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A FRACTIONAL ORDER HARDY INEQUALITY

BARTŁOMIEJ DYDA

Abstract. We investigate the following integral inequality:

$$\int_D \frac{|u(x)|^p}{\operatorname{dist}(x,D^c)^\alpha}dx \le c\int_D\int_D \frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}}dx\,dy, \quad u\in C_c(D),$$

where $\alpha, p > 0$ and $D \subset \mathbb{R}^d$ is a Lipschitz domain or its complement or a complement of a point.

1. Introduction and notation

Let $D \subset \mathbb{R}^d$, $d \ge 1$, be an open set and let $\delta_D(x) = \inf\{|x-y| : y \in D^c\}$. Let $0 < \alpha, p < \infty$. In this paper we study the following integral inequality of Hardy type:

$$\int_D \frac{|u(x)|^p}{\delta_D(x)^\alpha}dx \le c\int_D\int_D \frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}}dx\,dy, \quad \text{for all } u\in C_c(D), \tag{1}$$

where $c = c(D, \alpha, d, p)$, i.e., $c < \infty$ is a constant that depends only on D, α, d and p.

There are connections between the right hand side of (1) and Sobolev spaces $W^{\lambda,p}(D)$ of order $\lambda = \alpha/p$ (see, e.g., [KJF], [Br]) and also fractional derivatives ([OK]). These connections explain why we call (1) a Hardy inequality: it estimates an integral of a function by an integral of its derivative. The reader interested in classical Hardy inequalities is referred to [OK] and [KP], and to [F], [FK], [FW], where such inequalities are obtained by variational methods.

The one–dimensional case of (1) was investigated in [J], [G], [KT]; [KP] gives a survey of such one–dimensional results. Special multi–dimensional versions of (1) may be found in [HKP] for $D = \mathbb{R}^d \setminus \{0\}$ and $p > 1$, and in [CS] for a wider class of domains, but only for $p = 2$.

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Another motivation to study (1) comes from the theory of integro–differential operators. Let $\alpha \in (0,2)$ and define

$$\Delta_D^{\alpha/2} u(x) = \mathcal{A} \lim_{\varepsilon \to 0^+} \int_{D \cap \{|y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\, dy, \quad u \in C_c^\infty(D),\ x \in D,$$

where $\mathcal{A}$ is a positive constant. It is elementary to verify that the limit exists and that

$$-(\Delta_D^{\alpha/2} u, u) = \frac{1}{2}\mathcal{A} \int_D \int_D \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dx\, dy = \mathcal{E}(u,u).$$

$\mathcal{E}(u,u)$ is a special case of the integral on the right hand side of (1), with $p = 2 > \alpha$. It is important to notice that $\mathcal{E}$ is a Dirichlet form with core $C_c^\infty(D)$; see, e.g., Example 1.2.1 in [FOT]. The corresponding Markov process $\{X_t\}_{t \ge 0}$ on D is called censored stable process [BBC]. This process has the same intensity of jumps as the symmetric α–stable Lévy process in $\mathbb{R}^d$ [BG] except that jumps outside D are prohibited. It is shown in [CK] that (1) in the special case $p = 2$ ([CS]) yields important estimates for the Green function of the censored stable process.

We note that (1) is related to the existence of positive superharmonic functions for the censored process and the subcriticality of Schrödinger perturbations of $\Delta_D^{\alpha/2}$. Also, (1) gives an answer to the interesting question of whether $\{X_t\}$ approaches ∂D in finite time, which in turn is related to trace theorems in Sobolev spaces [BBC]. A discussion of those applications will be given in a forthcoming paper; see also [CS], [A], [TU].

The main goal of this paper is to prove the following theorem.

Theorem 1.1. *Let $\alpha > 0$ and $p > 0$. The Hardy inequality* (1) *holds true in each of the following cases:*

(T1) *D is a bounded Lipschitz domain and $\alpha > 1$;*
(T2) *D is a complement of a bounded Lipschitz domain, $\alpha \neq 1$ and $\alpha \neq d$;*
(T3) *D is a domain above the graph of a Lipschitz function $\mathbb{R}^{d-1} \to \mathbb{R}$ and $\alpha \neq 1$;*
(T4) *D is a complement of a point and $\alpha \neq d$.*

The existing proofs of the special cases of (1) mentioned above reduce (1) to the classical Hardy inequality by means of complex interpolation ([KT], [CS]) or by other means ([G], [KP]). Our method of proof of (1) is completely different. We use only elementary properties of the Lebesgue integral along with simple geometrical properties of the domains in question. This allows us to extend (1) to the present wide class of domains. Moreover, the method seems to have potential for further generalisations.

By means of a counterexample we prove that (1) is *false* if, e.g.,

(F1) D is a bounded Lipschitz domain and $\alpha \le 1$, $\alpha < p$, or
(F2) D is a complement of a compact set and $d = \alpha < p$.

The counterexamples are given in Section 2. In Section 3 we develop the technique needed to prove Theorem 1.1. The proof is given in Section 4, where we also discuss some possible extensions.

