ON SMOOTH SURFACES IN $\mathbb{P}^4$ CONTAINING A PLANE CURVE

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ABSTRACT. Let $\Sigma \subset \mathbb{P}^4$ be an integral hypersurface of degree s with a $(s-2)$-uple plane. We show that the degrees of smooth surfaces $S \subset \Sigma$ with $q(S) = 0$ are bounded by a function of s. We also show that if $S \subset \mathbb{P}^4$ is a smooth surface with $q(S) = 0$ and if S lies on a quartic hypersurface Σ such that $\dim(\mathrm{Sing}(\Sigma)) = 2$, then $\deg(S) \leq 40$.

1. Introduction

We work over an algebraically closed field of characteristic zero.

In this paper we study smooth surfaces S in $\mathbb{P}^4$ which contain a plane curve, P. The first part contains some generalities about the linear system $|H - P|$. In particular, we prove that its base locus has dimension zero and we describe it. In the second part we consider surfaces lying on a hypersurface of degree s containing a $(s\text{-}2)$-uple plane (we suppose $s \geq 4$). Indeed, if the surface does not lie on a hyperquadric, this implies that it contains a plane curve (Lemma 3.1). The main results are the following:

THEOREM 1.1. *Let $\Sigma \subset \mathbb{P}^4$ be an integral hypersurface of degree s containing a plane in its singular locus with multiplicity $(s-2)$. Then the degree of smooth surfaces $S \subset \Sigma$ with $q(S) = 0$ is bounded by a function of s.*

We then restrict to the case of regular surfaces lying on a hyperquartic with singular locus of dimension two. It turns out that, if $\deg(S) \geq 5$, the hyperquartic must have a double plane (Lemma 3.4). In this situation we can compute an effective bound.

THEOREM 1.2. *Let $S \subset \mathbb{P}^4$ be a smooth surface with $q(S) = 0$ and lying on a quartic hypersurface Σ, such that $\mathrm{Sing}(\Sigma)$ has dimension two. Then $d = \deg(S) \leq 40$.*

The assumption $q(S) = 0$ is due to technical reasons. In fact, we believe that it is not strictly necessary (see 3.12).

Theorem 1.2 is of some interest for the classification of surfaces that are not of general type, since in this case one only needs to consider surfaces lying on low degree hypersurfaces. For similar results concerning smooth surfaces on hyperquartics with isolated singularities see [2].

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2. Smooth surfaces containing a plane curve

Let $S \subset \mathbb{P}^4$ be a smooth, nondegenerate surface, of degree d, containing a plane curve, P, of degree p. If $p \geq 2$, there is a unique plane, Π, containing P. Otherwise, if P is a line, there are ∞^2 such planes, and we choose one of them and call it Π. We assume that P is the one-dimensional part of $\Pi \cap S$. We denote by δ the linear system cut out on S, residually to P, by the hyperplanes containing Π. Since, by Severi's theorem [5], $H^0(\mathcal{O}_S(1)) \simeq H^0(\mathcal{O}_{\mathbb{P}^4}(1))$ (we assume S is not a Veronese surface), $\delta = |H - P|$ if $p \geq 2$. If P is a line, δ is a pencil in the ∞^2 linear system $|H - P|$. Finally, we denote by Y_H the element of δ cut out by the hyperplane H, and by $C_H = P \cup Y_H$ the corresponding hyperplane section of S.

Lemma 2.1.

(i) *The curve P is reduced and the base locus of δ is empty or zero-dimensional and contained in Π. The general element $Y_H \in \delta$ is smooth off of Π and has no component in Π.*

(ii) *If $p = 1$, the linear system $|H - P|$ is base point free.*

Proof. (i) Clearly the base locus of δ is contained in Π. Assume an irreducible component of P, P_1, is in the base locus of δ. Then, for every H through Π, $C_H = H \cap S$ is singular along P_1. It follows that $T_xS \subset H$, for every $x \in P_1$. Since this holds for every H through Π, we get $T_xS = \Pi$ for all $x \in P_1$, but this contradicts Zak's theorem [6] which states that the Gauss map is finite. The same argument shows that P is reduced. We conclude by Bertini's theorem.

(ii) Assume P is a line. Clearly the base locus of $|H - P|$ is contained in P. Take $x \in P$. Now let H be a hyperplane containing P, but not containing T_xS. Then $C_H = P \cup Y_H$ is smooth at x, so $x \notin Y_H$. $\square$

Remark 2.2.   (i) If $p = 1$, $|H - P|$ is base point free and yields a morphism $f : S \to \mathbb{P}^2$, which is nothing else than the projection from the line P. If there is no plane curve on S in a plane through P, f is a finite morphism of degree $d - 2 + P^2$.

(ii) Let $S \subset \mathbb{P}^4$ be an elliptic scroll. Then S contains a one-dimensional family of cubic plane curves which are unisecants (see, e.g., [1]). If P is such

a cubic, and if H is a general hyperplane through P, then $H \cap S = P \cup f \cup f'$, where f, f' are two rulings. This shows that the general curve $Y_H \in |H - P|$ need not be irreducible.

