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COMMUTATORS WITH FINITE SPECTRUM

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Abstract. The properties of a bounded derivation d on a complex Banach algebra A such that the spectrum $\sigma([x, dx])$ is finite for all $x \in A$ are studied. In particular we show that, if $\sigma([x, dx])$ is a singleton for each x, then d maps into the radical of A.

Introduction

The Kleinecke–Shirokov theorem from 1956/57 states that, if a and b are elements of a (complex) Banach algebra A such that $[a, [a, b]] = 0$, then $[a, b] = ab - ba$ is quasinilpotent. This extends to bounded derivations in the following way. If $d\colon A \to A$ is a bounded derivation such that $[a, da] = 0$, then da is quasinilpotent; see, e.g., [6, Theorem 2.7.19]. It easily follows that, provided $[a, da] = 0$ for *all* $a \in A$, dA is contained in the radical $\operatorname{rad} A$ of A. (This statement also holds for unbounded derivations, however it is much more difficult to achieve in this case; compare [8] and [9].) Far less seems to be known if the spectrum $\sigma([a, da])$ of $[a, da]$ is merely assumed to be finite. In [2] and [4], the authors investigated conditions entailing that the range of a bounded derivation d lies in the socle $\operatorname{soc} A$ of a (semisimple) Banach algebra A. For example, the finiteness of the spectra $\sigma(dx)$ for all $x \in A$ turns out to be a necessary and sufficient condition.

One of the objectives of the present note is to discuss the consequences of such finiteness hypotheses for the behaviour of a bounded derivation d. In particular, we shall prove that, in a wide class of Banach algebras, if the spectrum of every commutator $[x, dx]$, $x \in A$, contains at most two elements, then d maps into the radical. It turns out that the techniques developed in [2], [4], [5] are well suited to describe bounded derivations d on A such that

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$\sigma([x, dx])$ is finite for all $x \in A$. This will be done in Section 2. In particular, we prove that, if d is a bounded derivation on A, then the two following properties are equivalent (where $\#S$ denotes the cardinality of a set S):

(a) $\#\sigma(dx) < \infty$ for all $x \in A$.

(b) $\#\sigma([x, dx]) < \infty$ for all $x \in A$.

We also refine the results in [4] by showing that, if (a) holds, then d maps into the radical $\operatorname{rad} A$, or $\operatorname{soc}(A/\operatorname{rad} A) \neq 0$ and for some element $a \in A$ such that $a + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$, $dx - [x, a] \in \operatorname{rad} A$ for all $x \in A$.

The paper is organised as follows. In the first section, which is purely algebraic in nature, we proceed with the study of dense algebras. Section 2 contains our main results. In Section 3, using essentially the same tools, we will briefly discuss the situation of generalised inner derivations.

1. The case of dense algebras

Throughout this section, $\mathcal{A}$ will be an algebra of linear operators on a complex vector space X acting densely with one-dimensional centraliser in $L(X)$. Let d be a derivation on $\mathcal{A}$. Following the terminology of [2], we shall say that d is *inner* if there is a linear operator T on X such that $dS = [S, T]$ for every $S \in \mathcal{A}$. Otherwise, d will be called *outer*. As usual, I denotes the identity mapping on X.

No topological considerations will enter the discussion in this section. In fact, instead of working with the full spectrum it will suffice to put the finiteness assumptions on the set $\sigma_p(T)$ of all eigenvalues of T despite the possibly infinite dimensions of X.

We recall the following result by Kaplansky, in the version proved in [1, Theorem 4.2.7], which will be one of the main tools in this paper.

THEOREM 1.1. *Let X be a complex vector space, and let T be a linear operator from X into X. Suppose that there exists an integer $n \geq 1$ such that $\zeta, T\zeta, \dots, T^n\zeta$ are linearly dependent for all $\zeta \in X$. Then T is algebraic of degree at most n.*

LEMMA 1.2. *Let T be a linear operator on X. Suppose that* $\dim X \geq 3$ *and that $\#\sigma_p([S, [S, T]]) \leq 2$ for each $S \in \mathcal{A}$. Then $T = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof. Assume that $\zeta, T\zeta, T^2\zeta, T^3\zeta$ are linearly independent vectors in X for some $\zeta \in X$. Then there is $S \in \mathcal{A}$ such that

$$S\zeta = 0, \quad ST\zeta = T^2\zeta, \;\; ST^2\zeta = 0, \; ST^3\zeta = -T\zeta.$$

This implies that

$$[S, [S, T]]\zeta = 0, \; [S, [S, T]]T\zeta = 2\,T\zeta, \; [S, [S, T]]T^2\zeta = -T^2\zeta$$

and hence $\{0, 2, -1\} \subseteq \sigma_p([S, [S, T]])$, which contradicts our assumption. Thus, by Theorem 1.1, there exist $r_1, r_2, r_3 \in \mathbb{C}$ such that $T^3 = r_1 I + r_2 T + r_3 T^2$. But

