

## SYMPLECTIC SURFACES AND GENERIC $j$ -HOLOMORPHIC STRUCTURES ON 4-MANIFOLDS

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ABSTRACT. It is a well known fact that every embedded symplectic surface  $\Sigma$  in a symplectic four-manifold  $(X^4, \omega)$  can be made  $J$ -holomorphic for some almost-complex structure  $J$  compatible with  $\omega$ . In this paper we investigate when such a structure  $J$  can be chosen generically in the sense of Taubes. The main result is stated in Theorem 1.2. As an application of this result we give examples of smooth and non-empty Seiberg-Witten and Gromov-Witten moduli spaces whose associated invariants are zero.

### 1. Introduction

To set up the background for the main theorem below, let  $C \subset X$  be a connected, symplectic surface embedded in the minimal symplectic 4-manifold  $X$  with symplectic form  $\omega$ . It is a well known fact that  $C$  can be made  $J$ -holomorphic for some almost-complex structure  $J$  compatible with  $\omega$ . This paper investigates when  $J$  can be chosen from a generic set of almost-complex structures. We start by recalling what *generic* means in our setting.

For a given  $E \in H_2(X; \mathbb{Z})$ , set

$$(1) \quad d = \frac{1}{2}(E^2 - K \cdot E),$$

where  $K$  is the canonical class of  $X$  associated to  $\omega$ . Define  $\mathcal{A}_d(X)$  as the set of pairs  $(J, \Omega)$  with  $J$  an almost-complex structure compatible with  $\omega$  and  $\Omega$  a set of  $d$  distinct points of  $X$ .  $\mathcal{A}_d(X)$  has the structure of a smooth manifold inherited from the Frechet manifold  $C^\infty(\text{End}(TX) \times \text{Sym}^d(X))$ .

Each  $J$ -holomorphic curve  $C$  comes equipped with a linear operator

$$D_C : C^\infty(N_C) \rightarrow C^\infty(N_C \otimes T^{0,1}C)$$

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obtained from the linearization of the generalized Cauchy-Riemann operator  $\overline{\partial}_C$ . Here  $N_C$  is the normal bundle of  $C$  in  $X$ . The operator  $D_C$  is elliptic and its (complex) index is given by  $d$  as defined in (1) with  $E = [C]$ . In the case when  $C$  contains all points of  $\Omega$ , let  $\text{ev}_\Omega : C^\infty(N_C) \rightarrow \oplus_{p \in \Omega} N_p$  be the evaluation map associated to  $\Omega$ . If  $d = 0$ , we say that  $D_C$  is non-degenerate if  $\text{Coker}(D_C) = \{0\}$ . In the case  $d > 0$ ,  $D_C$  is called non-degenerate if

$$(2) \quad D_C \oplus \text{ev}_\Omega : C^\infty(N_C) \rightarrow C^\infty(N_C \otimes T^{0,1}C) \oplus_{p \in \Omega} N_p$$

has trivial cokernel.

DEFINITION 1.1. A pair  $(J, \Omega) \in \mathcal{A}_m(X)$ ,  $m \geq 0$ , is said to be generic if the following five conditions are met for all  $E \in H_2(X; \mathbb{Z})$  for which the number  $d$  as defined by (1) is not greater than  $m$  (see [11] for more details, especially on the definition of  $n$ -non-degeneracy which is immaterial for the present discussion and thus omitted):

- (1) For a fixed class  $E \in H_2(X; \mathbb{Z})$ , there are only finitely many embedded  $J$ -holomorphic curves representing  $E$  and containing  $d$  points of  $\Omega$ .
- (2) For each  $J$ -holomorphic curve  $C$ , the operator  $D_C$  is non-degenerate.
- (3) There are no connected  $J$ -holomorphic curves representing the class  $E \in H_2(X; \mathbb{Z})$  and containing more than  $d$  points of  $\Omega$ .
- (4) There is an open neighborhood of  $(J, \Omega)$  in  $\mathcal{A}_m(X)$  such that each pair  $(J', \Omega')$  from that neighborhood satisfies conditions (1)–(3) above. Furthermore, the number of  $J'$ -holomorphic curves containing  $d$  points of  $\Omega'$  is constant as  $(J', \Omega')$  varies through this neighborhood.
- (5) If  $E^2 = K \cdot E = 0$  then each of the finitely many  $J$ -holomorphic curves in  $E$  containing  $d$  points of  $\Omega$  is  $n$ -non-degenerate for each positive integer  $n$ .