Since δ is a pencil and since the base locus, $\mathcal{B}$, is zero-dimensional, the degree of $\mathcal{B}$ is $(H - P)^2$. Now we give a geometric description of $\mathcal{B}$. Let $Z := \Pi \cap S$. Then Z is a 1-dimensional subscheme of Π (and also of S) and is composed by P and possibly by some 0-dimensional component, which may be isolated or embedded in P.

DEFINITION 2.3.   We define $\mathcal{R}$ as the residual scheme of Z with respect to P (in S). Hence $\mathcal{I}_\mathcal{R} = (\mathcal{I}_Z : \mathcal{I}_P)$.

Since $\mathcal{R} \subset Z$, we can view $\mathcal{R}$ as a subscheme of Π or of S.

LEMMA 2.4.   *We have $\mathcal{B} = \mathcal{R}$.*

Proof. We observe that $\mathcal{R} \subset \mathcal{B}$ and that $\deg(\mathcal{B}) = d - 2p + P^2$. Then we only have to compute $\deg(\mathcal{R})$.

Considering a section of $\omega_S(3)$ (which is always globally generated, by Kodaira's vanishing theorem and Castelnuovo-Mumford's lemma), we can associate to S a reflexive sheaf $\mathcal{F}$ of rank two and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4} \xrightarrow{t} \mathcal{F} \to \mathcal{I}_S(2) \to 0$$

such that $(t)_0 = S$. The singular locus of $\mathcal{F}$ is a divisor in $|2H + K|$ (because $\omega_S(3)$ is globally generated) and the Chern classes of $\mathcal{F}$ are $c_1 = 2$, $c_2 = d$.

We can restrict the sequence above to Π and get a section

$$0 \to \mathcal{O}_\Pi \xrightarrow{t_\Pi} \mathcal{F}_\Pi.$$

Clearly $P \subset (t_\Pi)_0$. Then dividing by an equation of P we get a non-zero section $\bar{t}_\Pi$ of $\mathcal{F}_\Pi(-p)$. We compute

$$\deg((\bar{t}_\Pi)_0) = c_2(\mathcal{F}_\Pi(-p)) = c_2(\mathcal{F}(-p)) = -2p + d + p^2.$$

The section $\bar{t}_\Pi$ will vanish on $\mathcal{R}$ and on the intersection with Π of the singular locus of $\mathcal{F}$, which is a curve $X \in |3H + K|$. Thus $(\bar{t}_\Pi)_0 = \mathcal{R} \cup (X \cap \Pi)$. When we restrict to Π we have $X \cap \Pi = X \cap P$ and we get

$$\#(X \cap \Pi) = (3H + K)P = 3p + PK.$$

It follows that

$$\deg(\mathcal{R}) = -5p + d + p^2 - PK.$$

Now we use adjunction to get $PK = p^2 - 3p - P^2$, and combining this with the previous equation we obtain the result.   □

REMARK 2.5.   There is a cheaper proof of this result. Using the argument
of [4], page 155, we infer that $\deg(\mathcal{R}) = d - 2p + P^2$.

Indeed, we can view $S \cap \Pi$ as the intersection of two hyperplane divisors
on S, H_1 and H_2, such that $H_1 \cap H_2 = \Pi$. Moreover, P is a Weil divisor
on the smooth surface S, and hence a Cartier divisor. Then we compute the
equivalence of P in the intersection $H_1 \cap H_2$, namely

$$(H_1 \cdot H_2)^P = (H_1 + H_2 - P) \cdot P = 2p - P^2.$$

This means that the "exceeding" curve P counts for $2p - P^2$ points in $H_1 \cap H_2$.
Thus the degree of its zero-dimensional component, $\mathcal{R}$, drops by $2p - P^2$. It
follows that $\deg(\mathcal{R}) = d - 2p + P^2$, and hence the result.

3. Degree s hypersurfaces with a $(s$-2$)$-uple plane

LEMMA 3.1.   If $S \subset \mathbb{P}^4$ is a smooth surface, lying on a degree s integral
hypersurface Σ with a $(s$-2$)$-uple plane, then S contains a plane curve or
$h^0(\mathcal{I}_S(2)) \neq 0$.

