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QUASI-ISOMORPHISMS OF A_∞-ALGEBRAS AND ORIENTED PLANAR TREES

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ABSTRACT. We construct quasi-isomorphisms of A-infinity algebras by using oriented planar trees and give an analog of Hodge decompositions for A-infinity algebras.

1. Introduction

In this paper we prove a minimal model theorem for A_∞-algebras in the sense of Kontsevich [3], and construct the following quasi-isomorphisms through oriented planar trees introduced by Stasheff [6]:

THEOREM 1.1. *Let (V, m) and (V', m') be A_∞-algebras which have harmonic projections and F an A_∞-morphism from (V, m) to (V', m'). If F is a quasi-isomorphism, then there is a quasi-isomorphism from (V', m') to (V, m) which induces the inverse of $(F_1)_*$ between the cohomology groups of the cochain complexes $(V[1], m_1)$ and $(V'[1], m'_1)$.*

In [4] Kontsevich gives a similar theorem for L_∞-algebras, which follows from the minimal model theorem in [3].

The content of our paper is as follows: Section 2 recalls some A_∞-algebraic notions. Section 3 contains the statement of the two main theorems of this paper; the proofs are given in Sections 6 and 7, respectively. Section 4 describes the minimal model theorem of A_∞-algebras, and Section 5 contains the proof of Theorem 1.1.

2. Preliminaries

We recall without proof some fundamental lemmas and propositions; see [2].

Let $V := \bigoplus_{k\in\mathbf{Z}} V^k$ be a graded vector space. We define $T^n(V) := V\otimes\cdots\otimes V$ (n times), and $T(V) := \bigoplus_{n\geq 1} T^n(V)$, and denote $v_1\otimes\cdots\otimes v_n \in T^n(V)$

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by $v_1 \cdots v_n$ for simplicity. Note that the grading of $v_1 \cdots v_n$ is the sum of the gradings of the $v_i's$. We define a linear map $\Delta : T(V) \to T(V) \otimes T(V)$ by

$$\Delta(v_1 \cdots v_n) := \sum_{i=1}^{n-1} (v_1 \cdots v_i) \otimes (v_{i+1} \cdots v_n).$$

For graded vector spaces V and V', let $f : T(V) \to T(V')$ be a linear map which satisfies the following conditions:

(i) $\Delta f = (f \otimes f)\Delta$.

(ii) f preserves the grading.

We have a natural projection $\pi : T(V') \to V'$ and define grading-preserving linear maps by

$$f_n := \pi \circ f|_{T^n(V)} : T^n(V) \to V', \ n = 1, 2, \cdots .$$

Lemma 2.1. *We have*

$$f(v_1 \cdots v_n) = \sum_{l=1}^{n} \sum_{h_1 + \cdots + h_l = n, h_i \geq 1} f_{h_1}(v_1 \cdots v_{h_1}) \cdots f_{h_l}(v_{h_1 + \cdots + h_{l-1}+1} \cdots v_n).$$

Conversely, a linear map $f : T(V) \to T(V')$ expressed in the above form with grading-preserving linear maps $f_n : T^n(V) \to V'$ satisfies (i) *and* (ii).

Let $m : T(V) \to T(V)$ be a linear map which satisfies the following conditions:

(iii) $(m \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} m)\Delta = \Delta m$, where $(\mathrm{id} \hat{\otimes} m)(x \otimes y) := (-1)^k x \otimes m(y)$ for x of grading k.

(iv) m increases the grading by 1.

We have a natural projection $\pi : T(V) \to V$ and define grading-1-increasing linear maps

$$m_n := \pi \circ m|_{T^n(V)} : T^n(V) \to V, \ n = 1, 2, \cdots .$$

Lemma 2.2. *We have*

$$m(v_1 \cdots v_n) = \sum_{l=1}^{n} \sum_{j=1}^{n-l+1} (-1)^{k_1 + \cdots + k_{j-1}} v_1 \cdots v_{j-1} m_l(v_j \cdots v_{j+l-1}) v_{j+l} \cdots v_n,$$

*where $v_i \in V^{k_i}$. Conversely, a linear map $m : T(V) \to T(V)$ expressed in the above form with grading-*1*-increasing linear maps $m_n : T^n(V) \to V$ satisfies* (iii) *and* (iv).

For a graded vector space $V = \bigoplus_{k \in \mathbf{Z}} V^k$, we introduce a new graded vector space $V[1]$ whose grading is defined by $(V[1])^k := V^{k+1}$. If $m : T(V[1]) \to T(V[1])$ satisfies (iii), (iv) and $mm = 0$, then we call (V, m) an A_∞-*algebra* and m an A_∞-*structure* of V. From Lemma 2.2 we obtain the following proposition; see [2]:

PROPOSITION 2.3. *$m : T(V[1]) \to T(V[1])$ is an A_∞-structure if and only if*

$$\sum_{l=1}^{n}\sum_{j=1}^{n-l+1}(-1)^{k_1+\cdots+k_{j-1}}m_{n-l+1}(x_1\cdots x_{j-1}m_l(x_j\cdots x_{j+l-1})x_{j+l}\cdots x_n)=0,$$

where $x_j \in (V[1])^{k_j}$.