The set of generic pairs  $(J, \Omega)$ , which we denote by  $\mathcal{J}_d^{\text{reg}}(X)$  (or simply by  $\mathcal{J}^{\text{reg}}(X)$  when no confusion is possible), is a Baire subset of  $\mathcal{A}_d(X)$ .

We are now ready to state our main result:

THEOREM 1.2. *Let  $(X, \omega)$  be a minimal symplectic 4-manifold and  $C$  a connected, embedded symplectic surface in  $X$  of genus  $g \geq 1$  and with  $C^2 \geq g - 1$ . Then for any  $\delta > 0$  there exists a generic pair  $(J_\delta, \Omega_\delta) \in \mathcal{J}^{\text{reg}}(X)$  and a connected  $J_\delta$ -holomorphic curve  $C_\delta$  inside the radius  $\delta$  tubular neighborhood of  $C$ , isotopic to  $C$ . Furthermore,  $C_\delta$  contains all  $d$  points of  $\Omega_\delta$ .*

COROLLARY 1.3. *The above theorem remains true if  $C = \sqcup C_i$  is a disjoint union of connected symplectic manifolds provided the condition  $C_i^2 \geq g_i - 1$  holds for each component  $C_i$ . That is, one can find a curve  $C_\delta = \sqcup C_{\delta,i}$ , where each  $C_{\delta,i}$  is an isotopic translate of  $C_i$  inside a radius  $\delta$  tubular neighborhood of  $C_i$ .*

REMARK 1.4. Whenever  $(J, \Omega)$  is a generic pair in the sense of Definition 1.1, the Gromov-Witten moduli space  $\mathcal{M}_X^{Gr}(E)$  is a smooth manifold of (real) dimension  $2d \geq 0$ . This together with the adjunction formula for a connected  $J$ -holomorphic curve  $C \in \mathcal{M}_X^{Gr}(E)$  implies that  $E^2 \geq g - 1$  (where  $g$  is the genus of  $C$ ). Conversely, given a connected symplectic curve  $C$  of genus  $g$  satisfying  $C^2 \geq g - 1$ , Theorem 1.2 shows that there are no other obstructions for the existence of a generic pair  $(J, \Omega)$  that makes  $C$  into a  $J$ -holomorphic curve.

REMARK 1.5. Suppose that  $(J, \Omega)$  is a generic pair and let  $C$  be a connected  $J$ -holomorphic curve of genus  $g$  and with  $[C] = E$ . The inequality  $E^2 \geq g - 1$  from the previous remark shows that  $J$ -holomorphic curves with negative square can only occur when  $E^2 = -1$  and  $g = 0$ . This case however is excluded if  $X$  is a minimal manifold (as is assumed in Theorem 1.2).

It is interesting to compare the result of Theorem 1.2 to the result proved in [3]. Expressed in our notation, among other results, it is shown in [3] that for  $C^2 \geq 2g - 1$  the operator  $D_C$  is surjective for *any* choice of an almost-complex structure  $J$  compatible with the symplectic form  $\omega$ . The improvement of the inequality in Theorem 1.2 comes at the twofold expense of first not being able to choose the almost-complex structure arbitrarily but rather from a dense (second-category) subset of almost-complex structures. Secondly, one may have to slightly “wiggle”  $C$  to get the desired curve. We also remark that the case of genus 0, which is excluded from Theorem 1.2, is completely covered by the results of [3].

The proof of Theorem 1.2 rests on the observation that the property of a  $J$ -holomorphic curve  $C$  to be generic with respect to a pair  $(J, \Omega) \in \mathcal{A}_d(X)$  is local in nature, that is, it only depends on the restriction of  $(J, \Omega)$  to a tubular neighborhood  $N(C)$  of the curve  $C$ . By the symplectic neighborhood theorem for four-manifolds (cf. [9]),  $N(C)$  is up to symplectomorphism determined by its volume and the square  $C^2$  of the curve  $C$ . Thus one is led to search for universal models of symplectic four-manifolds  $Y_{g,n}$  with a Gromov-Witten basic class  $E_{g,n} \in H_2(Y_{g,n}; \mathbb{Z})$  with  $E_{g,n} \cdot E_{g,n} = n$  and for which a connected genus  $g$   $J$ -holomorphic representative exists for all generic  $(J, \Omega)$ . These manifolds together with their Gromov-Witten invariants are discussed in Section 3.2 after a brief survey of the Seiberg-Witten theory of four-manifolds with  $b^+ = 1$ , which is given in Section 3.1. No originality is claimed on any of the facts stated in Section 3; they serve merely as a reminder and to set notation. The proof of Theorem 1.2 is then completed in Section 4. Section 2 gives applications of the main theorem.

