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ON DIRICHLET L-FUNCTIONS AND THE INDEX OF VISIBLE POINTS

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Abstract. We investigate the value distribution of the index of visible points with congruence constraints by estimating all moments of the index twisted by an arbitrary Dirichlet character with respect to a fixed modulus.

1. Introduction and statement of results

Surprising connections between various questions in number theory and problems in mathematical physics have been discovered in the past few decades by a number of authors. These connections have made it possible to employ powerful methods and ideas from number theory, such as spacings between Farey fractions pioneered in [10], [11], and [13], and furthered recently in [1], [2], [3], and [15], where estimates for Kloosterman sums are being used. For example, Boca, Gologan, and one of the authors [5], [6] recently solved a problem raised by Sinai on the free path length of the linear trajectory of a two-dimensional Euclidean billiard generated by the free motion of a billiard ball, subject to elastic reflections on the boundary of the unit square $[0,1]^2$ with small pockets of size ε removed at the four corners. The billiard problem has the mass point moving from the origin along a geodesic line in $[0,1]^2$ with constant speed and angle, until it collides with the boundary. At a smooth boundary point, the billiard ball reflects so that the tangential component of its velocity remains the same, while the normal component changes its sign. The trajectory between two such reflections is specular, and the motion ends when the billiard ball reaches one of the corner pockets. The method used in [5], [6] is number theoretical in nature, and it exploits the connection between billiards in $[0,1]^2$ and visible points in the plane, which in turn are related to Farey fractions. This further links the problem to the distribution of inverses in residue classes, in which the Kloosterman machinery [9] is used in a decisive way and ultimately solves the problem. The distribution is sensitive to the

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initial position of the particles, which in [5], [6] is fixed at the origin. If one fixes the initial position at an arbitrary rational point, the billiard problem is intimately related to the distribution of visible points with congruence constraints. An integer point P is said to be visible from a fixed point P_0 if the open line segment (P_0P) does not contain any integer point. Thus visibility of P from the origin is equivalent to the condition that its coordinates are relatively prime, whereas visibility from a more general fixed rational point P_0 naturally brings congruence constraints into the problem. A notion that plays an important role in the study of the local distribution of Farey fractions is that of the Farey index, recently introduced and studied by Hall and Shiu in [12], and furthered in [4]. In relation to these works, it would be interesting to investigate the distribution of the index of visible points with congruence constraints. We introduce some notation.

The Farey sequence $\mathfrak{F}_Q$ of order Q is the ascending sequence of fractions in the unit interval $(0,1]$ whose denominators do not exceed Q, i.e., $1/Q = \gamma_1 < \gamma_2 < \cdots < \gamma_{N_Q} = 1$. Thus the fraction $\gamma_i = b/s$ belongs to $\mathfrak{F}_Q$ if b and s are relatively prime and $0 < b \le s \le Q$. The number N_Q of terms in $\mathfrak{F}_Q$ is given by $N_Q = \varphi(1) + \varphi(2) + \cdots + \varphi(Q)$ and it is well known (see Theorem 330, p. 268, in [14]) that

$$N_Q = \frac{3Q^2}{\pi^2} + O(Q \log Q).$$

The essential property of $\mathfrak{F}_Q$, from which all the other properties follow, is that $rb - as = 1$ for any two consecutive fractions a/r and b/s. Now for any three consecutive Farey fractions, say,

$$\gamma_{i-1} = \frac{a}{r} < \gamma_i = \frac{b}{s} < \gamma_{i+1} = \frac{c}{t},$$

the ratio

$$\nu(\gamma_i) := \frac{r+t}{s} = \frac{a+c}{b}$$

is an integer called the index of the fraction $\gamma_i = b/s$. In particular we have $\nu(\gamma_1) = 1$ and $\nu(\gamma_{N_Q}) = 2Q$. As an example, the indices of $\mathfrak{F}_7$ are indicated in the following table:

γ_i	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	1
$\nu(\gamma_i)$	1	2	2	3	1	4	2	1	7	1	2	4	1	3	2	2	1	14

In the language of visible points, the index of a visible point (s,b) that lies inside or on the sides of the triangle with vertices $(0,0)$, $(Q,0)$, and (Q,Q), is meant to be the index of the Farey fraction b/s, and it has the following geometric interpretation: If we arrange the visible points that belong to the triangle in increasing order with respect to the slope of the ray from the origin (rotated counterclockwise) passing through them, then for any three consecutive visible points (r,a), (s,b), and (t,c), the index of the point (s,b)

is two times the area of the triangle joining the points (r, a) and (t, c) to the origin. (See [7], pp. 208–212, and [14], pp. 23–37.) In this way, the index is intrinsically related to the position of consecutive visible points.

In [12], the relation

$$\nu(\gamma_i) = \left[\frac{Q+r}{s}\right] \tag{1.1}$$

is obtained, from which

$$\left[\frac{2Q+1}{s}\right] - 1 \le \nu(\gamma_i) \le \left[\frac{2Q}{s}\right].$$

Hence if $s \mid 2Q+1$, then $\nu(\gamma_i) = [2Q/s]$. Otherwise, the index may take the two values $[2Q/s]$ and $[2Q/s] - 1$. From these observations, Hall and Shiu introduced the concept of Farey deficiency, which is the number $\delta(s)$ of Farey fractions γ_i with denominator s such that $\nu(\gamma_i) = [2Q/s] - 1$. Their investigation of the frequency of the two values of the index led them to the very remarkable facts

$$\sum_{\gamma_i \in \mathfrak{F}_Q} \nu(\gamma_i) = 3N_Q - 1$$

and

$$\sum_{s=1}^{Q} \delta(s) = Q(2Q+1) - N_{2Q} - 2N_Q + 1,$$

in addition to the asymptotic formula

$$\sum_{\gamma_i \in \mathfrak{F}_Q} \nu(\gamma_i)^2 = \frac{24Q^2}{\pi^2}\left(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{17}{8} + 2\gamma\right) + O(Q(\log Q)^2), \tag{1.2}$$

where ζ is the Riemann zeta-function and γ is Euler's constant.

In the present paper, we study the value distribution of the index of visible points which satisfy congruence constraints. In order to achieve this goal, we estimate all moments of the index twisted by an arbitrary Dirichlet character χ with respect to a fixed modulus k. We consider, for any positive integer l, any modulus k, any Dirichlet character χ modulo k, and any large positive integer Q, the lth moment

$$\mathcal{M}_l(\chi, Q) := \sum_{\gamma_i = \frac{b}{s} \in \mathfrak{F}_Q} \chi(s)\nu(\gamma_i)^l. \tag{1.3}$$

Let p denote a prime number. We can summarize our results as follows.

Theorem 1.1. *Fix a positive integer k and a Dirichlet character χ modulo k. Then for all large positive integers Q, we have:*

(i)

$$\mathcal{M}_1(\chi, Q) = \begin{cases} O_k(Q \log Q) & \text{if } \chi \neq \chi_0, \\ \dfrac{3Q^2\varphi(k)}{2kL(2,\chi_0)} + O_k(Q\log Q) & \text{if } \chi = \chi_0. \end{cases}$$

(ii)

$$\mathcal{M}_2(\chi, Q) = \begin{cases} \dfrac{4Q^2L(1,\chi)}{L(2,\chi)} + O_k(Q(\log Q)^2) & \text{if } \chi \neq \chi_0, \\ \dfrac{24Q^2}{\pi^2}\left(\log 2Q - \dfrac{\zeta'(2)}{\zeta(2)} - \dfrac{3}{2} + 2\gamma + \displaystyle\sum_{p|k}\dfrac{p\log p}{p^2-1}\right) & \\ \times \displaystyle\prod_{p|k}\dfrac{p}{p+1} - \dfrac{5Q^2\varphi(k)}{2kL(2,\chi_0)} + O_k(Q^{4/3}(\log Q)^{5/3}) & \text{if } \chi = \chi_0. \end{cases}$$

