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VARIATIONALLY COMPLETE ACTIONS ON NONNEGATIVELY CURVED MANIFOLDS

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ABSTRACT. As an application of a more general result on singular Riemannian foliations we prove that variationally complete actions on nonnegatively curved manifolds are hyperpolar.

1. Introduction

We shall prove the following theorem.

THEOREM A. *A variationally complete action on a complete Riemannian manifold with nonnegative sectional curvature is hyperpolar.*

The concepts used in the announcements of the two theorems stated in the introduction will be defined in Section 2.

Theorem A was proved in [6] for actions on Euclidean spaces and in [9] for actions on compact symmetric spaces. In the trivial case when the group acting only consists of the identity, Theorem A reduces to the well known fact that a complete Riemannian manifold without conjugate points and with nonnegative sectional curvature is flat. The proof below can be seen as a generalization of the proof of this case.

The proof of Theorem A uses very little about group actions. In Section 3 we will prove the following theorem that immediately implies Theorem A.

THEOREM B. *A singular Riemannian foliation without horizontal conjugate points in a complete Riemannian manifold with nonnegative sectional curvature admits flat sections.*

We would like to thank Burkhard Wilking for pointing out to us his preprint [14], which contains the essential tools used to prove the above theorems.

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2. Isometric actions and singular Riemannian foliations

Variationally complete actions were first introduced by Bott in [2] and later in a slightly different way in [3]. We will use the definition in [2].

Let M be a complete Riemannian manifold and G a connected Lie group acting on M by isometries. Let N be an orbit of the action and let γ be a geodesic that is orthogonal to N at time t_0. An *N-Jacobi field along* γ is a Jacobi field that is the variational vector field of a variation through geodesics all meeting N orthogonally at time t_0. The action of G on M is said to be *variationally complete* if the following holds for all orbits N of G and all geodesics γ meeting N orthogonally: Any N-Jacobi field J along γ that is tangent to some orbit of G different from N is the restriction to γ of a Killing field induced by the action of G.

Hyperpolar actions were introduced by Conlon in [5] using a terminology that is different from the one that we will adopt here.

Again let M be a complete Riemannian manifold and G a connected Lie group acting on M by isometries. Then the action of G on M is said to be *polar* if there is a complete submanifold Σ in M meeting all orbits in such a way that all intersections between Σ and an orbit are orthogonal. The submanifold Σ is called a *section.* It follows that a section is totally geodesic and that its dimension is equal to the cohomogeneity of the action. If there is a flat section, one says that the action is *hyperpolar.* Note that we assume neither that Σ is properly embedded nor that it is free of self-intersections.

Conlon proved in [5] that hyperpolar actions are variationally complete. The converse is not true. A trivial example of a variationally complete action that is not hyperpolar is the action of $G = \{e\}$ on a Riemannian manifold without conjugate points which is not flat.

We will now introduce singular Riemannian foliations and show how the concepts of variationally complete and hyperpolar actions can be generalized.

Following [11, p. 189], we say that a partition $\mathcal{F}$ of a Riemannian manifold M into disjoint connected submanifolds, called *leaves*, is a *singular Riemannian foliation* if it satisfies the following two conditions.

(1) The tangent space of the leaf through p is equal to $\{X_p \mid X \in \Xi_{\mathcal{F}}\}$ for all p in M, where $\Xi_{\mathcal{F}}$ denotes the space of vector fields X on M with the property that X_p lies in the tangent space of the leaf through p for all p in M.
(2) A geodesic that hits one leaf perpendicularly remains perpendicular to all leaves it passes through.

A partition $\mathcal{F}$ of M that only satisfies the condition in (2) is called a *transnormal system.*

Now let G be a connected Lie group acting on a Riemannian manifold M by isometries and let $\mathcal{F}$ be the partition of M into the orbits of G. Then it is not difficult to show that $\mathcal{F}$ is a singular Riemannian foliation; see [11, p. 188–189].

We now generalize the notion of variationally complete actions to singular Riemannian foliations.

Let M be a Riemannian manifold with a singular Riemannian foliation $\mathfrak{F}$. We say that $\mathfrak{F}$ is *without horizontal conjugate points* if the following is true for all leaves N and all geodesics γ meeting N perpendicularly. Any N-Jacobi field J along γ that is tangent to a leaf of $\mathcal{F}$ different from N is tangent to all leaves γ passes through.

