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CLIFFORD LINKS ARE THE ONLY MINIMIZERS OF THE ZONE MODULUS AMONG NON-SPLIT LINKS

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Abstract. The zone modulus is a conformally invariant functional over the space of two-component links embedded in $\mathbf{R}^3$ or $\mathbf{S}^3$. It is a positive real number and its lower bound is 1. Its main property is that the zone modulus of a non-split link is greater than $(1+\sqrt{2})^2$. In this paper, we will show that the only non-split links with modulus equal to $(1+\sqrt{2})^2$ are the *Clifford links*, that is, the conformal images of the standard geometric Hopf link.

0. Introduction

Langevin and O'Hara introduced in [1] a conformally invariant functional for knots, called the *measure of acyclicity*. It is the volume (with respect to a conformally invariant measure on the space of all round spheres) of the set of spheres that cut the knot in at least four points. There exists a constant C such that a curve with measure of acyclicity below C is the unknot. To prove this, they introduced a knot modulus called the *zone modulus*.

This work comes after O'Hara's definition in [3] of the concept of a *knot energy*. Roughly, a functional on the space of knots is an energy when it blows up near a self-intersection. An energy is also expected to possess thresholds such that a curve with energy lower than a particular threshold must belong to a particular knot type. A knot representative in a knot class that realizes the minimum energy provides the best shaped knot of its class.

One of the most famous knot energy functionals, introduced by O'Hara in [3], is

$$E(\gamma)=\iint\left\{\frac{1}{|\gamma(v)-\gamma(u)|^2}-\frac{1}{D(\gamma(u),\gamma(v))^2}\right\}|\gamma'(u)||\gamma'(v)|\,du\,dv,$$

where γ is an embedded curve and $D(\gamma(u),\gamma(v))$ denotes the length of the shortest path from $\gamma(u)$ to $\gamma(v)$ on γ. In [4] Freedman, He and Wang proved

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the conformal invariance of E and called E the *Möbius energy.* In the same paper they showed that the energy of a closed curve is always greater than or equal to 4 and that equality holds only for circles. They proved also that each prime knot class has an energy-minimizing representative, and that, given $m > 0$, there are finitely many knot types such that $E \leq m$. In [5], Kim and Kusner constructed explicit examples of knotted curves which are critical for E.

In [2], Langevin and the author proved that the minimum of the zone modulus over all non-split two-component links is $(\sqrt{2}+1)^2$. This minimum is attained by a special configuration of two circles called a *Clifford link*, defined as follows:

Definition 1. We say that a link is a *Clifford link* when it consists of two circles such that each sphere containing one of the circles is perpendicular to each of the spheres containing the other circle. Equivalently, a *Clifford link* is a conformal image of the standard geometric Hopf link.

In [4], Freedman, He and Wang defined the mutual Möbius energy of two curves as

$$E(\gamma_1, \gamma_2) = \iint \frac{|\gamma_1'(u)||\gamma_2'(v)|}{|\gamma_1(u) - \gamma_2(v)|^2}\ du\ dv.$$

Kim and Kusner showed in [5] that the standard geometric Hopf link is critical for E. In [7], He gave a geometric interpretation of the Euler-Lagrange equation for any E-critical pair of curves. He showed that there exists a pair of curves that minimizes E over all linked pairs of loops and that every such pair is ambiently isotopic to the Hopf link. As far as the author knows, it is still a conjecture that Clifford links are the only configurations that minimize the Möbius energy among two-component non-split links.

The purpose of the present paper is to solve the analogous conjecture for the zone modulus. We will show:

Theorem 1. *The two-component links that realize the minimum zone modulus among all non-split two-component links are the Clifford links.*

It should be noted that the standard geometric Hopf link or its conformal class, the Clifford links, seems to be a recurrent minimizer or maximizer of various functionals. For example, Kusner proved in [6] that the thickness of a non-split two-component link in $\mathbf{S}^3$ cannot exceed that of the standard geometric Hopf link, which equals $\pi/4$. In [2], we proved that the standard geometric Hopf is the only non-split two-component link with thickness $\pi/4$.