When $n = 1$, the above equation is

$$m_1m_1 = 0. \tag{1}$$

Hence we obtain a cochain complex $(V[1], m_1)$. Let (V, m) and (V', m') be A_∞-algebras. If $f : T(V[1]) \to T(V'[1])$ satisfies (i), (ii) and $m'f = fm$, then we call f an *A_∞-morphism*. From Lemma 2.1 and Lemma 2.2 we obtain the following proposition:

PROPOSITION 2.4. *$f : T(V[1]) \to T(V'[1])$ is an A_∞-morphism if and only if*

$$\sum_{l=1}^{n}\sum_{h_1+\cdots+h_l=n,h_j\geq 1} m'_l\big(f_{h_1}(x_1\cdots x_{h_1})\cdots f_{h_l}(x_{h_1+\cdots+h_{l-1}+1}\cdots x_n)\big)$$
$$=\sum_{l=1}^{n}\sum_{j=1}^{n-l+1}(-1)^{k_1+\cdots+k_{j-1}}f_{n-l+1}\big(x_1\cdots x_{j-1}m_l(x_j\cdots x_{j+l-1})x_{j+l}\cdots x_n\big),$$

where $x_i \in (V[1])^{k_i}$.

When $n = 1$, the above equation is

$$m'_1f_1 = f_1m_1. \tag{2}$$

Hence f_1 induces a homomorphism $(f_1)_*$ between the cohomology groups of $(V[1], m_1)$ and $(V'[1], m'_1)$. If $(f_1)_*$ is an isomorphism, then we call f a *quasi-isomorphism*.

LEMMA 2.5. *Let $f : T(V) \to T(V')$ satisfy* (i) *and* (ii)*. If $f_1 : V \to V'$ is an isomorphism of vector spaces, then f has an inverse $g : T(V') \to T(V)$, which also satisfies* (i) *and* (ii)*.*

Proof. We construct g_n inductively. Define $g_1 := f_1^{-1}$ and assume that we have obtained g_h for $h \leq k-1$. It is easy to see that $fg = gf = \mathrm{id}$ if and only if the following equations hold:

$$f_1\big(g_k(x_1\cdots x_k)\big)$$
$$=-\sum_{l=2}^{k} f_l\Big(\sum_{h_1+\cdots+h_l=k,k-1\geq h_j\geq 1} g_{h_1}(x_1\cdots x_{h_1})\cdots g_{h_l}(x_{h_1+\cdots+h_{l-1}+1}\cdots x_k)\Big).$$

Since f_1 is an isomorphism, $g_k(x_1 \cdots x_k)$ can be defined. Hence, by the inductive step, we obtain g_n for all n so that $fg = gf = \mathrm{id}$. □

From Lemma 2.5 we obtain the following lemma:

LEMMA 2.6. *Let f be an A_∞-morphism from (V, m) to (V', m'). If $f_1 : V[1] \to V'[1]$ is an isomorphism of vector spaces, then f is an isomorphism of A_∞-algebras, i.e., there is an A_∞-morphism $g : (V', m') \to (V, m)$ such that $fg = gf = \mathrm{id}$.*

3. Main constructions

We assume that an A_∞-algebra (V, m) has a grading-preserving linear map $\Pi : V[1] \to V[1]$ and a grading-1-decreasing linear map $H : V[1] \to V[1]$ such that

$$\Pi^2 = \Pi, \tag{3}$$

$$\mathrm{id} - \Pi = m_1 H + H m_1. \tag{4}$$

From (1) and (4) we obtain

$$m_1 \Pi = \Pi m_1. \tag{5}$$

Next, we introduce oriented planar trees; see Figure 1 below and [5]. An *oriented planar tree* is a finite, connected, simply connected and oriented 1-dimensional graph which has some tail vertices and exactly one root vertex. We call the number of edges coming into a vertex v the *arity* of v. The arity of an internal vertex is greater than or equal to 2, and that of a root vertex is 1. The number of edges starting from a tail vertex is 1; similarly the number of edges starting from an internal vertex is 1. We denote the number of internal vertices of an oriented planar tree T by $I(T)$.