The following proposition shows that Theorem A is a consequence of Theorem B.

PROPOSITION 2.1. *Let G be a Lie group acting on a complete Riemannian manifold by isometries and let $\mathcal{F}$ be the singular Riemannian foliation on M induced by the action. Then the action of G is variationally complete if and only if $\mathcal{F}$ is without horizontal conjugate points.*

Proof. Let N be an orbit of G that a geodesic γ meets perpendicularly, and let J be an N-Jacobi field that is tangent to an orbit different from N.

We first assume that the action is variationally complete. Then it is clear that $\mathcal{F}$ does not have horizontal conjugate points since J is the restriction to γ of a Killing field and hence tangent to all orbits that γ meets.

Conversely, assume that $\mathcal{F}$ is without horizontal conjugate points. Then we know that J is tangent to all orbits. We have to prove that J is the restriction of a Killing field induced by the action. Let $\gamma(t_0)$ be a point on γ that is not a focal point of N and let $\tilde{J}$ be a restriction of a Killing field to γ with the property that $\tilde{J}(t_0) = J(t_0)$. Then $\tilde{J}$ is an N-Jacobi field along γ, and it follows that the difference $S = J - \tilde{J}$ is an N-Jacobi field which vanishes in t_0. This implies that S vanishes identically since $\gamma(t_0)$ is not a focal point of N and it follows that J is the restriction of a Killing field to γ. □

We will say that a singular Riemannian foliation $\mathcal{F}$ *admits sections* (see [1] and [13]) if for every p in M there is a complete submanifold Σ in M containing p and meeting all leaves in $\mathcal{F}$ in such a way that all intersections between Σ and a leaf are orthogonal. It is clear that a hyperpolar group action gives rise to a singular Riemannian foliation admitting flat sections. It is also clear from the terminology that the submanifolds Σ will also here be called *sections*. It follows that the dimension of a section is equal to the minimal codimesnion of a leaf in $\mathcal{F}$.

A leaf in a singular Riemannian foliation $\mathcal{F}$ is said to be *singular* if its dimension is not maximal in $\mathcal{F}$, and *regular* otherwise. A point in M is said to be *singular* if it lies in a singular leaf, and *regular* otherwise. The set M_r of regular points is open, connected and dense; see [11, p. 197].

It is clear that the set Σ_r of regular points in a section Σ is open and one can prove that it is also dense. As a consequence it follows that Σ is totally geodesic. We omit the proofs since these facts will not be used here.

The following theorem is a slight generalization of the theorem of Conlon in [5].

THEOREM 2.2. *Let $\mathcal{F}$ be a singular Riemannian foliation in a complete Riemannian manifold admitting flat sections. Then $\mathcal{F}$ is without horizontal conjugate points.*

We omit the proof of Theorem 2.2 since it is easy. Notice that we could as well have assumed in the theorem that $\mathcal{F}$ admits sections without conjugate points.

We will now discuss some results that will be used in the proof of Theorem B.

For p in M_r we let $\mathcal{H}_p$ denote the normal space at p of the leaf through p. Note that the dimension of $\mathcal{H}_p$ does not depend on p in M_r since we are restricting ourselves to the set of regular points. It is also clear that $\mathcal{H}_p$ depends differentiably on p in M_r. The collection of subspaces $\{\mathcal{H}_p \mid p \in M_r\}$ is therefore a distribution over M_r, which we denote by $\mathcal{H}$. The tangent space of the orbit through a point p in M_r will be denoted by $\mathcal{V}_p$ and the corresponding distribution over M_r by $\mathcal{V}$. We will refer to $\mathcal{H}$ and $\mathcal{V}$ as the *horizontal* and *vertical distributions* of the singular foliation $\mathcal{F}$.

If the singular Riemannian foliation $\mathcal{F}$ admits sections, then the distribution $\mathcal{H}$ is clearly integrable and the integral manifolds are the intersections of the sections of $\mathcal{F}$ with M_r. We will prove in Section 3 that the horizontal distribution of a singular Riemannian foliation without horizontal conjugate points is integrable and then extend the integral manifolds through the singular points to complete sections.