## 2. Applications

As an application of Theorem 1.2, we give examples of symplectic manifolds with non-empty Seiberg-Witten and Gromov-Witten moduli spaces under generic conditions, whose associated invariants are zero. Such examples

can be found for the case where the dimension of the moduli space is zero as well as for the case of positive dimension.

EXAMPLE 1. Consider the elliptic surface  $E(n)$ . It has a symplectic section  $S_n$  with genus zero and square  $-n$ . Let  $F_i, i = 1, 2, \dots$ , be regular fibers of the elliptic fibration. Then the symplectic surface  $C_{n,m}$ , obtained by smoothing the surface  $S_n \cup F_1 \cup \dots \cup F_m$ , is a genus  $g_{n,m} = m$  surface of square  $2m - n$ . Choosing  $m \geq n - 1$  ensures the condition  $C_{n,m}^2 \geq g_{n,m} - 1$ . Theorem 1.2 provides a generic pair  $(J, \Omega) \in \mathcal{J}^{\text{reg}}(E(n))$  and a  $J$ -holomorphic curve  $C'_{n,m}$  in the class  $[C_{n,m}]$ . In particular, the moduli space  $\mathcal{M}_{E(n)}^{Gr}([C_{n,m}])$  for this generic pair  $(J, \Omega)$  is nonempty, while  $Gr_{E(n)}([C_{n,m}]) = 0$ . The dimension of the moduli space is

$$\dim_{\mathbb{R}} \mathcal{M}_{E(n)}^{Gr}([C_{n,m}]) = 2(m - n + 1).$$

EXAMPLE 2. Let  $\Sigma$  be a genus 2 Riemann surface and let  $X = \Sigma \times T^2$ . Choose the symplectic form  $\omega$  on  $X$  to be the sum of volume forms  $\omega_{\Sigma}$  and  $\omega_{T^2}$  on  $\Sigma$  and  $T^2$  for which  $\text{Vol}(\Sigma) = 1 = \text{Vol}(T^2)$ . Let  $C$  be the symplectic surface obtained by smoothing  $\Sigma \cup T^2$ . Then the genus of  $C$  is 3 and its square is 2; in particular,  $\dim \mathcal{M}_X^{Gr}([C]) = 0$  and  $\dim \mathcal{M}_X^{SW}(L) = 0$  for  $L = 2P \cdot D \cdot ([C]) - K$ .

Pick an almost-complex structure  $J \in \mathcal{J}^{\text{reg}}(X)$  ( $\Omega$  is just the empty set here and we suppress it from the notation) and a  $J$ -holomorphic curve  $C'$  in the class  $[C]$ . It is not hard, but somewhat tedious, to show that all  $J$ -holomorphic curves in  $[C]$  are connected curves of genus 3. To see this, consider the two possible alternatives:

- (1) There is a representative  $D'$  of  $[C]$  of the form  $D' = D'_1 \sqcup \dots \sqcup D'_n$  with  $D'_i \cdot D'_i = 1$  for  $i = 1, 2$  and  $D'_i \cdot D'_i = 0$  for  $i \geq 3$ . This is an immediate contradiction since classes of square 1 cannot exist on a manifold with even canonical class.
- (2) There is a representative  $D$  of  $[C]$  of the form  $D = D_1 \sqcup \dots \sqcup D_n$  with  $D_1^2 = 2$  and  $D_i^2 = 0$  for  $i \geq 2$ . This implies that  $g(D_i) = 0$  for  $i \geq 2$  and  $2 \leq g(D_1) \leq 3$ . The latter claim follows readily from the fact that the dimension  $\dim \mathcal{M}_X^{Gr}([D_1]) = 2(D_1^2 - g(D_1) + 1)$  is non-negative and from the adjunction formula for  $D$ . The case  $g(D_1) = 2$  leads (via the adjunction formula applied to  $[C]$ ) to  $[C] \cdot K = 0$ , which is a contradiction. Thus the only possibility is  $g(D_1) = 3$ , implying  $K \cdot D_1 = 2$ .

