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ON AN EXTENSION OF CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. We establish the L^p boundedness for a class of singular integral operators under the H^1 kernel condition. These operators were introduced by Duoandikoetxea and Rubio de Francia as an extension of the classical Calderón-Zygmund operators.

1. Introduction

In their well-known paper [1] Calderón and Zygmund treated the L^p boundedness problem of singular integral operators on $\mathbf{R}^n$ given by

$$T_\Omega : f \to \text{p.v.} \int_{\mathbf{R}^n} f(x-y) \frac{\Omega(y')}{|y|^n} dy, \tag{1.1}$$

where $y' = y/|y|$ for $y \neq 0$, $\Omega \in L^1(\mathbf{S}^{n-1})$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(y') d\sigma_n(y') = 0. \tag{1.2}$$

The measure $d\sigma_n$ in (1.2) is the normalized Lebesgue measure on $\mathbf{S}^{n-1}$.

It was shown in [1] that the L^p boundedness of T_Ω holds for $1 < p < \infty$ if $\Omega \in L \log L(\mathbf{S}^{n-1})$ and that the space $L \log L(\mathbf{S}^{n-1})$ cannot be replaced by any Orlicz space $L^\phi(\mathbf{S}^{n-1})$ with a monotonically increasing function ϕ satisfying $\phi(t) = o(t \log t)$, $t \to \infty$ (e.g., $L(\log L)^{1-\varepsilon}(\mathbf{S}^{n-1})$, $0 < \varepsilon \leq 1$). Using the method of rotations invented by Calderón and Zygmund, Connet and Ricci-Weiss obtained the following result independently:

THEOREM A ([4], [10]). *If $\Omega \in H^1(\mathbf{S}^{n-1})$, then T_Ω is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Here $H^1(\mathbf{S}^{n-1})$ represents the Hardy space over the unit sphere. Theorem A is an improvement over the result of Calderón and Zygmund because $H^1(\mathbf{S}^{n-1}) \supset L \log L(\mathbf{S}^{n-1})$. The condition $\Omega \in H^1(\mathbf{S}^{n-1})$ is a natural one in

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light of the general principle in harmonic analysis that H^1 is a natural substitute for L^1 (e.g., the Hilbert transform is bounded on L^p for $p > 1$ and on H^1, but not on L^1). In [5] Duoandikoetxea and Rubio de Francia introduced the following extension of the operators T_Ω:

Let $m, n \in \mathbf{N}$, $m \le n-1$, and $\mathcal{M}$ be a compact, smooth, m-dimensional manifold in $\mathbf{R}^n$. Suppose that $\mathcal{M} \cap \{rv : r > 0\}$ contains at most one point for any $v \in \mathbf{S}^{n-1}$. Let $\mathcal{C}(\mathcal{M})$ denote the cone $\{r\theta : r > 0,\, \theta \in \mathcal{M}\}$ equipped with the measure $ds(r\theta) = r^m dr d\sigma(\theta)$, where $d\sigma$ represents the induced Lebesgue measure on $\mathcal{M}$. For a locally integrable function in $\mathcal{C}(\mathcal{M})$ of the form

$$K(r\theta) = r^{-m-1} h(r)\Omega(\theta), \tag{1.3}$$

where Ω satisfies

$$\int_{\mathcal{M}} \Omega(\theta) d\sigma(\theta) = 0, \tag{1.4}$$

we define the corresponding singular integral operator $T_{\mathcal{M},\Omega}$ on $\mathbf{R}^n$ by

$$\begin{aligned}(T_{\mathcal{M},\Omega} f)(x) &= \text{p.v.} \int_{C(\mathcal{M})} f(x-y) K(y) ds(y) \\ &= \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \int_{\mathcal{M}} f(x - r\theta)\Omega(\theta) h(r) r^{-1} d\sigma(\theta) dr,\end{aligned} \tag{1.5}$$

initially for $f \in \mathcal{S}(\mathbf{R}^n)$.

Clearly the operators in (1.1) correspond to the case $m = n-1$ and $\mathcal{M} = \mathbf{S}^{n-1}$; i.e., $T_{\mathbf{S}^{n-1},\Omega} = T_\Omega$ when $h \equiv 1$.