$[S,[S,T-\lambda]]=[S,[S,T]]$ for each $\lambda\in\mathbb{C}$. Hence, without loss of generality, we can assume that $r_1\neq 0$. Suppose now that $\zeta, T\zeta, T^2\zeta$ are linearly independent vectors for some $\zeta\in X$. Then we can find $\alpha\in\mathbb{C}$ and $S\in\mathcal{A}$ such that

$$\alpha r_1+r_2+r_3=5,\ S\zeta=\alpha T\zeta,$$
$$ST\zeta=T\zeta,\ ST^2\zeta=T\zeta.$$

We check easily that

$$[S,[S,T]]\zeta=(1-2\alpha)T\zeta+\alpha T^2\zeta,$$
$$[S,[S,T]]T\zeta=-T\zeta+T^2\zeta,$$
$$[S,[S,T]]T^2\zeta=3\,T\zeta+T^2\zeta.$$

Since the matrix
$$\begin{pmatrix} 0 & 0 & 0\\ 1-2\alpha & -1 & 3\\ \alpha & 1 & 1\end{pmatrix}$$
has $0,2,-2$ as eigenvalues, it follows that $\#\sigma_p([S,[S,T]])\geq 3$, contrary to our assumption. Consequently, $T^2=s_1I+s_2T$ for some $s_1,s_2\in\mathbb{C}$. Replacing T by $T-(s_2/2)I$ if necessary, we may assume without loss of generality that $T^2\in\mathbb{C}I$.

Set $T^2=\alpha I$, $\alpha\in\mathbb{C}$. Assume that $\zeta, T\zeta$ are linearly independent for some $\zeta\in X$ and choose $\zeta'\in X$ such that $\zeta, T\zeta, \zeta'$ are linearly independent. In the case that $\alpha\neq 0$ we can assume that $\alpha=1/2$. Then there exists $S\in\mathcal{A}$ such that

$$S\zeta=0,\ ST\zeta=\zeta+2\,T\zeta,\ S\zeta'=\zeta'.$$

This entails that

$$[S,[S,T]]T\zeta=-2\,T\zeta,$$

and

$$[S,[S,T]](\zeta+T\zeta)=2(\zeta+T\zeta).$$

If $\zeta, T\zeta, \zeta', T\zeta'$ are linearly independent vectors, without loss of generality we may suppose that $ST\zeta'=\zeta'+T\zeta'$. Hence, $[S,[S,T]]\zeta'=0$. If there exist scalars $r_1,r_2,r_3\in\mathbb{C}$ such that $T\zeta'=r_1\zeta+r_2T\zeta+r_3\zeta'$, then $[S,[S,T]]\zeta'=r_1\zeta+r_2T\zeta$. Thus, for suitable scalars λ, λ',

$$[S,[S,T]](\zeta'+\lambda\zeta+\lambda' T\zeta)=0.$$

Consequently, $\{0,2,-2\}\subseteq\sigma_p([S,[S,T]])$, a contradiction.

Let us assume now that $\alpha=0$. Choose $S\in\mathcal{A}$ such that

$$S\zeta=\frac{1}{2}\,T\zeta,\quad ST\zeta=\zeta+T\zeta,\ S\zeta'=\zeta'.$$

Then

$$[S,[S,T]]\zeta=\zeta+2T\zeta,\ \ [S,[S,T]]T\zeta=-2\zeta-T\zeta.$$

The matrix $\left(\begin{smallmatrix} 1 & -2 \\ 2 & -1 \end{smallmatrix}\right)$ has eigenvalues $\pm\sqrt{3}$. Suppose that there exist scalars $r_1, r_2, r_3 \in \mathbb{C}$ such that $T\zeta' = r_1\zeta + r_2 T\zeta + r_3\zeta'$. Since $T^2 = 0$, we have $r_1 T\zeta + r_3 T\zeta' = 0$ and therefore

$$-r_1 T\zeta = r_3 T\zeta' = r_1 r_3 \zeta + r_2 r_3 T\zeta + r_3^2 \zeta'.$$

As $\{\zeta, T\zeta, \zeta'\}$ is linearly independent, it follows that $r_1 r_3 = r_2 r_3 - r_1 = r_3^2 = 0$, whence $r_3 = r_1 = 0$. Therefore $[S,[S,T]]\zeta' \in \mathbb{C}\zeta + \mathbb{C}T\zeta$. Hence, there exists $\zeta'' \in X \setminus \{0\}$ such that $[S,[S,T]]\zeta'' = 0$, and $\#\sigma_p([S,[S,T]]) \geq 3$. On the other hand, if the vectors $\zeta, T\zeta, \zeta', T\zeta'$ are linearly independent, we can moreover suppose that $ST\zeta' = \beta\zeta'$ for some scalar β. It follows that

$$[S,[S,T]]\zeta' = T\zeta' - \beta\zeta',$$
$$[S,[S,T]]T\zeta' = \beta T\zeta' - 2\beta^2\zeta'.$$

Since the matrix $\left(\begin{smallmatrix} -\beta & -2\beta^2 \\ 1 & \beta \end{smallmatrix}\right)$ has $i\beta$, $-i\beta$ as eigenvalues, for suitable β, we have $\#\sigma_p([S,[S,T]]) \geq 3$.

We conclude from this that $\{\zeta,\ T\zeta\}$ must be linearly dependent for all $\zeta \in X$, which entails that T is a scalar multiple of the identity as claimed. □

We will now drop the additional assumption in Lemma 1.2.