For sets A, $B \subset \mathbb{R}^d$ we write $\operatorname{dist}(A, B) = \inf\{|a - b| : a \in A,\ b \in B\}$. $|A|$ denotes the Lebesgue measure of A and we write $\operatorname{card} A$ for the number of its points. We always assume Borel measurability of the considered sets. When D is fixed, we often write $\delta_x = \delta_D(x)$ to abbreviate the notation. *Constants* are positive real numbers.

We note that if we replace D in (1) by $hD = \{hx : x \in D\}$ and u by $\tilde{u} = u^h$, where $u^h(x) = h^{\alpha-d}u(x/h)$ and $h > 0$, then the values of both sides of (1) remain unchanged.

2. Counterexamples

Let $0 < \alpha \le 1$, $\alpha < p$. Assume that D is bounded and satisfies the following condition:

(2) There are $\varepsilon \ge 0$ and $c < \infty$ such that for every $n \in \mathbb{N}$ there exists a set $E_n \subset \partial D$ such that $\operatorname{card} E_n \le cn^{d-\alpha-\varepsilon}$ and $\partial D \subset \bigcup_{z \in E_n} B(z, 1/(2n))$.

In the case when $\varepsilon = 0$ we will additionally assume that $\int_D (1/\delta_D(x)^\alpha)dx = \infty$. We will show that under these assumptions (1) is false; note that this happens, for example, if D is a bounded Lipschitz domain. Clearly, if D satisfies (2), then ∂D has finite $(d - \alpha - \varepsilon)$–dimensional Hausdorff measure. (The inverse implication is not generally true.)

The idea of the following counterexample concerning (F1) comes from [B].

For each n large enough we pick a function $u_n \in C_c^\infty(\mathbb{R}^d)$ satisfying the following conditions:

(i) $u_n = 1$ on $A_n = \{x : \delta_D(x) > 1/(2n)\}$;
(ii) $u_n = 0$ outside A_{2n};
(iii) $0 \le u_n \le 1$ and $|\nabla u_n| \le cn$ everywhere.

Here and below in this section c denotes a positive constant, depending on D, α and p, but not on n. We see that

$$\int_D\int_D \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{d+\alpha}} dx\, dy \le \left(\int_D\int_{D\setminus A_n} + \int_{D\setminus A_n}\int_D\right) dx\, dy$$
$$= 2\int_D\int_{D\setminus A_n} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{d+\alpha}}\, dx\, dy = 2I_n.$$

We pick E_n as in (2). Note that $D \setminus A_n \subset \bigcup_{z \in E_n} B(z, 1/n)$. We set $B_z = B(z, 1/n)$. We have

$$\begin{aligned} I_n &\le \sum_{z \in E_n} \int_D \int_{B_z} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{d+\alpha}} dx\, dy \\ &\le \sum_{z \in E_n} \sum_{l=0}^{\infty} \int_{B(z,(2l+2)/n) \setminus B(z,2l/n)} \int_{B_z} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{d+\alpha}} dx\, dy. \end{aligned}$$

For $l = 0$ the above double integral is

$$\begin{aligned} \int_{B(z,2/n)} \int_{B_z} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{d+\alpha}} dx\, dy &\le c^p \int_{B(z,2/n)} \int_{B_z} \frac{n^p |x-y|^p}{|x-y|^{d+\alpha}} dx\, dy \\ &\le c' \int_{B(z,2/n)} \int_{B(y,3/n)} n^p |x-y|^{p-d-\alpha} dx\, dy \\ &= c'' n^{\alpha - d}, \end{aligned}$$

and for $l = 1, 2, \ldots$ it is

$$\begin{aligned} &\int_{B(z,(2l+2)/n) \setminus B(z,2l/n)} \int_{B_z} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{d+\alpha}} dx\, dy \\ &\qquad \le |B(z,(2l+2)/n) \setminus B(z,2l/n)| \cdot |B_z| \cdot \frac{1}{((2l-1)/n)^{d+\alpha}} \\ &\qquad \le c n^{\alpha-d} \left(\frac{1}{l+1} \right)^{\alpha+1}. \end{aligned}$$

Thus

$$I_n \le c n^{\alpha-d} \sum_{z \in E_n} \sum_{l=0}^{\infty} \left(\frac{1}{l+1} \right)^{\alpha+1} \le c' n^{-\varepsilon}. \tag{3}$$

The left hand side of (1) is

$$\int_D \frac{|u_n(x)|^p}{\delta_D(x)^\alpha} dx \to \int_D \frac{1}{\delta_D(x)^\alpha} dx \qquad \text{as } n \to \infty. \tag{4}$$

We see that (3) and (4) give a contradiction to (1). In particular, (1) is false in the case (F1).

As far as (F2) is concerned, let $D = \mathbb{R}^d \setminus \{0\}$ and let $d = \alpha < p$. For each n we pick a function $v_n \in C_c^\infty(D)$ satisfying the following conditions:

- (i) $v_n(x) = 1$ for $2/n < |x| < 1$;
- (ii) $v_n(x) = 0$ for $|x| < 1/n$ or $|x| > 2$;
- (iii) $0 \le v_n \le 1$ and $|\nabla v_n| \le cn$ everywhere;
- (iv) $|\nabla v_n(x)| \le c$ for $|x| > 1$.