Concerning the L^p boundedness of $T_{\mathcal{M},\Omega}$, the following was obtained in [5]:

Theorem B ([5]). *Let $T_{\mathcal{M},\Omega}$ be given as in* (1.3)–(1.5). *Suppose that*

(i) $\Omega \in L^q(\mathcal{M}, d\sigma)$ *for some* $q > 1$*;*

(ii) $\sup_{R>0} \left(\frac{1}{R} \int_0^R |h(r)|^2 dr \right) < \infty$*;*

(iii) $\mathcal{M}$ *has a contact of finite order with every hyperplane.*

Then $T_{\mathcal{M},\Omega}$ extends to a bounded operator on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

The factor $h(\cdot)$ in K had been introduced earlier by R. Fefferman [9]. In light of Theorems A and B, the following question arises naturally:

Question. *Is $T_{\mathcal{M},\Omega}$ still bounded on L^p spaces if condition* (i) *in Theorem B is replaced by the weaker condition $\Omega \in H^1(\mathcal{M})$?*

For a general $\mathcal{M}$, the only previously known case is $p = 2$, where one can establish the L^2 boundedness of $T_{\mathcal{M},\Omega}$ for $\Omega \in H^1(\mathcal{M})$ by using the results in [7]. The main purpose of this article is to show that the L^p boundedness of $T_{\mathcal{M},\Omega}$ holds for the entire range $1 < p < \infty$. Namely, we have the following result:

THEOREM C. *Let $T_{\mathcal{M},\Omega}$ be given as in Theorem B, and let h and $\mathcal{M}$ satisfy* (ii) *and* (iii)*, respectively. If $\Omega \in H^1(\mathcal{M})$, then $T_{\mathcal{M},\Omega}$ extends to a bounded operator on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

REMARKS. (a) The L^p boundedness of $T_{\mathcal{M},\Omega}$ can be established for a limited range of p if the exponent 2 in condition (ii) is replaced by a smaller $\gamma > 1$ (see Theorem 4.1). The L^p boundedness for p outside this range is not known.

(b) If the finite-type condition (iii) is dropped, then the L^p boundedness of $T_{\mathcal{M},\Omega}$ may fail even for $\Omega \in L^\infty$ and $p = 2$, as evidenced by the convex curve $\mathcal{M} = \{(t, e^{-1/t}) : 0 \le t \le \delta\}$ and surface of revolution $\mathcal{M} = \{(x_1, x_2, e^{-1/(x_1^2+x_2^2)}) : 0 \le |(x_1, x_2)| \le \delta\}$. See Section 4 for more details.

The paper is organized as follows. In Section 2 we recall the definition and atomic decomposition of Hardy spaces on $\mathcal{M}$. The main estimates are established in Section 3. The proof of Theorem C will appear in Section 4, along with some further results. The authors thank the referee for some helpful comments.

2. Hardy spaces over $\mathcal{M}$

The Hardy spaces $H^p(\mathcal{M})$ can be defined by using the maximal operator

$$\mathcal{A} : f \to (\mathcal{A}f)(x) = \sup_{t>0} |u(t,x)|,$$

where $u(t,x)$ is the solution of the boundary value problem

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta_x)u = 0, & (t,x) \in \mathbf{R}^+ \times \mathcal{M}, \\ u(0,x) = f(x), & x \in \mathcal{M}. \end{cases} \tag{2.1}$$

Here Δ_x denotes the Laplace-Beltrami operator of $\mathcal{M}$.

DEFINITION 2.1. We define

$$H^p(\mathcal{M}) = \{f \in \mathcal{S}'(\mathcal{M}) : \quad \|\mathcal{A}f\|_{L^p(\mathcal{M})} < \infty\}.$$

For $f \in H^p(\mathcal{M})$ we set $\|f\|_{H^p(\mathcal{M})} = \|\mathcal{A}f\|_{L^p(\mathcal{M})}$.

From [2] it is known that

$$H^p(\mathcal{M}) = L^p(\mathcal{M}) \subset H^1(\mathcal{M}) \subset L^1(\mathcal{M}) \tag{2.2}$$

when $p > 1$. The inclusions in (2.2) are proper.