Proposition 1.3. *Let T be a linear operator on X with the property that $\#\sigma_p\left([S,[S,T]]\right) = 1$ for each $S \in \mathcal{A}$. Then $T = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof. By the first part of the proof of Lemma 1.2, it suffices to consider the case where $\dim X = 2$ and $T^2 \in \mathbb{C}I$. Writing T in triangular form,

$$T = \begin{pmatrix} \alpha_1 & \alpha_3 \\ 0 & \alpha_4 \end{pmatrix}, \ \ \alpha_i \in \mathbb{C},$$

we find that

$$(\alpha_1 + \alpha_4)\alpha_3 = 0 \quad \text{and} \quad \alpha_1^2 = \alpha_4^2.$$

Therefore, $T \in \mathbb{C}I$ or $\alpha_1 + \alpha_4 = 0$. Suppose the latter. Letting

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we obtain

$$[S,[S,T]] = \begin{pmatrix} -4\alpha_1 & -2\alpha_3 \\ -2\alpha_3 & 4\alpha_1 \end{pmatrix}.$$

Since $[S,[S,T]]$ has only one eigenvalue and trace zero, the determinant has to be zero as well; thus

$$4\alpha_1^2 + \alpha_3^2 = 0.$$

Letting

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

our assumption on the set of eigenvalues implies that

$$-4\alpha_1^2 + \alpha_3^2 = 0.$$

As a result $\alpha_1 = \alpha_3 = 0$, wherefore $T = 0$. This proves the claim. □

In the next lemma we consider double commutators with finitely many eigenvalues where the cardinality is uniformly bounded.

Lemma 1.4. *Let T be a linear operator on X, and let $n \in \mathbb{N}$. Suppose that $\#\sigma_p([S,[S,T]]) \le n$ for each $S \in \mathcal{A}$. Then $T = T_0 + \lambda I$, where $\lambda \in \mathbb{C}$ and T_0 is a finite rank operator.*

Proof. Assume that the set

$$\{\zeta_1, \ldots, \zeta_{n+1}, T\zeta_1, \ldots, T\zeta_{n+1}, T^2\zeta_1, \ldots, T^2\zeta_{n+1}\}$$

is linearly independent for some vectors $\zeta_1, \ldots, \zeta_{n+1}$ in X. Fix $\alpha_i, \beta_i \in \mathbb{C}$, $i = 1, \ldots, n+1$. Then, there exists $S \in \mathcal{A}$ such that

$$S\zeta_i = \alpha_i\zeta_i,\ ST\zeta_i = \beta_i\zeta_i, \text{ and } ST^2\zeta_i = 0.$$

We have

$$[S,[S,T]]\zeta_i = -\alpha_i\beta_i\zeta_i + \alpha_i^2 T\zeta_i$$

and

$$[S,[S,T]]T\zeta_i = -2\beta_i^2\zeta_i + \alpha_i\beta_i T\zeta_i.$$

Hence,

$$[S,[S,T]](\zeta_i + T\zeta_i) = (-\alpha_i\beta_i - 2\beta_i^2)\zeta_i + (\alpha_i^2 + \alpha_i\beta_i)T\zeta_i.$$

Choose α_i, β_i with $-\alpha_i\beta_i - 2\beta_i^2 = \alpha_i^2 + \alpha_i\beta_i$ and $\alpha_i^2 + \alpha_i\beta_i \ne \alpha_j^2 + \alpha_j\beta_j$ if $i \ne j$. Then we have $\#\sigma_p([S,[S,T]]) \ge n+1$. Thus, for each set of $n+1$ vectors $\zeta_1, \ldots, \zeta_{n+1}$, the set

$$\{\zeta_1, \ldots, \zeta_{n+1}, T\zeta_1, \ldots, T\zeta_{n+1}, T^2\zeta_1, \ldots, T^2\zeta_{n+1}\}$$

is linearly dependent. By Kaplansky's theorem, T is algebraic. Suppose now that, for some distinct elements λ_1, λ_2 in $\sigma_p(T)$, the spaces $\ker(T - \lambda_1)$ and $\ker(T - \lambda_2)$ are infinite dimensional. If $\xi_1, \ldots, \xi_r$, $\eta_1, \ldots, \eta_r$ are linearly independent vectors of X such that $T\xi_i = \lambda_1\xi_i$ and $T\eta_i = \lambda_2\eta_i$, then for each pair (α_i, β_i) of complex numbers, there is an element $S \in \mathcal{A}$ satisfying $S\xi_i = \alpha_i\eta_i$ and $S\eta_i = \beta_i\xi_i$. Hence,

$$[S,[S,T]]\xi_i = 2\alpha_i\beta_i(\lambda_1 - \lambda_2)\xi_i.$$

Therefore, by choosing suitable r, α_i, β_i, we get a contradiction. Consequently, there exists at most one $\lambda \in \sigma_p(T)$ with $\dim\ker(T - \lambda) = \infty$. Let V be the space satisfying $\ker(T-\lambda)^2 = \ker(T-\lambda) \oplus V$. If V is infinite dimensional, then we can take linearly independent vectors $\zeta_1, \ldots, \zeta_{n+1}$ in V. Since $V \cap \ker(T - \lambda) = \{0\}$, the set

$$\{\zeta_1, \ldots, \zeta_{n+1}, (T-\lambda)\zeta_1, \ldots, (T-\lambda)\zeta_{n+1}\}$$

is linearly independent. Since $(T-\lambda)^2\zeta_i = 0$, we can choose S as above replacing T by $T-\lambda I$, and in the same manner we obtain a contradiction. Consequently, $\ker(T-\lambda)$ has finite codimension in $\ker(T-\lambda)^2$. The remainder of the proof runs as in [4, Lemma 3.1] and thus we omit the details. $\square$