Similarly as before, we show that the left hand side of (1) with v_n in place of u is greater than $c\log n$, while the right hand side is bounded. In order to prove the latter, we write

$$\begin{aligned}\int_D\int_D \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{d+\alpha}}dx\,dy &\le 2\left(\int_{B(2/n)}\int_{B(3/n)}+\int_{B(2/n)}\int_{A(3/n,\infty)}\right)dx\,dy\\ &\quad+2\left(\int_{A(2/n,2)}\int_{A(1,3)}+\int_{A(2/n,2)}\int_{A(3,\infty)}\right)dx\,dy\\ &=2(J_n^1+J_n^2)+2(J_n^3+J_n^4),\end{aligned}$$

where $B(r)=B(0,r)$ and $A(r_1,r_2)=B(r_2)\backslash B(r_1)$. We replace $|v_n(x)-v_n(y)|$ by 1 in J_n^2 and J_n^4, by $cn|x-y|$ in J_n^1 and by $c|x-y|$ in J_n^3. An elementary calculation now shows that J_n^1, J_n^2, J_n^3 and J_n^4 are bounded.

Now let D be a complement of a compact set $K\subset B(0,R)$. We consider the dilations (see the remark at the end of Section 1) $\tilde v_n=v_n^{2Rn}$ of the functions v_n considered above. We have $\operatorname{supp}\tilde v_n\subset B(0,2R)^C$. The functions $\tilde v_n$ give a counterexample to (1) in the case (F2) because on the support of $\tilde v_n$ we have $|x|\le 2\delta_D(x)$.

3. Main estimates

Let $D\subset\mathbb{R}^d$ be an open set, $D\ne\emptyset$ and $D\ne\mathbb{R}^d$. Let $\Omega\subset D$. We fix a function $u\in C_c(D)$ and define

$$F=F(u,\Omega;R,S)=\left\{x\in\Omega:|u(x)|^p>\frac{2^{p+1}}{S\delta_x^d}\int_{B(x,R\delta_x)\cap\Omega}|u(x)-u(y)|^p dy\right\},$$

where R and S are some positive numbers. R stands for *range*, while S stands for *sensitivity*; we may, if necessary, enlarge R or make S smaller (in this sense R is large and S is small). Ω is an auxiliary set which allows us to localise our considerations; see the remark following Lemma 3.4 in this connection. Points from F will be called *flat*, because, on average, $u(y)$ is close to $u(x)$ on $B(x,R\delta_x)\cap\Omega$ for $x\in F$. However, we note here that $F\subset\operatorname{supp}u$; in particular, $\operatorname{dist}(F,D^c)>0$.

The following property should be compared with our goal (1).

Property 1.

$$\int_{\Omega\setminus F}\frac{|u(x)|^p}{\delta_x^\alpha}dx\le\frac{2^{p+1}R^{d+\alpha}}{S}\int_{\Omega\setminus F}\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}}dy\,dx.$$

Proof. For $x\in\Omega\setminus F$ we have

$$\begin{aligned}\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}}dy&\ge\int_{B(x,R\delta_x)\cap\Omega}\frac{|u(x)-u(y)|^p}{(R\delta_x)^{d+\alpha}}dy\\ &\ge\frac{S}{2^{p+1}R^{d+\alpha}}\frac{|u(x)|^p}{\delta_x^\alpha}\end{aligned}$$

and the property follows. $\square$

Property 2. *Let $x \in F$ and $E \subset B(x, R\delta_x) \cap \Omega$. Let $E^*(x) = \{y \in E : (1/2)|u(y)| \le |u(x)| \le (3/2)|u(y)|\}$. If $|E| \ge S\delta_x^d$, then*

$$|E^*(x)| \ge |E| - \frac{1}{2}S\delta_x^d \ge \frac{1}{2}|E|.$$

Proof. This follows immediately from the definition of the set F. $\square$

Property 3. *Let $E_1 \subset \Omega$. Let E_2 be a set such that $E_2 \subset B(x, R\delta_x) \cap \Omega$ and $|E_2| \ge S\delta_x^d$ for all $x \in E_1$. Then*

$$\int_{E_1 \cap F} \frac{|u(x)|^p}{\delta_x^\alpha} dx \le 2\left(\frac{3}{2}\right)^p \frac{|E_1|}{|E_2|}\left(\frac{\sup\{\delta_x : x \in E_2\}}{\inf\{\delta_x : x \in E_1\}}\right)^\alpha \int_{E_2} \frac{|u(y)|^p}{\delta_y^\alpha} dy.$$