Let $B_n(x,r) = \{y \in \mathbf{R}^n : |y - x| < r\}$.

DEFINITION 2.2. A function $a(\cdot)$ on $\mathcal{M}$ is called an H^1 atom if there are $\rho > 0$ and $\theta_0 \in \mathcal{M}$ such that

$$\begin{cases} \operatorname{supp}(a) \subset \mathcal{M} \cap B_n(\theta_0, \rho), \\ \int_{\mathcal{M}} a(\theta) d\sigma(\theta) = 0, \\ \|a\|_\infty \le \rho^{-m}. \end{cases} \tag{2.3}$$

The following can be established by using the arguments in [3]:

LEMMA 2.3. *If $\Omega \in H^1(\mathcal{M})$ and satisfies* (1.4)*, then there exist H^1 atoms $\{a_j\}$ and complex numbers $\{c_j\}$ such that*

$$\Omega = \sum_j c_j a_j$$

and

$$\|\Omega\|_{H^1(\mathcal{M})} \approx \sum_j |c_j|.$$

3. Main estimates

DEFINITION 3.1. A smooth mapping ϕ from an open set U in $\mathbf{R}^m$ into $\mathbf{R}^n$ is said to be of finite type at $u_0 \in U$ if, for every $\eta \in \mathbf{S}^{n-1}$, there exists a nonzero multi-index $\alpha = \alpha(\eta)$ such that

$$\left. \frac{\partial^\alpha [\eta \cdot \phi(u)]}{\partial u^\alpha} \right|_{u=u_0} \ne 0. \tag{3.1}$$

For a continuous mapping ϕ from a neighborhood of $\overline{B_m(0,1)}$ to $\mathbf{R}^n$, an integrable function $b(\cdot)$ on $B_m(0,1)$, and a measurable function h on $\mathbf{R}^+$, we define the family of measures $\{\sigma_{\phi,b,h,k} \mid k \in \mathbf{Z}\}$ on $\mathbf{R}^n$ by

$$\int_{\mathbf{R}^n} F(x) d\sigma_{\phi,b,h,k} = \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} F(r\phi(u)) r^{-1} b(u) h(r) du dr. \tag{3.2}$$

LEMMA 3.2. *Suppose that h satisfies* (ii) *in Theorem B. Then, for $1 < p < \infty$, there exists a constant $C_p > 0$ such that*

$$\left\| \left(\sum_{k \in \mathbf{Z}} |\sigma_{\phi,b,h,k} * g_k|^2 \right)^{1/2} \right\|_p \le C_p \|b\|_1 \left\| \left(\sum_{k \in \mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_p \tag{3.3}$$

holds for all continuous mappings ϕ and measurable functions $\{g_k\}$ on $\mathbf{R}^n$.

Proof. For $\xi \in \mathbf{R}^n$ we define the maximal operator M_ξ on $\mathbf{R}^n$ by

$$(M_\xi f)(x) = \sup_{k \in \mathbf{Z}} \left[2^{-k} \int_{2^k}^{2^{k+1}} |f(x + r\xi)| dr \right].$$

It follows from the L^p boundedness of the one-dimensional Hardy-Littlewood maximal operator that

$$\|M_\xi f\|_{L^p(\mathbf{R}^n)} \leq A_p \|f\|_{L^p(\mathbf{R}^n)} \tag{3.4}$$

for $1 < p \leq \infty$, where A_p is independent of ξ.

For $\{g_k\} \in L^p(\mathbf{R}^n, l^2)$, there exists a function $w \in L^{(p/2)'}(\mathbf{R}^n)$ such that $\|w\|_{(p/2)'} = 1$ and

$$\left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\phi,b,h,k} * g_k|^2 \right)^{1/2} \right\|_p^2 = \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} |\sigma_{\phi,b,h,k} * g_k|^2 \right) w(x)dx.$$