Since  $\omega \in H^2(X; \mathbb{Z})$  and  $\omega([C]) = 2$ , we see immediately that  $n \leq 2$ . Suppose thus that  $D = D_1 \sqcup D_2$ . Then from  $K \cdot D_1 = 2$  we see that  $[D_1] = [\Sigma] + a[T^2] + F$ , where  $F \in H_2(X; \mathbb{Z})$  is generated by classes obtained from cross-products of 1-cycles on  $\Sigma$  with 1-cycles on  $T^2$ . This forces  $[D_2] = (1 - a)[T^2] - F$ . Notice that  $F \cdot \Sigma = F \cdot T^2 = \omega \cdot F = 0$ . From  $D_1^2 = 2$  we infer that  $2a + F^2 = 2$  and from  $D_2^2 = 0$  we get  $F^2 = 0$ . Thus  $a = 1$  and so  $[D_1] = [\Sigma] + [T^2] + F$  and  $[D_2] = -F$ .

This leads to a contradiction, since now  $\omega(D_2) = 0$  and so  $D_2$  cannot be a  $J$ -holomorphic curve.

Each point in  $\mathcal{M}_X^{Gr}([C])$  gives rise to a Seiberg-Witten monopole in  $\mathcal{M}_X^{SW}(L)$  with  $L = 2P \cdot D \cdot ([\Sigma])$  (see [12]). It was shown in [4] that each such monopole is a smooth point in the moduli space for large enough values of  $r$  in the Taubes perturbation form  $\mu_0 = F_0^+ - ir\omega/8$ . In other words, the pair of metric and perturbation forms  $(g, \mu_0)$  (with  $g$  being the metric induced by  $\omega$  and  $J$ ) is a generic pair for the Seiberg-Witten theory for the  $\text{Spin}^c$ -structure  $L$  and as such gives rise to a smooth moduli space. On the other hand,  $SW_X(L) = 0$ , as can be seen in a number of ways. (For example, introduce the “twisted” symplectic form  $\omega' = 1 \cdot 1\omega_\Sigma + \omega_{T^2}$ . Then  $L \cdot \omega' > K \cdot \omega'$ , which according to [10] implies that  $L$  cannot be a basic class.)

### 3. Preliminaries

**3.1. Seiberg-Witten theory on manifolds with  $b^+ = 1$ .** Let  $X$  be a 4-manifold with  $b^+ = 1$ . For a given  $\text{Spin}^c$ -structure  $W = W^+ \oplus W^-$ , with determinant  $L = \det(W^+) \in H^2(X; \mathbb{Z})$ , the Seiberg-Witten invariant depends on a choice of a chamber inside the space  $\text{Met} \times i\Omega^{2,+}$ . Here  $\text{Met}$  is the space of Riemannian metrics on  $X$ . The two chambers are divided by a (real) codimension 1 wall of pairs  $(g, \mu)$ , defined by the equation

$$\frac{i\mu}{2\pi} \wedge \omega_g - L \wedge \omega_g = 0,$$

where  $\omega_g$  is a generator of the positive forward cone in  $H^2(X; \mathbb{Z})$ . In the case where  $X$  is symplectic, we agree to always choose  $\omega_g$  to be the symplectic form.

The Seiberg-Witten equations do not admit reducible solutions if  $(g, \mu)$  does not lie on the wall. We denote the two chambers by  $\mathcal{C}^-(L)$  and  $\mathcal{C}^+(L)$  according to the sign of the expression

$$\left\langle \frac{i\mu}{2\pi} \wedge \omega_g - L \wedge \omega_g, [X] \right\rangle.$$

We will denote the Seiberg-Witten invariant by  $SW_X^\pm(L)$  according to the choice of chamber  $\mathcal{C}^\pm(L)$  from which the pair  $(g, \mu)$  used in calculating the invariant was taken. The number  $SW_X^+(L) - SW_X^-(L)$  is called the *wall crossing number* and it is well understood (see, for example, [6]). The special case relevant to the present situation is stated in the following theorem (Corollary 1.4 in [6]):

**THEOREM 3.1.** *Let  $X$  be an  $S^2$ -bundle over a Riemann surface  $\Sigma$  of genus  $g$ . Let  $E \in H^2(X; \mathbb{Z})$  with  $(2E + c_1(X))^2 \geq 2e_X + 3\sigma_X$ . Then the wall crossing number is*

$$SW_X^+(L) - SW_X^-(L) = \pm \left( \frac{2E + c_1(X)}{2} [S^2] \right)^g,$$

where  $[S^2]$  is the fiber class.

**3.2. The Gromov-Witten invariants of  $Y_0$  and  $Y_1$ .** This section describes the spaces  $Y_{g,n}$  mentioned in the Introduction as well as their Gromov-Witten basic classes  $E_{g,n}$ . As it turns out, it suffices to consider only two symplectic manifolds  $Y_0$  and  $Y_1$  by letting  $Y_{g,2n} = Y_0$  and  $Y_{g,2n-1} = Y_1$ . The main results of this section, Corollaries 3.3 and 3.4, are well known and their proofs can be found in the literature (see, e.g., [2]). They are included here for the continuity of the argument and for the benefit of the reader, but no originality is claimed.