(iii)

$$\mathcal{M}_3(\chi, Q) = \frac{8Q^3L(2,\chi)}{L(3,\chi)} + O(Q^2\log Q).$$

(iv) *For any* $l \geq 4$,

$$\mathcal{M}_l(\chi, Q) = \frac{2^lQ^lL(l-1,\chi)}{L(l,\chi)} + O(Q^{l-1}).$$

As a corollary, we obtain asymptotic formulas for the moments of the index of Farey fractions $\gamma_i = b/s$ with denominator s in an arithmetic progression. For any $s \equiv u \pmod{k}$, we consider the lth moment

$$\mathcal{M}_l(u,k,Q) := \sum_{\substack{\gamma_i = \frac{b}{s} \in \mathfrak{F}_Q \\ s \equiv u \pmod{k}}} \nu(\gamma_i)^l. \tag{1.4}$$

Corollary 1.2. *Fix positive integers k and u with* $\gcd(u,k) = 1$. *Then for all large positive integers Q, we have:*

(i)

$$\mathcal{M}_1(u,k,Q) = \frac{3Q^2}{2kL(2,\chi_0)} + O_k(Q\log Q).$$

(ii)

$$\mathcal{M}_2(u,k,Q) = \frac{24Q^2}{\pi^2\varphi(k)}\left(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma + \sum_{p|k}\frac{p\log p}{p^2-1}\right)\prod_{p|k}\frac{p}{p+1}$$
$$- \frac{5Q^2}{2kL(2,\chi_0)} + \frac{4Q^2}{\varphi(k)}\sum_{\chi\neq\chi_0}\frac{\overline{\chi}(u)L(1,\chi)}{L(2,\chi)} + O_k(Q^{4/3}(\log Q)^{5/3}).$$

(iii)

$$\mathcal{M}_3(u,k,Q) = \frac{8Q^3}{\varphi(k)} \sum_{\chi} \frac{\overline{\chi}(u)L(2,\chi)}{L(3,\chi)} + O(Q^2 \log Q).$$

(iv) *For any* $l \geq 4$,

$$\mathcal{M}_l(u,k,Q) = \frac{2^l Q^l}{\varphi(k)} \sum_{\chi} \frac{\overline{\chi}(u)L(l-1,\chi)}{L(l,\chi)} + O(Q^{l-1}).$$

It should be remarked that the asymptotics above indicate that higher moments are biased toward some arithmetic progressions. For example, if $s \equiv 1 \pmod 4$ and $s \equiv 3 \pmod 4$, then

$$\lim_{Q\to\infty} \frac{\mathcal{M}_1(3,4,Q)}{\mathcal{M}_1(1,4,Q)} = 1 \quad \text{and} \quad \lim_{Q\to\infty} \frac{\mathcal{M}_2(3,4,Q)}{\mathcal{M}_2(1,4,Q)} = 1;$$

but

$$\lim_{Q\to\infty} \frac{\mathcal{M}_3(3,4,Q)}{\mathcal{M}_3(1,4,Q)} = \frac{\frac{L(2,\chi_0)}{L(3,\chi_0)} - \frac{L(2,\chi)}{L(3,\chi)}}{\frac{L(2,\chi_0)}{L(3,\chi_0)} + \frac{L(2,\chi)}{L(3,\chi)}} = 0.107456\ldots,$$

where χ_0 and χ are the principal and non-principal Dirichlet characters modulo 4.

2. The first moment

In this section we prove part (i) of Theorem 1.1. Our starting point is formula (2.1) from [12]:

$$T(s) := \sum_{\gamma_i = \frac{b}{s} \in \mathfrak{F}_Q} \nu(\gamma_i) = \frac{2}{s} \sum_{\substack{r=Q-s+1 \\ (r,s)=1}}^{Q} r = 2\sum_{d|s} \mu(d) \left[\frac{Q}{d}\right] - \varphi(s) + \varepsilon(s), \tag{2.1}$$

where $\varepsilon(1) = 1$ and $\varepsilon(s) = 0$ for $s > 1$. We have

$$\begin{aligned} \mathcal{M}_1(\chi,Q) &= \sum_{s\leq Q} \chi(s) T(s) \\ &= 2\sum_{s\leq Q} \chi(s) \sum_{d|s} \mu(d) \left[\frac{Q}{d}\right] - \sum_{s\leq Q} \chi(s)\varphi(s) + 1. \end{aligned} \tag{2.2}$$

Suppose that $\chi \neq \chi_0$. Using $[y] = y + O(1)$, interchanging summations, and employing the Pólya-Vinogradov inequality,

$$\begin{aligned}
\sum_{s\le Q}\chi(s)\sum_{d|s}\mu(d)\left[\frac{Q}{d}\right] &= \sum_{s\le Q}\chi(s)\sum_{d|s}\mu(d)\left(\frac{Q}{d}+O(1)\right) \\
&= Q\sum_{d\le Q}\frac{\mu(d)}{d}\sum_{\substack{s\le Q\\ d|s}}\chi(s) + O\left(\sum_{s\le Q}\tau(s)\right) \\
&= Q\sum_{d\le Q}\frac{\chi(d)\mu(d)}{d}\sum_{l\le\left[\frac{Q}{d}\right]}\chi(l) + O(Q\log Q). \\
&= O(\sqrt{k}\log k\cdot Q\log Q).
\end{aligned} \tag{2.3}$$

(Assuming the Generalized Riemann Hypothesis, one can use the sharper version of the Pólya-Vinogradov inequality provided in [17].) Next, using $\varphi(n) = \sum_{d|n}\frac{\mu(d)n}{d}$ and rearranging, we get

$$\begin{aligned}
\sum_{s\le Q}\chi(s)\varphi(s) &= \sum_{d\le Q}\frac{\mu(d)}{d}\sum_{\substack{s\le Q\\ d|s}}\chi(s)s \\
&= \sum_{d\le Q}\chi(d)\mu(d)\sum_{l\le\left[\frac{Q}{d}\right]}\chi(l)l \\
&= \sum_{d\le Q}\chi(d)\mu(d)\sum_{j\le\left[\frac{Q}{d}\right]}\sum_{j\le l\le\left[\frac{Q}{d}\right]}\chi(l).
\end{aligned} \tag{2.4}$$

Applying the Pólya-Vinogradov inequality to the inner-most sum on the far right side, we see that

$$\sum_{s\le Q}\chi(s)\varphi(s) \ll \sqrt{k}\log k\cdot Q\log Q,$$

and inserting this and (2.3) into (2.2) finishes the case $\chi \neq \chi_0$.

Now suppose that $\chi = \chi_0$. Using

$$\sum_{n\le x}\chi_0(n) = \frac{\varphi(k)x}{k} + O(\tau(k)), \tag{2.5}$$

we get

$$\sum_{s\le Q}\chi_0(s)\sum_{d|s}\mu(d)\left[\frac{Q}{d}\right] = \frac{Q^2\varphi(k)}{k}\sum_{d\le Q}\frac{\chi_0(d)\mu(d)}{d^2} + O(\tau(k)Q\log Q),$$

and since

$$\sum_{d\le Q}\frac{\chi_0(d)\mu(d)}{d^2} = \sum_{d=1}^{\infty}\frac{\chi_0(d)\mu(d)}{d^2} + O_k\left(\frac{1}{Q}\right)$$

and

$$\sum_{d=1}^{\infty}\frac{\chi_0(d)\mu(d)}{d^2}=\prod_p\left(1-\frac{\chi_0(p)}{p^2}\right)=\prod_{p\nmid k}\left(1-\frac{1}{p^2}\right)=\frac{1}{L(2,\chi_0)},$$

we have

$$\sum_{s\le Q}\chi_0(s)\sum_{d\mid s}\mu(d)\left[\frac{Q}{d}\right]=\frac{Q^2\varphi(k)}{kL(2,\chi_0)}+O(\tau(k)Q\log Q). \tag{2.6}$$