Thus, by Hölder's inequality and (3.4),

$$\begin{aligned}
&\left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\phi,b,h,k} * g_k|^2 \right)^{1/2} \right\|_p^2 \\
&\quad \leq C\|b\|_1 \sum_{k\in\mathbf{Z}} 2^{-k} \int_{\mathbf{R}^n} \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} |g_k(x - r\phi(u))|^2 |b(u)w(x)| dudrdx \\
&\quad = C\|b\|_1 \int_{B_m(0,1)} |b(u)| \left[\sum_{k\in\mathbf{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} \int_{\mathbf{R}^n} |g_k(x)|^2 |w(x + r\phi(u))| dxdr \right] du \\
&\quad \leq C\|b\|_1 \int_{B_m(0,1)} \left[\int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} |g_k(x)|^2 \right) (M_{\phi(u)} w)(x) dx \right] |b(u)| du \\
&\quad \leq C\|b\|_1^2 \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_p^2,
\end{aligned}$$

which proves the lemma. □

LEMMA 3.3. *Suppose that ϕ is smooth and of finite type at every point in $\overline{B_m(0,1)}$ and h satisfies* (ii) *in Theorem B. Then there exists a $\delta > 0$ such that*

$$|\hat{\sigma}_{\phi,b,h,k}(\xi)| \leq C(2^k|\xi|)^{-\delta} \|b\|_2 \tag{3.5}$$

holds for $\xi \in \mathbf{R}^n$ and $k \in \mathbf{Z}$.

In order to prove Lemma 3.3, we recall the following result from [7]:

LEMMA 3.4. *Let $\Psi : \mathbf{R} \times \mathbf{R}^m \times [0,1]^l \to \mathbf{R}$ be a smooth function and let $\psi \in C_0^\infty(\mathbf{R}^{m+1})$. For $\lambda \in \mathbf{R}$ and $\theta \in [0,1]^l$ define the operator $S_{\theta,\lambda}$ by*

$$(S_{\theta,\lambda} f)(t) = \int_{\mathbf{R}^m} e^{i\lambda\Psi(t,y,\theta)} \psi(t,y) f(y) dy. \tag{3.6}$$

Suppose that for each $(t, y) \in \mathrm{supp}(\psi)$ *and* $\theta \in [0,1]^l$ *there are* $k \in \mathbf{N}$ *and* $\alpha \in (\mathbf{N} \cup \{0\})^m$ *with* $|\alpha| \geq 1$ *such that* $\partial_t^k \partial_y^\alpha \Psi(t, y, \theta) \neq 0$. *Then there is a* $\sigma > 0$ *independent of* λ *and* θ *such that*

$$\|S_{\theta,\lambda} f\|_{L^{p'}(\mathbf{R})} \leq C_p (1 + |\lambda|)^{-\sigma/p'} \|f\|_{L^p(\mathbf{R}^m)}$$

for $p \in (1, 2]$ *and* $f \in L^p(\mathbf{R}^m)$. *The constant* C_p *is independent of* λ, θ *and* f.

Proof of Lemma 3.3. By (3.2)

$$|\hat{\sigma}_{\phi,b,h,k}(\xi)| = \left| \int_{2^k}^{2^{k+1}} \left(\int_{B_m(0,1)} e^{-ir\xi\cdot\phi(u)} b(u) du \right) r^{-1} h(r) dr \right|.$$

Let $\Psi(t, u, \theta) = t(\theta \cdot \phi(u))$. By writing

$$r\xi \cdot \phi(u) = (2^k |\xi|) \Psi(2^{-k} r, u, \xi/|\xi|)$$

and applying Lemma 3.4, there exists a $\sigma > 0$ such that

$$\begin{aligned} |\hat{\sigma}_{\phi,b,h,k}(\xi)| &\leq 2^{-k/2} \left(\int_{2^k}^{2^{k+1}} |h(r)|^2 dr \right)^{1/2} (1 + 2^k |\xi|)^{-\sigma/2} \|b\|_2 \\ &\leq C (2^k |\xi|)^{-\delta} \|b\|_2, \end{aligned}$$

where $\delta = \sigma/2 > 0$. □

By using the arguments in [7, pp. 140–142], we also have the following result:

Lemma 3.5. *Let* $b(\cdot)$ *be a function satisfying* $\mathrm{supp}(b) \subset B_m(0, \rho)$ *and* $\|b\|_\infty \leq \rho^{-m}$ *for some* $\rho < 1$. *Suppose that* h *satisfies* (ii) *in Theorem B. Then there exists a constant* $C > 0$ *such that*

$$\left| \int_{2^k}^{2^{k+1}} \left(\int_{B_m(0,\rho)} e^{-ir[Q(u) + \sum_{|\beta|=s} d_\beta u^\beta]} b(u) du \right) r^{-1} h(r) dr \right| \leq C \left(2^k \rho^s \sum_{|\beta|=s} |d_\beta| \right)^{-1/(4s)} \tag{3.7}$$

holds for all polynomials $Q : \mathbf{R}^m \to \mathbf{R}$ *with* $\deg(Q) < s$ *and* $\{d_\beta\} \subset \mathbf{R}$. *The constant* C *is independent of* ρ.

The following lemma is essentially Lemma 5.2 in [6], which has its roots in [5] and [8]. While condition (iv) below is weaker than the corresponding one in Lemma 5.2 of [6], the proofs are the same (see also [5, p. 544]).

Lemma 3.6. *Let* $l, n \in \mathbf{N}$ *and* $\{\sigma_{s,k} : 0 \leq s \leq l \text{ and } k \in \mathbf{Z}\}$ *be a family of measures on* $\mathbf{R}^n$ *with* $\sigma_{0,k} = 0$ *for every* $k \in \mathbf{Z}$. *Let* $\{\alpha_{sj} : 1 \leq s \leq l$ *and* $1 \leq$

$j \le 2\} \subset \mathbf{R}^+$, $\{\eta_s : 1 \le s \le l\} \subset \mathbf{R}^+ \backslash \{1\}$, $\{M_s : 1 \le s \le l\} \subset \mathbf{N}$, and $L_s : \mathbf{R}^n \to \mathbf{R}^{M_s}$ be linear transformations for $1 \le s \le l$. Suppose that

(i) *$\|\sigma_{s,k}\| \le 1$ for $k \in \mathbf{Z}$ and $1 \le s \le l$;*

(ii) *$|\hat{\sigma}_{s,k}(\xi)| \le C(\eta_s^k |L_s \xi|)^{-\alpha_{s2}}$ for $\xi \in \mathbf{R}^m, k \in \mathbf{Z}$ and $1 \le s \le l$;*

(iii) *$|\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s-1,k}(\xi)| \le C(\eta_s^k |L_s \xi|)^{\alpha_{s1}}$ for $\xi \in \mathbf{R}^n, k \in \mathbf{Z}$ and $1 \le s \le l$;*

(iv) *for some $p_0 > 2$ there exists a $C > 0$ such that*

$$\left\| \left(\sum_{k \in \mathbf{Z}} |\sigma_{s,k} * g_k|^2 \right)^{1/2} \right\|_{L^{p_0}(\mathbf{R}^n)} \le C \|f\|_{L^{p_0}(\mathbf{R}^n)}$$

for all $\{g_k\} \in L^{p_0}(\mathbf{R}^n, l^2)$ and $1 \le s \le l$.

Then for every $p \in (p_0', p_0)$, there exists a positive constant C_p such that

$$\| \sum_{k \in \mathbf{Z}} \sigma_{l,k} * f \|_{L^p(\mathbf{R}^m)} \le C_p \|f\|_{L^p(\mathbf{R}^m)} \tag{3.8}$$

and

$$\left\| \left(\sum_{k \in \mathbf{Z}} |\sigma_{l,k} * f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^m)} \le C_p \|f\|_{L^p(\mathbf{R}^m)} \tag{3.9}$$

hold for all $f \in L^p(\mathbf{R}^m)$. The constant C_p is independent of the linear transformations $\{L_s\}_{s=1}^l$.