Let  $\Sigma$  be any Riemann surface of genus  $g \geq 2$ . Define the  $Y_0$  and  $Y_1$  to be

$$Y_0 = \Sigma \times S^2 \quad \text{and} \quad Y_1 = Y_0 \#_{F_0=F_1} (S^2 \tilde{\times} S^2).$$

Here,  $S^2 \tilde{\times} S^2$  denotes the twisted  $S^2$  bundle over  $S^2$ . It is diffeomorphic to  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ . As  $Y_1$  is obtained by taking the fiber sum of two  $S^2$  fibrations, it itself inherits the structure of an  $S^2$  fibration over  $S^2$ .

To calculate the Gromov-Witten invariants of  $Y_i$ , we invoke Taubes' theorem relating the Gromov-Witten invariants to the Seiberg-Witten invariants, which often prove easier to calculate. The following theorem can be found in [12].

**THEOREM 3.2.** *Let  $(X, \omega)$  be a symplectic 4-manifold with  $b^+ = 1$ . Let  $\mu_0 = F_{A_0}^+ - ir\omega/8 \in i\Omega^{2,+}$  (where  $A_0$  is a certain connection on the canonical line bundle) and let  $g$  be any generic metric compatible with the symplectic form. Then, for any  $E \in H^2(X; \mathbb{Z})$ , the Seiberg-Witten invariant of  $X$  for the  $\text{Spin}^c$ -structure  $W_E^+ = E \oplus (E \otimes K^{-1})$ , calculated with the metric  $g$  and the perturbation form  $\mu_0$  with  $r \gg 1$ , is equal to the Gromov-Witten invariant for the class  $E$ .*

The Seiberg-Witten invariants for both  $Y_0$  and  $Y_1$  are calculated in much the same way. We will only give the calculation for  $Y_0$  explicitly and indicate the minute differences that occur for  $Y_1$ .

The main input for calculating the Seiberg-Witten invariants of  $Y_0$  and  $Y_1$  are the wall crossing formula and the existence of metrics with positive scalar curvature.

Let  $g_\Sigma$  and  $g_{S^2}$  be metrics on  $\Sigma$  and  $S^2$  with constant scalar curvature and with volumes equal to  $4\pi(g - 1)$  and  $4\pi$ , respectively. It follows from the Gauss-Bonnet theorem that the scalar curvatures  $s_\Sigma$  and  $s_{S^2}$  of these metrics are

$$s_\Sigma = -1 \quad \text{and} \quad s_{S^2} = 1.$$

Denote by  $\omega_\Sigma$  and  $\omega_{S^2}$  the volume forms induced by  $g_\Sigma$  and  $g_{S^2}$  and define the symplectic form  $\omega_{\lambda,\varepsilon}$  on  $Y_0$  to be

$$(3) \quad \omega_{\lambda,\varepsilon} = \lambda \cdot \omega_\Sigma + \varepsilon \cdot \omega_{S^2}.$$

The positive parameters  $\lambda, \varepsilon > 0$  will be chosen later;  $\varepsilon$  should be thought of as being small. The product metric

$$g_{\lambda, \varepsilon} = \lambda g_{\Sigma} \oplus \varepsilon g_{S^2}$$

on  $Y_0$  is compatible with  $\omega_{\lambda, \varepsilon}$  and its scalar curvature  $s_{\lambda, \varepsilon}$  is

$$s_{\lambda, \varepsilon} = -\frac{1}{\lambda} + \frac{1}{\varepsilon}.$$

Our first condition on the parameters  $\lambda$  and  $\varepsilon$  will be that  $\varepsilon < \lambda$ , ensuring that  $s_{\lambda, \varepsilon} > 0$ . (The choice of the second condition is deferred to Section 4.)

With  $\omega_{\lambda, \varepsilon}$  chosen as in (3), the canonical class  $K_0$  of  $Y_0$  is easily calculated from the adjunction formula and from the fact that both  $\Sigma \times \{pt\}$  and  $\{pt\} \times S^2$  are symplectic submanifolds of  $Y_0$ . One finds that

$$K_0 = (2g - 2)\bar{S} - 2\bar{\Sigma} \in H^2(Y_0; \mathbb{Z}),$$

where  $\bar{S} = P \cdot D \cdot ([S^2])$  and  $\bar{\Sigma} = P \cdot D \cdot ([\Sigma])$ .