O'Neill associated two tensors to Riemannian submersions in [12], one of which will turn out to be very convenient for us in proving the integrability of the horizontal distribution in the proof of Theorem B. In our case the Riemannian submersion in question will be the canonical projection of M_r onto the space $M_r/\mathcal{F}$ of leaves endowed with the quotient metric.

We now define the O'Neill tensor that is relevant for the proof. Let X be a tangent vector in TM_r. Then X can be written as an orthogonal sum of a vector X^v in $\mathcal{V}$ and a vector X^h in $\mathcal{H}$. We will call X^v the *vertical* and X^h the *horizontal component of* X. Now let X and Y be vector fields (locally defined) on M_r. Then we set

$$A_X Y = (\nabla_{X^h} Y^h)^v + (\nabla_{X^h} Y^v)^h.$$

It is easy to see that A is a tensor of type $(1, 2)$. It is one of the two fundamental tensors that O'Neill associated to a Riemannian submersion. We list some properties of the O'Neill tensor A from [12].

(1) $A_X(\mathcal{H}) \subset \mathcal{V}$ and $A_X(\mathcal{V}) \subset \mathcal{H}$ for all X.
(2) A_X is a skew symmetric operator on the tangent spaces of M_r.
(3) If X and Y are horizontal vector fields, then

$$A_X Y = \frac{1}{2}[X, Y]^v.$$

Property (3) implies that $\mathcal{H}$ is integrable if and only if $A_X(\mathcal{H}) = 0$ for horizontal X. It follows from (1) and (2) that $A_X(\mathcal{H}) = 0$ if and only if $A_X(\mathcal{V}) = 0$.

3. The proof of Theorem B

We will assume in this section that $\mathcal{F}$ is a singular Riemannian foliation of M without horizontal conjugate points. Our first goal will be to show that the O'Neill tensor A_X vanishes on M_r. For the proof we will need the tensor $A_{\dot\gamma(t)}$ along a geodesic γ that passes through M_r. If $\gamma(t)$ is singular, then $A_{\dot\gamma(t)}$ is not yet defined. This difficulty was overcome by Wilking in [14] in a more general setting.

Let γ be a complete geodesic in M that is orthogonal to one and hence to all leaves of $\mathcal{F}$ it meets. We will denote the leaf through $\gamma(t)$ by N_t and assume that N_0 is regular. We let $\mathcal{J}$ denote the set of all N_t-Jacobi fields along γ for all t. Let $\mathcal{K}$ denote the subset of $\mathcal{J}$ consisting of Jacobi fields J with the property that $J(t)$ is tangent to N_t for all t.

PROPOSITION 3.1. *We have*

$$T_{\gamma(t)} N_t = \{J(t) \mid J \in \mathcal{K}\}$$

for all t.

Proof. In the following let U_t denote a sufficiently small neighborhood of $\gamma(t)$ in N_t. Let v be a tangent vector in $T_{\gamma(t_0)} N_{t_0}$ and let $\alpha(s)$ in U_{t_0} be a curve passing through $\gamma(0)$ tangent to v. Let U_{t_1} for $t_1 > t_0$ be within a tubular neighborhood of U_{t_0} and let $\pi : U_{t_1} \to U_{t_0}$ be the orthogonal projection. Let $\beta(s)$ be a curve in U_{t_1} that π projects onto $\alpha(s)$. Let γ_s be the variation of γ defined by letting $\gamma_s|_{[t_0,t_1]}$ be the shortest connection between $\alpha(s)$ and $\beta(s)$ with $||\dot\gamma_s|| = ||\dot\gamma||$. Let J be the corresponding Jacobi field. Clearly $v = J(t_0)$ and $J \in \mathcal{J}$. The Jacobi field J is an N_{t_0}-Jacobi field that is also tangent to the leaf N_{t_1}. Hence J is in $\mathcal{K}$ and the claim in the proposition follows. □

If $\gamma(t)$ lies in M_r, then the vertical space $\mathcal{V}_{\gamma(t)}$ was defined above as the tangent space $T_{\gamma(t)} N_t$. We set $\mathcal{V}_{\gamma(t)} = \mathcal{V}_t$. By Proposition 3.1 we have

$$\mathcal{V}_t = \{J(t) \ \mid \ J \in \mathcal{K}\}$$

Following Wilking [14] we now extend the definition of $\mathcal{V}_t$ to all real numbers t by setting

$$\mathcal{V}_t = \{J(t) \ \mid \ J \in \mathcal{K}\} \oplus \{J'(t) \ \mid \ J \in \mathcal{K}, \ J(t) = 0\}.$$