Proof. Let Π be the $(s$-2$)$-uple plane in Σ and let H be a hyperplane con-
taining Π. Then $H \cap \Sigma = (s - 2)\Pi \cup Q$, where Q is a quadric surface and
$C_H = S \cap H \subset (s - 2)\Pi \cup Q$. If $\dim(C_H \cap \Pi) = 0$, then $C_H \subset Q$, i.e.,
$h^0(\mathcal{I}_{C_H}(2)) \neq 0$. From the exact sequence

$$0 \to \mathcal{I}_S(1) \to \mathcal{I}_S(2) \to \mathcal{I}_{C_H}(2) \to 0$$

we get $h^0(\mathcal{I}_S(2)) \neq 0$ (because $h^1(\mathcal{I}_S(1)) = 0$ by Severi's theorem). Therefore
we can assume $\dim(C_H \cap \Pi) = 1$ and this is equivalent to saying that S
contains a plane curve.   □

NOTATIONS 3.2.   Let $\Sigma \subset \mathbb{P}^4$ be an integral hypersurface of degree s con-
taining a plane, Π, in its singular locus, with multiplicity $(s - 2)$. Let $S \subset \Sigma$
be a smooth surface. If $h^0(\mathcal{I}_S(2)) \neq 0$, then $d := \deg(S) \leq 2s$. From now on
we assume $h^0(\mathcal{I}_S(2)) = 0$. By Lemma 3.1, $\dim(S \cap \Pi) = 1$ and we denote by
P the 1-dimensional component of $\Pi \cap S$. Also we let $p := \deg(P)$.

We assume $q(S) = 0$. This assumption implies that every hyperplane
section $C = H \cap S$ is linearly normal in $H \simeq \mathbb{P}^3$.

If H is a hyperplane through Π, we denote by $C = Y_H \cup P$ the hyperplane
section $H \cap S$. We have $C \subset \Sigma \cap H = (s - 2)\Pi \cup Q_H$, where Q_H is a quadric
surface. By Lemma 2.1, if H is general, $Y_H \subset Q_H$. If we restrict to Π,
the surfaces $q_H = Q_H \cap \Pi$ form, as H varies, a family of conics in Π. Let
us set $\mathcal{B}_q = \bigcap_{H \supset \Pi} q_H$, where $\mathcal{B}_q$ is the base locus of the conics q_H. Since
$Y_H \cap \Pi \subset Q_H \cap \Pi = q_H$, we have $\mathcal{R} \subset \mathcal{B}_q$.

Recall that if

$$\mu = c_2(\mathcal{N}_S(-s)) = d(d + s(s - 4)) - s(2\pi - 2)$$

(where π is the sectional genus of S), then, by Lemma 1 of [3],

$$0 \le \mu \le (s-1)^2 d - D(3H + K),$$

where D is the one-dimensional part of the intersection of S with $\mathrm{Sing}(\Sigma)$. In our situation $P \subset D$, so

$$\mu \le (s-1)^2 d - P(3H + K) = (s-1)^2 d - 3p - PK.$$

By adjunction we compute $P^2 + PK = p^2 - 3p$ and therefore

$$\mu \le (s-1)^2 d - p^2 + P^2 = s(s-2)d - p^2 + 2p + r$$

(since $r = d - 2p + P^2$).

LEMMA 3.3.   *With the notations above, the base locus $\mathcal{B}_q$ of the conics q_H is (s-1)-uple for Σ.*

Proof. We assume the plane Π is given by $x_0 = x_1 = 0$. Thus if $\phi = 0$ is an equation of Σ, we have $\phi \in (x_0, x_1)^{s-2}$. We can write, for example,

$$\phi = \sum_{i=0}^{s-2} Q_i(x_0, x_1, x_2, x_3, x_4) x_0^i x_1^{s-2-i},$$

where the Q_i are quadratic forms.

The general hyperplane H_α containing Π has an equation of the form $x_0 = \alpha x_1$, $\alpha \in k$. We consider $\phi_{|H_\alpha}$, the equation of the surface $\Sigma \cap H_\alpha$:

$$\phi_{|H_\alpha} = \sum_{i=0}^{s-2} Q_i(\alpha x_1, x_1, x_2, x_3, x_4) \alpha^i x_1^{s-2}$$

$$= x_1^{s-2} \sum_{i=0}^{s-2} Q_i(\alpha x_1, x_1, x_2, x_3, x_4) \alpha^i.$$

Clearly

$$\sum_{i=0}^{s-2} Q_i(\alpha x_1, x_1, x_2, x_3, x_4) \alpha^i = 0$$

is an equation defining Q_H for the hyperplane H_α. Let $x = (0 : 0 : x_2 : x_3 : x_4)$ be a point in $\mathcal{B}_q$. Hence

$$\sum_{i=0}^{s-2} Q_i(x) \alpha^i = 0$$

for all $\alpha \in k$, and this implies that $Q_i(x) = 0$.

Now, we see that the $(s-2)$-th derivatives of ϕ all vanish at a point $x \in \mathcal{B}_q$; equivalently, x is a $(s-1)$-uple point for Σ. $\square$

LEMMA 3.4. *If $S \subset \mathbb{P}^4$ is a smooth surface with $q(S) = 0$, lying on a quartic hypersurface Σ having singular locus of dimension two, then, if $\deg(S) \geq 5$, the component of dimension two in $\mathrm{Sing}(\Sigma)$ is a plane (or a union of planes) and S contains a plane curve.*

Proof. Let us suppose that $\mathrm{Sing}(\Sigma)$ contains an irreducible surface of degree > 1. Then the general hyperplane section $S \cap H = C$ lies on $F = \Sigma \cap H$, which is a quartic surface of $\mathbb{P}^3$ having an irreducible curve of degree > 1 in its singular locus. From the classification of quartic surfaces in $\mathbb{P}^3$ it follows that such a surface is a projection of a (nondegenerate) quartic surface $F' \subset \mathbb{P}^4$ (we will say that F is not "linearly normal").

