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THE ISOMETRIC EXTENSION OF THE INTO MAPPING FROM A $\mathcal{L}^\infty(\Gamma)$-TYPE SPACE TO SOME BANACH SPACE

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ABSTRACT. We give some conditions under which an "into" isometric mapping from the unit sphere of an $\mathcal{L}^\infty(\Gamma)$-type space (in particular, the atomic AM-space) to the unit sphere of some Banach space can be (real) linearly extended.

1. Introduction

After the extension problem of isometries between unit spheres was posed by D. Tingley in [8], almost all of papers concerning this problem considered only "onto"(surjective) mappings between two spheres (see [1], [3]).

In [2], we first considered the isometric extension problem of "into" mappings between two unit spheres. In [9], some conditions were given under which an isometry between unit spheres of "atomic" AL^p-spaces ($1 < p < \infty, p \neq 2$) can be linearly isometrically extended. Moreover, in [5], Z. Hou obtained an affirmative answer for an "into" isometry between the unit spheres of arbitrary AL^p-spaces without any condition. In [4], we considered the isometric extension problem of "onto" mappings between two unit spheres of ℓ^∞-type spaces.

In the present paper, we will obtain some natural and useful conditions under which an isometry from the unit sphere of an $\mathcal{L}^\infty(\Gamma)$-type space into the unit sphere of some Banach space E can be (real) linearly isometrically extended. Here, an $\mathcal{L}^\infty(\Gamma)$-type space is a normed space of functions on an index set Γ equipped with the sup norm. For example, the spaces $\ell^\infty(\Gamma)$, $c(\Gamma)$ and $c_0(\Gamma)$ (in particular, ℓ^∞, c and c_0) are all $\mathcal{L}^\infty(\Gamma)$-type spaces (or $\mathcal{L}^\infty(\Gamma)$ spaces, in brief).

In this paper, all spaces are over the real field.

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2. Some lemmas

We first give a lemma which is similar to Lemma 2.1 in [2].

LEMMA 1. *Let E be a normed space, V_0 be an isometric mapping from the unit sphere of $\mathcal{L}^\infty(\Gamma)$ into the unit sphere $S(E)$. If $-V_0[S(\mathcal{L}^\infty(\Gamma))] \subset V_0[S(\mathcal{L}^\infty(\Gamma))]$, then*

$$V_0(-x) = -V_0(x) \quad \forall\, x \in S(\mathcal{L}^\infty(\Gamma)).$$

Proof. First, we will show that $V_0(-e_\gamma) = -V_0(e_\gamma)$ for all $\gamma \in \Gamma$. In fact, for each $\gamma \in \Gamma$ and $\gamma' \neq \gamma$ ($\gamma' \in \Gamma$), by the hypothesis on V_0, we have $V_0 x = -V_0 e_\gamma$, $V_0 x' = -V_0 e_{\gamma'}$ and $V_0 y' = -V_0(-e_{\gamma'})$ (where, x, x' and y' are elements in $S(\mathcal{L}^\infty(\Gamma))$). From the equalities

$$\|x - e_\gamma\| = \|V_0(x) - V_0(e_\gamma)\| = \| - 2V_0 e_\gamma\| = 2$$

and (similarly) $\|x' - e_{\gamma'}\| = 2$ we immediately get, by the definition of norm in $\mathcal{L}^\infty(\Gamma)$, that

$$x(\gamma) = -1, \quad x'(\gamma') = -1. \tag{1}$$

Moreover, notice that

$$\|x - x'\| = \|V_0 x - V_0 x'\| = \| - V_0 e_\gamma + V_0 e_{\gamma'}\| = \|e_{\gamma'} - e_\gamma\| = 1,$$

which implies by (1) that

$$x(\gamma') \leq 0, \quad \forall\, \gamma' \neq \gamma, \gamma' \in \Gamma. \tag{2}$$