For a semisimple algebra A the socle $\operatorname{soc} A$ of A is defined as the sum of all minimal left ideals of A. If there are no minimal left ideals in A, then $\operatorname{soc} A = 0$ by definition. Suppose that $\mathcal{A}$ is a dense algebra of linear operators on X and that $T \in \mathcal{A}$ is a finite rank operator; then $T \in \operatorname{soc}\mathcal{A}$. This follows from [3, Theorem 3.3].

Proposition 1.5. *Let d be an inner derivation on $\mathcal{A}$. Suppose that there exists $n \in \mathbb{N}$ such that $\#\sigma_p([S, dS]) \leq n$ for each $S \in \mathcal{A}$. Then there is $T_0 \in \operatorname{soc}\mathcal{A}$ such that $dS = [S, T_0]$ for all $S \in \mathcal{A}$.*

Proof. Let the inner derivation d be implemented by the operator T; without loss of generality we assume that $d \neq 0$ and that X is infinite dimensional. By the above lemma, there exists a finite rank operator T_0 on X such that $T = T_0 + \lambda I$ for some $\lambda \in \mathbb{C}$.

Write $T_0 = \sum_{j=1}^{r} \xi_j \otimes \varphi_j$ for a linearly independent set $\{\xi_1, \dots, \xi_r\}$ of vectors in X and linear functionals $\varphi_1, \dots, \varphi_r$ on X. Choose vectors $\eta_1, \dots, \eta_r$ in X such that $\{\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_r\}$ is linearly independent. There are S, $R \in \mathcal{A}$ such that

$$S\xi_j = \eta_j,\ R\xi_j = 0, \quad \text{and} \quad R\eta_j = \xi_j \qquad (1 \leq j \leq r).$$

Then,

$$\begin{aligned} R\,[S, T_0]\zeta &= R\big(\varphi_1(\zeta)S\xi_1 + \cdots + \varphi_r(\zeta)S\xi_r \\ &\qquad\qquad - \varphi_1(S\zeta)\xi_1 - \cdots - \varphi_r(S\zeta)\xi_r\big) \\ &= \varphi_1(\zeta)R\eta_1 + \cdots + \varphi_r(\zeta)R\eta_r \\ &= T_0\zeta \end{aligned}$$

for all $\zeta \in X$. Hence $T_0 = R\,[S, T_0] = R\,[S, T] = R\,dS \in \mathcal{A}$ so that T_0 is a finite rank operator in $\mathcal{A}$ and therefore, by the above remark, belongs to $\operatorname{soc}\mathcal{A}$. $\square$

Remark. Using [4, Lemma 3.1], we can deduce a similar fact for inner derivations d with the property that there is $n \in \mathbb{N}$ with $\#\sigma_p(dS) \leq n$ for each $S \in \mathcal{A}$.

In the last result of this section, we will make use of the following "Jacobson density theorem with derivations" obtained in [5, Theorem 3.6].

THEOREM 1.6. *Let d be a derivation of $\mathcal{A}$. Then the following are equivalent:*

(a) *d is outer.*
(b) *For a linearly independent set $\{\xi_1, \dots, \xi_n\} \subseteq X$ and arbitrary sets $\{\eta_1, \dots, \eta_n\}$, $\{\zeta_1, \dots, \zeta_n\} \subseteq X$ there exists $S \in \mathcal{A}$ such that*

$$S\xi_i = \eta_i \quad \text{and} \quad (dS)\xi_i = \zeta_i \qquad (1 \le i \le n).$$

PROPOSITION 1.7. *Let $n \in \mathbb{N}$, and let d be a derivation of $\mathcal{A}$ such that $\#\sigma_p([S, dS]) \le n$ for all $S \in \mathcal{A}$. Then d is inner.*

Proof. If X is finite dimensional, it is well known that d is inner. Suppose now that d is outer. Then we can find linearly independent vectors $\xi_1, \dots, \xi_{n+1}$, $\xi_1', \dots, \xi_{n+1}'$ in X and, by Theorem 1.6, we can choose $S \in \mathcal{A}$ such that

$$S\xi_i = 0, \;\; S\xi_i' = i\xi_i, \;\; (dS)\xi_i = \xi_i' \qquad (1 \le i \le n+1).$$

Then $[S, dS]\,\xi_i = i\xi_i$ for all i, wherefore $\#\sigma_p([S, dS]) \ge n+1$. This contradiction completes the proof. $\square$