Proof. Assume $E_1 \cap F \ne \emptyset$. We fix $\eta > 1$ and we pick $x_0 \in E_1 \cap F$ such that $\sup_{E_1 \cap F} |u| \le \eta |u(x_0)|$. We have $|u(x_0)| \le (3/2)|u(y)|$ for $y \in E_2^*(x_0)$. Hence

$$\begin{aligned}
\int_{E_1 \cap F} \frac{|u(x)|^p}{\delta_x^\alpha} dx &\le |E_1 \cap F| \cdot \left(\sup_{E_1 \cap F} |u|\right)^p \cdot \frac{1}{\inf\{\delta_x^\alpha : x \in E_1 \cap F\}} \\
&\le \frac{|E_1 \cap F|}{|E_2^*(x_0)|} \cdot \left(\frac{\sup\{\delta_y : y \in E_2^*(x_0)\}}{\inf\{\delta_x : x \in E_1\}}\right)^\alpha \left(\frac{3}{2}\eta\right)^p \int_{E_2^*(x_0)} \frac{|u(y)|^p}{\delta_y^\alpha} dy.
\end{aligned}$$

By Property 2, $|E_2^*(x_0)| \ge (1/2)|E_2|$, and so we get

$$\int_{E_1 \cap F} \frac{|u(x)|^p}{\delta_x^\alpha} dx \le 2\frac{|E_1|}{|E_2|}\left(\frac{\sup\{\delta_x : x \in E_2\}}{\inf\{\delta_x : x \in E_1\}}\right)^\alpha \left(\frac{3}{2}\eta\right)^p \int_{E_2} \frac{|u(y)|^p}{\delta_y^\alpha} dy.$$

The proof is completed by letting $\eta \to 1$. $\square$

Property 3 enables us to "sweep out" the part of the integral on the left hand side of (1) which is over F to the complement of F, where in turn we can use Property 1.

We will eventually come to the point where the geometry of the domain D plays an essential role; the following lemma will be used in Section 4 to prove a Hardy inequality for domains which are complements of bounded sets. The main idea in proving the lemma is to use an appropriate "stopping time" argument.

Lemma 3.1. *Suppose $0 < \alpha < d$ and $r > 0$. There exists a constant $c = c(\alpha, d, p)$ such that for all functions $u \in C_c(\mathbb{R}^d)$ we have*

$$\int_{B(0,r)^c} \frac{|u(x)|^p}{|x|^\alpha} dx \le c \int_{B(0,r)^c} \int_{B(0,r)^c} \frac{|u(x) - u(y)|^p}{|x - y|^{d+\alpha}} dy\, dx.$$

Proof. We let $D = \mathbb{R}^d \setminus \{0\}$. We may assume that $u \in C_c(D)$, because neither side of the above inequality depends on the values of u on $B(0,r)$. We define $\Omega = B(0,r)^c$ and $F = F(u,\Omega; R, S)$, where $R = 2^{m+1} + 1$, $S = 2^{(m-1)d} \cdot |B(0,2) \setminus B(0,1)|$, and m is a natural number to be determined later. We define $A_n = B(0, 2^{n+1}r) \setminus B(0, 2^n r)$ for $n = 0, 1, \ldots$.

We fix $j \in \{0, 1, \ldots\}$. Note that $E_1 = A_j$ and $E_2 = A_{j+m}$ satisfy the assumptions in Property 3. Thus we get

$$\int_{A_j \cap F} \frac{|u(x)|^p}{|x|^\alpha} dx \le 2 \left(\frac{3}{2}\right)^p \frac{2^{jd}}{2^{(j+m)d}} \left(\frac{2^{j+m+1}}{2^j}\right)^\alpha \int_{A_{j+m}} \frac{|u(y)|^p}{|y|^\alpha} dy = \gamma \int_{A_{j+m}} \frac{|u(y)|^p}{|y|^\alpha} dy, \tag{5}$$

where $\gamma = 2^{m(\alpha-d)} 2^{1+\alpha} (3/2)^p < 1$ for sufficiently large m (this condition defines m). We write

$$\int_{A_{j+m}} \frac{|u(y)|^p}{|y|^\alpha} dy = \left(\int_{A_{j+m} \setminus F} + \int_{A_{j+m} \cap F} \right) \frac{|u(y)|^p}{|y|^\alpha} dy$$

and we repeat (5) with $j+m$ in place of j, then with $j+2m$ in place of j, and so on. We eventually stop iterating (5) when the considered set $A_{j+km} \cap F$ becomes empty. We obtain

$$\int_{\Omega \cap F} \frac{|u(x)|^p}{|x|^\alpha} dx \le \left(\sum_{n=1}^\infty \gamma^n \right) \cdot \sum_{j=0}^\infty \int_{A_{j+m} \setminus F} \frac{|u(y)|^p}{|y|^\alpha} dy.$$

Hence by Property 1 we have

$$\int_\Omega \frac{|u(x)|^p}{|x|^\alpha} dx = \left(\int_{\Omega \setminus F} + \int_{\Omega \cap F} \right) \frac{|u(x)|^p}{|x|^\alpha} dx \le \left(\sum_{n=0}^\infty \gamma^n \right) \int_{\Omega \setminus F} \frac{|u(y)|^p}{|y|^\alpha} dy$$
$$\le \frac{1}{1-\gamma} \frac{2^{p+1} R^{d+\alpha}}{S} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+\alpha}} dy\, dx. \qquad \square$$

For example, if $\alpha = 1$ and $p = d = 2$, then we may take $m = 4$ in the proof of Lemma 3.1 and we get $c = 2 \cdot 33^3/(21\pi) < 1090$.