Let us recall briefly how to obtain this fact. If a quartic surface, $F \subset \mathbb{P}^3$, contains a conic or a twisted cubic in its singular locus, then one easily sees that F is rational. The general plane section is a quartic plane curve with two (or three) nodes. Hence on a smooth model F'' it is a smooth curve, H, of genus one (or zero) and with $H^2 = 4$. From the exact sequence

$$0 \to \mathcal{O}_{F''} \to \mathcal{O}_{F''}(H) \to \mathcal{O}_H(H) \to 0,$$

we get $h^0(\mathcal{O}_{F''}(H)) > 4$. By the way, arguing in a similar way, one can prove that a degree s surface in $\mathbb{P}^3$ with a $(s-1)$-uple line is a rational, "non linearly normal" surface.

Returning to our proof, the curve $C \subset F$ (which is smooth) is the isomorphic projection of a curve $C' \subset F'$, since $h^0(\mathcal{O}_C(1)) = h^0(\mathcal{O}_{C'}(1))$, and since C is linearly normal, we conclude that C' is degenerate, and hence contained in a hyperplane section of F'. It follows that $d = \deg(C') \leq 4$. So we may assume that the singular locus of Σ does not contain irreducible surfaces of degree > 1. Thus $\mathrm{Sing}(\Sigma)$ contains a plane, say Π, which is double in Σ. Indeed, Σ cannot have a triple plane, for otherwise $F = \Sigma \cap H$ would be a quartic surface in $\mathbb{P}^3$ with a triple line, and we can argue as before because such a surface is not linearly normal in $\mathbb{P}^3$. By Lemma 3.1, S contains a plane curve. $\square$

Proof of Theorems 1.1 and 1.2. We distinguish between different cases, according to the behaviour of the curves q_H. Note that it is not possible that $q_H = 0$ for every H. Indeed, if it were so, Π would be an $(s\text{-}1)$-uple for Σ. Then for all hyperplanes $H \supset \Pi$, $\Sigma \cap H = (s-1)\Pi \cup \Pi_H$, where Π_H is a plane. With notations as above we would get that $Q_H = \Pi \cup \Pi_H$, but we know by Lemma 2.1 that, if H is general, Y_H does not have any component in Π, so $Y_H \subset \Pi_H$ is a plane curve and $h^0(\mathcal{I}_C(2)) \neq 0$, which is absurd.

So we are left with the following possibilities. The conics may move, i.e., vary as H varies, so that at least two of them intersect properly. Then $\dim(\mathcal{B}_q) = 0$. Conversely, they may all be equal to a fixed conic q or they can be all reducible and contain a fixed line D, while the remaining line is moving. Observe that there are always two possibilities: the one-dimensional

part of $\mathcal{B}_q$ may or may not be contained in S. We start the proof by showing that $h^1(\mathcal{I}_C(2)) = 0$, where $C = Y_H \cup P$. Indeed, if this is so, then from

$$0 \to \mathcal{I}_S(1) \to \mathcal{I}_S(2) \to \mathcal{I}_C(2) \to 0$$

we obtain $h^1(\mathcal{I}_S(2)) = 0$. Using

$$0 \to \mathcal{I}_S(2) \to \mathcal{I}_S(3) \to \mathcal{I}_C(3) \to 0$$

and the fact that $h^0(\mathcal{I}_C(3)) \neq 0$, we get that $h^0(\mathcal{I}_S(3)) \neq 0$, and this implies $d \leq 3s$.

The proof will follow from the lemmas below.

LEMMA 3.5.   If $p_a(Y_H) \leq 2(d - p - 4)$ and if $r \leq 4$, then d is bounded by a function of s. More precisely, if $s = 4$, $d \leq 40$.