PROPOSITION 3.2. *The dimension of* $\mathcal{V}_t$ *does not depend on* t *and* $\mathcal{V}_t = T_{\gamma(t)}N_t$ *when* $\gamma(t)$ *is regular. In particular,* $\dim \mathcal{V}_t = \dim \mathcal{K}$. *Furthermore, the bundle* $\mathcal{V}_\gamma$ *over* $\mathbf{R}$ *with fibers* $\mathcal{V}_t$ *is smooth.*

Proof. We only make a few remarks on the proof.

The space $\{J'(t_0) \mid J \in \mathcal{K},\ J(t_0) = 0\}$ is trivial if $\gamma(t_0)$ is regular. To see this assume that there is a $J \in \mathcal{K}$ such that $J(t_0) = 0$, $J'(t_0) \neq 0$ and $\gamma(t_0)$ is regular. Then $\gamma(t_0)$ is a focal point along γ of all leaves N_t. This leads to a contradiction since N_{t_0} is regular.

In the proof of the smoothness of the bundle $\mathcal{V}_\gamma$ the following observation is essential. If $J(t)$ is a smooth vector field along γ that vanishes in t_0, then the vector field I defined by setting $I(t) = \frac{1}{t-t_0}J(t)$ if $t \neq t_0$ and $I(t_0) = J'(t_0)$ is smooth. (Assume that u is a C^k-function defined on a neighborhood I of t_0 satisfying $u(t_0) = 0$. Then there is a C^{k-1}-function v on I such that $u(t) = (t - t_0)v(t)$ for all $t \in I$.) □

We let $\mathcal{H}_t$ denote the orthogonal complement of $\mathcal{V}_t$ in $T_{\gamma(t)}M$ for every t in $\mathbf{R}$ and denote the corresponding bundle by $\mathcal{H}_\gamma$. We extend the definition of *vertical* and *horizontal components* to the splitting $T_{\gamma(t)}M = \mathcal{V}_t \oplus \mathcal{H}_t$ for all t and continue to denote the components of X by X^v and X^h, respectively.

If $\gamma(t)$ lies in M_r we set $A_t = A_{\dot\gamma(t)}$, where A is the O'Neill tensor defined in Section 2. We now extend the definition of A_t to all t. Let $X(t)$ be a vector field along γ. We set

$$A_t(X(t)) = ((X^h)'(t))^v + ((X^v)'(t))^h.$$

This definition clearly agrees with the one above if $\gamma(t)$ is regular. The tensor A_t is clearly skew symmetric for all t.

Now let J be a Jacobi field in $\mathcal{J}$ and set

$$Y = J^h.$$

If J is in $\mathcal{K}$, then Y vanishes identically. If Y vanishes at two different regular points, then it vanishes everywhere and J belongs to $\mathcal{K}$ by the definition of singular Riemannian foliations without horizontal conjugate points. (Notice that if Y vanishes at two different points, one of which, say t_0, is singular, then it does not necessarily follow that $J(t_0)$ lies in $T_{\gamma(t_0)}N_{t_0}$, only that it is contained in the larger space $\mathcal{V}_{t_0}$.)

The strategy of the proof will be the following. If A_t is not identically zero, then there is a Jacobi field J in $\mathcal{J}$ such that the corresponding field Y does not vanish identically although it vanishes at two different regular points, contradicting that $\mathcal{F}$ does not have horizontal conjugate points.

We will need a differential equation that a vector field of the type $Y = J^h$ satisfies. Such an equation is derived in [14]. If we let ∇_t^h denote the induced connection in $\mathcal{H}_\gamma$, then the equation is

$$(\nabla_t^h)^2 Y(t) + (R(Y(t), \dot\gamma(t))\dot\gamma(t))^h - 3A_t^2 Y(t) = 0.$$