We now consider Lipschitz domains. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we write $x = (\tilde{x}, x_d)$, where $\tilde{x} = (x_1, \ldots, x_{d-1})$. We make the convention that $\tilde{x} = 0$ for $x \in \mathbb{R}$ and, correspondingly, $\mathbb{R}^0 = \{0\}$. The reader may easily check that our considerations below apply also in the case when $d = 1$. We assume that D is a Lipschitz domain with *localisation radius* r_0 and *Lipschitz constant* λ; i.e., D is an open set and for each $z \in \partial D$ there are an isometry T_z of $\mathbb{R}^d$ and a Lipschitz function $\varphi_z \colon \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant not greater than λ such that

$$T_z(D) \cap B(T_z(z), r_0) = \{x : x_d > \varphi_z(\tilde{x})\} \cap B(T_z(z), r_0).$$

Let $z \in \partial D$. To simplify the notation we assume in what follows that T_z is an identity mapping. Otherwise we can apply the considerations below to the Lipschitz domain $T_z(D)$ and the point $T_z(z)$, and then come back to D and z using T_z^{-1}. For $x \in \mathbb{R}^d$ we put

$$V_z(x) = |x_d - \varphi_z(\tilde{x})|.$$

$V_z(x)$ is the "vertical" distance from x to the graph of φ_z. For a set $E \subset \mathbb{R}^{d-1}$ and $\rho > 0$ we define

$$Q_z(E, \rho) = \{x \in D : \tilde{x} \in E,\ 0 < V_z(x) \leq \rho\}.$$

We call $Q_z(E, \rho)$ a "box" with base E and height ρ. In the case when $E = K_\rho = \{x \in \mathbb{R}^{d-1} : |x_i - z_i| \leq \rho/2 \text{ for } i = 1, 2, \ldots, d-1\}$ we simply write $Q_z(\rho)$ instead of $Q_z(E, \rho)$ and we call $Q_z(\rho)$ a Lipschitz box; see Figure 1 below. Now we fix

$$\rho < \frac{r_0}{2}\left(\frac{d-1}{4} + \left(\frac{\lambda}{2}\sqrt{d-1} + 1\right)^2\right)^{-1/2}.$$

It may be checked that for such ρ we have $Q_z(\rho) \subset D \cap B(z, r_0/2)$. It is also easy to see that

$$\frac{V_z(x)}{\sqrt{1+\lambda^2}} \leq \delta_x \leq V_z(x) \qquad \text{for } x \in Q_z(\rho) \subset D \cap B(z, r_0/2). \tag{6}$$

For $j = 0, 1, \ldots$ we consider the usual dyadic decompositions of the base $E = K_\rho$ of $Q_z(\rho)$ into the union of $(d-1)$–dimensional cubes E_j^i indexed by $i = 1, 2, \ldots, 2^{j(d-1)}$. The cubes E_j^i have disjoint interiors and sides of length $2^{-j}\rho$. Such a decomposition gives rise to sets $Q_j^i = Q_z(E_j^i, \rho)$, $i = 1, 2, \ldots, 2^{j(d-1)}$, of the same height ρ.

We define

$$A_n = Q_z(K_\rho, \rho/2^n) \setminus Q_z(K_\rho, \rho/2^{n+1}), \qquad \text{for } n = 0, 1, \ldots. \tag{7}$$

The sets A_n are mutually disjoint and $\bigcup_{n=0}^\infty A_n = Q_z(\rho)$. Thus we have a decomposition of $Q_z(\rho)$ into a union of sets $A_j \cap Q_j^i$ whose pairwise intersections are of Lebesgue measure zero. Furthermore, we have

$$\frac{\rho}{2^j}\frac{1}{2\sqrt{1+\lambda^2}} \leq \delta_x \leq \frac{\rho}{2^j}, \qquad \text{for } x \in A_j \cap Q_j^i; \tag{8}$$

$$|A_j \cap Q_j^i| = \frac{1}{2}\left(\frac{\rho}{2^j}\right)^d; \tag{9}$$

and for $x \in A_k \cap Q_j^i$, $y \in A_j \cap Q_j^i$, where $k \geq j$,

$$|x - y| \leq \frac{\rho}{2^j}\sqrt{d - 1 + (\lambda\sqrt{d-1} + 1)^2}. \tag{10}$$

In the next result we consider $Q = Q_z(\rho)$ defined above.