The inner-most sum on the far right side of (2.4) is treated in the same way. Using

$$\sum_{j\le l\le\left[\frac{Q}{d}\right]}\chi_0(l)=\frac{\varphi(k)}{k}\left(\left[\frac{Q}{d}\right]-j\right)+O(\tau(k)),$$

we see that

$$\begin{aligned}\sum_{s\le Q}\chi_0(s)\varphi(s)&=\frac{\varphi(k)}{k}\sum_{d\le Q}\chi_0(d)\mu(d)\left(\frac{Q^2}{d^2}+O\left(\frac{Q}{d}\right)\right)\\&\quad-\frac{\varphi(k)}{k}\sum_{d\le Q}\chi_0(d)\mu(d)\sum_{j\le\left[\frac{Q}{d}\right]}j+O\left(\tau(k)\sum_{d\le Q}\sum_{j\le\left[\frac{Q}{d}\right]}1\right)\\&=\frac{Q^2\varphi(k)}{kL(2,\chi_0)}+O(Q\log Q)-\frac{\varphi(k)}{2k}\sum_{d\le Q}\chi_0(d)\mu(d)\\&\quad\times\left(\frac{Q^2}{d^2}+O\left(\frac{Q}{d}\right)\right)+O(\tau(k)Q\log Q)\\&=\frac{Q^2\varphi(k)}{2kL(2,\chi_0)}+O(\tau(k)Q\log Q),\end{aligned}$$

and inserting this and (2.6) into (2.2) finishes the case $\chi=\chi_0$.

3. Estimates for the deficiency

In this section we provide asymptotic formulas for the deficiency.

THEOREM 3.1. *Fix a positive integer k and a Dirichlet character χ modulo k. Then for all large positive integers Q, we have:*

$$\sum_{s=1}^{Q}\chi(s)\delta(s)=\begin{cases}O(\sqrt{k}\log k\cdot Q(\log Q)^2) & \text{if } \chi\ne\chi_0,\\ \dfrac{2Q^2}{\prod_{p\mid k}(1+1/p)}-\dfrac{3Q^2\varphi(k)}{kL(2,\chi_0)} & \\ \qquad +O\left(\dfrac{k2^{\omega(k)}Q^{4/3}(\log Q)^{5/3}}{\varphi(k)}\right) & \text{if } \chi=\chi_0.\end{cases}$$

Proof. Let $s(\gamma_i) = s$. Since $\nu(\gamma_i)$ takes at most two values, $[2Q/s]$ and $[2Q/s] - 1$, with the lower value $\delta(s)$ times, we have

$$T(s) = (\varphi(s) - \delta(s))\left[\frac{2Q}{s}\right] + \delta(s)\left(\left[\frac{2Q}{s}\right] - 1\right) = \varphi(s)\left[\frac{2Q}{s}\right] - \delta(s). \tag{3.1}$$

Hence

$$\sum_{s\le Q} \chi(s)\delta(s) = \sum_{s\le Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right] - \sum_{s\le Q} \chi(s)T(s). \tag{3.2}$$

Now,

$$\sum_{s\le Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right] = \sum_{d\le Q} \frac{\mu(d)}{d} \sum_{\substack{s\le Q\\ d\mid s}} \chi(s)s\left[\frac{2Q}{s}\right] = \sum_{d\le Q} \chi(d)\mu(d)S_\chi\left(\frac{2Q}{d}\right),$$

where

$$S_\chi(y) := \sum_{l\le\left[\frac{y}{2}\right]} \chi(l)l\left[\frac{y}{l}\right].$$

We apply Abel summation to the sum $S_\chi(y)$. Define $A(n) := \sum_{m\le n}\chi(m)$. Let $b(m) = m\,[y/m]$ if $m \le [y/2]$, and let $b(m) = 0$ if $m \ge [y/2] + 1$. Then

$$\begin{aligned} S_\chi(y) &= \sum_{n\le\left[\frac{y}{2}\right]} \chi(n)b(n) = \sum_{n\le\left[\frac{y}{2}\right]} (A(n) - A(n-1))b(n) \\ &= \sum_{n\le\left[\frac{y}{2}\right]} A(n)b(n) - \sum_{n\le\left[\frac{y}{2}\right]-1} A(n)b(n+1) \\ &= \sum_{n\le\left[\frac{y}{2}\right]} A(n)(b(n) - b(n+1)) \end{aligned}$$

and

$$|b(n) - b(n+1)| \le \frac{y}{n+1} + n\left(\left[\frac{y}{n}\right] - \left[\frac{y}{n+1}\right]\right).$$

Suppose that $\chi \ne \chi_0$. By the Pólya-Vinogradov inequality $A(n) \ll \sqrt{k}\log k$ and $S_\chi(y) \ll \sqrt{k}\log k \cdot y\log y$. Hence

$$\sum_{s\le Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right] \ll \sqrt{k}\log k \cdot Q(\log Q)^2, \tag{3.3}$$

and inserting this into (3.2) and applying Theorem 1.1 (i) finishes the case $\chi \ne \chi_0$.

Next, since $[2Q/s] = 1$ throughout the range $Q < s \le 2Q$, extending the range from $1 \le s \le Q$ to $1 \le s \le 2Q$ gives us

$$\sum_{s\le Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right] = \sum_{s\le 2Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right] - \sum_{Q<s\le 2Q} \chi(s)\varphi(s),$$

where

$$\sum_{Q<s\le 2Q}\chi(s)\varphi(s) = \sum_{d\le 2Q}\frac{\mu(d)}{d}\sum_{\substack{Q<s\le 2Q\\ d|s}}\chi(s)s \tag{3.4}$$
$$= \sum_{d\le 2Q}\chi(d)\mu(d)\sum_{\left[\frac{Q}{d}\right]+1\le l\le\left[\frac{2Q}{d}\right]}\chi(l)l$$
$$= \sum_{d\le 2Q}\chi(d)\mu(d)\sum_{j\le\left[\frac{2Q}{d}\right]}\ \sum_{\max\left(\left[\frac{Q}{d}\right]+1,j\right)\le l\le\left[\frac{2Q}{d}\right]}\chi(l).$$

Suppose that $\chi = \chi_0$. Applying (2.5) to the inner-most sum on the far right side of (3.4), we see that the sum $\sum_{Q<s\le 2Q}\chi(s)\varphi(s)$ is

$$\sum_{d\le 2Q}\chi_0(d)\mu(d)\sum_{j\le\left[\frac{2Q}{d}\right]}\left(\frac{\varphi(k)}{k}\left(\left[\frac{2Q}{d}\right]-\max\left(\left[\frac{Q}{d}\right]+1,j\right)\right)+O(\tau(k))\right)$$
$$= \frac{2Q\varphi(k)}{k}\sum_{d\le 2Q}\frac{\chi_0(d)\mu(d)}{d}\left(\frac{2Q}{d}+O(1)\right)+O(Q\log Q)$$
$$- \frac{\varphi(k)}{k}\sum_{d\le 2Q}\chi_0(d)\mu(d)\left(\sum_{\left[\frac{Q}{d}\right]+1\le j\le\left[\frac{2Q}{d}\right]} j+\sum_{j\le\left[\frac{Q}{d}\right]}\left(\left[\frac{Q}{d}\right]+1\right)\right)$$
$$+ O(\tau(k)Q\log Q)$$
$$= \frac{4Q^2\varphi(k)}{k}\sum_{d\le 2Q}\frac{\chi_0(d)\mu(d)}{d^2}-\frac{\varphi(k)}{k}\sum_{d\le 2Q}\chi_0(d)\mu(d)\left(\frac{5Q^2}{2d^2}+O\left(\frac{Q}{d}\right)\right)$$
$$+ O(\tau(k)Q\log Q)$$
$$= \frac{3Q^2\varphi(k)}{2kL(2,\chi_0)}+O(\tau(k)Q\log Q).$$