This formula can be best understood by comparing it with the O'Neill formula for the sectional curvature of the quotient space M/G. Let X and Y form an orthonormal basis of the two-plane σ in $\mathcal{H}_p$, where p is a regular point. We set $\sigma^* = d\pi(\sigma)$, where $\pi : M \to M/G$ is the canonical projection. Then by [12, p. 465] the sectional curvature of σ^* is

$$\langle (R(Y, X)X)^h - 3A_X^2 Y, Y\rangle = \langle R(Y, X)X, Y\rangle + 3\|A_X Y\|^2,$$

where we have used the skew symmetry of A_X. We now define a differentiable family $\mathcal{R}(t)$ of endomorphisms of $\mathbf{R}^k$ that are self-adjoint with respect to the canonical inner product on $\mathbf{R}^k$. Let $X_1, \ldots, X_k$ be vector fields in $\mathcal{H}_t$ along γ which are parallel with respect to ∇_t^h and form an orthonormal basis of $\mathcal{H}_{\gamma(t)}$ together with $\dot\gamma(t)$ for every t. Now let $v(t) = (v_1(t), \ldots, v_k(t))$ be a curve in $\mathbf{R}^k$. We associate to $v(t)$ the horizontal vector field $\Phi v(t) = v_1(t)X_1(t) + \cdots + v_k(t)X_k(t)$ along γ. We get a map Φ which is clearly a bijection between curves in $\mathbf{R}^k$ and horizontal vector fields along γ. Now we set

$$\mathcal{R}(t)v(t) = \Phi^{-1}[(R(Y(t), \dot\gamma(t))\dot\gamma(t))^h - 3A_t^2 Y(t)],$$

where we have set $Y(t) = \Phi v(t)$. It is clear that $\mathcal{R}(t)$ is self-adjoint and that

$$\langle \mathcal{R}(t)v(t), v(t)\rangle \geq 0$$

for all t and all v. The differential equation

$$v''(t) + \mathcal{R}(t)v(t) = 0$$

is an example of a Morse-Sturm system; see [7] and [8]. Associated to the Morse-Sturm system is the index form

$$I_{a,b}(v, w) = \int_a^b \langle v', w'\rangle - \langle \mathcal{R}(t)v, w\rangle \, dt,$$

where we assume v and w to belong to $H_0^1([a, b]; \mathbf{R}^k)$, the space of absolutely continuous paths with square integrable derivative and vanishing in a and b.

The idea is now to show that the index of $I_{a,b}$ is positive on an appropriate interval $[a, b]$ if A_t does not vanish identically, and then use that to arrive at the contradiction that there is a Jacobi field J in $\mathcal{J}$ such that the corresponding vector field Y vanishes in two different regular points without J belonging to $\mathcal{K}$.

Now assume that there is a t_0 such that A_{t_0} does not vanish. Then there is a vector v_0 in $\mathbf{R}^k$ such that $\langle \mathcal{R}(t_0)v_0, v_0\rangle > 0$ and

$$C = \int_{t_0-1}^{t_0+1} \langle \mathcal{R}(t)v_0, v_0\rangle \, dt > 0.$$

Now there is a possibly quite large number $N > 0$ and a smooth real-valued function ϕ on $\mathbf{R}$ that vanishes outside of $[t_0 - N, t_0 + N]$, is identically equal to 1 on $[t_0-1, t_0+1]$ and satisfies

$$\int_{t_0-N}^{t_0+N} \phi'(t)^2\, dt < C.$$

Set $v(t) = \phi(t)v_0$. Then clearly

$$I_{a,b}(v,v) < 0,$$

where $a = t_0 - N$ and $b = t_0 + N$, and we have proved that $I_{a,b}$ has a positive index.

Now the Morse-Sturm Oscillation Theorem (see [7] and [8]), which is an analogue of the Morse Index Theorem, tells us that there is a nonvanishing solution $v(t)$ of

$$v''(t) + \mathcal{R}(t)v(t) = 0$$

such that $v(a) = 0$ and $v(c) = 0$ for some c in the open interval (a, b). We let Y denote the $\mathcal{H}$-valued vector field $Y = \Phi v(t)$ along γ. Then the nonvanishing vector field Y satisfies the differential equation

$$(\nabla_t^h)^2 Y(t) + (R(Y(t), \dot\gamma(t))\dot\gamma(t))^h - 3A_t^2 Y(t) = 0$$

and vanishes in a and c. We can assume that both $\gamma(a)$ and $\gamma(c)$ are regular after moving a slightly if necessary.