Proof. We have

$$\pi = p_a(Y_H) + \frac{(p-1)(p-2)}{2} + d - p - r - 1,$$

so

$$\pi - 1 \leq 3(d - p) + \frac{p^2 - 3p}{2} - 9 - r.$$

Since

$$\mu \leq s(s-2)d - p^2 + 2p + r,$$

and, on the other hand,

$$\mu = d(d + s^2 - 4s) - 2s(\pi - 1),$$

this yields

$$\pi - 1 \geq \frac{d^2 - 2sd + p^2 - 2p - r}{2s}.$$

Now comparing the lower and the upper bounds on $\pi - 1$, we obtain

$$d^2 - 8sd + p^2(1 - s) + p(9s - 2) + 18s + r(2s - 1) \leq 0,$$

and since $r \geq 0$ this becomes

$$d^2 - 8sd + p^2(1 - s) + p(9s - 2) + 18s \leq 0.$$

This implies

$$(*) \qquad\qquad\qquad d \leq 4s + \sqrt{\Delta},$$

where

$$\Delta = 16s^2 + p^2(s - 1) - p(9s - 2) - 18s.$$

A short calculation shows that

$$\sqrt{\Delta} \leq p\sqrt{s - 1} + 4s$$

for all $s \geq 0$. In conclusion,

$$d \leq 8s + p\sqrt{s - 1}.$$

We again use the relation

$$0 \leq \mu \leq s(s-2)d - p^2 + 2p + 4.$$

Together with the above bound on d, this becomes

$$s(s-2)(8s + p\sqrt{s-1}) \geq p^2 - 2p - 4.$$

This implies that p is bounded by a function of s. We conclude since $d \leq 8s + p\sqrt{s-1}$.

If $s = 4$, we can give a better bound for $\sqrt{\Delta}$. Indeed,

$$\sqrt{\Delta} \leq p\sqrt{3} - 8$$

if $p \geq 19$, and thus

$$d \leq 8 + p\sqrt{3}.$$

The same relation used above now gives

$$8d \geq p^2 - 2p - 4,$$

and hence

$$p^2 - 2p - 8\sqrt{3}p - 68 \leq 0,$$

which implies $p \leq 19$ and consequently, by $(*)$, $d \leq 40$. On the other hand, if $p \leq 18$, then, again by $(*)$, we have $d \leq 39$. $\square$

LEMMA 3.6. *If $r \leq 4$ and if $\mathcal{R}$ does not contain three collinear points, then d is bounded by a function of s. In particular, if $s = 4$, $d \leq 40$.*

Proof. Assume first Q_H is a smooth quadric surface. We have

$$Y_H \cap \Pi = Y_H \cap P + \mathcal{R},$$

so

$$0 \to \mathcal{I}_C(2) \to \mathcal{I}_P(2) \to \mathcal{O}_{Y_H}(\mathcal{R} + 1) \to 0.$$

The curve Y_H has bidegree (a, b), $a \leq b$. We may assume $a \geq 4$, for otherwise $p_a(Y_H) \leq 2(d - p - 4)$ and we conclude by Lemma 3.5.

Thus Y_H is linearly normal. We have

$$h^0(\mathcal{O}_{Y_H}(1 + \mathcal{R})) = 4$$

if and only if $\mathcal{R}$ gives independent conditions to $\omega_{Y_H}(-1)$. This is equivalent to saying that $\mathcal{R}$ gives independent conditions to the curves of bidegree $(a - 3, b - 3)$. If $a = b = 4$, then $\deg(Y_H) = d - p = 8$ and using

$$s(s-2)d - p^2 + 2p + 4 \geq 0$$

we get

$$0 \leq -d^2 + d(18 + s(s-2)) - 76.$$

This shows that d is bounded by a function of s. In particular, if $s = 4$, $d \leq 22$. So we may assume $a \geq 4$, $b \geq 5$. Since $r \leq 4$ and no three points of

$\mathcal{R}$ are collinear, the curves of bidegree $(a - 3, b - 3)$ separate the points of $\mathcal{R}$. It follows that the map

$$H^0(\mathcal{I}_P(2)) \to H^0(\mathcal{O}_{Y_H}(1 + \mathcal{R}))$$

is surjective, and hence

$$h^1(\mathcal{I}_C(2)) = 0.$$

As noted before, this implies $d \leq 3s$.

Now we suppose Q_H is an irreducible quadric cone (recall that every reduced curve on a quadric cone is a.C.M.). If $d - p$ is even, then Y_H is a complete intersection $((d - p)/2, 2)$ and

$$\omega_{Y_H} \cong \mathcal{O}_{Y_H}\left(\frac{d - p}{2} - 2\right).$$

So if $(d - p)/2 - 3 \geq 3$, arguing as above, we get

$$h^0(\mathcal{O}_{Y_H}(1 + \mathcal{R})) = 4.$$

On the other hand, if this condition is not satisfied, then $d - p \leq 11$, i.e., $p \geq d - 11$. Recall that

$$0 \leq \mu \leq s(s - 2)d - p^2 + 2p + 4.$$

It follows that

$$(d - 11)(d - 13) \leq s(s - 2)d + 4$$

and, for fixed s, this implies that d is bounded. If $s = 4$, we have

$$d^2 - 32d + 139 \leq 0,$$

which yields $d \leq 26$.

If $d - p$ is odd, then Y_H is linked to a line L by a complete intersection T of type $((d - p + 1)/2, 2)$. Since L can be any ruling of Q_H, we may assume $L \cap \mathcal{R} = \emptyset$. The exact sequence of liaison

$$0 \to \mathcal{I}_T\left(\frac{d - p - 5}{2}\right) \to \mathcal{I}_L\left(\frac{d - p - 5}{2}\right) \to \omega_{Y_H}(-1) \to 0$$

shows that the divisors of $\omega_{Y_H}(-1)$ are cut on Y_H by surfaces of degree $\delta = (d - p - 5)/2$, containing L but not T, residually to $L \cap Y_H$. We may consider surfaces of the form $H_1 \cup \ldots \cup H_\delta$, where H_1 contains L and where $H_2, \ldots, H_\delta$ are general planes. It follows that our condition is satisfied if $\delta - 1 \geq 3$. If $\delta \leq 3$, then $p \geq d - 11$ and we conclude as above.