2. The general case

We start this section by drawing an immediate consequence of Proposition 1.3. Denoting by $\operatorname{rad} A$ the Jacobson radical of a Banach algebra A, we let $\mathcal{Z}(A)$ stand for the *centre modulo the radical*, i.e.,

$$\mathcal{Z}(A) = \{y \in A \mid [x, y] \in \operatorname{rad} A \text{ for all } x \in A\}.$$

THEOREM 2.1. *Let A be a Banach algebra, and let $a \in A$. Suppose that $\#\sigma([x, [x, a]]) = 1$ for all $x \in A$. Then $a \in \mathcal{Z}(A)$.*

Proof. Let P be a primitive ideal of A and denote by A_P the associated dense algebra of bounded operators on a Banach space X_P. Let x_P be the image of $x \in A$ in A_P. By hypothesis, the spectrum of each double commutator $[x_P, [x_P, a_P]]$ is a singleton for every $x \in A$. By Proposition 1.3, a_P is a scalar multiple of the identity on X_P, thus $[x_P, a_P] = 0$ for all $x \in A$. As a result, $[x, a]$ is contained in every primitive ideal of A and so $a \in \mathcal{Z}(A)$. $\square$

An analogous result under the assumption $\#\sigma([x, a]) = 1$ for all $x \in A$ can be found in [1, Theorem 5.2.1].

The next lemma is essentially a consequence of Baire's category theorem and its proof runs exactly as in [4, Lemma 2.1].

LEMMA 2.2. *Let T be a continuous linear mapping on a Banach algebra A satisfying $\#\sigma([x, Tx]) < \infty$ for all $x \in A$. Then there exists a positive integer n such that $\#\sigma([x, Tx]) \le n$ for all $x \in A$.*

We need one more preparation for the main result of this article.

LEMMA 2.3. *Let d be a continuous derivation on the Banach algebra A. Suppose that there exists $n \in \mathbb{N}$ such that $\#\sigma([x, dx]) \leq n$ for all $x \in A$. Then there exist at most a finite number of primitive ideals P_i such that $dA \nsubseteq P_i$.*

Proof. First recall that, if P is a primitive ideal of A, then $dP \subseteq P$ [6, Proposition 2.7.22]. Hence d induces a continuous derivation d_P on the Banach algebra A/P and using Propositions 1.7 and 1.5 we deduce that d_P is inner with implementing element contained in $\operatorname{soc} A/P$. If P has codimension one, then clearly $d_P = 0$ and so $dA \subseteq P$. Suppose now that there exist distinct primitive ideals $P_1, \dots, P_m,\ P'_1, \dots, P'_r,\ m, r \in \mathbb{N}$, of A such that $dA \nsubseteq P_i,\ dA \nsubseteq P'_j,\ \dim A/P_i \geq 3$ and $\dim A/P'_j = 2$, where $i = 1, \dots, m,\ j = 1, \dots, r$. Let $\pi_1, \dots, \pi_m,\ \pi'_1, \dots, \pi'_r$ be irreducible representations of A on Banach spaces $X_1, \dots, X_m,\ X'_1, \dots, X'_r$, respectively, such that $\ker \pi_i = P_i$ and $\ker \pi'_j = P'_j$ for all i and j.

Let $a_1, \dots, a_m$ be in A such that, for each $x \in A$,

$$d_{P_i}(\pi_i(x)) = \pi_i(xa_i - a_i x).$$

For $1 \leq i \leq m$, choose $\zeta_i \in X_i$ such that ζ_i and $\pi_i(a_i)\zeta_i$ are linearly independent. Since $\dim\ A/P_i \geq 3$, there exists $\zeta'_i \in X_i$ such that the vectors $\zeta_i, \pi_i(a_i)\zeta_i, \zeta'_i$ are linearly independent. Applying the extended Jacobson density theorem [7, p. 283] we get the existence of an element $x \in A$ with the property that

$$\pi_i(x)\zeta_i = 0, \quad \pi_i(xa_i)\zeta_i = \zeta'_i, \quad \pi_i(x)\zeta'_i = i\zeta_i \qquad (1 \leq i \leq m).$$

Consequently, $\pi_i([x, dx])\zeta_i = i\zeta_i, i = 1, \dots, m$, implying that $\{1, \dots, m\} \subseteq \sigma([x, dx])$. Therefore, $m \leq n$.

For each $1 \leq j \leq r$, we have $\bigcap_{k \neq j} P'_k \nsubseteq P'_j$ (since each P'_k has codimension two and P'_j is prime), hence $\pi'_j\big(\bigcap_{k \neq j} P'_k\big)$ is a non-zero ideal of $\pi'_j(A)$ and therefore a dense algebra of operators on X'_j. Proposition 1.3 entails that there exists $x_j \in \bigcap_{k \neq j} P'_k$ such that $\#\sigma([\pi'_j(x_j), d_{P'_j}\pi'_j(x_j)]) = 2$. Replacing x_j by λx_j, for $\lambda \in \mathbb{C}$ suitable, we can assume that there exist $\xi_j \in X'_j$ such that $[\pi'_j(x_j), d_{P'_j}\pi'_j(x_j)]\xi_j = j\xi_j$. Take $x = x_1 + \cdots + x_r$. We have $\pi'_j([x, dx])\xi_j = j\xi_j$, hence $\{1, \dots, r\} \subseteq \sigma([x, dx])$ and $r \leq n$. This completes the proof. □

Our main result characterises the continuous derivations satisfying the finiteness of the spectrum condition as those which, modulo the radical, can be implemented by an element in the socle modulo the radical.