We will now show that there is a Jacobi field J in $\mathcal{J}$ such that $J^h = Y$. We consider the leaf N_a through $\gamma(a)$. There is an N_a-Jacobi field J along γ such that $J(a) = 0$ and $J'(a) = \nabla_t^h Y(a)$. Set $\hat Y = J^h$. Then Y and $\hat Y$ are both solutions of the same second order differential equation and satisfy $Y(a) = \hat Y(0)$ as well as $\nabla_t^h Y(a) = \nabla_t^h \hat Y(a)$. By the uniqueness of such solutions we have that $Y = \hat Y$. This finishes the proof that A_t vanishes identically and hence that the horizontal distribution $\mathcal{H}$ is integrable over M_r.

Before we continue we remark that the arguments we have been going through can clearly be used to prove the following proposition.

PROPOSITION 3.3. *A complete Riemannian manifold without conjugate points and with nonnegative curvature is flat.*

We now need to extend the integral manifolds of $\mathcal{H}$ through the singular points and show that the singular Riemannian foliation $\mathcal{F}$ admits sections. If $\mathcal{F}$ is induced by an isometric action, then it is proved in Appendix A in [10] that this can be done. There is also a similar result in [2] for general singular Riemannian foliations, but we cannot apply it because of a compactness assumption which is not necessarily satisfied here.

We continue to assume that $\mathcal{F}$ is a singular Riemannian foliation without horizontal conjugate points. We will also use the same notation as above. In particular, we let γ denote a geodesic orthogonal to the leaves it meets, we denote its horizontal bundle by $\mathcal{H}_\gamma$, and so on. We assume that γ passes through some regular points. Then we have the following proposition.

PROPOSITION 3.4.

(i) *A point* $\gamma(t_0)$ *is a focal point of* N_t *along* γ *if and only if* N_{t_0} *is a singular leaf. In particular, the singular points on* γ *are isolated.*

(ii) *The induced covariant connection* ∇_t^h *in* $\mathcal{H}_\gamma$ *coincides with the covariant derivative* ∇_t *along* γ*.*

(iii) *Let* J *be a Jacobi field along* γ *such that* $J(t_0)$ *and* $J'(t_0)$ *lie in* $\mathcal{H}_{t_0}$*. Then* $J(t) \in \mathcal{H}_t$ *for all* t*.*

(iv) *The integral manifolds of the horizontal distribution* $\mathcal{H}$ *over* M_r *are totally geodesic.*

Proof. (i) Let J be an N-Jacobi field that vanishes in a point t_0. Then J is tangent to N_{t_0} and it follows from the definition of singular Riemannian foliations without horizontal conjugate points that J is tangent to all leaves γ passes through, i.e., $J \in \mathcal{K}$. If $\gamma(t_0)$ is a regular point, then $J'(t_0) = 0$; see the proof of Proposition 3.2. It follows that $\gamma(t_0)$ is a singular point if it is a focal point.

Conversely assume that $\gamma(t_0)$ is a singular point. Then $\dim N_{t_0} < \dim \mathcal{K}$; see Proposition 3.2. Hence there is a nonvanishing Jacobi field J in $\mathcal{K}$ with $J(t_0) = 0$. It follows that $\gamma(t_0)$ is a focal point of N_t.

The focal points of N_t along γ are isolated as a consequence of the Morse Index Theorem. It follows that the singular points are isolated along γ.

(ii) Let X be a vector field along γ with values in $\mathcal{H}_\gamma$. Since $A_t = 0$ we have that $X'(t)^v = 0$ for all t by the definition of A_t. This proves the claim.

(iii) It follows from (ii) that there are parallel vector fields $X_1, \ldots, X_k$ along γ with values in $\mathcal{H}_\gamma$ forming an orthonormal base of $\mathcal{H}_t$ for every t. It also follows from (ii) that $R(X_i, \dot\gamma)\dot\gamma$ is a linear combination of the $X_1, \ldots, X_k$. This implies that we can restrict the Jacobi equation along γ to vector fields with values in $\mathcal{H}_\gamma$. This implies the claim.

(iv) This follows from (ii). □

We are now in a position to finish the proof of Theorem B.

Let $U \subset M_r$ be a connected integral manifold of $\mathcal{H}$ and let $p \in U$. We set

$$f = \exp_p|_{\mathcal{H}_p} : \mathcal{H}_p \to M.$$

We will prove that f is a complete section of $\mathcal{F}$, which is then clearly an extension of U since U is totally geodesic by Proposition 3.4 (iv).