We construct a tree $\overline{T}$ from an oriented planar tree T as in Figure 2, by inserting a new vertex as the midpoint of each internal edge of T. Because the arity of a new vertex is one, $\overline{T}$ is not an oriented planar tree *in our sense*. Then we assign Π to each tail vertex and to the root vertex, $-H$ to each new vertex and m_k to each internal vertex of arity k, and we define a map $m_{n,T} : T^n(V[1]) \to V[1]$ by the compositions of the maps along the oriented edges of $\overline{T}$. For example, if $\overline{T}$ is as in Figure 2,

$$m_{5,T}(x_1 \cdots x_5) := \Pi m_2((-Hm_2)(\Pi(x_1)(-Hm_3)(\Pi(x_2)\Pi(x_3)\Pi(x_4)))\Pi(x_5)).$$

DEFINITION 3.1. We define grading-1-increasing linear maps $\tilde{m}_n : T^n(V[1]) \to V[1]$ by

- $\tilde{m}_1 := m_1$,
- $\tilde{m}_n := \sum_T m_{n,T},\ n \geq 2$,

where the sum is over the oriented planar trees with n tail vertices.

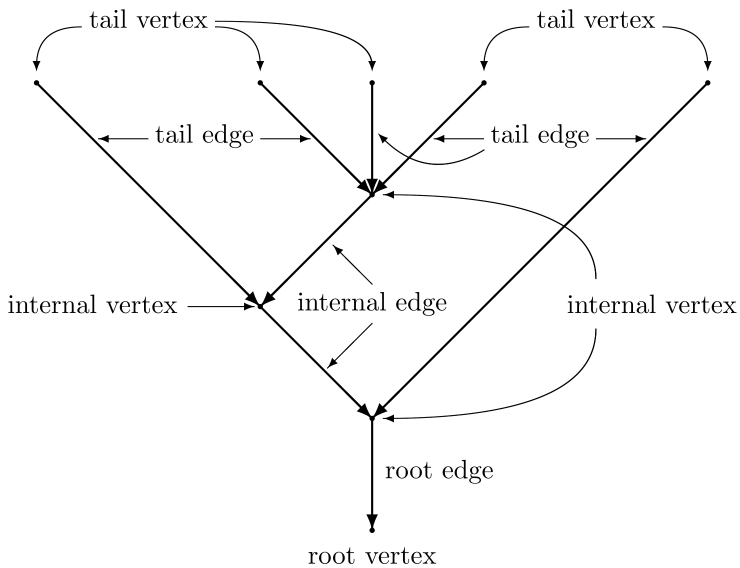


FIGURE 1

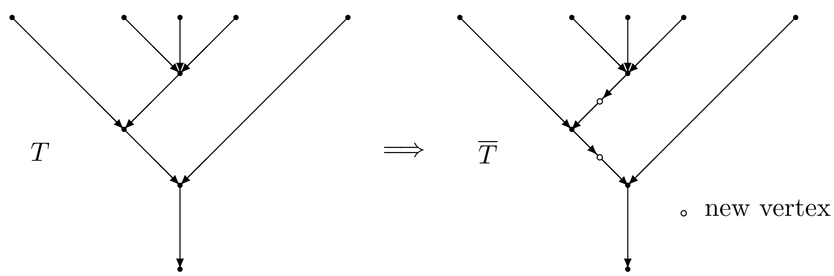


FIGURE 2

Since we assign Π to each tail vertex and to the root vertex of $\overline{T}$, we obtain:

LEMMA 3.2. $\Pi\tilde{m}_n(x_1\cdots x_n)=\tilde{m}_n(\Pi x_1\cdots\Pi x_n)$.

The following is the first main theorem in this paper:

THEOREM 3.3. *The maps $\tilde{m}_n$, $n=1,2,\dots$, define an A_∞-structure $\tilde{m}$ of V.*

Now we assign Π to each tail vertex, $-H$ to each new vertex and to the root vertex and m_k to each internal vertex of arity k, and we define a map $g_{n,T} : T^n(V[1]) \to V[1]$ by the compositions of the maps along the oriented edges of $\overline{T}$, i.e., we replace Π of $m_{n,T}$ at the root vertex of $\overline{T}$ by $-H$.

Definition 3.4. We define grading-preserving linear maps $g_n : T^n(V[1]) \to V[1]$ by

- $g_1 := \mathrm{id}$,
- $g_n := \sum_T g_{n,T},\ n \geq 2$,

where the sum is over the oriented planar trees with n tail vertices.

The following is the second main theorem in this paper:

Theorem 3.5. *The maps $g_n,\ n = 1, 2, \ldots,$ define an A_∞-morphism $g : (V, \tilde{m}) \to (V, m)$.*

Using Lemma 2.6 and $g_1 = \mathrm{id}$, we obtain the following corollary:

Corollary 3.6. *The A_∞-algebra $(V, \tilde{m})$ is isomorphic to the A_∞-algebra (V, m).*

By (3) we obtain $V[1] = \mathrm{Im}\,\Pi \oplus \mathrm{Im}(\mathrm{id} - \Pi)$, and define

$$B[1] := \mathrm{Im}\,\Pi, \quad C[1] := \mathrm{Im}(\mathrm{id} - \Pi).$$

Since we assign Π to each root vertex of an oriented planar tree in the definition of $\tilde{m}_n,\ n \geq 2$, we obtain $\mathrm{Im}\,\tilde{m}_n \subset B[1],\ n \geq 2$. Moreover, from (5) and $\tilde{m}_1 := m_1$ we obtain $\mathrm{Im}\,\tilde{m}_1|_{B[1]} \subset B[1]$. Hence, the following lemma holds:

Lemma 3.7. *$(B, \tilde{m}|_{B[1]})$ is an A_∞-algebra.*

Definition 3.8. We define grading-preserving linear maps $i_n : T^n(B[1]) \to V[1]$ by

- $i_1 := \mathrm{id}\,|_{B[1]}$,
- $i_n := 0,\ n \geq 2$.