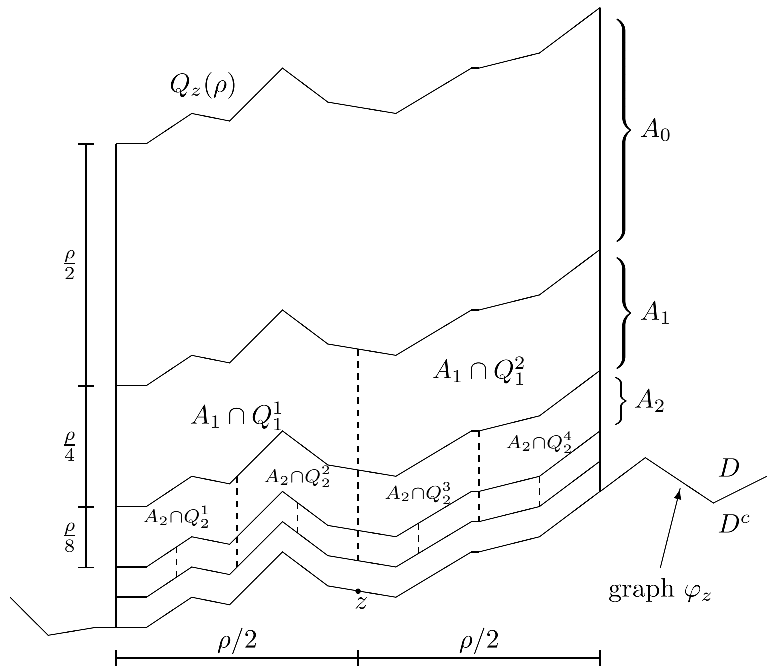

FIGURE 1. Lipschitz box $Q_z(\rho)$, $d = 2$.

LEMMA 3.2. *Let $\alpha > 1$. There exists a constant $c = c(\lambda, \alpha, d, p)$ such that for all functions $u \in C_c(D)$ we have*

$$\int_Q \frac{|u(x)|^p}{\delta_x^\alpha} dx \le c \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{d+\alpha}} dy\, dx.$$

Proof. We fix $m \in \mathbb{N}$ and a function $u \in C_c(D)$. We pick $\Omega = Q$ and let $F = F(u, Q; R, S)$, where $R = 2\sqrt{1+\lambda^2}(d - 1 + (\lambda\sqrt{d-1} + 1)^2)^{1/2}$ and $S = 2^{-m-1}$.

Consider $j \in \{0, 1, \dots\}$ and the sets $A_j \cap Q_j^i$ defined above. The numbers R and S were selected in order to ensure that $E_1 = A_j \cap Q_j^i$ and $E_2 = A_{j+m} \cap Q_j^i$ satisfy the assumptions in Property 3; see (8), (9) and (10) in this connection. Thus we have

$$\begin{aligned}\int_{(A_j \cap Q_j^i)\cap F} \frac{|u(x)|^p}{\delta_x^\alpha} dx &\le 2\left(\frac{3}{2}\right)^p 2^m (2\sqrt{1+\lambda^2}2^{-m})^\alpha \int_{A_{j+m}\cap Q_j^i} \frac{|u(y)|^p}{\delta_y^\alpha} dy \\ &= \gamma \int_{A_{j+m}\cap Q_j^i} \frac{|u(y)|^p}{\delta_y^\alpha} dy,\end{aligned}$$

where $\gamma = 2^{m(1-\alpha)} \cdot 2^{1+\alpha}(\sqrt{1+\lambda^2})^\alpha (3/2)^p < 1$ for sufficiently large m (this condition defines m). After summing over $i = 1, \ldots, 2^{j(d-1)}$ we obtain

$$\int_{A_j \cap F} \frac{|u(x)|^p}{\delta_x^\alpha} dx \le \gamma \int_{A_{j+m}} \frac{|u(y)|^p}{\delta_y^\alpha} dy.$$

We now proceed in exactly the same manner as at the end of the proof of Lemma 3.1 and get

$$\int_\Omega \frac{|u(x)|^p}{\delta_x^\alpha} dx \le \frac{1}{1-\gamma} \frac{2^{p+1} R^{d+\alpha}}{S} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}} dy\, dx. \qquad \square$$

LEMMA 3.3. *Suppose $\alpha > d$ and $r > 0$. There exist a constant $c = c(\alpha, d, p)$ and a natural number $m = m(\alpha, d, p)$ such that for all functions $u \in C_c(\mathbb{R}^d)$ we have*

$$\int_{B^c} \frac{|u(x)|^p}{|x|^\alpha} dx \le c \left(\int_{B^c} \int_{B^c} \frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}} dy\, dx + \int_A \frac{|u(x)|^p}{|x|^\alpha} dx \right),$$

where $B = B(0, 2^m r)$, $A = B(0, 2^m r) \setminus B(0, r)$.

Proof. As in the proof of Lemma 3.1, we let $D = \mathbb{R}^d \setminus \{0\}$, $u \in C_c(D)$, $\Omega = B(0,r)^c$ and $F = F(u, \Omega; R, S)$, where $R = (2^m + 2)/2^m$ and $S = 2^{-(m+1)d} \cdot |B(0,2) \setminus B(0,1)|$ (where m will be defined later). We define $A_n = B(0, 2^{n+1} r) \setminus B(0, 2^n r)$ for $n = 0, 1, \ldots$.