The following is the main result of this section:

THEOREM 3.7. *Let ϕ and h be given as in Lemma 3.3. Suppose that, for some $u_0 \in B_m(0, 1/2)$ and $\rho < 1/2$, $b(\cdot)$ satisfies the following:*

$$\begin{cases} \operatorname{supp}(b) \subset B_m(u_0, \rho), \\ \|b\|_\infty \le \rho^{-m}, \\ \int_{B_m(u_0,\rho)} b(u) du = 0. \end{cases} \tag{3.10}$$

Then, for $1 < p < \infty$, there exists a $A_p > 0$ such that

$$\left\| \sum_{k \in \mathbf{Z}} \sigma_{\phi,b,h,k} * f \right\|_p \le A_p \|f\|_p \tag{3.11}$$

holds for $f \in L^p(\mathbf{R}^n)$. The constant A_p is independent of u_0 and ρ.

Proof. By Lemma 3.3, there exists a $\delta > 0$ such that

$$|\hat{\sigma}_{\phi,b,h,k}(\xi)| \le C(2^k |\xi|)^{-\delta} \rho^{-m/2}. \tag{3.12}$$

Let $l = [m/(2\delta)] + 1$. We define a sequence of mappings $\{\Phi^s\}_{s=0}^l$ by

$$\Phi^l = \phi = (\phi_1, \ldots, \phi_n)$$

and

$$\Phi^s(u) = \left(\sum_{|\beta| \le s} \frac{1}{\beta!} \frac{\partial^\beta \phi_1(u_0)}{\partial u^\beta} (u - u_0)^\beta, \dots, \sum_{|\beta| \le s} \frac{1}{\beta!} \frac{\partial^\beta \phi_n(u_0)}{\partial u^\beta} (u - u_0)^\beta \right)$$

for $s = 0, 1, \dots, l - 1$. Let

$$\sigma_{s,k} = \sigma_{\Phi^s, b, h, k}$$

for $0 \le s \le 1$ and $k \in \mathbf{Z}$. By its definition and Lemma 3.2, the family of measures $\{\sigma_{s,k}\}$ satisfies conditions (i) and (iv) in Lemma 3.6, for any $p_0 > 2$. For $j = 1, \dots, n$, let

$$d_{j,\beta} = \frac{1}{\beta!} \frac{\partial^\beta \phi_j(u_0)}{\partial u^\beta}.$$

By (3.12) and Lemma 3.5, we have

$$|\hat{\sigma}_{l,k}(\xi)| \le C(2^k \rho^l |\xi|)^{-\delta}$$

and

$$|\hat{\sigma}_{s,k}(\xi)| \le C \left[2^k \rho^s \sum_{|\beta| = s} |\sum_{j=1}^n d_{j\beta} \xi_j| \right]^{-1/(4s)}$$

for $1 \le s \le l - 1$, $k \in \mathbf{Z}$ and $\xi \in \mathbf{R}^n$. On the other hand, we have

$$\begin{aligned} |\hat{\sigma}_{l,k}(\xi) - \hat{\sigma}_{l-1,k}(\xi)| &\le C 2^k |\xi| \int_{B_m(u_0,\rho)} |\phi(u) - \Phi^{l-1}(u)| |b(u)| du \\ &\le C(2^k |\xi| \rho^l) \end{aligned}$$

and

$$\begin{aligned} |\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s-1,k}(\xi)| &\le C 2^k \int_{B_m(u_0,\rho)} |\xi \cdot (\Phi^s(u) - \Phi^{s-1}(u))| |b(u)| du \\ &\le C 2^k \rho^s \sum_{|\beta| = s} |\sum_{j=1}^n d_{j\beta} \xi_j| \end{aligned}$$

for $1 \le s \le l - 1$, $k \in \mathbf{Z}$ and $\xi \in \mathbf{R}^n$. In addition, it follows from the cancellation property in (3.10) that $\sigma_{0,k} = 0$ for $k \in \mathbf{Z}$. One then obtains (3.11) by invoking Lemma 3.6. Theorem 3.7 is proved. □

4. Conclusion

Proof of Theorem C. By Theorem B and Lemma 2.3, it suffices to prove the L^p boundedness of $T_{\mathcal{M},a}$ when a is an atom satisfying (2.3) with a sufficiently small ρ. By the smoothness and compactness of $\mathcal{M}$ we may assume that there is a smooth mapping ϕ from a neighborhood of $\overline{B_m(0,1)}$ into $\mathbf{R}^n$ such that

(i) $\theta_0 \in \phi(B_m(0, 1/2))$ and $\mathcal{M} \cap B_n(\theta_0, \rho) \subset \phi(B_m(0, 1)) \subset \mathcal{M}$;
(ii) the vectors $\partial\phi/\partial u_1, \dots, \partial\phi/\partial u_m$ are linearly independent for each $u \in \overline{B_m(0,1)}$;

(iii) ϕ is of finite type at every point in $\overline{B_m(0,1)}$ (see [11, p. 350]).