Hence

$$\sum_{Q<s\le 2Q}\chi(s)\varphi(s) = \frac{3Q^2\varphi(k)}{2kL(2,\chi_0)}+O(\tau(k)Q\log Q). \tag{3.5}$$

Combining all estimates in (3.2) and applying Theorem 1.1 (i), we obtain

$$\sum_{s\le Q}\chi_0(s)\delta(s) = \sum_{s\le 2Q}\chi_0(s)\varphi(s)\left[\frac{2Q}{s}\right]-\frac{3Q^2\varphi(k)}{kL(2,\chi_0)}+O(\tau(k)Q\log Q). \tag{3.6}$$

The sum on the right side is

$$\sum_{s\le 2Q}\chi_0(s)\varphi(s)\sum_{\substack{n\le 2Q\\ s|n}}1 = \sum_{n\le 2Q}\sum_{s|n}\chi_0(s)\varphi(s) = \sum_{n\le 2Q}g_{\chi_0}(n), \tag{3.7}$$

where

$$g_{\chi_0}(n) = \sum_{m|n} \chi_0(m)\varphi(m)$$

satisfies $|g_{\chi_0}(n)| \le n$.

Let the letters s, σ, t, and T be the usual symbols used in the theory of the Riemann zeta-function. We write

$$G_k(s) = \sum_{n=1}^{\infty} \frac{\chi_0(n)\varphi(n)}{n^s} = \sum_{\substack{n=1\\(n,k)=1}}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \prod_{p|k} \frac{1-\frac{1}{p^{s-1}}}{1-\frac{1}{p^s}}.$$

Then the Dirichlet series for $g_{\chi_0} = \chi_0\varphi * 1$ is

$$G(s) = \sum_{n=1}^{\infty} \frac{g_{\chi_0}(n)}{n^s} = \zeta(s)G_k(s) = \zeta(s-1) \prod_{p|k} \frac{1-\frac{1}{p^{s-1}}}{1-\frac{1}{p^s}},$$

where

$$\left| \prod_{p|k} \frac{1-\frac{1}{p^{s-1}}}{1-\frac{1}{p^s}} \right| \le \frac{k2^{\omega(k)}}{\varphi(k)}$$

uniformly for $\sigma \ge 1$. Employing Perron's formula

$$\sum_{n\le x} f(n) = \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \frac{x^s F(s)}{s}\, ds + R(T),$$

where $F(s)$ is the Dirichlet series for $f(n)$ and

$$|R(T)| \le \frac{x^\alpha}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^\alpha \left|\log \frac{x}{n}\right|},$$

we put $x = 2Q + 1/2$ and sum over $n \le 2Q$, so that

$$\sum_{n\le 2Q} g_{\chi_0}(n) = \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \frac{(2Q+\frac{1}{2})^s G(s)}{s}\, ds + R(T) \tag{3.8}$$

and

$$|R(T)| \ll \frac{Q^\alpha}{T} \sum_{n=1}^{\infty} \frac{n^{1-\alpha}}{\left|\log \frac{2Q+\frac{1}{2}}{n}\right|}.$$

Let $\alpha = 2 + 1/\log Q$. Following the arguments given in [8] (see pp. 106–107), we decompose the sum above into three subsums extended over the following sets of n: $n \le Q$, $Q < n \le 3Q$, and $n > 3Q$. For values of n which satisfy $n \le Q$ or $n > 3Q$ it is immediately clear that

$$\left|\log \frac{2Q+\frac{1}{2}}{n}\right| > \log \frac{3}{2}.$$

Hence the first and last subsums are $\ll 1/(\alpha-2)$. For values of n which satisfy $Q < n \le 3Q$ the middle subsum is

$$\ll Q^{1-\alpha} \sum_{-Q<m\le Q} \frac{1}{\left|\log \frac{2Q+\frac{1}{2}}{2Q+m}\right|} \ll Q^{2-\alpha} \sum_{-Q<m\le Q} \frac{1}{|m-\frac{1}{2}|} \ll Q^{2-\alpha}\log Q.$$

Therefore

$$|R(T)| \ll \frac{Q^\alpha}{T}\left(\frac{1}{\alpha-2} + Q^{2-\alpha}\log Q\right) \ll \frac{Q^2\log Q}{T}.$$

To evaluate the integral in (3.8), we deform the contour, as explained in [19]. We shift the portion $|t| \le T$ of this path of integration to the left of the line $\mathrm{Re}(s) = \alpha$, thereby replacing it by a rectangular path joining the points $1 \pm iT$ and $\alpha \pm iT$. Since the integrand is holomorphic on and within this contour, we have by Cauchy's residue theorem

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \frac{(2Q+\frac{1}{2})^s G(s)}{s}\,ds = \frac{(2Q+\frac{1}{2})^2}{2\prod_{p|k}(1+1/p)} + \sum_{j=1}^{3} I_j.$$

The main contribution is due to the residue of the simple pole at the point $s = 2$. The integrals I_1 and I_3 are along the horizontal segments $[\alpha-iT, 1-iT]$ and $[1+iT, \alpha+iT]$, respectively, and the integral I_2 is over the vertical segment $[1-iT, 1+iT]$.

We proceed to estimate the integral along our modified contour. Since $|\zeta(s)| \ll T^{(1-\sigma)/2}\log T$ if $0 \le \sigma \le 1$, and $\ll \log T$ if $1 \le \sigma \le 2$ (see Theorem 1.9, p. 25, in [16]), we see that

$$\begin{aligned}|I_1|, |I_3| &\ll \frac{k2^{\omega(k)}}{\varphi(k)}\int_1^\alpha \frac{|(2Q+\frac{1}{2})^{\sigma+iT}||\zeta(\sigma-1+iT)|}{|\sigma+iT|}\,d\sigma \\ &\ll \frac{k2^{\omega(k)}(QT^{1/2}\log T + Q^2\log T)}{\varphi(k)T}.\end{aligned}$$

Next, we have

$$|I_2| \ll \frac{k2^{\omega(k)}}{\varphi(k)}\int_{-T}^{T} \frac{|(2Q)^{1+it}||\zeta(it)|}{|1+it|}\,dt \ll \frac{k2^{\omega(k)}QT^{1/2}(\log T)^2}{\varphi(k)}.$$

Collecting all estimates and choosing $T = Q^{2/3}/(\log Q)^{2/3}$, we obtain

$$\sum_{n\le 2Q} g_{\chi_0}(n) = \frac{2Q^2}{\prod_{p|k}(1+1/p)} + O\left(\frac{k2^{\omega(k)}Q^{4/3}(\log Q)^{5/3}}{\varphi(k)}\right),$$

and the required result follows by inserting this into (3.6). □

4. The second moment

In this section we prove part (ii) of Theorem 1.1. We have

$$\chi(s) \sum_{\gamma_i = \frac{b}{s} \in \mathfrak{F}_Q} \nu(\gamma_i)^2 = \chi(s)\varphi(s)\left[\frac{2Q}{s}\right]^2 - \chi(s)\delta(s)\left(2\left[\frac{2Q}{s}\right] - 1\right).$$

Then

$$\mathcal{M}_2(\chi, Q) = X_\chi(Q) - 2Y_\chi(Q) + \sum_{s \le Q} \chi(s)\delta(s),$$

where

$$X_\chi(Q) := \sum_{s \le Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right]^2 \quad \text{and} \quad Y_\chi(Q) := \sum_{s \le Q} \chi(s)\delta(s)\left[\frac{2Q}{s}\right].$$

As in the proof of Theorem 3.1, we extend the range from $1 \le s \le Q$ to $1 \le s \le 2Q$ to obtain

$$X_\chi(Q) = \sum_{s \le 2Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right]^2 - \sum_{Q < s \le 2Q} \chi(s)\varphi(s).$$