We will label  $\text{Spin}^c$ -structures of  $Y_0$  by elements  $E \in H^2(Y_0; \mathbb{Z})$  by letting  $W_E$  be the  $\text{Spin}^c$ -structure with  $W_E^+ = E \oplus (E \otimes K^{-1})$ . Thus the determinant line bundle  $L = \det(W_E^+)$  is equal to  $2E - K$ . We label the corresponding Seiberg-Witten moduli spaces by  $\mathcal{M}_{Y_0}^{\pm}(L)$ , the sign again depending upon the chamber  $\mathcal{C}^{\pm}(L)$  determined by the metric and perturbation.

For  $a, b \in \mathbb{Z}$ , let  $E = a\bar{\Sigma} + b\bar{S}$  and consider the  $\text{Spin}^c$ -structure  $W_E$ . The dimension of the Seiberg-Witten moduli space is given by

$$\dim_{\mathbb{R}} \mathcal{M}^{\pm}(E) = \frac{1}{4} (L^2 - K_0^2) = 2b(a + 1) - a(2g - 2).$$

In order for the  $\text{Spin}^c$ -structure  $W_E$  to have nonzero Seiberg-Witten invariant, the dimension of the moduli space needs to be non-negative. In the case when  $a = 1$  (the case of interest to us) the above formula together with the observation that  $E^2 = 2b$  leads to the following necessary condition for the nonvanishing of the invariant:

$$E^2 \geq g - 1.$$

Consider now  $E = \bar{\Sigma} + b\bar{S}$  with  $E^2 = 2b \geq g - 1$  and let  $L = 2E - K$ . It is easy to see that

$$(4) \quad \langle L \wedge \omega, [Y_0] \rangle = 32\pi\lambda(g - 1) + 16\pi\varepsilon(b - g + 1).$$

Two pairs of a metrics and perturbation forms will play a role in the subsequent discussion:

- (1)  $(g, \mu) = (g_{\lambda, \varepsilon}, 0)$ : By our choice  $\lambda > \varepsilon$  and by the restriction  $2b \geq g - 1$ , the right-hand side of (4) is positive:

$$32\pi\lambda(g - 1) + 16\pi\varepsilon(b - g + 1) \geq 16\pi(g - 1)(2\lambda - \frac{1}{2}\varepsilon) > 0.$$

This means that the pair  $(g_{\lambda, \varepsilon}, 0)$  lies in the chamber  $\mathcal{C}^-(L)$ .

- (2)  $(g, \mu) = (g_0, \mu_0)$ : Here  $g_0$  is any generic metric (but still compatible with  $\omega_{\lambda, \varepsilon}$ ) and  $\mu_0$  is Taubes' perturbation form

$$\mu_0 = F_{A_0}^+ - \frac{ir\omega}{8}.$$

It is easily checked that for large enough  $r$ , the pair  $(g_0, \mu_0)$  lies in  $\mathcal{C}^+(L)$  (for any  $\text{Spin}^c$ -structure).

By the positivity of  $s_{\lambda, \varepsilon}$  we have that  $SW_{Y_0}^-(L) = 0$ , which together with Theorems 3.1 and 3.2 immediately gives the following corollary.

**COROLLARY 3.3.** *For  $g \geq 1$ , let  $E_{g, 2n} = \overline{\Sigma} + n\overline{S^2} \in H^2(Y_0; \mathbb{Z})$  with  $E_{g, 2n}^2 \geq g - 1$ . Then*

$$Gr_{Y_0}(E_{g, 2n}) = \pm 2^g.$$

While the discussion preceding Corollary 3.3 was for the case  $g \geq 2$ , it is not hard to see that it remains valid in the case  $g = 1$ . The changes that need to be made to the analysis preceding the corollary are the following: Choose the product metric on  $\Sigma = T^2$  so that its scalar curvature is zero. Choose  $\omega_{\lambda, \varepsilon}$  and  $g_{\lambda, \varepsilon}$  as before and observe that  $s_{\lambda, \varepsilon} = 1/\varepsilon$ , which is positive for  $\varepsilon > 0$ . The rest of the discussion goes over verbatim and so establishes the validity of Corollary 3.3 in the case  $g = 1$  as well.