Lemma 3.9. *The maps $i_n,\ n = 1, 2, \ldots,$ define an A_∞-morphism $i : (B, \tilde{m}|_{B[1]}) \to (V, \tilde{m})$.*

Definition 3.10. We define grading-preserving linear maps $p_n : T^n(V[1]) \to B[1]$ by

- $p_1 := \Pi$,
- $p_n := 0,\ n \geq 2$.

Lemma 3.11. *The maps $p_n,\ n = 1, 2, \ldots,$ define an A_∞-morphism $p : (V, \tilde{m}) \to (B, \tilde{m}|_{B[1]})$.*

Proof. By Lemma 3.2 we have

$$\begin{aligned}
&p\tilde{m}(x_1\cdots x_n)\\
&=p\Big(\sum_{l=1}^{n}\sum_{j=1}^{n-l+1}(-1)^{k_1+\cdots+k_{j-1}}x_1\cdots x_{j-1}\tilde{m}_l(x_j\cdots x_{j+l-1})x_{j+l}\cdots x_n\Big)\\
&=\sum_{l=1}^{n}\sum_{j=1}^{n-l+1}(-1)^{k_1+\cdots+k_{j-1}}\Pi x_1\cdots \Pi x_{j-1}\Pi\tilde{m}_l(x_j\cdots x_{j+l-1})\Pi x_{j+l}\cdots \Pi x_n\\
&=\sum_{l=1}^{n}\sum_{j=1}^{n-l+1}(-1)^{k_1+\cdots+k_{j-1}}\Pi x_1\cdots \Pi x_{j-1}\tilde{m}_l(\Pi x_j\cdots \Pi x_{j+l-1})\Pi x_{j+l}\cdots \Pi x_n\\
&=\tilde{m}(\Pi x_1\cdots\Pi x_n)\\
&=\tilde{m}|_{B[1]}p(x_1\cdots x_n),
\end{aligned}$$

where $x_j\in (V[1])^{k_j}$. Thus p is an A_∞-morphism from $(V,\tilde{m})$ to $(B,\tilde{m}|_{B[1]})$. □

Lemma 3.12. *$i:(B,\tilde{m}|_{B[1]})\to (V,\tilde{m})$ and $p:(V,\tilde{m})\to(B,\tilde{m}|_{B[1]})$ are quasi-isomorphisms.*

Since we assign Π to each tail vertex of oriented planar trees in the definition of $\tilde{m}_n,\ n\geq 2$, we obtain Im $\tilde{m}_n|_{C[1]}=0,\ n\geq 2$. Moreover, from (5) and $\tilde{m}_1:=m_1$ we obtain Im $\tilde{m}_1|_{C[1]}\subset C[1]$. Hence, the following lemma holds:

Lemma 3.13. *$(C,\tilde{m}|_{C[1]})$ is an A_∞-algebra.*

Definition 3.14. We define grading-preserving linear maps
$j_n:T^n(C[1])\to V[1]$ by

- $j_1:=\mathrm{id}\,|_{C[1]}$,
- $j_n:=0,\ n\geq 2$.

Lemma 3.15. *The maps $j_n,\ n=1,2,\ldots,$ define an A_∞-morphism $j:(C,\tilde{m}|_{C[1]})\to(V,\tilde{m})$.*

Definition 3.16. We define grading-preserving linear maps
$q_n:T^n(V[1])\to C[1]$ by

- $q_1:=\mathrm{id}-\Pi$,
- $q_n:=0,\ n\geq 2$.

Lemma 3.17. *The maps $q_n,\ n=1,2,\ldots,$ define an A_∞-morphism $q:(V,\tilde{m})\to(C,\tilde{m}|_{C[1]})$.*