On the other hand, from

$$\|y' + e_{\gamma'}\| = \|y' - (-e_{\gamma'})\| = \|V_0(y') - V_0(-e_{\gamma'})\| = \| - 2V_0(-e_{\gamma'})\| = 2$$

and

$$\|x - y'\| = \|V_0 x - V_0 y'\| = \| - V_0 e_\gamma + V_0(-e_{\gamma'})\| = \| - e_{\gamma'} - e_\gamma\| = 1$$

we get $y'(\gamma') = 1$ and

$$x(\gamma') \geq 0, \quad \forall \gamma' \neq \gamma, \gamma' \in \Gamma. \tag{3}$$

From (1), (2) and (3) we obtain $x = -e_\gamma$. Thus we have proved that

$$V_0(-e_\gamma) = -V_0(e_\gamma), \quad \forall\, \gamma \in \Gamma. \tag{4}$$

Now, we complete the proof of the lemma. For each $x \in \mathcal{L}^\infty(\Gamma)$, by the hypothesis on V_0, let $V_0 y = -V_0 x$ (where y is some element in $\mathcal{L}^\infty(\Gamma)$). From the equalities (noticing (4))

$$\|e_\gamma + y\| = \|y - (-e_\gamma)\| = \| - V_0 x - V_0(-e_\gamma)\| = \|V_0 e_\gamma - V_0 x\| = \|e_\gamma - x\|$$

and

$$\|e_\gamma - y\| = \|V_0 e_\gamma - V_0 y\| = \|V_0 e_\gamma + V_0 x\| = \|V_0 x - V_0(-e_\gamma)\| = \|e_\gamma + x\|$$

we get

$$x(\gamma) \le 0 \Rightarrow \|e_\gamma + y\| = 1 + |x(\gamma)| \Rightarrow y(\gamma) = |x(\gamma)| = -x(\gamma)$$

and

$$x(\gamma) > 0 \Rightarrow \|e_\gamma - y\| = 1 + |x(\gamma)| \Rightarrow y(\gamma) = -|x(\gamma)| = -x(\gamma), \quad \forall\, \gamma \in \Gamma.$$

Thus we obtain that $y = -x$, which completes the proof. □

LEMMA 2. *Let Y be a normed space, $y_1, y_2, \dots, y_n$ be in the unit sphere $S(Y)$. If for every $\theta_k = \pm 1$ $(1 \le k \le n)$,*

$$\|\theta_1 y_1 + \theta_2 y_2 + \cdots + \theta_m y_m\| = 1 \quad (1 \le m \le n), \tag{5}$$

then for every $\lambda_k \in \mathbb{R}$ $(1 \le k \le n)$,

$$\left\| \sum_{k=1}^{n} \lambda_k y_k \right\| = \max_{1 \le k \le n} |\lambda_k|. \tag{6}$$

Proof. Without loss of generality, we may assume that $\lambda_k \ne 0$ $(1 \le k \le n)$ and $|\lambda_1| = \max_{1 \le k \le n} |\lambda_k|$.

By the Hahn-Banach theorem, there exists y_1^* in the unit sphere $S(Y^*)$ such that

$$y_1^*(y_1) = \|y_1\| = 1, \quad y_1^*(y_k) = 0 \;\; (2 \le k \le n) \tag{7}$$

by hypothesis (5). Thus we get

$$\left\| \sum_{k=1}^{n} \lambda_k y_k \right\| \ge \left| y_1^* \left(\sum_{k=1}^{n} \lambda_k y_k \right) \right| = |\lambda_1| = \max_{1 \le k \le n} |\lambda_k|. \tag{8}$$

On the other hand, notice that every normed space Y can be embedded linearly and isometrically into a $C(\Omega)$ space with Ω being a compact subset of the unit ball of Y^* (see, for example, Corollary 2.6.22 in [7]). So, we can consider Y as a linear subspace $C(\Omega)$, and we get by (5) that

$$|y_1(t)| + |y_2(t)| + \cdots + |y_n(t)| \le 1, \quad \forall\, t \in \Omega,$$

and

$$\left| \left(\sum_{k=1}^{n} \lambda_k y_k \right)(t) \right| \le \sum_{k=1}^{n} |\lambda_k y_k(t)| \le \max_{1 \le k \le n} |\lambda_k|, \quad \forall\, t \in \Omega.$$

Thus,

$$\left\| \sum_{k=1}^{n} \lambda_k y_k \right\| \le \max_{1 \le k \le n} |\lambda_k|. \tag{9}$$