Suppose that $\chi \neq \chi_0$. Applying the Pólya-Vinogradov inequality to the far right side of (3.4), we obtain

$$\sum_{Q < s \le 2Q} \chi(s)\varphi(s) \ll \sqrt{k} \log k \cdot Q \log Q \tag{4.1}$$

and

$$X_\chi(Q) = \sum_{s \le 2Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right]^2 + O(\sqrt{k} \log k \cdot Q \log Q).$$

By (3.3), the sum above is

$$\sum_{s \le 2Q} \chi(s)\varphi(s)\left[\frac{2Q}{s}\right]\left(\left[\frac{2Q}{s}\right] + 1\right) + O(\sqrt{k} \log k \cdot Q(\log Q)^2). \tag{4.2}$$

Let

$$f_\chi(n) := \sum_{s \mid n} \frac{\chi(s)\varphi(s)}{s}.$$

Then the sum in (4.2) becomes

$$2 \sum_{s \le 2Q} \frac{\chi(s)\varphi(s)}{s} \sum_{\substack{n \le 2Q \\ s \mid n}} n = 2 \sum_{n \le 2Q} n f_\chi(n). \tag{4.3}$$

Hence

$$X_\chi(Q) = 2 \sum_{n \le 2Q} n f_\chi(n) + O(\sqrt{k} \log k \cdot Q(\log Q)^2),$$

and by Theorem 3.1

$$\mathcal{M}_2(\chi, Q) = 2\sum_{n\le 2Q} nf_\chi(n) - 2Y_\chi(Q) + O(\sqrt{k}\log k \cdot Q(\log Q)^2).$$

We now consider the sum $Y_\chi(Q)$. By (2.1) and (3.1)

$$\delta(s) = \varphi(s)\left[\frac{2Q}{s}\right] - T(s) = \varphi(s)\left(\left[\frac{2Q}{s}\right]+1\right) - 2\sum_{d|s}\mu(d)\left[\frac{Q}{d}\right] - \varepsilon(s).$$

Hence

$$\delta(s) = \varphi(s)\left(\left[\frac{2Q}{s}\right]+1\right) - \frac{2Q\varphi(s)}{s} + O(\tau(s))$$

and, since

$$\sum_{s\le Q}\frac{\tau(s)}{s} \ll (\log Q)^2,$$

we get

$$Y_\chi(Q) = \sum_{s\le Q}\chi(s)\varphi(s)\left[\frac{2Q}{s}\right]\left(\left[\frac{2Q}{s}\right]+1\right) - 2Q\sum_{s\le Q}\frac{\chi(s)\varphi(s)}{s}\left[\frac{2Q}{s}\right] + O(Q(\log Q)^2). \tag{4.4}$$

Extending the range of these sums from $s\le Q$ to $s\le 2Q$ and applying (4.1) and (4.3), we find that the first sum on the right side of (4.4) is

$$\sum_{s\le 2Q}\chi(s)\varphi(s)\left[\frac{2Q}{s}\right]\left(\left[\frac{2Q}{s}\right]+1\right) - 2\sum_{Q<s\le 2Q}\chi(s)\varphi(s) = 2\sum_{n\le 2Q} nf_\chi(n) + O(\sqrt{k}\log k\cdot Q\log Q)$$

and the second sum there is

$$\sum_{s\le 2Q}\frac{\chi(s)\varphi(s)}{s}\left[\frac{2Q}{s}\right] - \sum_{Q<s\le 2Q}\frac{\chi(s)\varphi(s)}{s}.$$

Now,

$$\sum_{s\le 2Q}\frac{\chi(s)\varphi(s)}{s}\left[\frac{2Q}{s}\right] = \sum_{s\le 2Q}\frac{\chi(s)\varphi(s)}{s}\sum_{\substack{n\le 2Q\\ s|n}} 1 = \sum_{n\le 2Q} f_\chi(n)$$

and, by the Pólya-Vinogradov inequality,

$$\sum_{Q<s\le 2Q}\frac{\chi(s)\varphi(s)}{s} = \sum_{Q<s\le 2Q}\frac{\chi(s)}{s}\sum_{d|s}\frac{\mu(d)s}{d} = \sum_{d\le 2Q}\frac{\mu(d)}{d}\sum_{\substack{Q<s\le 2Q\\ d|s}}\chi(s) \ll \sqrt{k}\log k\cdot\log Q.$$

Hence

$$Y_\chi(Q) = 2 \sum_{n \le 2Q} (n - Q) f_\chi(n) + O(Q(\log Q)^2)$$

and

$$\mathcal{M}_2(\chi, Q) = 2 \sum_{n \le 2Q} (2Q - n) f_\chi(n) + O(\sqrt{k} \log k \cdot Q(\log Q)^2), \tag{4.5}$$

and it remains to examine the sum here.

The Dirichlet series for $f_\chi(n)$ is given by

$$\sum_{n=1}^{\infty} \frac{f_\chi(n)}{n^s} = \frac{\zeta(s) L(s, \chi)}{L(s+1, \chi)}.$$

Employing

$$\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{x^{s+1}}{s(s+1)} \, ds = \max(x - 1, 0), \quad x > 0,$$

and noting that the Dirichlet series converges absolutely on $\mathrm{Re}(s) = 2$, we see that by term by term integration

$$\begin{aligned} \sum_{n \le 2Q} (2Q - n) f_\chi(n) &= \sum_{n=1}^{\infty} \max\left(\frac{2Q}{n} - 1, 0\right) n f_\chi(n) \\ &= \frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{(2Q)^{s+1} \zeta(s) L(s, \chi)}{s(s+1) L(s+1, \chi)} \, ds. \end{aligned}$$

The integrand has a simple pole at the points $s = 0$ and $s = 1$. We deform the path of integration from $2 - iT$ to $2 + iT$ to the union of line segments $s = 2 + it$ ($|t| \ge T$), $s = \sigma \pm iT$ ($0 < \sigma \le 2$), $s = it$ ($-T \le t \le T$), and a cut to the right avoiding the point $s = 0$, so as to go horizontally from $2 - iT$ to $-iT$, vertically from $-iT$ to $-i\rho$, around a semicircle γ_0 of radius $\rho = 1/\log Q$ circling the point 0, vertically from $i\rho$ to iT, and horizontally from iT to $2 + iT$. Hence, by Cauchy's residue theorem

$$\sum_{n \le 2Q} (2Q - n) f_\chi(n) = \frac{2Q^2 L(1, \chi)}{L(2, \chi)} + \sum_{j=1}^{7} I_j.$$

The integrals I_1 and I_7 are along the vertical segments $(2 - i\infty, 2 - iT]$ and $[2 + iT, 2 + i\infty)$, respectively, on which $\sigma = 2$. The integrand is $\ll_k Q^3/t^2$ and

$$|I_1|, |I_7| \ll_k \int_T^{\infty} \frac{Q^3}{t^2} \, dt \ll_k \frac{Q^3}{T}.$$

The integrals I_2 and I_6 are along the horizontal segments $[2 - iT, -iT]$ and $[iT, 2 + iT]$, respectively, on which $s = \sigma \pm iT$. We have $|\zeta(s)| \ll T^{(1/2)+\varepsilon}$, $|L(s, \chi)| \ll_k T^{(1/2)+\varepsilon}$, and $|L(s+1, \chi)| \gg_k 1/\log T$ (see [18]), so that

$$|I_2|, |I_6| \ll_k Q^3 T^{-1+3\varepsilon}.$$

The integral I_4 is along the curve γ_0. Since the factors in the integrand are bounded, we have

$$|I_4| \ll_k Q \int_{\gamma_0} \frac{|ds|}{|s|} \ll_k Q \log Q \cdot \text{length}(\gamma_0) \ll_k Q.$$

The integrals I_3 and I_5 are along the vertical segments $[-iT, -i\rho]$ and $[i\rho, iT]$, respectively. We employ the (asymmetric) functional equations for $\zeta(s)$ and $L(s,\chi)$ on the line $\sigma = 0$. For all s we have

$$\zeta(s) = \zeta(1-s) 2^s \pi^{s-1} \Gamma(1-s) \sin \frac{\pi s}{2}.$$

Now any character χ modulo k is induced by a primitive character χ^* modulo d, for some $d \mid k$, and for all s we have in the notation (12) of [8] (p. 71) $\mathfrak{a} := \mathfrak{a}(\chi^*) = 0$ if $\chi^*(-1) = 1$, and $\mathfrak{a} = 1$ if $\chi^*(-1) = -1$:

$$i^{\mathfrak{a}} L(s, \chi^*) = \tau(\chi^*) L(1-s, \overline{\chi^*}) 2^s \pi^{s-1} k^{-s} \Gamma(1-s) \sin \frac{\pi(s+\mathfrak{a})}{2},$$

where $\tau(\chi^*)$ is the Gaussian sum associated with χ^* and $|\tau(\chi^*)| = \sqrt{k}$.