We fix $j \ge m$. Note that $E_1 = A_j$ and $E_2 = A_{j-m}$ satisfy the assumptions in Property 3. Thus we get

$$\begin{aligned} (11) \qquad \int_{A_j \cap F} \frac{|u(x)|^p}{|x|^\alpha} dx &\le 2 \left(\frac{3}{2}\right)^p \frac{2^{jd}}{2^{(j-m)d}} \left(\frac{2^{j-m+1}}{2^j} \right)^\alpha \int_{A_{j-m}} \frac{|u(y)|^p}{|y|^\alpha} dy \\ &= \gamma \int_{A_{j-m}} \frac{|u(y)|^p}{|y|^\alpha} dy, \end{aligned}$$

where $\gamma = 2^{m(d-\alpha)} 2^{1+\alpha} (3/2)^p < 1$ for sufficiently large m (this condition defines m). We write

$$\int_{A_{j-m}} \frac{|u(y)|^p}{|y|^\alpha} dy = \left(\int_{A_{j-m} \setminus F} + \int_{A_{j-m} \cap F} \right) \frac{|u(y)|^p}{|y|^\alpha} dy$$

and we repeat (11) with $j - m$ in place of j, then with $j - 2m$ in place of j, and so on; we stop when $j - km \in \{0, 1, \ldots, m-1\}$. We obtain

$$\int_{B^c \cap F} \frac{|u(x)|^p}{|x|^\alpha} dx \le \left(\sum_{n=1}^\infty \gamma^n \right) \cdot \left(\sum_{j=m}^\infty \int_{A_{j-m} \setminus F} \frac{|u(y)|^p}{|y|^\alpha} dy + \int_{A \cap F} \frac{|u(y)|^p}{|y|^\alpha} dy \right).$$

Hence by Property 1 we have

$$\begin{aligned}\int_{B^c}\frac{|u(x)|^p}{|x|^\alpha}dx &= \left(\int_{B^c\setminus F}+\int_{B^c\cap F}\right)\frac{|u(x)|^p}{|x|^\alpha}dx\\ &\le \left(\sum_{n=0}^\infty \gamma^n\right)\int_{B^c\setminus F}\frac{|u(y)|^p}{|y|^\alpha}dy+\left(\sum_{n=1}^\infty \gamma^n\right)\int_{A\cap F}\frac{|u(y)|^p}{|y|^\alpha}dy\\ &\le \frac{1}{1-\gamma}\frac{2^{p+1}R^{d+\alpha}}{S}\int_{B^c}\int_{B^c}\frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}}dy\,dx\\ &\qquad+\frac{\gamma}{1-\gamma}\int_A\frac{|u(y)|^p}{|y|^\alpha}dy. \qquad \square\end{aligned}$$

As we see, the proof of Lemma 3.3 is a modification of that of Lemma 3.1, but the "sweeping out" procedure can terminate at the support of u. The following lemma may be proved by an analogous modification of the proof of Lemma 3.2. The notation is as explained before Lemma 3.2. We also define $Q_k=\bigcup_{n=k}^\infty A_n$ for $k=1,2,\dots$; see (7).

LEMMA 3.4. *Let $\alpha<1$. There exist $m=m(\lambda,\alpha,d,p)$ and $c=c(\lambda,\alpha,d,p)$ such that for all functions $u\in C_c(D)$ we have*

(12)
$$\int_{Q_m}\frac{|u(x)|^p}{\delta_x^\alpha}dx\le c\left(\int_{Q_m}\int_{Q_m}\frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}}dy\,dx+\int_{Q\setminus Q_m}\frac{|u(x)|^p}{\delta_x^\alpha}dx\right).$$

Note that in Lemmas 3.2 and 3.4 we only need that D be locally Lipschitz, namely, $Q_z(\rho)$ should be a Lipschitz box and (6) should hold for $x\in Q_z(\rho)$. We also note that the critical values $\alpha=d$ and $\alpha=1$ were excluded from our considerations in Lemmas 3.1 and 3.3 and Lemmas 3.2 and 3.4, respectively; see also Section 2.

4. Final conclusions and further results

Proof of Theorem 1.1. Let $u\in C_c(D)$. Assume (T4). If $\alpha<d$, then we pick (a small) $r>0$ such that $\operatorname{supp} u\cap B(0,r)=\emptyset$ and we see that the inequality in Lemma 3.1 is stronger than (1). Similarly, if $\alpha>d$ and $r>0$ is such that $\operatorname{supp} u\cap B(0,2^m r)=\emptyset$, where m is the constant of Lemma 3.3, then the inequality in Lemma 3.3 is stronger than (1).

Assume now (T3). Let $z\in\partial D$. We take ρ large enough, as we may, so that $\operatorname{supp} u\subset Q_z(\rho)$ in the case when $\alpha>1$, and $\operatorname{supp} u\subset Q_m$ in the case when $\alpha<1$, where m is the constant of Lemma 3.4. Then the inequalities in Lemmas 3.2 and 3.4 are stronger than (1).