The result follows from (8) and (9). □

LEMMA 3. *Let E be a normed space, V_0 be an isometric mapping from the unit sphere $S(\mathcal{L}^\infty(\Gamma))$ into the unit sphere $S(E)$. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ be mutually disjoint subsets of the index set Γ. Suppose that the following conditions hold:*

(i) $\|\sum_{k=1}^n \theta_k V_0(\chi_{\Gamma_k})\| = 1$ *for every* $\theta_k = \pm 1$ $(1 \le k \le n)$. *(Here, χ_{Γ_k} is the characteristic function of Γ_k, $1 \le k \le n$.)*

(ii) *If* $V_0 x = \sum_{k=1}^n \lambda_k V_0(\chi_{\Gamma_k})$, *then* $x = \sum_{k=1}^n \lambda'_k \chi_{\Gamma_k} + x_0$ *with* $\operatorname{supp} x_0 \subset (\bigcup_{k=1}^n \Gamma_k)^c$.

(iii) $-V_0[S(\mathcal{L}^\infty(\Gamma))] \subset V_0[S(\mathcal{L}^\infty(\Gamma))]$.

Then we have that $\lambda'_k = \lambda_k \quad (1 \le k \le n)$ *and* x_0 *is zero element.*

Proof. Using hypothesis (i) and Lemma 2, we get, for all $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathbb{R}$,

$$\left\| \sum_{k=1}^n \lambda_k V_0(\chi_{\Gamma_k}) \right\| = \max_{1 \le k \le n} |\lambda_k|. \tag{10}$$

Without loss of generality, we may assume that $\lambda'_1 \neq 0$ in the hypothesis (ii). By (10), we have

$$\begin{aligned} \left\| V_0 x + \frac{\lambda'_1}{|\lambda'_1|} V_0(\chi_{\Gamma_1}) \right\| &= \left\| \sum_{k=2}^n \lambda_k V_0(\chi_{\Gamma_k}) + \left(\lambda_1 + \frac{\lambda'_1}{|\lambda'_1|}\right) V_0(\chi_{\Gamma_1}) \right\| \\ &= \max_{2 \le k \le n} \left(|\lambda_k|, |\lambda_1 + \frac{\lambda'_1}{|\lambda'_1|}| \right) \\ &= \max_{2 \le k \le n} \left(|\lambda_k|, 1 + \frac{\lambda'_1 \lambda_1}{|\lambda'_1|} \right). \end{aligned} \tag{11}$$

Using the assumption on V_0 and Lemma 1, we can continue the above equalities by

$$\begin{aligned} &= \left\| x + \frac{\lambda'_1}{|\lambda'_1|} \chi_{\Gamma_1} \right\| = \left\| \sum_{k=1}^n \lambda'_k \chi_{\Gamma_k} + x_0 + \frac{\lambda'_1}{|\lambda'_1|} \chi_{\Gamma_1} \right\| \\ &= \max_{2 \le k \le n} \left(|\lambda'_k|, |\lambda'_1 + \frac{\lambda'_1}{|\lambda'_1|}|, \|x_0\| \right) \\ &= 1 + |\lambda'_1| \quad (> 1). \end{aligned} \tag{12}$$

By (11) and (12), since $|\lambda_k| \le 1$ $(2 \le k \le n)$, we have

$$1 + \frac{\lambda'_1 \lambda_1}{|\lambda'_1|} = 1 + |\lambda'_1|.$$

Hence we have $\lambda'_1 = \lambda_1$. Similarly, we obtain $\lambda'_k = \lambda_k$ $(2 \le k \le n)$.