Theorem 2.4. *Let d be a continuous derivation on a Banach algebra A. Then the following conditions are equivalent:*

(a) $\#\sigma(dx) < \infty$ *for all* $x \in A$.
(b) $\#\sigma([x, dx]) < \infty$ *for all* $x \in A$.
(c) *There exists* $a \in A$ *such that* $a + \operatorname{rad} A \in \operatorname{soc}\bigl(A/\operatorname{rad} A\bigr)$ *and* $dx - [x, a] \in \operatorname{rad} A$ *for all* $x \in A$.

Proof. For the proof of the implications (c)⇒(a) and (c)⇒(b) observe that $\sigma(x) = \sigma(x + \operatorname{rad} A)$ for all $x \in A$, d leaves the radical invariant, and that every element of the socle has finite spectrum.

Suppose that (b) is true and that the set Γ of all primitive ideals P of A such that $dA \nsubseteq P$ is non-empty; otherwise, there is nothing to prove. By Lemmas 2.2 and 2.3, Γ is finite, say $\Gamma = \{P_1, \ldots, P_r\}$. Without loss of generality, we may assume that $P_j \nsubseteq P_i$ for $j \neq i$, $1 \leq i, j \leq r$. Since $dA \subseteq \bigcap_{P \notin \Gamma} P$ and each P_i is a prime ideal, it follows that $\bigcap_{P \neq P_i} P \nsubseteq P_i$ and thus, for each i, $\bigcap_{P \neq P_i} P + P_i$ is a non-zero ideal of A/P_i and therefore a dense algebra of bounded operators on some Banach space X_i. Let d_i be the induced derivation on A/P_i. Of course, $\#\sigma([x + P_i, d_i(x + P_i)]) < n$ for some $n \in \mathbb{N}$ and all $x \in A$. According to Propositions 1.7 and 1.5, there exists $a_i \in \bigcap_{P \neq P_i} P$ such that $a_i + P_i \in \operatorname{soc}(A/P_i)$ and $dx + P_i = [x, a_i] + P_i$ for every $x \in A$. Put $a = a_1 + \cdots + a_r$. Clearly, $a + P_i = a_i + P_i$ for $i = 1, \ldots, r$ and $a \in P$ for $P \in \operatorname{Prim}(A) \setminus \Gamma$. By [4, Proposition 2.2], $a + \operatorname{rad} A \in \operatorname{soc}(A/\operatorname{rad} A)$. Moreover, for every $P \in \operatorname{Prim}(A)$ and for all $x \in A$, $([x, a] - dx) + P = 0$. Thus, we have $dx - [x, a] \in \operatorname{rad} A$ for all $x \in A$ and (c) is proved.

It remains to prove that (a) implies (c). To this end, we repeat the same arguments, using [4, Lemmas 2.1 and 3.1]. □

Corollary 2.5. *Every derivation d on a semisimple Banach algebra A such that $\#\sigma([x, dx]) < \infty$ for all $x \in A$ maps into the socle of A.*

Proof. It is well known that every derivation on a semisimple Banach algebra is continuous [6, Theorem 5.2.28]. Hence the result follows directly from Theorem 2.4. □

Corollary 2.6. *Every bounded derivation d on a Banach algebra A such that $\#\sigma([x, dx]) = 1$ for all $x \in A$ maps into the radical of A.*

Proof. Since d is bounded, it leaves $\operatorname{rad} A$ invariant so that we may assume that A is semisimple. By Theorem 2.4, there is $a \in \operatorname{soc} A$ such that $dx = [x, a]$ for all $x \in A$. Applying Theorem 2.1 we conclude that a belongs to the centre of A, wherefore $d = 0$. □

Theorem 2.7. *Let d be a non-zero derivation on a semisimple Banach algebra A. Suppose that $\#\sigma([x, dx]) \leq 2$ for all $x \in A$. Then A contains a finite-dimensional ideal I such that $dA \subseteq I$.*

Proof. Assume that A is not finite-dimensional. By Theorem 2.4, and since d is automatically bounded, there exists $a \in \operatorname{soc} A$ such that $dx = [x, a]$ for all $x \in A$. Let $\{P_1, \ldots, P_r\}$ be the set of all primitive ideals in A such that $dA \nsubseteq P_i$. Using Lemma 1.2, we deduce that $\dim A/P_i < \infty$. Since A is not finite-dimensional, $\bigcap_{i=1}^{r} P_i \neq 0$. Hence, there exist primitive ideals P such that $P \neq P_i$ for all i. Let us consider

$$\varphi\colon \bigcap_{P \neq P_i} P \longrightarrow A/P_1 \oplus \cdots \oplus A/P_r,$$
$$a \longmapsto (a + P_1, \ldots, a + P_r).$$

Clearly, φ is injective. Hence, $I = \bigcap_{P \neq P_i} P$ is a finite-dimensional ideal of A and $dA \subseteq I$. □

Corollary 2.8. *Let A be a prime semisimple infinite dimensional Banach algebra. Let d be a non-zero derivation on A. Then there exists $x \in A$ such that $\#\sigma([x, dx]) \geq 3$.*

Proof. Whenever I is a finite-dimensional ideal in a semisimple Banach algebra A, the identity e of I is a central idempotent in A such that $A = I \oplus (1 - e)A$. Therefore the statement follows from Theorem 2.7. □

It follows that, if d is a bounded derivation on a Banach algebra A such that $A/\operatorname{rad} A$ is prime and infinite dimensional, the hypothesis $\#\sigma([x, dx]) \leq 2$ for all $x \in A$ implies that $dA \subseteq \operatorname{rad} A$.