We finish this section by showing that an analogous result holds for  $Y_1$ . In  $Y_1$ , let  $\Sigma' = \Sigma \# S \subseteq Y_0 \#_{F_0=F_1} (S^2 \tilde{\times} S^2)$  with  $S = \mathbb{C}\mathbb{P}^1 \subseteq \mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2} \cong S^2 \tilde{\times} S^2$ . Let  $F$  denote a fiber of the fibration  $Y_1 \rightarrow S^2$ . The canonical class  $K_1$  of  $Y_1$  is

$$K_1 = (2g - 1)\overline{F} - 2\overline{\Sigma}'_0, \quad \overline{F} = P \cdot D \cdot ([F]), \quad \overline{\Sigma}' = P \cdot D \cdot ([\Sigma']).$$

As in the case of  $Y_0$ , consider  $E = a\overline{\Sigma}'_0 + b\overline{F} \in H^2(Y_1; \mathbb{Z})$ . The dimension for the Seiberg-Witten moduli space for the  $\text{Spin}^c$ -structure  $W_E$  is

$$\dim \mathcal{M}_{Y_1}^{SW}(L) = 2b(a + 1) - 2a(g - 2).$$

In the case when  $a = 1$ , the necessary condition for the nonvanishing of  $SW_{Y_1}^\pm(L)$  (with  $L = 2E - K_1$ ) becomes

$$E^2 = 2b + 1 \geq g - 1.$$

It is a known fact (cf. [5]) that ruled surfaces admit metrics of positive scalar curvature. The rest of the discussion for  $Y_1$  proceeds now in much the same way as that given for  $Y_0$  and one arrives at the following analogue of Corollary 3.3:

**COROLLARY 3.4.** *Let  $E_{g, 2n+1} = E = \overline{\Sigma}_0 + n\overline{F} \in H^2(Y_1; \mathbb{Z})$  with  $E^2 \geq g - 1$ . Then*

$$Gr_{Y_1}(E) = \pm 2^g.$$



**4. Proof of Theorem 1.2**

We now proceed to the proof of Theorem 1.2. Let  $C$  be an embedded, connected, symplectic submanifold of  $(X^4, \omega)$  of genus  $g \geq 1$  and with square  $[C]^2 = n \geq g - 1$ . Assume in addition that  $n = 2k$  is even; the case where  $n$  is odd is treated in much the same way by replacing  $Y_0$  below by  $Y_1$ . Let  $N(C)$  be a tubular neighborhood of  $C$  in  $X$  and let  $\text{Vol}(C)$  be the volume of  $C$ .

On the other hand, let  $D$  be any of the (at least)  $2^g$   $J'$ -holomorphic curves in  $Y_0$  in the class  $[\Sigma] + k[S^2]$  for the choice of a generic pair  $(J', \Omega')$  on  $Y_0$ . This last statement uses Corollary 3.3 (or Corollary 3.4 in the case of  $n = 2k - 1$ ). Adjust the choices of  $\lambda$  and  $\varepsilon$  so that  $\text{Vol}(D) = \text{Vol}(C)$  (in addition to  $\lambda > \varepsilon > 0$ ). Let  $N(D)$  be a tubular neighborhood of  $D$  in  $Y_0$  containing no other  $J'$ -holomorphic curves besides  $D$ .

By the symplectic neighborhood theorem for 4-manifolds (cf. [9], Exercise 3.30), the tubular neighborhood of a connected, embedded symplectic surface is up to symplectomorphism determined by the square and volume of the surface. We would like to say that the pairs  $(N(C), \omega|_{N(C)})$  and  $(N(D), \omega_{\lambda, \varepsilon}|_{N(D)})$  are symplectomorphic via a symplectomorphism  $\varphi : N(C) \rightarrow N(D)$  taking  $C$  to  $D$ . There is one potential problem with this approach, namely a priori all of the at least  $2^g$   $J'$ -holomorphic curves in the class  $[\Sigma] + k[S^2]$  in  $Y_0$  may be disconnected. Fortunately, the opposite extreme is true as the next lemma shows.

**LEMMA 4.1.** *Let  $(J', \Omega')$  be a generic pair on  $Y_i$  and let  $D$  be an embedded  $J'$ -holomorphic curve in  $Y_i$  containing  $\Omega'$ . Suppose that  $D$  represents the homology class  $[\Sigma] + k[S^2]$  in the case  $i = 0$  and represents the class  $[\Sigma'] + k[F]$  in the case  $i = 1$ . Then  $D$  is connected.*

*Proof.* Assume to the contrary that we can write  $D$  as a disjoint union  $D = D_1 \sqcup D_2$ . We will show that one of the two components has fundamental class zero.

*Case  $i = 0$ :* Let  $[D_1] = a[\Sigma] + b[S^2]$  and  $D_2 = c[\Sigma] + d[S^2]$ . Since  $a + c = 1$  we can assume that  $a \geq 1$ . We will first show that in fact  $a = 1$  and thus  $c = 0$ .