Finally, notice that for each $\Gamma_0 \subset (\bigcup_{k=1}^n \Gamma_k)^c$, and suppose that $\lambda'_0 \neq 0$ and $x_0 = \lambda'_0 \chi_{\Gamma_0} + x_{00}$ with $\operatorname{supp} x_{00} \subset (\bigcup_{k=0}^n \Gamma_k)^c$. Then, similarly to the above argument, from the contradiction

$$\left\|x + \frac{\lambda_0'}{|\lambda_0'|}\chi_{\Gamma_0}\right\| = 1 + |\lambda_0'| \ > 1 = \left\|V_0 x + \frac{\lambda_0'}{|\lambda_0'|}V_0(\chi_{\Gamma_0})\right\|, \tag{13}$$

we obtain $x_0 = \theta$. This completes this proof. □

3. Main results

THEOREM 1. *Let E be a Banach space, V_0 be an isometric mapping from the unit sphere $S(\mathcal{L}^\infty(\Gamma))$ into the unit sphere $S(E)$. Then V_0 can be extended to a linear isometry defined on the whole space $\mathcal{L}^\infty(\Gamma)$ if and only if the following conditions hold:*

(i) *For every x_1 and x_2 in $S(\mathcal{L}^\infty(\Gamma))$, λ_1 and λ_2 in $\mathbb{R}$,*

$$\|\lambda_1 V_0 x_1 + \lambda_2 V_0 x_2\| = 1 \implies \lambda_1 V_0 x_1 + \lambda_2 V_0 x_2 \in V_0[S(\mathcal{L}^\infty(\Gamma))].$$

(ii) *If $V_0(x) = \sum_{k=1}^n \lambda_k V_0(\chi_{\Gamma_k})$, then $x = \sum_{k=1}^n \lambda_k' \chi_{\Gamma_k} + x_0$. Here $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are mutually disjoint subsets of Γ and* $\operatorname{supp} x_0 \subset (\bigcup_{k=1}^n \Gamma_k)^c$.

Proof. If V_0 can be extended to a linear isometry on the whole $\mathcal{L}^\infty(\Gamma)$, it is clear that the conditions (i) and (ii) hold.

Conversely, assume that both (i) and (ii) hold. Firstly, by the equality

$$\sum_{k=1}^n \lambda_k V_0 x_k = \left\|\sum_{k=1}^{n-1} \lambda_k V_0 x_k\right\| \sum_{k=1}^{n-1} \frac{\lambda_k}{\left\|\sum_{k=1}^{n-1} \lambda_k V_0 x_k\right\|} V_0 x_k + \lambda_n V_0 x_n,$$

we get by induction that

$$\left\|\sum_{k=1}^n \lambda_k V_0 x_k\right\| = 1 \implies \sum_{k=1}^n \lambda_k V_0 x_k \in V_0[S(\mathcal{L}^\infty(\Gamma))], \tag{14}$$
$$\forall\, x_k \in S(\mathcal{L}^\infty(\Gamma)), \lambda_k \in \mathbb{R} \quad (1 \le k \le n),\ n \in \mathbb{N}.$$

Secondly, we shall show that for each $n \in \mathbb{N}$, all mutually disjoint characteristic functions $\chi_{\Gamma_1}, \chi_{\Gamma_2}, \ldots, \chi_{\Gamma_n}$ and $\theta_k = \pm 1 \quad (1 \le k \le n)$,

$$\|\theta_1 V_0(\chi_{\Gamma_1}) + \theta_2 V_0(\chi_{\Gamma_2}) + \cdots + \theta_n V_0(\chi_{\Gamma_n})\| = 1. \tag{15}$$

Indeed, we will prove (15) by induction. For $n = 2$, this is easy to verify using the fact that V_0 is isometric and Lemma 1. Now assume that (15) holds for $n = m - 1$. By (14), there exists $\hat{x} \in S(\mathcal{L}^\infty(\Gamma))$ such that

$$V_0 \hat{x} = \sum_{k=1}^{m-1} \theta_k V_0(\chi_{\Gamma_k}). \tag{16}$$

Now let $n = m$, and suppose that (15) does not hold. Without loss of generality, let $\|\sum_{k=1}^m \theta_k V_0(\chi_{\Gamma_k})\| > 1$. Then it follows by Lemma 1 that

$$\|\hat{x} + \theta_m \chi_{\Gamma_m}\| = \|V_0 \hat{x} + \theta_m V_0(\chi_{\Gamma_m})\| > 1.$$