3. On generalised inner derivations

A linear mapping $d\colon A \to A$ is called a *generalised inner derivation* if $dx = ax + xb$ for some $a, b \in A$. In this section we shall investigate to what extent the results obtained above for commutators extend to this more general setting.

Proposition 3.1. *Let X be a complex vector space, and let $\mathcal{A}$ be an algebra of linear operators acting densely on X such that its centraliser in $L(X)$ is one-dimensional. Suppose that there are linear operators T, R on X such that $\#\sigma(TS + SR) = 1$ for all $S \in \mathcal{A}$. If $\dim X \geq 3$, then $T \in \mathbb{C}I$ and $R = -T$.*

Proof. Assume that the vectors $\zeta, R\zeta, R^2\zeta$ are linearly independent for some vector $\zeta \in X$. Choose $S \in \mathcal{A}$ such that

$$S\zeta = 0, \quad SR\zeta = 0, \quad SR^2\zeta = R\zeta.$$

Then, $(TS + SR)\zeta = 0$ and $(TS + SR)R\zeta = R\zeta$. This contradicts the assumption on the spectrum of $TS + SR$, hence $R^2 = \lambda_0 I + \lambda_1 R$ for some scalars λ_0, λ_1 by Theorem 1.1. Now assume that $\{\zeta, R\zeta, T\zeta, RT\zeta\}$ is linearly independent for some vector ζ. Then we can choose $S \in \mathcal{A}$ such that

$$S\zeta = 0,\ ST\zeta = 0,\ SR\zeta = 0 \text{ and } SRT\zeta = T\zeta.$$

It follows that $(TS + SR)\zeta = 0$ and $(TS + SR)T\zeta = T\zeta$, a contradiction. Next, suppose that the vectors $\zeta, R\zeta, T\zeta$ are linearly independent for some $\zeta \in X$. Write $RT\zeta = \alpha_0\zeta + \alpha_1 R\zeta + \alpha_2 T\zeta$ for some scalars $\alpha_0, \alpha_1, \alpha_2$. Take S in $\mathcal{A}$ such that $S\zeta = 0, ST\zeta = 2\zeta$ and $SR\zeta = \zeta$. Then the corresponding matrix representation of $TS + SR$ with respect to $\{\zeta, T\zeta\}$ is

$$\begin{pmatrix} 1 & \alpha_1 + 2\alpha_2 \\ 0 & 2 \end{pmatrix}.$$

Hence $\{1, 2\} \subseteq \sigma(TS+SR)$, a contradiction. Suppose that $\{\zeta, T\zeta\}$ is linearly independent for some $\zeta \in X$. By our assumption, $\dim X \geq 3$. Therefore, we can choose η in X such that the set $\{\zeta, R\zeta, \eta\}$ is linearly independent. If the set $\{\zeta, R\zeta, \eta, R\eta\}$ is linearly independent, we obtain a contradiction by considering S so that

$$S\zeta = 0,\ S\eta = 0,\ SR\zeta = \zeta \text{ and } SR\eta = 2\eta,$$

since, in this case, $(TS + SR)\zeta = \zeta$ and $(TS + SR)\eta = 2\eta$. Thus $R\eta = h_1\zeta + h_2 R\zeta + h_3\eta$ for some scalars $h_1, h_2, h_3 \in \mathbb{C}$. Choose $S \in \mathcal{A}$ such that

$$S\eta = 0,\ \ S\zeta = 0,\ \ SR\zeta = \zeta.$$

Then, $(TS + SR)\eta = h_2\zeta$ and $(TS + SR)\zeta = \zeta$. It follows that $(TS + SR)(-h_2\zeta + \eta) = 0$. Again, this yields a contradiction. Therefore, for each $\zeta \in X$, the vectors $\zeta, R\zeta$ are linearly dependent and so $R \in \mathbb{C}I$. Set $R = \lambda I$ for some scalar λ. Then, $TS+SR = (T+\lambda I)S$ for all $S \in \mathcal{A}$. Finally, assume that for some $\zeta \in X$, the vectors ζ, $(T+\lambda I)\zeta$ are linearly independent. Choose S so that $S\zeta = 0$ and $S(T + \lambda I)\zeta = \zeta$. Then

$$(T + \lambda I)S\zeta = 0,\ \ (T + \lambda I)S(T + \lambda I)\zeta = (T + \lambda I)\zeta,$$

which is not possible. As a result, $T + \lambda I = \lambda' I$ for some scalar $\lambda' \in \mathbb{C}$. Since $\lambda' S = (T + \lambda I)S$ has singleton spectrum for all S, it follows that $\lambda' = 0$ and so $T = -R \in \mathbb{C}I$. $\square$

The following example shows that the assumption on the dimension of X cannot be relaxed in general.