Proof. We can prove this lemma in a similar fashion as Lemma 3.11. □

Lemma 3.18. *The cohomology of $(C[1],\tilde{m}_1)$ vanishes.*

Proof. Since $(\tilde{m}_1 H + H\tilde{m}_1)\Pi = (\mathrm{id} - \Pi)\Pi = 0$, we have $H\tilde{m}_1\Pi = -\tilde{m}_1 H\Pi$. Take an element $(\mathrm{id} - \Pi)x \in C[1]$ such that $\tilde{m}_1(\mathrm{id} - \Pi)x = 0$. Then $H\tilde{m}_1 x = H\tilde{m}_1\Pi x = -\tilde{m}_1 H\Pi x$. So we can conclude

$$(\mathrm{id} - \Pi)x = \tilde{m}_1 Hx + H\tilde{m}_1 x = \tilde{m}_1 Hx - \tilde{m}_1 H\Pi x = \tilde{m}_1 H(\mathrm{id} - \Pi)x$$

and

$$(\mathrm{id} - \Pi)x = (\mathrm{id} - \Pi)^2 x = (\mathrm{id} - \Pi)\tilde{m}_1 H(\mathrm{id} - \Pi)x = \tilde{m}_1(\mathrm{id} - \Pi)H(\mathrm{id} - \Pi)x,$$

which implies that the cohomology of $(C[1], \tilde{m}_1)$ vanishes. □

4. Minimal model theorem for A_∞-algebras

We now state the minimal model theorem for A_∞-algebras. We first recall harmonic forms of Hodge decompositions.

Let (V, m) be an A_∞-algebra. If $m_1 = 0$, then we call (V, m) *minimal*. If $m_n = 0,\ n \geq 2$, and the cohomology group of the cochain complex $(V[1], m_1)$ vanishes, then we call (V, m) *linear contractible*. Note that $(C, \tilde{m}|_{C[1]})$ is linear contractible. If (V, m) has linear maps Π and H such that $m_1|_{B[1]} = 0$, then we call Π a *harmonic projection*. Note that $m_1|_{B[1]} \neq 0$ in general; for example, if $\Pi = \mathrm{id}$ and $H = 0$, then Π and H satisfy (3) and (4), but $m_1|_{B[1]} \neq 0$ in general.

THEOREM 4.1 (Minimal model theorem for A_∞-algebras). *If (V, m) has a harmonic projection, then $(B[1], \tilde{m}|_{B[1]})$ is minimal and $(C[1], \tilde{m}|_{C[1]})$ is linear contractible.*

5. Proof of Theorem 1.1

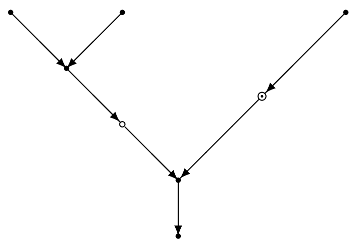
In Theorem 1.1 we obtain the following sequence of quasi-isomorphisms:

$$(B, \tilde{m}|_{B[1]}) \xrightarrow{i} (V, \tilde{m}) \xrightarrow{g} (V, m) \xrightarrow{F} (V', m') \xrightarrow{(g')^{-1}} (V', \tilde{m'}) \xrightarrow{p'} (B', \tilde{m'}|_{B'[1]}).$$

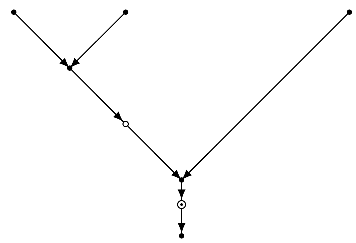
Since our A_∞-algebras have harmonic projections, $(B, \tilde{m}|_{B[1]})$ and $(B', \tilde{m'}|_{B'[1]})$ are minimal, and hence the linear map $\left(p' \circ (g')^{-1} \circ F \circ g \circ i\right)_1 : B[1] \to B'[1]$ is an isomorphism of vector spaces. Therefore, by Lemma 2.6, $K = \left(p' \circ (g')^{-1} \circ F \circ g \circ i\right)^{-1} : (B', \tilde{m'}|_{B[1]}) \to (B, \tilde{m}|_{B[1]})$ is an isomorphism of A_∞-algebras, and we obtain the following sequence of quasi-isomorphisms:

$$(V', m') \xrightarrow{(g')^{-1}} (V', \tilde{m'}) \xrightarrow{p'} (B', \tilde{m'}|_{B'[1]}) \xrightarrow{K} (B, \tilde{m}|_{B[1]}) \xrightarrow{i} (V, \tilde{m}) \xrightarrow{g} (V, m).$$

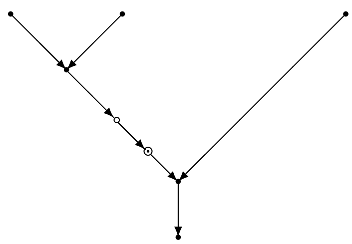
Hence the map $G : (V', m') \to (V, m)$ defined by $g \circ i \circ K \circ p' \circ (g')^{-1}$ is a quasi-isomophism, as claimed in the theorem.



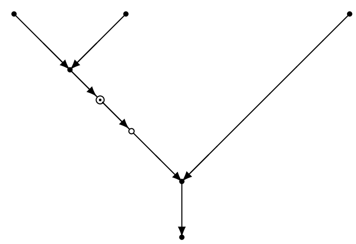
type (I) tail edge



type (II) root edge



type (III) + internal edge



type (IV) − internal edge

FIGURE 3

6. Proof of Theorem 3.3

We prove that the maps $\tilde{m}_n : T^n(V[1]) \to V[1],\ n = 1, 2, \cdots$, in Definition 3.1 satisfy the equations in Proposition 2.3.