Thus there is a smooth function $J(u)$ such that

$$\int_{\phi(B_m(0,1))} F d\sigma = \int_{B_m(0,1)} F(\phi(u))J(u)du$$

for any integrable function F on $\mathcal{M}$. We have

$$T_{\mathcal{M},a} f = \sum_{k\in\mathbf{Z}} \sigma_{\phi,b,h,k} * f,$$

where $b(u) = a(\phi(u))J(u)\chi_{B_m(0,1)}$. Let $u_0 = \phi^{-1}(\theta_0)$. It follows from (i)–(iii) that

$$\begin{cases} \mathrm{supp}(b) \subset B_m(u_0, c\rho), \\ \|b\|_\infty \le C\rho^{-m}, \\ \int_{\mathbf{R}^m} b(u)du = 0. \end{cases}$$

By applying Theorem 3.7, we obtain the L^p boundedness of $T_{\mathcal{M},a}$ with a bound independent of ρ for $1 < p < \infty$. Theorem C is proved. □

Theorem C admits the following generalization.

THEREOM 4.1. *Let $T_{\mathcal{M},\Omega}$ be given as in* (1.3)–(1.5). *Suppose that*

(i) $\Omega \in H^1(\mathcal{M}, d\sigma)$*;*

(ii) $\sup_{R>0} \left(\frac{1}{R}\int_0^R |h(r)|^\gamma dr\right) < \infty$ *for some* $\gamma > 1$*;*

(iii) *$\mathcal{M}$ has a contact of finite order with every hyperplane.*

Then $T_{\mathcal{M},\Omega}$ extends to a bounded operator on $L^p(\mathbf{R}^n)$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

As usual, the pointwise existence of $T_{\mathcal{M},\Omega} f$ for f in L^p spaces can be established by considering the following maximal truncated singular integral:

$$(T^*_{\mathcal{M},\Omega} f)(x) = \sup_{\varepsilon>0} \left| \int_\varepsilon^\infty \int_{\mathcal{M}} f(x - r\theta)\Omega(\theta)h(r)r^{-1}d\sigma(\theta)dr \right|. \tag{4.1}$$

THEOREM 4.2. *Let $\Omega, \mathcal{M}$ be given as in Theorem 4.1 and $h \in L^\infty(\mathbf{R}^+)$. Then the operator $T^*_{\mathcal{M},\Omega}$ given in* (4.1) *is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

The proofs are omitted.

We conclude the paper by addressing the failure of L^2 boundedness of $T_{\mathcal{M},\Omega}$ in the absence of the finite-type assumption. By letting

$$\mathcal{M} = \{(x_1, x_2, e^{-1/(x_1^2+x_2^2)}) : 0 \le |(x_1, x_2)| \le 2\delta_0\}$$

for some $\delta_0 > 0$ and selecting a suitable Ω (which can be L^∞ or better), the L^2 unboundedness would follow if the following holds:

$$\limsup_{\varepsilon\to 0,\, N\to\infty} \left| \int_1^N \int_\varepsilon^{\delta_0} \cos(re^{-1/s^2}) s ds \frac{dr}{r} \right| = \infty. \tag{4.2}$$

Since

$$\int_a^b \frac{\cos u}{u} du \geq -\ln(a) - 4$$

holds for $0 < a < 1 < b$, for $N > e^{1/\varepsilon^2}$ we have

$$\int_1^N \int_\varepsilon^{\delta_0} \cos(re^{-1/s^2}) sds \frac{dr}{r} \geq \ln(1/\varepsilon) - C(\delta_0),$$

which implies (4.2).

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