#}(z)^2K(\log 1/|z|))dA(z)\le C_1+B\|K\|_\infty.\end{aligned}$$

This proves Theorem 3.5(ii).

(iii) By Lappan and Xiao [14], there exist functions f_1 and f_2 in $\mathcal{N}$ such that

$$c_0=\inf_{z\in\Delta}(1-|z|^2)(f_1^{\#}(z)+f_2^{\#}(z))>0. \tag{3.6}$$

If $Q_K^{\#}=\mathcal{N}$ or $Q_K^{\#}=\mathcal{B}^{\#}\supset\mathcal{N}$, we have

$$\begin{aligned}\infty &>\sup_{a\in\Delta}\iint_\Delta(f_1^{\#}(z)^2+f_2^{\#}(z)^2)K(g(z,a))dA(z)\\ &\ge\frac{1}{2}\iint_\Delta(f_1^{\#}(z)+f_2^{\#}(z))^2K(g(z,0))dA(z)\\ &\ge(c_0^2/2)\iint_\Delta(1-|z|^2)^{-2}K(g(z,0))dA(z)\\ &=\pi c_0^2\int_0^1(1-r^2)^{-2}K(\log(1/r))rdr.\end{aligned}$$

Hence (2.5) holds which finishes the proof of Theorem 3.5(iii). □

REMARK 6. There is an analogue of (3.6) for Bloch functions with the spherical derivatives $f_1^\#$ and $f_2^\#$ replaced by $|f_1'|$ and $|f_2'|$ (cf. Proposition 5.4 in Ramey and Ullrich [16]), which could have been used in the proof of the necessity of (2.5) in Theorem 2.3 instead of the direct argument which we used. The proofs of Proposition 5.4 in [16] and Theorem X are related. The proof of (3.6) in [14] is different.

Finally we consider the classes

$$\mathcal{B}_0^\# = \left\{ f \in M(\Delta) : \lim_{|a|\to 1} \iint_{\Delta(a,r)} f^\#(z)^2 \, dA(z) = 0 \quad \text{for some } r \in (0,1) \right\},$$

$$Q_{K,0}^\# = \left\{ f \in M(\Delta) : \lim_{|a|\to 1} \iint_{\Delta} f^\#(z)^2 K(g(z,a)) dA(z) = 0 \right\},$$

$$\mathcal{N}_0 = \left\{ f \in M(\Delta) : (1-|z|^2) f^\#(z) \to 0, \quad |z| \to 1 \right\},$$

and the spherical Dirichlet class

$$\mathcal{D}^\# = \left\{ f \in M(\Delta) : \iint_{\Delta} f^\#(z)^2 \, dA(z) < \infty \right\}.$$

By Lemma 3.2 in Yamashita [22] we have $\mathcal{N}_0 = \mathcal{B}_0^\#$. Arguing as in the proof of Theorem 2.4, we deduce:

THEOREM 3.6. $Q_{K,0}^\# \subset \mathcal{B}_0^\# = \mathcal{N}_0$.

THEOREM 3.7. *If* (2.5) *holds, then* $Q_{K,0}^\# = \mathcal{N}_0$.

REMARK 7. It suffices to prove that $\mathcal{N}_0 \subset Q_{K,0}^\#$. We deduce this using the same argument as in the first part of the proof of Theorem 2.5. We note that in this argument, the growth of K at infinity is unimportant since we have $\mathcal{N}_0 = \mathcal{B}_0^\#$.

We have the following analogue of Corollary 2.4(ii) and Theorem 2.7. Our classes are not linear and we cannot use the open mapping theorem.

THEOREM 3.8.

(i) *If* $K(0) > 0$, *then* $\mathcal{D}^\# = Q_K^\#$.

(ii) $\mathcal{D}^\# \subset Q_{K,0}^\#$ *if and only if* $K(0) = 0$.

(iii) *Assume that* $Q_K^\# \neq Q_{K,0}^\#$. *If* $\mathcal{D}^\# = Q_K^\#$, *then* $K(0) > 0$.

(iv) *If* $\mathcal{D}^\# = Q_K^\# = Q_{K,0}^\#$, *then* $K(0) = 0$.

We do not know whether the situation in (iv) can ever occur (see the question after Corollary 2.7).

Proof of Theorem 3.8. To prove (i), we assume that $K(0) > 0$ and note that $\mathcal{D}^\# \subset \mathcal{B}_0^\# = \mathcal{N}_0 \subset \mathcal{N}$. If K is bounded, it is clear that $Q_K^\# = \mathcal{D}^\#$. If K is unbounded, we use Theorem 3.3 and the fact that $Q_{K_1}^\# = \mathcal{D}^\#$ (we use the notation of Theorem 3.3) to deduce that $Q_K^\# = \mathcal{N} \cap Q_{K_1}^\# = \mathcal{N} \cap \mathcal{D}^\# = \mathcal{D}^\#$. This finishes the proof of (i).

The proof of (ii) uses the same argument as the proof of Theorem 2.7 except that we again use the fact that $\mathcal{D}^\# \subset \mathcal{B}_0^\# = \mathcal{N}_0$.

To prove (iii), we note that our assumptions imply that $D^\# \not\subset Q_{K,0}^\#$ and use (ii).

If the assumptions of (iv) hold, we have $\mathcal{D}^\# \subset Q_{K,0}^\#$, and the conclusion follows from (ii). □

Corollary 3.6. *$\mathcal{D}^\# \subset Q_{p,0}^\#$ for all p, $0 < p < \infty$.*

Acknowledgements. The second-named author is supported in part by the Väisälä and Wihuri Funds, Finland. He wishes to express his gratitude to the Department of Mathematics at Uppsala University, Sweden, for its hospitality during his visit in Uppsala in September, 2000. The authors would like to thank the referee for his/her helpful comments.

Note added in proof. Shortly before receiving the Galley proofs of this paper, the second author learned with deep regret that his collaborator, Professor Matts Essén, passed away on May 9, 2003. He dedicates this paper to the memory of Matts Essén.

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