1. Preliminary definitions and known facts

We will recall in this section the definition of the zone modulus of a two-component link and some results of [2].

1.1. The modulus of a zone between two spheres. We first define the modulus of a zone between two disjoint spheres, which we call for simplicity the modulus of two spheres.

DEFINITION 2. Given two disjoint spheres S_1 and S_2 in $\mathbf{R}^3$, let us choose a conformal transformation that makes the two spheres concentric with radii $R_2 > R_1$. Then the *modulus* $\mu(S_1, S_2)$ of the two spheres is the ratio $R_2/R_1 > 1$.

We can express the modulus in terms of the cross-ratio. Recall that the cross-ratio of four collinear points is defined as

$$\mathrm{Cr}(x_1, x_2, x_3, x_4) = (x_1 - x_3)(x_2 - x_4)/(x_2 - x_3)(x_1 - x_4).$$

The cross-ratio is invariant by any homography of the line. We can extend its definition to four concircular points as follows: The cross-ratio of four points on a circle is the cross-ratio of the four image points by a stereographic projection of the circle onto a line.

Two disjoint spheres S_1 and S_2 generate a pencil of spheres with limit points. It is the set of spheres perpendicular to all the circles perpendicular to both S_1 and S_2. The limit points are the two points of intersection of these circles. Consider a circle perpendicular both to S_1 and S_2 as in Figure 1. It contains the limit points l_1 and l_2 of the pencil generated by S_1 and S_2 and intersects each S_i in two points. Let us take two of these points, p_1 and p_2, such that l_1, p_1, p_2, l_2 are in this order on the circle.

Let I be a Möbius transformation that sends l_2 to infinity. The spheres $I(S_1)$ and $I(S_2)$ are now concentric and we have

$$\mathrm{Cr}(I(p_1), I(p_2), I(l_2), I(l_1)) = R_2/R_1,$$

where R_1 and R_2 are the radii of $I(S_1)$ and $I(S_2)$. By definition, we have $\mu(S_1, S_2) = R_2/R_1$. Thus

$$\mu(S_1, S_2) = \mathrm{Cr}(p_1, p_2, l_2, l_1).$$

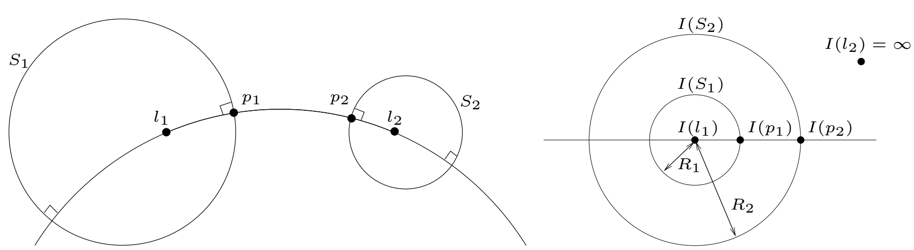

FIGURE 1. Modulus in term of cross-ratio.

REMARK 1. Let P be a plane and S a sphere disjoint from P as in Figure 2. The abscissa λ of the limit point of the pencil generated by S and P is $\sqrt{ab}$. Then,

$$\mu(P,S) = \mathrm{Cr}(0,a,\lambda,-\lambda) = \frac{\sqrt{ab}+a}{\sqrt{ab}-a}.$$

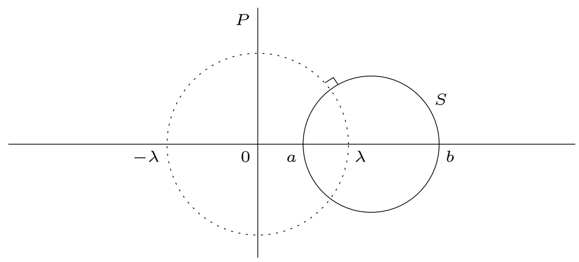

FIGURE 2. Modulus of a sphere and a plane.