If Q_H is the union of two distinct planes, then Y_H is the union of two distinct plane curves. We have

$$p_a(Y_H) \geq \left(\frac{d - p}{2} - 1\right)\left(\frac{d - p}{2} - 2\right) - 1,$$

because the minimal value for the arithmetical genus of a union of two plane curves of global degree δ is achieved when each curve has degree $\delta/2$ and the two components do not intersect. Consequently,

$$\pi - 1 \geq \frac{d^2 + p^2 - 2pd - 6d + 6p + 4}{4} + \frac{p^2 - 3p + 2}{2} + d - p - r - 2.$$

We may assume that the general hyperplane section of S does not lie on a cubic surface (otherwise $h^0(\mathcal{I}_S(3)) \neq 0$ and $d \leq 3s$), so $\pi - 1 \leq d^2/8$. Comparing these two inequalities (and using $r \leq 4$) we obtain

$$6p^2 - 8p - 4dp + d^2 - 4d - 32 \leq 0.$$

If $d \geq 25$, no value of p can satisfy this inequality, so $d \leq 24$ (for all s). $\square$

COROLLARY 3.7.   *If* $\dim(\mathcal{B}_q) = 0$, *then* $r \leq 4$ *and* d *is bounded by a function of* s. *If* $s = 4$, $d \leq 40$.

Proof. Since $\mathcal{B}_q$ is the intersection of the conics q_H, $\mathcal{I}_{\mathcal{B}_q}(2)$ is globally generated. Hence $\mathcal{B}_q$ is contained in a complete intersection of two conics. Recalling that $\mathcal{R} \subset \mathcal{B}_q$, it follows that $r \leq 4$ and that $\mathcal{R}$ does not contain three collinear points. We conclude by Lemma 3.6. $\square$

LEMMA 3.8.   *Assume* $\dim(\mathcal{B}_q) = 1$, *that* $\mathcal{B}_q$ *contains a line* D *and that* $D \not\subset S$. *Then* $d \leq s$.

Proof. Under these assumptions, we claim that the general curve C is smooth. Indeed, let $|L|$ be the linear system cut on S by the hyperplanes containing D and let $B = D \cap S = \{p_1, \ldots, p_r\}$. Clearly B is the base locus of $|L|$ and the general element of $|L|$ is smooth out of B. If all curves in $|L|$ were singular at a point $p_i \in B$, then $T_{p_i} S \subset H$ for all $H \supset D$. Since the intersection of all $H \supset D$ is nothing but D, this is absurd. The same holds for all $p \in B$. It follows that the singular curves in $|L|$ form a closed subset of $|L|$.

Since D is contained in the $\mathcal{B}_q$, D is $(s-1)$-uple for Σ (see 3.3). Let H be a general hyperplane through D. Then $F = \Sigma \cap H$ is a degree s surface of $\mathbb{P}^3$ with a line, D, of multiplicity $(s-1)$. Such a surface is a projection of a degree s surface $F' \subset \mathbb{P}^4$. We have $S \cap H = C \subset F$ and we may assume C smooth and irreducible. Moreover, since $q(S) = 0$, C is linearly normal in $\mathbb{P}^3$. Now C is the isomorphic projection of a degree d curve $C' \subset F'$ (in particular, $\mathcal{O}_{C'}(1) \cong \mathcal{O}_C(1)$). Hence C' is degenerate in $\mathbb{P}^4$, and this implies $d \leq s$. $\square$

LEMMA 3.9.   *Assume that the one-dimensional part of* $\mathcal{B}_q$ *is a line* D *and that* $D \subset S$. *Then* $r \leq 1$ *and Lemma 3.6 applies.*

Proof. In this case $q_H = D \cup D_H$ and the $D_H{}'s$ are moving. The base locus of the $D_H{}'s$, $\mathcal{D}$, is either empty or a point, b. If $\mathcal{D} = \emptyset$, then $Y_H \cap \Pi \subset P$ and it follows that $r = 0$. Hence we assume from now on that $\mathcal{D} = \{b\}$.

If $b \in D$, we have $\mathcal{B}_q = D \cup \eta_b$, where η_b is the first infinitesimal neigh-bourhood of b in Π. Let $x \in Y_H \cap \Pi$ for a general H and let ξ_x be the zero-dimensional subscheme of $Y_H \cap \Pi$ supported at x. We then have:

CLAIM.   Let $x \in Y_H \cap \Pi$. If $\xi_x \not\subset P$, then $x = b$ and, moreover, $\xi_x \subset \eta_b$ if $b \in D$.