Example. In $M_2(\mathbb{C})$, the algebra of all complex 2×2 matrices, let us consider $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = R$. For each $S = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in M_2(\mathbb{C})$, $TS + SR = \left(\begin{smallmatrix} z & w+x \\ 0 & z \end{smallmatrix}\right)$. Hence, $\#\sigma(TS + SR) = 1$.

THEOREM 3.2. *Let A be a semisimple Banach algebra, and let a, b be in A. Suppose that $\#\sigma(ax + xb) = 1$ for all $x \in A$. Then there exists an at most four-dimensional ideal I of A such that $ax + xb \in I$ for each $x \in A$. Moreover, there exists $u \in Z(A)$ such that the elements $a-u$ and $b+u$ belong to I.*

Proof. Assume towards a contradiction that there exist primitive ideals P_1, P_2 of A such that $b + P_i \notin \mathbb{C}$. Let π_1, π_2 be irreducible representations of A on Banach spaces X_1, X_2, respectively, such that $\ker \pi_i = P_i$, $i = 1, 2$. Then we can choose $\zeta_i \in X_i$ such that the vectors $\zeta_i, \pi_i(b)\zeta_i$ are linearly independent for $i = 1, 2$. By the extended Jacobson density theorem, there exists $x \in A$ such that

$$\pi_i(x)\zeta_i = 0, \quad \pi_i(x)\pi_i(b)\zeta_i = 2^i\zeta_i \qquad (i = 1, 2).$$

Hence, $\pi_i(ax+xb)\zeta_i = 2^i\zeta_i$ and $\{2, 4\} \subseteq \sigma(ax+xb)$, a contradiction. Therefore there is at most one primitive ideal P' such that $b+P' \notin \mathbb{C}$, and by the above proposition, A/P' is four-dimensional (and thus a simple algebra). Let Γ be the set of primitive ideals distinct from P'. For each $P \in \Gamma$, $b + P \in \mathbb{C}$. Hence, as in the proof of the above proposition, $a + P = -b + P$ and $ax + xb \in \bigcap_{P\in\Gamma} P$ for all $x \in A$. If P' does not exist, then $b = -a$ and $a \in Z(A)$, since $\bigcap_{P\in\Gamma} P = 0$. If P' exists, then $\bigcap_{P\in\Gamma} P \oplus P' = A$ as A/P' is simple and $\bigcap_{P\in\Gamma} P \nsubseteq P'$. So take $a = a_1 + u$, $b = b_1 + v$, where a_1, $b_1 \in \bigcap_{P\in\Gamma} P$ and $u, v \in P'$. If $P \in \Gamma$, then $a + P = u + P = -v + P \in \mathbb{C}$. Hence, $u + v \in \bigcap_{P\in\Gamma} P \cap P' = 0$ and so $v = -u$ and $ax + xb \in \bigcap_{P\in\Gamma} P$ yields that $ux - xu \in \bigcap_{P\in\Gamma} P \cap P' = 0$ for all x. Therefore $u \in Z(A)$ and $ax+xb \in \bigcap_{P\in\Gamma} P$. Set $I = \bigcap_{P\in\Gamma} P$. Then $\dim I \le 4$ since $I = 0$ or $I \simeq A/P'$. This completes the proof. □

Our final result is deduced from Theorem 3.2 in the same way as Corollary 2.8 from Theorem 2.7.

COROLLARY 3.3. *Let A be a semisimple Banach algebra, and let d be a non-zero generalised inner derivation on A. Suppose that $\#\sigma(dx) = 1$ for all $x \in A$. Then A contains a central idempotent e such that $dA \subseteq eA$ and eA has dimension 4.*

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References

[1] B. Aupetit, *A primer on spectral theory*, Universitext, Springer-Verlag, New York, 1991. MR **92c:**46001

[2] M. Brešar, *Derivations mapping into the socle. II*, Proc. Amer. Math. Soc. **126** (1998), 181–187. MR **98c:**46088

[3] M. Brešar and D. Eremita, *The lower socle and finite rank elementary operators*, Comm. Algebra **31** (2003), 1485–1497. MR **2004b:**16020

[4] M. Brešar and P. Šemrl, *Derivations mapping into the socle*, Math. Proc. Cambridge Philos. Soc. **120** (1996), 339–346. MR **96m:**46085

[5] ______, *On locally linearly dependent operators and derivations*, Trans. Amer. Math. Soc. **351** (1999), 1257–1275. MR **99e:**47039

[6] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs. New Series, vol. 24, The Clarendon Press Oxford University Press, New York, 2000. MR **2002e:**46001

[7] J. M. G. Fell and R. S. Doran, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. 1*, Pure and Applied Mathematics, vol. 125, Academic Press Inc., Boston, MA, 1988. MR **90c:**46001

[8] M. Mathieu and V. Runde, *Derivations mapping into the radical. II*, Bull. London Math. Soc. **24** (1992), 485–487. MR **93f:**46073

[9] M. P. Thomas, *The image of a derivation is contained in the radical*, Ann. of Math. (2) **128** (1988), 435–460. MR **90d:**46075

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