REMARK 2. As a consequence, if P is a plane and S_1 and S_2 are two spheres with the same radius and if S_1 is closer to the plane than S_2, then we have $\mu(P,S_1) < \mu(P,S_2)$.

REMARK 3. As another consequence, if a sphere S of constant radius approaches a plane P, without intersecting it, then the modulus of P and S tends to 1. Indeed, if $b-a$ is constant and a tends to 0, then $\mu(P,S)$ tends to 1.

REMARK 4. Let S_1, S_2 and S_3 be three disjoint spheres. Suppose the open 3-ball bounded by S_2 contains S_3, but is disjoint from S_1. Then $\mu(S_1,S_2) < \mu(S_1,S_3)$.

This can be proved by performing a conformal transformation that turns S_1 into a plane and computing the two cross-ratios.

1.2. The zone modulus of a link. Let K_1 and K_2 be two embedded curves in $\mathbf{S}^3$.

DEFINITION 3. A pair (S_1,S_2) of spheres is said to be *non-trivial* for K_1 and K_2 if they are disjoint and if, for each sphere, there is at least one point of K_1 and one point of K_2 on it.

DEFINITION 4. The *zone modulus* of K_1 and K_2 is the supremum of the moduli of all non-trivial pairs of spheres for K_1 and K_2.

The main result of [2] is the following:

THEOREM 2. *Two linked curves have a zone modulus greater than or equal to* $(1+\sqrt{2})^2$.

1.3. Trisecants. The following lemma is a concise rewriting of results of [2].

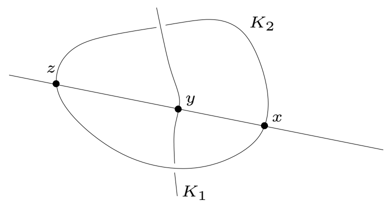

FIGURE 3. A trisecant.

LEMMA 1. *Let K_1 and K_2 be two linked curves such that K_1 goes through infinity and let x be a point of K_2. There exists a straight line L through x that cuts K_1 in $y \neq \infty$ and K_2 again in z (see Figure 3). We call such a line a* trisecant *through x. If the zone modulus of K_1 and K_2 equals $(1+\sqrt{2})^2$, then y is the midpoint between x and z and there is no other point of intersection between L and K_1 or K_2.*

Trisecants may be seen as a conformal version of quadrisecants for two linked curves. This subject goes back to 1933 (see Pannwitz's work in [8]). A more modern treatment appears in Kuperberg's paper [9] and Denne's thesis.

2. Proof of Theorem 1

Let K_1 and K_2 be two linked curves. Two cases may occur:

(1) For every point x on each curve, the other curve is contained in a sphere perpendicular at x to the first curve.
(2) On one of the curves, say K_1, there exists a point x_1 such that no sphere perpendicular at x_1 to K_1 contains K_2.

If the first case occurs, there exist two points x_1 and x_2 on K_1 and two distinct spheres S_1 and S_2 containing K_2 and perpendicular at x_1 and x_2 to K_1. Thus K_2 is the round circle intersection of S_1 and S_2. For the same reasons, K_1 is also a round circle. Since K_1 is perpendicular to S_1 and S_2, it is perpendicular to each sphere going through $S_1 \cap S_2 = K_2$. Thus each sphere containing K_1 is perpendicular to each sphere containing K_2, so according to Definition 1, K_1 and K_2 form a Clifford link and the theorem is proved in the first case.

To conclude the proof, it is enough to prove that the second case never occurs when $\text{modulus}(K_1, K_2) = (1+\sqrt{2})^2$. We will suppose the contrary and show in the remainder of this section that this is impossible.

From now on, we suppose that modulus$(K_1, K_2) = (1+\sqrt{2})^2$ and that there exists a point x_1 on K_1 such that no sphere perpendicular at x_1 to K_1 contains K_2. By a suitable Möbius transformation, we send x_1 to infinity and the tangent at x_1 to a vertical line. The spheres perpendicular to K_1 at x_1 are now all the horizontal planes. Then there exist two distinct horizontal planes P_{top} and P_{bottom} tangent to K_2 such that K_2 lies between these planes.