Proof of the Claim. We have $\xi_x \subset S \cap \Pi$. If $\xi_x \not\subset P$, then its residual scheme with respect to P is nonempty and so is, a fortiori, the residual scheme of $Z = S \cap \Pi$ with respect to P, namely $\mathcal{R}$. So $\mathcal{R}$ has a component, $\mathcal{R}_x$, supported at x. Since $\mathcal{R} \subset \mathcal{B}_q$, we conclude that $x = b$ or $x \in D$.

If $x = b$ and $b \notin D$, we are done. So we assume $x \in D$. Since $\xi_x \subset q_H$, if $x \neq D \cap D_H$, then $\xi_x \subset D \subset P$, which is absurd. Thus $x = D \cap D_H$. If $b \in D$, this implies $x = b$ and $\xi_x \subset \eta_b$ (because $\xi_x \subset q_H$). So we may assume $b \notin D$. In this case the $D_H{}'s$ have no base point on D. Thus, if H is general, then $\mathcal{R} \cap D \cap D_H = \emptyset$, which is a contradiction since $x \in \mathcal{R} \cap D \cap D_H$.   □

We return to the proof of the lemma. If $\mathcal{D} = \{b\}$ and $b \notin D$, then $Y_H \cap \Pi \subset P$ but for at most one point (b), so $Y_H P \geq d - p - 1$ and $r \leq 1$.

If $\mathcal{D} = \{b\}$ and $b \in D$, then for all $x \in Y_H \cap \Pi$, $\xi_x \subset \eta_b$, so the residual scheme of ξ_x with respect to D is contained in the residual scheme of η_b with respect to D, which is b. This shows that $Y_H P \geq d - p - 1$, and hence $r \leq 1$.   □

LEMMA 3.10.   *Assume that $\mathcal{B}_q$ is a conic q ($q_H = q$ for all H). If $q \subset S$, then $r = 0$ and Lemma 3.6 applies.*

Proof. In this case $q \subset P$. Since $Y_H \cap \Pi \subset q_H$, we have $Y_H \cap \Pi \subset P$, and hence $Y_H P = d - p$, i.e., $r = 0$.   □

LEMMA 3.11.   *Assume that $\mathcal{B}_q$ is a conic q and that $q \not\subset S$. Then $d \leq \max\{s, 20\}$.*

Proof. If no component of q is contained in S (i.e., in P), then $Y_H \cap \Pi = Y_H \cap q$ is fixed (otherwise, as H varies, the points of $Y_H \cap \Pi$ will cover a component of q). So $Y_H \cap q = \mathcal{R}$, i.e., $d - p = r$. Since $r = d - 2p + P^2$, we get $P^2 = p$ and $Y_H P = (H - P)P = 0$. This means that $C_H = Y_H \cup P$ is disconnected, which is absurd.

It follows that $q = D \cup L$ with $D \subset S$ and $q \not\subset S$. If $L \neq D$, we have $L \subset \mathcal{B}_q$, $L \not\subset S$ and we conclude that $d \leq s$ thanks to Lemma 3.8.

So we may assume $q = 2D$, $D \subset P \subset S$, but $2D \not\subset S$ ($2D$ means D doubled in Π). In this case, for all H, $q_H = 2D$, so Q_H is tangent to Π along D. This implies that, for a general H, Q_H is either a cone or the union of two distinct planes through D. In the latter case, $Y_H = P_1 \cup P_2$ and

$Y_H D = P_1 D + P_2 D = d - p$. Since $Y_H D \subset Y_H P$, it follows that $r = 0$, and we conclude with Lemma 3.6.

From now on we assume that, for a general H, Q_H is a cone and D a ruling of Q_H. If $d - p$ is even, Y_H is a complete intersection $((d - p)/2, 2)$, then

$$p_a(Y_H) = \frac{d^2 - 2pd - 4d + p^2 + 4p + 4}{4}$$

and so

$$\pi - 1 = \frac{d^2 - 2pd - 4d + p^2 + 4p + 4}{4} + \frac{p^2 - 3p + 2}{2} + d - p - r - 2.$$

Now $Y_H \cap D \subset Y_H \cap P$, so

$$Y_H D = \frac{d - p}{2} \leq d - p - r = Y_H P,$$

i.e., $r \leq (d - p)/2$, and it follows that

$$\pi - 1 \geq \frac{d^2 - 2pd - 4d + p^2 + 4p + 4}{4} + \frac{p^2 - 3p + 2}{2} + \frac{d - p}{2} - 2.$$

Now comparing this expression with $\pi - 1 \leq d^2/8$ (we can suppose as usual $h^0(\mathcal{I}_C(3)) = 0$) we get

$$6p^2 - 8p - 4dp + d^2 - 4d \leq 0.$$

If $d \geq 21$, there are no values of p satisfying this inequality. Therefore $d \leq 20$.