We first show that f is an immersion. We need to show that

$$(d\exp_p)_v w \neq 0$$

for all nonvanishing v and w in $\mathcal{H}_p$. We use that $J(t) = (d\exp_p)_{tv} tw$ is a Jacobi field along the horizontal geodesic $\gamma(t) = \exp_p tv$ satisfying $J(0) = 0$ and $J'(0) = w$ in $\mathcal{H}_p$. It follows from Proposition 3.4 (iii) that $J(t)$ lies in $\mathcal{H}_t$ for all t. Now it follows that $J(t) \neq 0$ for $t \neq 0$ since $\mathcal{F}$ does not have horizontal conjugate points. This implies that f is an immersion.

We now show that all intersections between f and leaves are perpendicular or equivalently that

$$T_v = \{(d\exp_p)_v w |\, w \in \mathcal{H}_p\}$$

is perpendicular to the leaf through $q = \exp_p v$ for all v in $\mathcal{H}_p$. The arguments that we used to prove that f is an immersion show that $T_v = \mathcal{H}_1$, where $\mathcal{H}_1$ is the horizontal space at $\gamma(1)$ along the geodesic $\gamma(t) = \exp_p tv$. The space $\mathcal{H}_1$ is contained in the normal space of the leaf through q. Hence it follows that all intersections between f and leaves are perpendicular.

We finally have to show that the image of f meets all leaves of $\mathcal{F}$. The reader should keep in mind that we do not assume that the leaves of $\mathcal{F}$ are properly embedded, an assumption that would considerably simplify the proof.

The discussion on p. 192 in [11] implies the following facts that we will use below:

There is for every q in M an $\epsilon > 0$ such that the following two properties are satisfied.

(i) The distance function between a given point r in the open ball $B_\epsilon(q)$ and a connected component $(N_r \cap B_{3\epsilon}(q))^\circ$ of $N_r \cap B_{3\epsilon}(q)$ takes on its minimum for every leaf N_r through a point r in $B_\epsilon(q)$.

(ii) The distance from $\hat{r}$ to $(N \cap B_{3\epsilon}(q))^\circ$ is constant as $\hat{r}$ moves in the connected component $(N_r \cap B_{3\epsilon}(q))^\circ$ of $N_r \cap B_{3\epsilon}(q)$ containing r, where r is an arbitrary point in $B_\epsilon(q)$.

Let q be a point in M and let $\epsilon > 0$ be as above. We assume that the image of f contains a point in $B_\epsilon(q)$. Our goal is to show that all leaves that meet $B_\epsilon(q)$ also meet the image of f. It follows from Proposition 3.4 (i) that there is a regular point r in $B_\epsilon(q) \cap \mathrm{image}(f)$. Let s be an arbitrary point in $B_\epsilon(q)$ and let $\hat{r}$ be a point in $(N_r \cap B_{3\epsilon}(q))^\circ$ with minimal distance to s. Set $\rho = d(\hat{r}, s) = d(s, (N_r \cap B_{3\epsilon}(q))^\circ)$. Then by (ii) above there is a point $\hat{s} \in (N_s \cap B_{3\epsilon}(q))^\circ$ such that the distance between r and $(N_s \cap B_{3\epsilon}(q))^\circ$ is $\rho = d(r, \hat{s})$. Let $\gamma : \mathbf{R} \to M$ be a geodesic such that $\gamma|_{[0,1]}$ is a shortest connection between r and $\hat{s}$ and hence also between r and $(N_s \cap B_{3\epsilon}(q))^\circ$. Then γ meets N_s perpendicularly. It follows that γ meets all leaves it passes through perpendicularly and that $\gamma'(0) \in \mathcal{H}_r$ since r is a regular point. The geodesic γ lies in the image of f since its tangent space at r coincides with $\mathcal{H}_r$ and f is totally geodesic. This means that f hits the leaf N_s and we have proved that all leaves that meet $B_\epsilon(q)$ are intersected by the image of f.

What we have just proved implies that the union $\mathcal{U}$ over the leaves that the image of f meets is open. The same argument can be used to prove that the complement of $\mathcal{U}$ is open since a ball $B_\epsilon(q)$ as in (i) and (ii) is contained in $\mathcal{U}$ if it meets it. It follows that $M = \mathcal{U}$.

This finishes the proof of Theorem B. □

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