Let T be an oriented planar tree with n tail vertices. We denote by $E(\overline{T})$ the set of the edges of $\overline{T}$. We take an edge $e \in E(\overline{T})$, insert a new vertex at the midpoint of e and denote the new tree by $\overline{T}_e$; there are four types of such trees $\overline{T}_e$ as shown in Figure 3, where the new vertex is indicated by a small circle with a dot in the center. We assign m_1 to the new vertex and assign the same maps of $m_{n,T}$ to the other vertices. Then we define a map $m_{n,\overline{T}_e} : T^n(V[1]) \to V[1]$ by the compositions of the maps along the oriented edges of $\overline{T}_e$.

Next, we define $\mathrm{sgn}(\overline{T}_e, x_1 \cdots x_n) \in \{-1, 1\}$ as follows. Let $e \in E(\overline{T})$. If the trace of oriented edges starting from i-th tail vertex, $1 \le i \le j$, does not go through e and the trace of oriented edges starting from $(j+1)$-th tail vertex goes through e, then we define

$$\mathrm{sgn}(\overline{T}_e, x_1 \cdots x_n) := (-1)^{k_1 + \cdots + k_j},$$

where $x_i \in (V[1])^{k_i}$. For example, for the type (I) tree in Figure 3 we have $\mathrm{sgn}(\overline{T}_e, x_1x_2x_3) = (-1)^{k_1+k_2}$.

Definition 6.1. We define degree-2-increasing linear maps $\hat{m}_n : T^n(V[1]) \to V[1],\ n \geq 2$, by

$$\hat{m}_n(x_1 \cdots x_n) := \sum_{\overline{T}_e} \mathrm{sgn}(\overline{T}_e, x_1 \cdots x_n) m_{n,\overline{T}_e}(x_1 \cdots x_n).$$

By using $\hat{m}_n$ we will prove that the maps $\tilde{m}_n,\ n = 1, 2, \cdots$, satisfy the equations in Proposition 2.3. We take an edge E of an oriented planar tree T and a new vertex at the midpoint of E. We take an edge $E_+ \in E(\overline{T})$ whose starting point is the new vertex and an edge $E_- \in E(\overline{T})$ whose end point is the new vertex. Note that $m_{n,\overline{T}_{E_+}}$ and $m_{n,\overline{T}_{E_-}}$ are linear maps corresponding to type (III) and type (IV), respectively, and that $\mathrm{sgn}(\overline{T}_{E_+}, x_1 \cdots x_n) = \mathrm{sgn}(\overline{T}_{E_-}, x_1 \cdots x_n)$. From (4) we obtain

$$m_{n,\overline{T}_{E_+}} + m_{n,\overline{T}_{E_-}} = m^{\Pi}_{n,T,E} - m^{\mathrm{id}}_{n,T,E},$$

where $m^{\Pi}_{n,T,E}$ is the map in which we replace $-H$ of $m_{n,T}$ at the midpoint of E by Π and $m^{\mathrm{id}}_{n,T,E}$ is the map in which we replace $-H$ of $m_{n,T}$ at the midpoint of E by id. Hence we obtain

$$\begin{aligned}\hat{m}_n(x_1 \cdots x_n) = &\sum_{\overline{T}_e,\ \text{type (I), (II)}} \mathrm{sgn}(\overline{T}_e, x_1 \cdots x_n) m_{n,\overline{T}_e}(x_1 \cdots x_n)\\ &+ \sum_T \sum_{E,\ \text{internal edge}} \mathrm{sgn}(\overline{T}_{E_\pm}, x_1 \cdots x_n) m^{\Pi}_{n,T,E}(x_1 \cdots x_n)\\ &- \sum_T \sum_{E,\ \text{internal edge}} \mathrm{sgn}(\overline{T}_{E_\pm}, x_1 \cdots x_n) m^{\mathrm{id}}_{n,T,E}(x_1 \cdots x_n).\end{aligned}$$

Proposition 6.2. *We have*

$$\begin{aligned}&- \sum_T \sum_{E,\ \text{internal edge}} \mathrm{sgn}(\overline{T}_{E_\pm}, x_1 \cdots x_n) m^{\mathrm{id}}_{n,T,E}(x_1 \cdots x_n)\\ &= \sum_{\overline{T}_e} \mathrm{sgn}(\overline{T}_e, x_1 \cdots x_n) m_{n,\overline{T}_e}(x_1 \cdots x_n).\end{aligned}$$