If $d - p$ is odd, then Y_H is linked to a line by a complete intersection $((d - p + 1)/2, 2)$ and we obtain

$$p_a(Y_H) = \frac{d^2 - 2dp + p^2 - 4d + 4p + 3}{4}.$$

Since

$$Y_H D = \frac{d - p + 1}{2} \leq Y_H P = d - p - r,$$

we have $r \leq (d - p - 1)/2$. Hence we can write

$$\pi - 1 \geq \frac{d^2 - 2dp + p^2 - 4d + 4p + 3}{4} + \frac{p^2 - 3p + 2}{2} + \frac{d - p + 1}{2} - 2.$$

If we compare this with $\pi - 1 \leq d^2/8$ and argue as before, we obtain $d \leq 20$. $\square$

Theorems 1.1 and 1.2 now follow from 3.5–3.11. $\square$

REMARK 3.12.   Actually, we believe that there are very few smooth surfaces on such hypersurfaces. For example, consider the following situation:

Assume that the blowing-up of Π, $\tilde{\Sigma} \to \Sigma$, yields a desingularization of Σ, so we have a double covering $T \to \Pi$ and $\tilde{S}$ mapping to S. Since T and $\tilde{S}$ are two divisors on the smooth threefold $\tilde{\Sigma}$, if they intersect, they intersect along a curve. We conclude that $S \cap \Pi = P$ and that all the points of $Y_H \cap \Pi$ lie on P.

Now assume that for general H, Q_H is a smooth quadric. Observe that the Q_H are parametrized by a smooth rational curve ($\simeq \mathbb{P}^1$). Let $\mathcal{P}$ denote the curve parametrizing the rulings of the quadrics Q_H. We get a degree two covering $f : \mathcal{P} \to \mathbb{P}^1$ which is ramified at the points corresponding to singular Q_H. Assume $\mathcal{P}$ is irreducible. With this assumption the curve $Y_H \subset Q_H$ has bidegree (a, a) (otherwise following the a ruling would yield a section of the covering, which is impossible since $g(\mathcal{P}) > 0$ because f is ramified in more than two points).

Now consider the exact sequence of residuation with respect to Π:

$$0 \to \mathcal{I}_{Y_H}(-1) \to \mathcal{I}_C \to \mathcal{I}_{P,\Pi} \to 0.$$

Since Y_H is a.C.M., it follows that $C = Y_H \cup P$ is also a.C.M. ($H^1_*(\mathcal{I}_C) = 0$ from the exact sequence above). Hence S is a.C.M. and

$$h^0(\mathcal{I}_S(3)) \geq h^0(\mathcal{I}_C(3)) \neq 0.$$

This implies $d(S) \leq 3s$. (Notice that we did not assume $q(S) = 0$.) Observe that the assumption that S is smooth is necessary in order to apply Lemma 2.1 and to conclude that $C = Y_H \cup P$ with $Y_H \subset Q_H$.

REMARK 3.13. There exist integral hypersurfaces in $\mathbb{P}^4$ such that the degree of the smooth surfaces contained in them is bounded. Indeed, it is enough to take a non linearly normal hypersurface in $\mathbb{P}^4$, recalling that the only non linearly normal smooth surface in $\mathbb{P}^4$ is the Veronese surface. The simplest example is the Segre cubic hypersurface. The previous results seem to indicate that this behaviour can happen also on some linearly normal hypersurfaces. From a "codimension two" point of view this is in contrast with the following proposition.

PROPOSITION 3.14. *Let $S \subset \mathbb{P}^3$ be an integral surface. Then S contains smooth curves of arbitrarily high degree.*

Proof. If S has singular locus of dimension ≤ 0, this follows from Bertini. If $\mathrm{Sing}(S)$ has dimension 1, we consider the normalization $p : \tilde{S} \to S$ of S. Then $\dim(\mathrm{Sing}(\tilde{S})) \leq 0$. Let C be the non-normal locus in S, $D = p^{-1}(C)$. Let δ be a very ample linear system on $\tilde{S}$. The general $X \in \delta$ is smooth and does not pass through any singular point of $\tilde{S}$. We want to show that for $X \in \delta$ general, $p_| : X \to S$ is an embedding. Since p is an isomorphism outside D, we only have to consider the points in $X \cap D$. Let $x \in C$. The curves of δ passing through two points of $p^{-1}(x)$ form a subspace of codimension 2. Letting x vary in C, we see that the curves of δ intersecting a fibre $p^{-1}(x)$ in more than one point constitute a subspace of codimension ≥ 1. Hence, for general $X \in \delta$, $p_| : X \to S$ is injective.

Since there are only finitely many points where dp has rank zero, we may assume that for $y \in D$, $dp_y : T_y\tilde{S} \to T_{p(y)}S$ has rank one. The curves of δ

passing through y and having tangent direction $\mathrm{Ker}(dp_y)$ at y form a subspace of codimension 2 of δ. Letting y vary in D we get a subspace of codimension 1. So for general $X \in \delta$, $dp_|$ is everywhere injective. $\hfill\square$

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