Let $\tilde{K}_1$ denote $K_1 \setminus \infty$. Let $x_2 \in K_2$. By Lemma 1, there exists a trisecant L through x_2 which cuts $\tilde{K}_1$ in a point x_3 and K_2 again in a point x_4. The point x_3 is the midpoint between x_2 and x_4. The following lemma shows that K_2 is trapped between spheres in particular position with L.

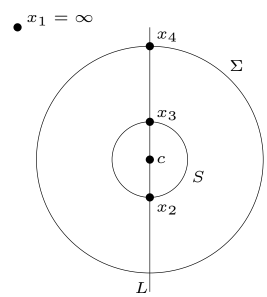

Figure 4. The spheres Σ and S.

Lemma 2. *Let c be the midpoint between x_2 and x_3. Let Σ and S be the spheres centered at c with Σ going through x_4 and S going through x_2 and x_3 (see Figure 4). The curve K_2 lies between Σ and S.*

Proof. Suppose that there exists a point x on K_2 outside the zone bounded by S and Σ. Then x is either outside Σ or inside S; see Figure 5. We will show that there exists a non-trivial pair of spheres of modulus strictly greater than $(1+\sqrt{2})^2$, contradicting our assumption that modulus$(K_1, K_2) = (1+\sqrt{2})^2$.

When x is outside Σ, consider the line L' through c and x and the plane P' through x that is perpendicular to L'. Since P' contains $x_1 \in K_1$ and $x \in K_2$, the pair (S, P') is non-trivial. Let a and b be the two points of intersection of S with L'. By Remark 1, $\mu(S, P')$ is a function of the abscissa of a and b on L' if x marks the origin. With x outside Σ, we have $|b-a| < |x-a|$. Therefore, $\mu(S, P') > (1+\sqrt{2})^2$.

When x is inside S, consider the sphere S' through x that is tangent to S at x_3 and the plane P through x_4 that is perpendicular to L. Since S' contains $x_3 \in K_1$ and $x \in K_2$, the pair (S', P) is non-trivial. By Remark 4, $\mu(S', P) > \mu(S, P) = (1+\sqrt{2})^2$. □

Corollary 1. *The curves K_1 and K_2 are perpendicular to L.*

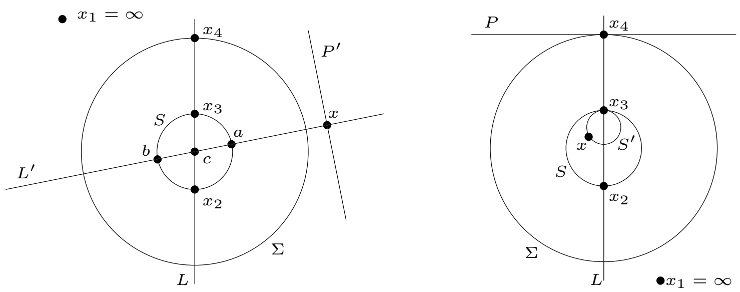

FIGURE 5. A point x of K_2 outside Σ or inside S exhibits a non-trivial pair of spheres whose modulus is too large.

Proof. Let c_1 be the midpoint between x_2 and x_3 and let c_2 be the midpoint between x_3 and x_4. Let Σ_1 and S_1 be the spheres centered at c_1 such that Σ_1 goes through x_4 and S_1 goes through x_2 and x_3. Let Σ_2 and S_2 be the spheres centered at c_2 such that Σ_2 goes through x_2 and S_2 goes through x_3 and x_4 (see Figure 6).

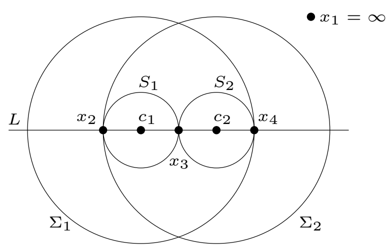

FIGURE 6. The four spheres that enclose K_2.