Assume (T1). We put $D_1=\{x\in D:\delta_x\ge\tilde\rho\}$ and $D_2=\{x\in D:\delta_x<\tilde\rho\}$, where $\tilde\rho$ is small enough. By Lemma 3.2 we get

$$\int_{D_2}\frac{|u(x)|^p}{\delta_x^\alpha}dx\le Nc\int_{D_2}\int_{D_2}\frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha}}dy\,dx, \tag{13}$$

since D_2 may be covered by sets of the form $Q_{z_k}(\rho)$ such that every $x \in D_2$ belongs to at most $N = N(\lambda, d)$ sets of type $Q_{z_k}(\rho)$. This is possible for sufficiently small $\tilde{\rho} < \rho$, and such a $\tilde{\rho}$ may be chosen to depend only on λ, r_0 and d. We take $F = F(u, D; R, S)$, where R and S are any positive numbers such that the assumptions in Property 3 are satisfied for $E_1 = D_1$ and $E_2 = D_2$ (e.g., $R = \operatorname{diam} D/\rho$ and $S = |D_2|/\rho^d$). By Property 3 we have

$$\int_{D_1 \cap F} \frac{|u(x)|^p}{\delta_x^\alpha} dx \le c \int_{D_2} \frac{|u(x)|^p}{\delta_x^\alpha} dx. \tag{14}$$

Furthermore, from Property 1 we get

$$\int_{D_1 \setminus F} \frac{|u(x)|^p}{\delta_x^\alpha} dx \le c \int_{D_1 \setminus F} \int_D \frac{|u(x) - u(y)|^p}{|x-y|^{d+\alpha}} dy\, dx. \tag{15}$$

Now (13), (14) and (15) give (1).

We finally consider (T2) and assume that D is a complement of a bounded Lipschitz domain and $\alpha > 1$, $\alpha \ne d$. Let M be such that $D \subset B(0, M)$ and let m be a natural number; in a similar manner as before we prove that

$$\int_{D_{2^{m+1}M}} \frac{|u(x)|^p}{\delta_x^\alpha} dx \le c \int_{D_{2^{m+1}M}} \int_{D_{2^{m+1}M}} \frac{|u(x) - u(y)|^p}{|x-y|^{d+\alpha}} dy\, dx, \tag{16}$$

where $D_{2^{m+1}M} = D \cap B(0, 2^{m+1}M)$. Now (1) follows easily from Lemma 3.1 if $\alpha < d$ and from Lemma 3.3 applied to $r = 2M$ if $\alpha > d$ (note that $|x| \le 2\delta_x$ if $x \in B(0, 2M)^c$).

The proof of (1) when D is a complement of a bounded Lipschitz domain and $\alpha < 1$ is similar and will be omitted. $\square$

Note that if (1) holds with $c = c_i$ for disjoint open sets D_i, then (1) holds also for $\bigcup_i D_i$ with $c = \sup_i c_i$, because the left hand side of (1) is a σ-additive function of the domain and the right hand side is superadditive. Now let $\alpha > 1$. By Theorem 1.1 the inequality (1) holds for $D_1 = (0, 1) \subset \mathbb{R}$ with constant $c_1 = c_1(\alpha)$ and for $D_2 = (0, \infty)$ with constant $c_2 = c_2(\alpha)$. Using dilations (see the remark at the end of Section 1), translations and a reflection $x \mapsto -x$ we see that (1) holds for every interval with the same constant c_1 and for every half-line with the same constant c_2. Thus we get (1) for every open set $D \subset \mathbb{R}$ and $\alpha > 1$.

Let us mention that if $\alpha < 1$ and D is a bounded Lipschitz domain, then

$$\int_D \frac{|u(x)|^p}{\delta_D(x)^\alpha} dx \le c \left(\int_D \int_D \frac{|u(x) - u(y)|^p}{|x-y|^{d+\alpha}} dx\, dy + \int_D |u(x)|^p dx \right) \tag{17}$$

for all $u \in C_c(D)$. This is proved by omitting the factor $\delta_x^{-\alpha}$ in the second integral of right hand side of (12). The details are similar to the proof of part (T1) of Theorem 1.1 and will be omitted. As communicated to the author by Z.-Q. Chen, (17) may be found in [CS]. By the counterexamples in Section 2 the additional term on the right hand side of (17) is necessary and (1) is not true in this case, even for C^∞ domains.

REMARK. The reader may have noticed that the right hand side of (1) is infinite for nonzero $u \in C_c^1(D)$ if $\alpha \geq p$. However, when $u \notin C_c^1(D)$, the right hand side of (1) may be finite. Consider, for example, the function u on the line that is equal to the Cantor function on $[0,1]$, symmetric with respect to $x=1$, and vanishes outside of $[0,2]$. Taking $D=(-1,3)$, $0<\alpha<1-\log 2/\log 3$, and an arbitrary $p>0$, one easily checks that the right hand side of (1) is finite, which corresponds to the fact that a.e. u is locally constant. For this reason we do not exclude the case $\alpha \geq p$ from our considerations, even though the application mentioned in the Introduction pertain to $C_c^\infty(D)$. This discussion is related to the problem whether for an arbitrary measurable function u the finiteness of the right hand side of (1) implies that u is essentially constant on D. For example, if $p=\alpha=1$, then this implication is true; see [Br], [H]. Similar results may be found in [P]. Finally let us note that the function u considered above gives a negative answer to Problem 2 in [Br].

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Institute of Mathematics, Wrocław University of Technology, Wrbrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail address: `bdyda@im.pwr.wroc.pl`