Putting $\sigma = 0$ and noting $\overline{\chi^*} = \overline{\chi}^*$, we obtain

$$|\zeta(it)| \ll \left| \Gamma(1-it) \sin \frac{it\pi}{2} \right| |\zeta(1+it)|, \tag{4.6}$$

$$|L(it, \chi^*)| \ll_k \left| \Gamma(1-it) \sin \frac{it\pi}{2} \right| |L(1-it, \overline{\chi}^*)|.$$

Using

$$L(s, \chi^*) = L(s, \chi) \prod_{p|k} \left(1 - \frac{\chi^*(p)}{p^s}\right)^{-1},$$

we get

$$|L(it, \chi)| \le 2^{\omega(k)} |L(it, \chi^*)| \ll_k \left| \Gamma(1-it) \sin \frac{it\pi}{2} \right| |L(1-it, \overline{\chi}^*)|,$$
$$|L(1-it, \overline{\chi}^*)| \ll_k |L(1-it, \overline{\chi})|,$$

so that

$$|L(it, \chi)| \ll_k \left| \Gamma(1-it) \sin \frac{it\pi}{2} \right| |L(1-it, \overline{\chi})|.$$

We distinguish between two cases. If $\mathfrak{a} = 0$, then the formulas

$$\Gamma(1+z) = z\Gamma(z),$$
$$|\Gamma(z)| = \sqrt{\frac{\pi}{\xi \sinh \pi \xi}} \quad (z = i\xi,\ \xi \ne 0 \text{ real})$$

(see Problem 7, p. 259, in [21]) give us

$$\left| \Gamma(1-it) \sin \frac{it\pi}{2} \right| = \sqrt{\frac{|t|\pi}{2} \tanh \frac{|t|\pi}{2}} \ll \min(|t|, \sqrt{|t|}).$$

Then noting that $|L(1-it,\overline{\chi})| = |L(1+it,\chi)|$, the integrand is

$$\ll_k \frac{Q|\Gamma(1-it)\sin\frac{it\pi}{2}|^2|\zeta(1+it)|}{|t|(|t|+1)} \ll_k \frac{Q\min(t^2,|t|)|\zeta(1+it)|}{|t|(|t|+1)}$$

and

$$|I_3|,|I_5| \ll_k Q\max_{\rho\le t\le T}|\zeta(1+it)|\Big(1+\int_1^T\frac{dt}{t}\Big) \ll_k Q(\log T+\log Q)\log T.$$

Collecting all estimates and choosing $T = Q^3$, we obtain

$$\sum_{n\le 2Q}(2Q-n)f_\chi(n) = \frac{2Q^2L(1,\chi)}{L(2,\chi)} + O_k(Q(\log Q)^2),$$

and the required result follows by inserting this into (4.5).

If $\mathfrak{a}=1$, then

$$|L(it,\chi)| \ll_k \left|\Gamma(1-it)\sin\frac{(1+it)\pi}{2}\right||L(1-it,\overline{\chi})|,$$

$$\left|\Gamma(1-it)\sin\frac{(1+it)\pi}{2}\right| = \sqrt{\frac{|t|\pi}{2}\coth\frac{|t|\pi}{2}} \ll \max(1,\sqrt{|t|}).$$

The integrand is

$$\ll_k \frac{Q|\Gamma(1-it)\sin\frac{it\pi}{2}||\Gamma(1-it)\sin\frac{(1+it)\pi}{2}||\zeta(1+it)|}{|t|(|t|+1)}$$
$$\ll_k \frac{Q\min(|t|,\sqrt{|t|})\max(1,\sqrt{|t|})|\zeta(1+it)|}{|t|(|t|+1)}$$

and again

$$|I_3|,|I_5| \ll_k Q(\log T)^2 \ll_k Q(\log Q)^2.$$

This finishes the case $\chi\ne\chi_0$.

Now suppose that $\chi=\chi_0$. By Theorem 3.1

$$\mathcal{M}_2(\chi_0,Q) = X_{\chi_0}(Q) - 2Y_{\chi_0}(Q) + \frac{2Q^2}{\prod_{p|k}(1+1/p)} - \frac{3Q^2\varphi(k)}{kL(2,\chi_0)} + O\Big(\frac{k2^{\omega(k)}Q^{4/3}(\log Q)^{5/3}}{\varphi(k)}\Big).$$

By (3.5)

$$X_{\chi_0}(Q) = \sum_{s\le 2Q}\chi_0(s)\varphi(s)\left[\frac{2Q}{s}\right]^2 - \frac{3Q^2\varphi(k)}{2kL(2,\chi_0)} + O(\tau(k)Q\log Q).$$

By (3.7) and (4.3), the sum above is

$$\sum_{s\le 2Q}\chi_0(s)\varphi(s)\left[\frac{2Q}{s}\right]\left(\left[\frac{2Q}{s}\right]+1\right)-\sum_{s\le 2Q}\chi_0(s)\varphi(s)\left[\frac{2Q}{s}\right]$$
$$=2\sum_{n\le 2Q} nf_{\chi_0}(n)-\frac{2Q^2}{\prod_{p|k}(1+1/p)}+O\left(\frac{k2^{\omega(k)}Q^{4/3}(\log Q)^{5/3}}{\varphi(k)}\right).$$

Hence

$$X_{\chi_0}(Q)=2\sum_{n\le 2Q} nf_{\chi_0}(n)-\frac{2Q^2}{\prod_{p|k}(1+1/p)}-\frac{3Q^2\varphi(k)}{2kL(2,\chi_0)}$$
$$+O\left(\frac{k2^{\omega(k)}Q^{4/3}(\log Q)^{5/3}}{\varphi(k)}\right).$$

To treat the sum Y_{χ_0} in (4.4) we extend the range from $1\le s\le Q$ to $1\le s\le 2Q$ and apply (3.5) and (4.3), to see that

$$\sum_{s\le Q}\chi_0(s)\varphi(s)\left[\frac{2Q}{s}\right]\left(\left[\frac{2Q}{s}\right]+1\right)$$
$$=\sum_{s\le 2Q}\chi_0(s)\varphi(s)\left[\frac{2Q}{s}\right]\left(\left[\frac{2Q}{s}\right]+1\right)-2\sum_{Q<s\le 2Q}\chi_0(s)\varphi(s)$$
$$=2\sum_{n\le 2Q} nf_{\chi_0}(n)-\frac{3Q^2\varphi(k)}{kL(2,\chi_0)}+O(\tau(k)Q\log Q).$$