Proof. We take an oriented planar tree T and an internal edge e of T. Then we remove the edge e to decompose T into two pieces and glue these two pieces together at the vertices which were the starting point and the end point of e. We thus obtain a new oriented planar tree $C_e(T)$, which we call the *contraction* of T at e. Note that $I(C_e(T)) = I(T) - 1$. Fix an oriented planar tree T' with n tail vertices and fix an internal vertex v of T'. Let $\{T^i\}$

be the set of the oriented planar trees such that $C_{E^i}(T^i) = T'$ with the end point of E^i corresponding to v. Consider the sum

$$-\sum_{T^i} \operatorname{sgn}(\overline{T^i}_{E^i}, x_1 \cdots x_n) m^{\mathrm{id}}_{n,T^i,E^i}(x_1 \cdots x_n).$$

Since m is an A_∞-structure, by Proposition 2.3, the sum is

$$\sum_{e \in E(\overline{T'})} \operatorname{sgn}(\overline{T'}_e, x_1 \cdots x_n) m_{n,\overline{T'}_e}(x_1 \cdots x_n),$$

where e has v as a starting or end point. By considering the above sums for all oriented trees and all internal vertices, we obtain the identities asserted in the proposition. □

The right hand side of the identity in Proposition 6.2 is $\hat{m}_n$. Hence

$$\begin{aligned} 0 = &\sum_{\overline{T}_e,\ \text{type (I), (II)}} \operatorname{sgn}(\overline{T}_e, x_1 \cdots x_n) m_{n,\overline{T}_e}(x_1 \cdots x_n) \\ &+ \sum_T \sum_{E,\ \text{internal edge}} \operatorname{sgn}(\overline{T}_{E_\pm}, x_1 \cdots x_n) m^{\Pi}_{n,T,E}(x_1 \cdots x_n). \end{aligned}$$

On the other hand, from (5) and $\tilde{m}_1 := m_1$, we obtain

$$\begin{aligned} &\sum_{\overline{T}_e,\ \text{type (I)}} \operatorname{sgn}(\overline{T}_e, x_1 \cdots x_n) m_{n,\overline{T}_e}(x_1 \cdots x_n) \\ &\quad = \sum_T \sum_{j=1}^n (-1)^{k_1 + \cdots + k_{j-1}} m_{n,T}(x_1 \cdots m_1(x_j) \cdots x_n) \\ &\quad = \sum_{j=1}^n (-1)^{k_1 + \cdots + k_{j-1}} \tilde{m}_n(x_1 \cdots \tilde{m}_1(x_j) \cdots x_n) \end{aligned}$$

and

$$\begin{aligned} &\sum_{\overline{T}_e,\ \text{type (II)}} \operatorname{sgn}(\overline{T}_e, x_1 \cdots x_n) m_{n,\overline{T}_e}(x_1 \cdots x_n) \\ &\quad = \sum_T m_1\big(m_{n,T}(x_1 \cdots x_n)\big) = \tilde{m}_1\big(\tilde{m}_n(x_1 \cdots x_n)\big). \end{aligned}$$

Let T_1 and T_2 be oriented planar trees. By gluing the root vertex of T_2 to the j-th tail vertex of T_1, we obtain an oriented planar tree denoted by $T_1 \circ_j T_2$.

From (3) we get

$$\sum_{T}\sum_{E,\text{ internal edge}} \operatorname{sgn}(\overline{T}_{E_\pm}, x_1\cdots x_n) m^{\Pi}_{n,T,E}(x_1\cdots x_n)$$
$$= \sum_{T}\sum_{T=T_1\circ_j T_2} (-1)^{k_1+\cdots+k_{j-1}} m_{n-l+1,T_1}(x_1\cdots m_{l,T_2}(x_j\cdots x_{j+l-1})\cdots x_n)$$
$$= \sum_{l=2}^{n-1}\sum_{j=1}^{n-l+1} (-1)^{k_1+\cdots+k_{j-1}} \tilde{m}_{n-l+1}(x_1\cdots \tilde{m}_l(x_j\cdots x_{j+l-1})\cdots x_n).$$

Summing up the above equations, we obtain

$$\sum_{l=1}^{n}\sum_{j=1}^{n-l+1} (-1)^{k_1+\cdots+k_{j-1}} \tilde{m}_{n-l+1}(x_1\cdots x_{j-1}\tilde{m}_l(x_j\cdots x_{j+l-1})x_{j+l}\cdots x_n) = 0.$$

By Proposition 2.3 this means that $\tilde{m}_n,\ n=1,2,\ldots,$ define an A_∞-structure $\tilde{m}$ of V. This completes the proof of Theorem 3.3.

7. Proof of Theorem 3.5

We prove that the functions $g_n : T^n(V[1]) \to V[1],\ n=1,2,\cdots$, in Definition 3.4 satisfy the equations in Proposition 2.4.

Let T be an oriented planar tree. We define maps $g_{n,\overline{T}_e} : T^n(V[1]) \to V[1]$ by replacing Π of $m_{n,\overline{T}_e}$ at the root vertex of $\overline{T}_e$ by $-H$.