By Lemma 2, K_2 must lie between Σ_1 and S_1 and between Σ_2 and S_2. Therefore K_2 must be tangent to S_1 and Σ_2 at x_2 and tangent to S_2 and Σ_1 at x_4. Therefore K_2 is perpendicular to L.

We can now choose a Möbius transformation that keeps L fixed and that exchanges x_1 with x_2. The same argument with K_1 and K_2 interchanged shows that K_1 is also perpendicular to L. □

COROLLARY 2. *The trisecant L through x_2 is unique.*

Proof. Suppose, to the contrary, that there exists another trisecant $\tilde{L}$ through x_2 which cuts $\tilde{K}_1$ in $\tilde{x_3}$ and K_2 again in $\tilde{x_4}$. For convenience, let us work in the plane that contains L and L' (see Figure 7). Let c be the midpoint between x_2 and x_3 and let C be the circle through x_4 centered at c. By Lemma 2, $\tilde{x_4}$ lies in the interior of C. Therefore we have $|x_2 - \tilde{x_4}| < |x_2 - x_4|$. Analogously, if we consider $\tilde{c}$ the midpoint between x_2 and $\tilde{x_3}$ and let $\tilde{C}$ be

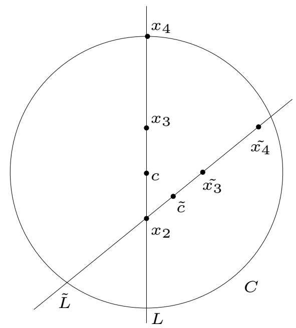

FIGURE 7. Uniqueness of the trisecant through x_2.

the circle through $\tilde{x_4}$ centered at $\tilde{c}$, then we have $|x_2 - x_4| < |x_2 - \tilde{x_4}|$. This is a contradiction. □

As a corollary, by moving the point x_2 on K_2, we can define a map $F : K_2 \to K_1$ that sends x_2 to x_3 and a map $G : K_2 \to K_2$ that sends x_2 to x_4. More precisely:

DEFINITION 5. Let x be any point of K_2. There exists a unique trisecant L through x that cuts $\tilde{K}_1$ and K_2 again. We define $F(x)$ to be the point where $\tilde{K}_1$ intersects L and $G(x)$ to be the point other than x where K_2 intersects L.

LEMMA 3. *The maps F and G are continuous.*

Proof. Let $x \in K_2$ and let x_n be a sequence of points of K_2, which converges to x. The curve K_2 is compact, so the sequence $y_n = G(x_n)$ has at least one point of accumulation a in K_2. Let y_{u_n} be a subsequence converging to a and let L_n denote the trisecant through x_{u_n}. These lines cut $\tilde{K}_1$ in a sequence of points $z_{u_n} = F(x_{u_n})$. Since $z_{u_n} = (x_{u_n} + y_{u_n})/2$, the sequence z_{u_n} converges to a point $z = (x+a)/2$ of $\tilde{K}_1$. Hence there exists a line L that cuts $\tilde{K}_1$ in z and K_2 in x and a and that is therefore the unique trisecant through x. Thus, there exists only one accumulation point of the sequence y_n which converges to $y = G(x)$. Therefore G is continuous. Since x_n and y_n are both convergent, z_n converges to the point $z = F(x)$, and therefore F is continuous. □

LEMMA 4. *The map G is a homeomorphism of K_2 with no fixed points such that $G \circ G(x) = x$.*

Proof. Let x and y be two points of K_2 such that $G(x) = G(y) = z$. This means that there exists a trisecant L through $x, F(x)$ and z, and another trisecant L' through $y, F(y)$ and z. Since there exists only one trisecant through z, we must have $L = L'$. By Lemma 1, K_2 intersects L in exactly two distinct points. Since $x \neq z$, we must have $x = y$. The map G is therefore one-to-one.