Further,

$$\sum_{s\le Q}\frac{\chi_0(s)\varphi(s)}{s}\left[\frac{2Q}{s}\right]=\sum_{s\le 2Q}\frac{\chi_0(s)\varphi(s)}{s}\left[\frac{2Q}{s}\right]-\sum_{Q<s\le 2Q}\frac{\chi_0(s)\varphi(s)}{s}$$
$$=\sum_{n\le 2Q} f_{\chi_0}(n)-\sum_{Q<s\le 2Q}\frac{\chi_0(s)\varphi(s)}{s},$$

where

$$\sum_{Q<s\le 2Q}\frac{\chi_0(s)\varphi(s)}{s}=\sum_{Q<s\le 2Q}\chi_0(s)\sum_{d|s}\frac{\mu(d)}{d}$$
$$=\sum_{d\le 2Q}\frac{\chi_0(d)\mu(d)}{d}\sum_{\frac{Q}{d}<l\le\left[\frac{2Q}{d}\right]}\chi_0(l)$$
$$=\sum_{d\le 2Q}\frac{\chi_0(d)\mu(d)}{d}\left(\frac{\varphi(k)}{k}\left(\left[\frac{2Q}{d}\right]-\frac{Q}{d}\right)+O(\tau(k))\right)$$
$$=\frac{Q\varphi(k)}{kL(2,\chi_0)}+O(\tau(k)\log Q).$$

Hence

$$Y_{\chi_0}(Q) = 2\sum_{n\le 2Q}(n-Q)f_{\chi_0}(n) - \frac{Q^2\varphi(k)}{kL(2,\chi_0)} + O(Q(\log Q)^2)$$

and, altogether,

$$\mathcal{M}_2(\chi_0,Q) = 2\sum_{n\le 2Q}(2Q-n)f_{\chi_0}(n) - \frac{5Q^2\varphi(k)}{2kL(2,\chi_0)} + O\biggl(\frac{k2^{\omega(k)}Q^{4/3}(\log Q)^{5/3}}{\varphi(k)}\biggr). \tag{4.7}$$

The Dirichlet series for $f_{\chi_0}(n)$ is

$$\sum_{n=1}^{\infty}\frac{f_{\chi_0}(n)}{n^s} = \frac{\zeta(s)^2}{\zeta(s+1)}\prod_{p\mid k}\frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{s+1}}},$$

where

$$\biggl|\prod_{p\mid k}\frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{s+1}}}\biggr| \le \frac{k2^{\omega(k)}}{\varphi(k)}$$

uniformly for $\sigma \ge 0$, so that

$$\sum_{n\le 2Q}(2Q-n)f_{\chi_0}(n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{(2Q)^{s+1}\zeta(s)^2}{s(s+1)\zeta(s+1)}\prod_{p\mid k}\frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{s+1}}}\,ds.$$

We now take the path of integration to be the infinite broken line that goes horizontally from $2-iT$ to $-iT$, vertically from $-iT$ to iT, and horizontally from iT to $2+iT$. We can integrate through the point $s=0$ because the pole at $s=0$ is removed by the zero of $\frac{1}{\zeta(s+1)}$. By Cauchy's residue theorem the integral above is $\sum_{j=1}^{5} I_j$ plus the residue left by the double pole at $s=1$, which is given by

$$\frac{12Q^2}{\pi^2}\biggl(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma + \sum_{p\mid k}\frac{p\log p}{p^2-1}\biggr)\prod_{p\mid k}\frac{p}{p+1}.$$

The integrals I_1 and I_5 are along the vertical segments $(2-i\infty, 2-iT]$ and $[2+iT, 2+i\infty)$, respectively, on which $\sigma = 2$. The integrand is $\ll \frac{k2^{\omega(k)}Q^3}{\varphi(k)t^2}$ and

$$|I_1|, |I_5| \ll \frac{k2^{\omega(k)}Q^3}{\varphi(k)}\int_T^{\infty}\frac{dt}{t^2} \ll \frac{k2^{\omega(k)}Q^3}{\varphi(k)T}.$$

The integrals I_2 and I_4 are along the horizontal segments $[2-iT, -iT]$ and $[iT, 2+iT]$, respectively, on which $s = \sigma + it$. We have $|\zeta(s)| \ll T^{(1/2)+\varepsilon}$ and

$|\zeta(s+1)| \gg \frac{1}{\log T}$ (see [18]), so that

$$|I_2|, |I_4| \ll \frac{k2^{\omega(k)}Q^3T^{-1+3\varepsilon}}{\varphi(k)}.$$

The integral I_3 is along the vertical segment $[-iT, iT]$, on which $\sigma = 0$. We employ (4.6) and $|\zeta(1-it)| = |\zeta(1+it)|$ to see that the integrand is

$$\ll \frac{k2^{\omega(k)}Q\min(t^2, |t|)|\zeta(1+it)|}{\varphi(k)|t|(|t|+1)}$$

and

$$|I_3| \ll \frac{k2^{\omega(k)}Q}{\varphi(k)}\left(1 + \int_1^T \frac{|\zeta(1+it)|}{t}\,dt\right) \ll \frac{k2^{\omega(k)}Q(\log T)^2}{\varphi(k)}.$$

Collecting all estimates and selecting $T = Q^3$, we obtain

$$\sum_{n\le 2Q}(2Q-n)f_{\chi_0}(n) = \frac{12Q^2}{\pi^2}\left(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma + \sum_{p|k}\frac{p\log p}{p^2-1}\right) \times \prod_{p|k}\frac{p}{p+1} + O\left(\frac{k2^{\omega(k)}Q(\log Q)^2}{\varphi(k)}\right),$$

and inserting this into (4.7) proves part (ii) of Theorem 1.1.

5. Higher moments

In this section we prove parts (iii) and (iv) of Theorem 1.1. We apply (1.1) and the binomial theorem to (1.3) to see that

$$\begin{aligned}
\mathcal{M}_l(\chi, Q) &= \sum_{s\le Q}\chi(s)\sum_{\gamma_i=\frac{b}{s}\in\mathfrak{F}_Q}\nu(\gamma_i)^l \qquad (5.1)\\
&= \sum_{s\le Q}\chi(s)\sum_{\substack{Q-s<r\le Q\\(r,s)=1}}\left[\frac{Q+r}{s}\right]^l\\
&= \sum_{s\le Q}\chi(s)\sum_{\substack{Q-s<r\le Q\\(r,s)=1}}\left(\left(\frac{Q+r}{s}\right)^l + O\left(\left(\frac{Q+r}{s}\right)^{l-1}\right)\right)\\
&= \sum_{s\le Q}\chi(s)\sum_{\substack{Q-s<r\le Q\\(r,s)=1}}\left(\frac{Q+r}{s}\right)^l + O(\mathcal{M}_{l-1}(\mathbb{1}, Q)),
\end{aligned}$$

where

$$\mathcal{M}_{l-1}(\mathbb{1}, Q) = \sum_{s\le Q}\sum_{\substack{Q-s<r\le Q\\(r,s)=1}}\left[\frac{Q+r}{s}\right]^{l-1}$$

with $\mathbb{1}$ being the trivial character: $\mathbb{1}(n) = 1$, for all $n \geq 1$. Note that

$$\sum_{s=1}^{\infty} \frac{\chi(s)\varphi(s)}{s^l} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \sum_{\substack{s=1 \\ d|s}}^{\infty} \frac{\chi(s)}{s^{l-1}} = \sum_{d=1}^{\infty} \frac{\chi(d)\mu(d)}{d^l} \sum_{m=1}^{\infty} \frac{\chi(m)}{m^{l-1}} = \frac{L(l-1,\chi)}{L(l,\chi)}.$$

Next, we observe that

$$\begin{aligned} \sum_{s\leq Q}\Bigg|\sum_{Q-s<r\leq Q}\left(\left(\frac{Q+r}{s}\right)^l - \left(\frac{2Q}{s}\right)^l\right)\Bigg| &\leq \sum_{s\leq Q}\sum_{Q-s<r\leq Q} \frac{(2Q)^l-(Q+r)^l}{s^l} \\ &\ll_l \sum_{s\leq Q} \frac{(Q-r)Q^{l-1}}{s^{l-1}} \\ &\ll_l \sum_{s\leq Q} \frac{Q^{l-1}}{s^{l-2}}, \end{aligned}$$

which is $\ll_l Q^2 \log Q$ if $l = 3$, and $\ll_l Q^{l-1}$ if $l \geq 4$. It follows that

$$\begin{aligned} \mathcal{M}_3(\chi, Q) &= \frac{8Q^3 L(2,\chi)}{L(3,\chi)} + O(Q^2 \log Q) + \mathcal{M}_2(\mathbb{1}, Q) \\ &= \frac{8Q^3 L(2,\chi)}{L(3,\chi)} + O(Q^2 \log Q) \end{aligned}$$

using part (ii) of Theorem 1.1. This proves part (iii) of Theorem 1.1. For $l \geq 4$ we have

$$\begin{aligned} \mathcal{M}_l(\chi, Q) &= 2^l Q^l \sum_{s=1}^{\infty} \frac{\chi(s)\varphi(s)}{s^l} + O(Q^{l-1}) + O(\mathcal{M}_{l-1}(\mathbb{1}, Q)) \\ &= \frac{2^l Q^l L(l-1,\chi)}{L(l,\chi)} + O(Q^{l-1}). \end{aligned}$$

This proves part (iv) of Theorem 1.1.