Definition 7.1. We define degree-1-increasing linear maps $\hat{g}_n : T^n(V[1]) \to V[1],\ n \geq 2$, by

$$\hat{g}_n(x_1\cdots x_n) := \sum_{\overline{T}_e} \operatorname{sgn}(\overline{T}_e, x_1\cdots x_n) g_{n,\overline{T}_e}(x_1\cdots x_n).$$

In a similar fashion as in the previous section, we obtain

$$0 = \sum_{\overline{T}_e,\text{ type (I), (II)}} \operatorname{sgn}(\overline{T}_e, x_1\cdots x_n) g_{n,\overline{T}_e}(x_1\cdots x_n)$$
$$+ \sum_{T}\sum_{E,\text{ internal edge}} \operatorname{sgn}(\overline{T}_{E_\pm}, x_1\cdots x_n) g^{\Pi}_{n,T,E}(x_1\cdots x_n).$$

On the other hand, from (5) and $\tilde{m}_1 := m_1$, we obtain

$$\begin{aligned}&\sum_{\overline{T}_e,\ \text{type (I)}} \operatorname{sgn}(\overline{T}_e, x_1\cdots x_n) g_{n,\overline{T}_e}(x_1\cdots x_n)\\&\quad= \sum_T \sum_{j=1}^{n} (-1)^{k_1+\cdots+k_{j-1}} g_{n,T}(x_1\cdots m_1(x_j)\cdots x_n)\\&\quad= \sum_{j=1}^{n} (-1)^{k_1+\cdots+k_{j-1}} g_n(x_1\cdots \tilde{m}_1(x_j)\cdots x_n).\end{aligned}$$

From (4) and $g_1 := \mathrm{id}$, we obtain

$$\begin{aligned}&\sum_{\overline{T}_e,\ \text{type (II)}} \operatorname{sgn}(\overline{T}_e, x_1\cdots x_n) g_{n,\overline{T}_e}(x_1\cdots x_n)\\&\quad= -m_1\big(g_n(x_1\cdots x_n)\big)\\&\qquad- \sum_{l\geq 2, h_1+\cdots+h_l=n, h_j\geq 1} m_l\big(g_{h_1}(x_1\cdots x_{h_1})\cdots g_{h_l}(x_{h_1+\cdots h_{l-1}+1}\cdots x_n)\big)\\&\qquad+ g_1\big(\tilde{m}_n(x_1\cdots x_n)\big).\end{aligned}$$

Moreover, from (3) we obtain

$$\begin{aligned}&\sum_T \sum_{E,\ \text{internal edge}} \operatorname{sgn}(\overline{T}_{E_\pm}, x_1\cdots x_n) g^{\Pi}_{n,T,E}(x_1\cdots x_n)\\&\quad= \sum_T \sum_{T=T_1\circ_j T_2} (-1)^{k_1+\cdots+k_{j-1}} g_{n-l+1,T_1}(x_1\cdots m_{l,T_2}(x_j\cdots x_{j+l-1})\cdots x_n)\\&\quad= \sum_{l=2}^{n-1} \sum_{j=1}^{n-l+1} (-1)^{k_1+\cdots+k_{j-1}} g_{n-l+1}(x_1\cdots \tilde{m}_l(x_j\cdots x_{j+l-1})\cdots x_n).\end{aligned}$$

Summing up the above equations, we obtain

$$\begin{aligned}0 &= -\sum_{l=1}^{n} \sum_{h_1+\cdots+h_l=n, h_j\geq 1} m_l\big(g_{h_1}(x_1\cdots x_{h_1})\cdots g_{h_l}(x_{h_1+\cdots h_{l-1}+1}\cdots x_n)\big)\\&\quad+ \sum_{l=1}^{n} \sum_{j=1}^{n-l+1} (-1)^{k_1+\cdots+k_{j-1}} g_{n-l+1}(x_1\cdots \tilde{m}_l(x_j\cdots x_{j+l-1})\cdots x_n).\end{aligned}$$

By Proposition 2.4 this means that $g_n,\ n = 1, 2, \ldots,$ define an A_∞-morphism g from $(V, \tilde{m})$ to (V, m). This completes the proof of Theorem 3.5.

Appendix A. Expression of g in Lemma 2.5 by trees

In Lemma 2.5 we constructed g_k inductively. In this appendix, we exhibit a construction for g_k by using oriented planar trees. Let T be an oriented planar tree, and assign f_1^{-1} to each vertex, $-f_1^{-1}$ to each new vertex and to the root vertex, and f_k to each vertex of arity k. We define a map $g_{n,T} : T^n(V') \to V$ by the compositions of the maps along the oriented edges of $\overline{T}$.

Definition A.1. We define degree-preserving linear maps $g_n : T^n(V') \to V$ by

- $g_1 := f_1^{-1}$,
- $g_n := \sum_T g_{n,T},\ n \geq 2$.

The sum is over the oriented planar trees with n tail vertices.

Lemma A.2. *The maps g_n, $n = 1, 2, \ldots$, define the inverse of f in Lemma 2.5.*

The proof of this lemma is left to the reader.

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