Let x be a point of K_2 and $y = G(x)$. The line through x and y is the unique trisecant through y. Hence $G(y) = x$. □

LEMMA 5. *The curve K_2 is symmetric about a vertical line. The image $F(K_2)$ is a segment of this line.*

Proof. Recall that P_{top} and P_{bottom} are distinct horizontal planes that are tangent to K_2, such that K_2 lies between P_{top} and P_{bottom}. Let t_2 be a point of $K_2 \cap P_{top}$ and $t_4 = G(t_2)$. Let b_2 be a point of $K_2 \cap P_{bottom}$ and $b_4 = G(b_2)$. Choose an orientation on K_2 such that t_2, b_2 and t_4 are in this order on K_2. The image by F of the arc joining t_2 to t_4 is a continuous path δ of K_1 that contains $F(b_2) = b_3$. Thus δ joins $F(t_2) = t_3$ to $F(t_4) = t_3$ through b_3. But since K_1 is a simple curve through infinity, δ is described twice. Thus for every point $z \in K_1$ between t_3 and b_3 there exist at least two distinct points x and y on the arc of K_2 joining t_2 to t_4 such that $F(x) = F(y) = z$. Since G is orientation preserving, $G(x)$ is on the arc of K_2 joining $G(t_2) = t_4$ to $G(t_4) = t_2$. Thus $G(x) \neq y$. The trisecants L through x and z and L' through y and z are distinct. By Corollary 1, L and L' are perpendicular to K_1. Since the tangent to K_1 at x_1 has been chosen to be a vertical line, L and L' are horizontal. The plane containing L and L' is therefore horizontal and perpendicular to K_1 at z. Thus, the tangent to K_1 at z is vertical. The arc of K_1 between t_3 and b_3 is therefore a segment of a vertical line. For any $x \in K_2$, the points x and $G(x)$ are symmetric about this line since $F(x)$ is the midpoint of x and $G(x)$. □

LEMMA 6. *The length between a point of K_2 and its image under F is constant.*

Proof. Let $\gamma(t)$ be a parametrization of K_2. We have:

$$\frac{d}{dt}|F(\gamma(t)) - \gamma(t)|^2 = 2\langle (F \circ \gamma)'(t) - \gamma'(t), F(\gamma(t)) - \gamma(t)\rangle$$

By Corollary 1, $F(\gamma(t)) - \gamma(t)$ is perpendicular to K_1 and K_2. Since $(F \circ \gamma)'(t)$ is the tangent to K_1 and $\gamma'(t)$ the tangent to K_2, we have

$$\frac{d}{dt}|F(\gamma(t)) - \gamma(t)|^2 = 0. \qquad \square$$

Let us summarize the situation: K_2 lies between two horizontal planes on a cylinder whose axis is a vertical line which coincides with K_1 in the region between the two planes (see Figure 8).

This configuration is in contradiction with Lemma 2. Indeed, the component K_2 is not contained in the interior of the sphere going through t_4 centered at the midpoint of t_2 and t_3.

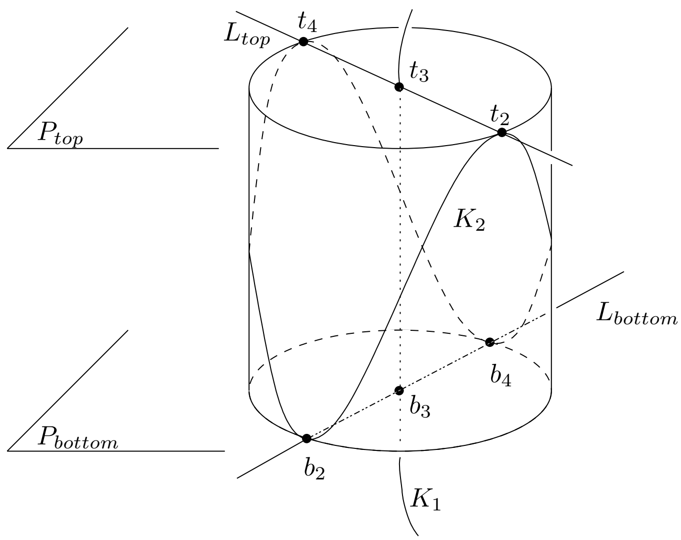

FIGURE 8. The shape of K_2.

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