6. Proof of Corollary 1.2

In this section we prove Corollary 1.2. In view of the identity

$$\frac{1}{\varphi(k)} \sum_{\chi} \chi(\overline{u}s) = \begin{cases} 1 & \text{if } s \equiv u \pmod{k}, \\ 0 & \text{otherwise} \end{cases}$$

and $u\overline{u} \equiv 1 \pmod{k}$, we have by (1.4)

$$\begin{aligned}\mathcal{M}_l(u,k,Q) &= \frac{1}{\varphi(k)} \sum_{\gamma_i = \frac{b}{s} \in \mathfrak{F}_Q} \nu(\gamma_i)^l \sum_{\chi} \chi(\overline{u}s) \\ &= \frac{1}{\varphi(k)} \sum_{\chi} \chi(\overline{u}) \sum_{\gamma_i = \frac{b}{s} \in \mathfrak{F}_Q} \chi(s)\nu(\gamma_i)^l \\ &= \frac{1}{\varphi(k)} \sum_{\chi} \chi(\overline{u}) \mathcal{M}_l(\chi, Q).\end{aligned}$$

Noting $\chi_0(\overline{u}) = 1$ and $\overline{\chi}(u) = \chi(\overline{u})$, we record that if $l = 1$ then Theorem 1.1 yields

$$\begin{aligned}\mathcal{M}_1(u,k,Q) &= \frac{3Q^2}{2kL(2,\chi_0)} + O\bigg(\frac{\tau(k)Q\log Q}{\varphi(k)}\bigg) + O\bigg(\sum_{\chi \neq \chi_0} \frac{\sqrt{k}\log k \cdot Q\log Q}{\varphi(k)}\bigg) \\ &= \frac{3Q^2}{2kL(2,\chi_0)} + O_k(Q\log Q).\end{aligned}$$

If $l = 2$, then Theorem 1.1 gives us

$$\begin{aligned}\mathcal{M}_2(u,k,Q) &= \frac{24Q^2}{\pi^2\varphi(k)}\bigg(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma + \sum_{p\mid k} \frac{p\log p}{p^2-1}\bigg) \prod_{p\mid k} \frac{p}{p+1} \\ &\quad - \frac{5Q^2}{2kL(2,\chi_0)} + O\bigg(\frac{k2^{\omega(k)}Q^{4/3}(\log Q)^{5/3}}{\varphi(k)^2}\bigg) \\ &\quad + \frac{4Q^2}{\varphi(k)} \sum_{\chi\neq\chi_0} \frac{\overline{\chi}(u)L(1,\chi)}{L(2,\chi)} + O\bigg(\sum_{\chi\neq\chi_0} \frac{\sqrt{k}\log k \cdot Q(\log Q)^2}{\varphi(k)}\bigg) \\ &= \frac{24Q^2}{\pi^2\varphi(k)}\bigg(\log 2Q - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma + \sum_{p\mid k} \frac{p\log p}{p^2-1}\bigg) \prod_{p\mid k} \frac{p}{p+1} \\ &\quad - \frac{5Q^2}{2kL(2,\chi_0)} + \frac{4Q^2}{\varphi(k)} \sum_{\chi\neq\chi_0} \frac{\overline{\chi}(u)L(1,\chi)}{L(2,\chi)} + O_k(Q^{4/3}(\log Q)^{5/3}).\end{aligned}$$

The asymptotic formulas for $\mathcal{M}_3(u,k,Q)$ and $\mathcal{M}_l(u,k,Q)$ for $l \geq 4$ are obtained in a similar way. The details are left to the reader.

References

[1] V. Augustin, F. P. Boca, C. Cobeli, and A. Zaharescu, *The h-spacing distribution between Farey points*, Math. Proc. Cambridge Philos. Soc. **131** (2001), 23–38. MR 1833071 (2002h:11017)

[2] F. P. Boca, C. Cobeli, and A. Zaharescu, *Distribution of lattice points visible from the origin*, Comm. Math. Phys. **213** (2000), 433–470. MR 1785463 (2001j:11094)

[3] ———, *A conjecture of R. R. Hall on Farey points*, J. Reine Angew. Math. **535** (2001), 207–236. MR 1837099 (2002f:11127)

[4] F. P. Boca, R. N. Gologan, and A. Zaharescu, *On the index of Farey sequences*, Q. J. Math. **53** (2002), 377–391. MR 1949149 (2004c:11021)

[5] ______, *The average length of a trajectory in a certain billiard in a flat two-torus*, New York J. Math. **9** (2003), 303–330. MR 2028172 (2004m:37065)

[6] ______, *The statistics of the trajectory of a certain billiard in a flat two-torus*, Comm. Math. Phys. **240** (2003), 53–73. MR 2004979 (2004f:37007)

[7] H. S. M. Coxeter, *Introduction to geometry*, John Wiley & Sons Inc., New York, 1961. MR 0123930 (23 #A1251)

[8] H. Davenport, *Multiplicative number theory*, third ed., Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 2000. MR 1790423 (2001f:11001)

[9] T. Estermann, *On Kloosterman's sum*, Mathematika **8** (1961), 83–86. MR 0126420 (23 #A3716)

[10] R. R. Hall, *A note on Farey series*, J. London Math. Soc. (2) **2** (1970), 139–148. MR 0253978 (40 #7191)

[11] ______, *On consecutive Farey arcs. II*, Acta Arith. **66** (1994), 1–9. MR 1262649 (95f:11074)

[12] R. R. Hall and P. Shiu, *The index of a Farey sequence*, Michigan Math. J. **51** (2003), 209–223. MR 1960929 (2004j:11019)

[13] R. R. Hall and G. Tenenbaum, *On consecutive Farey arcs*, Acta Arith. **44** (1984), 397–405. MR 777016 (86e:11087)

[14] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, fifth ed., The Clarendon Press Oxford University Press, New York, 1979. MR 568909 (81i:10002)

[15] M. N. Huxley and A. A. Zhigljavsky, *On the distribution of Farey fractions and hyperbolic lattice points*, Period. Math. Hungar. **42** (2001), 191–198. MR 1832705 (2002d:11022)

[16] A. Ivić, *The Riemann zeta-function*, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1985, The theory of the Riemann zeta-function with applications. MR 792089 (87d:11062)

[17] H. L. Montgomery and R. C. Vaughan, *Exponential sums with multiplicative coefficients*, Invent. Math. **43** (1977), 69–82. MR 0457371 (56 #15579)

[18] ______, *Multiplicative Number Theory: I. Classical Theory*, Cambridge University Press, Cambridge, to appear.

[19] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press, London, 1939.

[20] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., Oxford University Press, New York, 1986. MR 882550 (88c:11049)

[21] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., Cambridge University Press, Cambridge, 1952. MR 0178117 (31 #2375)

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