Hence there is an index $\gamma_m \in \Gamma_m$ such that

$$|\hat{x}(\gamma_m) + \theta_m| > 1. \tag{17}$$

Moreover, by the induction assumption and (14), there exists $\tilde{x} \in S(\mathcal{L}^\infty(\Gamma))$ such that

$$V_0\tilde{x} = -\sum_{k=2}^{m-1} \theta_k V_0(\chi_{\Gamma_k}) + \theta_m V_0(\chi_{\Gamma_k}). \tag{18}$$

By Lemma 3, (18) implies

$$\tilde{x} = -\sum_{k=2}^{m-1} \theta_k \chi_{\Gamma_k} + \theta_m \chi_{\Gamma_k}. \tag{19}$$

Thus, by Lemma 1, (17) and (19) we have

$$\|V_0\hat{x} + V_0\tilde{x}\| = \|\hat{x} + \tilde{x}\| \geq |\hat{x}(\gamma_m) + \tilde{x}(\gamma_m)| = |\hat{x}(\gamma_m) + \theta_m| > 1, \tag{20}$$

while, by (16) and (18), we also have

$$\|V_0\hat{x} + V_0\tilde{x}\| = \|\theta_1 V_0(\chi_{\Gamma_1}) + \theta_m V_0(\chi_{\Gamma_m})\| = \|\theta_1\chi_{\Gamma_1} + \theta_m\chi_{\Gamma_m}\| = 1. \tag{21}$$

The contradiction between (20) and (21) proves (15).

Now, using Lemma 2 and Lemma 3, we obtain that for each $n \in \mathbb{N}$, all mutually disjoint subsets $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ of the index set Γ, and any $\lambda_1^0, \lambda_2^0, \ldots, \lambda_n^0$ in $\mathbb{R}$ with $\max_{1\leq k\leq n} |\lambda_k^0| = 1$,

$$V_0\left(\sum_{k=1}^{n} \lambda_k^0 \chi_{\Gamma_k}\right) = \sum_{k=1}^{n} \lambda_k^0 V_0(\chi_{\Gamma_k}).$$

That is, V_0 is linear on the subset which consists of all simple functions of the unit sphere $S(\mathcal{L}^\infty(\Gamma))$.

Finally, we similarly define a mapping on the subspace X consisting of all simple functions of $\mathcal{L}^\infty(\Gamma)$ as follows:

$$V_1x = V_1\left(\sum_{k=1}^{n} \lambda_k \chi_{\Gamma_k}\right) = \sum_{k=1}^{n} \lambda_k V_0(\chi_{\Gamma_k}), \quad \forall\, x = \sum_{k=1}^{n} \lambda_k \chi_{\Gamma_k} \in X\ (\subset \mathcal{L}^\infty(\Gamma)).$$

By Lemma 2, we have

$$\|V_1x\| = \max_{1\leq k\leq n} |\lambda_k| = \|x\|, \quad \forall\, x = \sum_{k=1}^{n} \lambda_k \chi_{\Gamma_k} \in X.$$

That is, V_1 is a linear isometry on X. Notice that the subspace X is dense in $\mathcal{L}^\infty(\Gamma)$, V_1 is isometric on X, and the space E is complete. Hence V_1 has a unique linearly isometric extension V to the whole space $\mathcal{L}^\infty(\Gamma)$. Then it is easy to see that V is an extension of V_0. This completes the proof. □

In particular, using the fact that the space c_0 has a Schauder basis, we get the following theorem:

THEOREM 2. *Let E be a Banach space, V_0 be an isometric mapping from the unit sphere $S(c_0)$ into the unit sphere $S(E)$. Then V_0 can be extended to a linear isometry defined on the whole space c_0 if and only if the following condition holds:*

($*$) *For all x_1 and x_2 in $S(c_0)$, λ_1 and λ_2 in $\mathbb{R}$,*

$$\|\lambda_1 V_0 x_1 + \lambda_2 V_0 x_2\| = 1 \implies \lambda_1 V_0 x_1 + \lambda_2 V_0 x_2 \in V_0[S(c_0)].$$

Proof. The proof is similar to the proof of Theorem 1. We only notice that the simple functions $\sum_{k=1}^n \lambda_k e_k$, $\lambda_k \in \mathbb{R}$, $n \in \mathbb{N}$, are dense in the space c_0, and if $V_0 x = \sum_{k=1}^n \lambda_k V_0 e_k$, then x must be of the form

$$x = \sum_{k=1}^{n} \lambda'_k e_k + x_0$$

with $\operatorname{supp} x_0 \subset (\{k \mid 1 \le k \le n\})^c$. Then it is easy to prove the result. □

From the above theorems, we immediately get the following corollaries.