It is a well known fact that for generic almost-complex structures,  $J$ -holomorphic curves intersect non-negatively (see [7]). Observe also that the manifolds  $Y_i$  are minimal and so Remark 1.5 applies (which excludes the existence of  $J'$ -holomorphic curves with negative square). We know by Corollary 3.3 that for  $N$  large enough the class  $[\Sigma] + N[S^2]$  has  $J$ -holomorphic representatives. Thus we get

$$\begin{aligned}
 (5) \quad [D_2] \cdot ([\Sigma] + N[S^2]) \geq 0 &\implies cN + d \geq 0 \\
 &\implies (1 - a)N + d \geq 0 \\
 &\implies 1 + \frac{d}{N} \geq a \geq 1 \\
 &\implies a = 1 \text{ and } c = 0.
 \end{aligned}$$

Since  $D_1$  and  $D_2$  are disjoint, we find that  $0 = [D_1] \cdot [D_2] = d$ , which shows that  $[D_2] = 0$ .

*Case  $i = 1$ :* Let  $[D_1] = a[\Sigma'] + b[F]$  and  $D_2 = c[\Sigma'] + d[F]$ . Since as before we have  $a + c = 1$ , so we can again assume that  $a \geq 1$ . By Corollary 3.4 we know that the class  $[\Sigma'] + N[F]$  has  $J$ -holomorphic representatives for all sufficiently large  $N$ . Then arguing as above we have

$$\begin{aligned}
 (6) \quad [D_2] \cdot ([\Sigma'] + N[F]) \geq 0 &\implies c + cN + d \geq 0 \\
 &\implies (1 - a)(N + 1) + d \geq 0 \\
 &\implies 1 + \frac{d}{N + 1} \geq a \geq 1 \\
 &\implies a = 1 \text{ and } c = 0.
 \end{aligned}$$

The fact  $0 = [D_1] \cdot [D_2] = d$  completes the proof. □

Use  $\varphi$  together with  $J'$  on  $N(D)$  to induce an almost-complex structure (also denoted by  $J'$ ) on  $N(C)$ . Extend  $J'$  to all of  $X$  in an arbitrary manner and denote it by  $J''$ . Let  $\Omega''$  denote the set  $\varphi^{-1}(\Omega')$ .

Observe that  $(J'', \Omega'') \in \mathcal{J}_d^{\text{reg}}(N(C))$ , but it could happen that  $(J'', \Omega'') \notin \mathcal{J}_d^{\text{reg}}(X)$  as there may be other  $J''$ -holomorphic curves in  $X$  for which the operator defined in (2) is not surjective. However, generic pairs  $(J, \Omega)$  on  $X$  are dense in  $\mathcal{A}_d(X)$  and so we can find, in an arbitrarily small neighborhood of  $(J'', \Omega'')$ , a pair  $(J, \Omega)$  that is generic. The following standard proposition completes the proof of Theorem 1.2.

**PROPOSITION 4.2.** *Let  $\varepsilon > 0$  be arbitrary. Then there exists  $\delta > 0$  such that if*

$$\text{dist}[(J, \Omega), (J'', \Omega'')] < \delta,$$

*then there exists a  $J$ -holomorphic curve  $C'$  in an  $\varepsilon$  tubular neighborhood of  $C$ .*

*Proof.* This is a direct consequence of the fourth point in the definition of genericity applied to the two pairs  $(J'', \Omega'')|_{N(C)}$  and  $(J, \Omega)|_{N(C)}$ . By construction,  $(J'', \Omega'') \in \mathcal{J}_d^{\text{reg}}(N(C))$ , and clearly

$$\text{dist}[(J, \Omega)|_{N(C)}, (J'', \Omega'')|_{N(C)}] \leq \text{dist}[(J, \Omega), (J'', \Omega'')].$$

This completes the proof of the proposition as well as that of Theorem 1.2. □

*Proof of Corollary 1.3.* The proof of Corollary 1.3 proceeds in much the same way. For each component  $C_i$  of  $C$ , one finds a generic pair  $(J'_i, \Omega'_i)$  on a tubular neighborhood  $N(C_i)$  of  $C_i$ . One extends the almost-complex structures  $J'_i$  to an arbitrary almost-complex structure  $J''$  on  $X$  and defines  $\Omega'' = \sqcup \Omega''_i$ , where the  $\Omega''_i$  are defined in the same way as  $\Omega''$  in the proof of Theorem 1.2. The analogue of Proposition 4.2 completes the proof of Corollary 1.3. □

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