COROLLARY 1. *Let E be a Banach space, V_0 be a surjective isometric mapping from the unit sphere $S(\mathcal{L}^\infty(\Gamma))$ onto the unit sphere $S(E)$. Then V_0 can be extended to a linear isometry defined on the whole space $\mathcal{L}^\infty(\Gamma)$ if and only if the following condition holds:*

($**$) *If $V_0(x) = \sum_{k=1}^n \lambda_k V_0(\chi_{\Gamma_k})$, then $x = \sum_{k=1}^n \lambda'_k \chi_{\Gamma_k} + x_0$. Here $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are mutually disjoint subsets of Γ, and $\operatorname{supp} x_0 \subset (\bigcup_{k=1}^n \Gamma_k)^c$.*

COROLLARY 2. *Let E be a Banach space, V_0 be a surjective isometric mapping from the unit sphere $S(c_0)$ onto the unit sphere $S(E)$. Then V_0 can be extended to a linear isometry defined on the whole space c_0.*

Note that Corollary 2 generalizes the result of [4] when the $\ell^\infty(\Gamma)$-type space is c_0.

Recall that, by Kakutani's representation theorem (see Theorem 1.b.6 of [6]), an AM-space(abstract M-space) is isometric and lattice isomorphic to a sublattice of $C(\Omega)$ space for some compact Hausdorff space Ω. Hence we immediately have the following conclusion:

COROLLARY 3. *Corollary 1 still holds if we replace the space $\mathcal{L}^\infty(\Gamma)$ by an atomic AM-space.*

From Theorem 1 we can also get the main result in [4]:

COROLLARY 4. *Let V_0 be a surjective isometric mapping from the unit sphere $S(\mathcal{L}^\infty(\Gamma))$ onto itself. Then V_0 can be extended to a linear isometry defined on the whole space $\mathcal{L}^\infty(\Gamma)$.*

Proof. We only need to check condition (ii) in Theorem 1. Indeed, if $\sum_{k=1}^{n} \lambda_k V_0(\chi_{\Gamma_k})$ is in the unit sphere $S(\mathcal{L}^\infty(\Gamma))$, (where $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are mutually disjoint subsets of the index set Γ), then there exists x in the unit sphere $S(\mathcal{L}^\infty(\Gamma))$ such that $V_0 x = \sum_{k=1}^{n} \lambda_k V_0(\chi_{\Gamma_k})$ because V_0 is a surjective mapping.

Using the same technique as in the proof of Lemma 3, for each $x = x(\gamma)$, if $\gamma^{(1)} \in \Gamma_1$, we can obtain, similar to (11) and (12) above, that

$$
\begin{aligned}
\left\| V_0 x + \frac{\lambda_1}{|\lambda_1|} V_0(e_{\gamma^{(1)}}) \right\| &= 1 + |\lambda_1| = \left\| x + \frac{\lambda_1}{|\lambda_1|} e_{\gamma^{(1)}} \right\| \\
&= \sup_{\gamma \neq \gamma^{(1)}} \left(|x(\gamma)|, |x(\gamma^{(1)}) + \frac{\lambda_1}{|\lambda_1|}| \right) \\
&= \sup_{\gamma \neq \gamma^{(1)}} \left(|x(\gamma)|, |1 + \frac{x(\gamma^{(1)})\lambda_1}{|\lambda_1|} \right),
\end{aligned} \tag{22}
$$

which implies that $x(\gamma^{(1)}) = \lambda_1$. Similarly, we obtain $x(\gamma^{(k)}) = \lambda_k$ for each $\gamma^{(k)} \in \Gamma_k$ $(2 \leq k \leq n)$.

Thus, using the equality (13) near the end in the proof of Lemma 3, we immediately obtain that $x = \sum_{k=1}^{n} \lambda_k \chi_{\Gamma_k}$. This completes this proof. □

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