ON THE AVERAGE OF THE SCALAR CURVATURE OF MINIMAL HYPERSURFACES OF SPHERES WITH LOW STABILITY INDEX

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ABSTRACT. In this paper we show that if the stability index of M is equal to $n+2$, then the average of the function $|A|^2$ is less than or equal to $n-1$. Moreover, if this average is equal to $n-1$, then M must be isometric to a Clifford minimal hypersurface.

1. Introduction

Let $M \subset S^n$ be a non totally geodesic compact minimal hypersurface embedded in the n-dimensional unit sphere S^n. Notice that for any $p \in M$ the tangent space of M at p, T_pM, is an $(n-1)$-dimensional subspace of $\mathbb{R}^{n+1}$. This implies that, up to a sign, there exists a unique unit vector $\nu(p)$ in the orthogonal complement of T_pM, such that $\langle \nu(p), p \rangle = 0$. When M is orientable, we can choose $\nu(p)$ so that it defines a smooth map $\nu : M \to S^n$. This map is known as the Gauss map.

It is known that the embeddedness of M implies its orientability [3].

It is not difficult to show that the image of the differential of the Gauss map at p, $d\nu_p : T_pM \to T_pS^n$, is contained in T_pM. Therefore the map $A_p(v) = -d\nu_p(v)$ is a linear map from T_pM to T_pM; this map is known as the shape operator of M at p. It can be shown that the shape operator is symmetric. Therefore there are $n-1$ real eigenvalues of A_p. These eigenvalues are known as the principal curvatures of M at p. Let us denote these eigenvalues by $\kappa_1(p), \ldots, \kappa_{n-1}(p)$. The minimality of M is equivalent to the condition $\kappa_1(p) + \cdots + \kappa_{n-1}(p) = 0$ for all $p \in M$. The function $|A|^2 : M \to \mathbb{R}$ is given by $|A|^2(p) = \kappa_1^2(p) + \cdots + \kappa_{n-1}^2(p)$. As we will see later, the study of this function is important in order to deduce properties of the hypersurface M.

Let us denote the space of smooth functions of M by $C^\infty(M)$ and the Laplacian on M by $\Delta : C^\infty(M) \to C^\infty(M)$. The stability operator $J : C^\infty(M) \to C^\infty(M)$ is defined by $J(f) = -\Delta f - |A|^2 f - (n-1)f$.

The stability index of M, $\mathrm{ind}(M)$, is defined as the number of negative eigenvalues (counted with multiplicity) of the stability operator. The theory of elliptic operators on compact manifolds guarantees that this index is finite. Since the spectrum of the Laplacian on Euclidean spheres is known, a direct verification shows that the stability index of M is 1 when M is a totally geodesic minimal hypersurface of S^n, i.e., when

$$M = \{x \in \mathbb{R}^{n+1} : |x| = 1 \quad \text{and} \quad \langle x, w \rangle = 0\},$$

where w is a nonzero fixed vector in $\mathbb{R}^{n+1}$. In this case the function $|A|^2 : M \to \mathbb{R}$ vanishes identically. Also, a direct verification shows that the stability index of M is $n + 2$ when M is a Clifford minimal hypersurface of S^n, i.e., when M is isometrically a set of the form

$$M_{kl} = \left\{ (x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{l+1} : |x|^2 = \frac{k}{n-1} \quad \text{and} \quad |y|^2 = \frac{l}{n-1} \right\},$$

where k and l are positive integers such that $k + l = n - 1$. For minimal Clifford hypersurfaces the function $|A|^2 : M \to \mathbb{R}$ takes the value $n - 1$ at every point in M. It can be shown that the stability index for any other minimal hypersurface in S^n is greater than or equal to $n + 2$ (see, e.g., [7]). A natural conjecture in this direction is:

CONJECTURE A. *The only minimal immersed hypersurfaces $M \subset S^n$ with stability index $n + 2$ are the minimal Clifford hypersurfaces.*

For surfaces, i.e., when $n = 3$, this conjecture was proved by F. Urbano [12]. Urbano's result will also follow as a corollary of the main result in this paper.

Regarding the function $|A|^2 : M \to \mathbb{R}$ defined on compact minimal hypersurfaces of S^n, it is known that if this function is constant, then this constant must be greater than or equal to $n - 1$ (see, e.g., [11]). It is also known that if this function is constant and the constant is equal to $n - 1$, the dimension of M, then M must be a minimal Clifford hypersurface of S^n (see, e.g., [2], [4]). In [11] we posed the following conjecture which, if true, would give a far-reaching generalization of the properties of the function $|A|^2 : M \to \mathbb{R}$ described above.

CONJECTURE B. *Let M be a non totally geodesic compact minimal embedded hypersurface of S^n. If the function $|A|^2 : M \to \mathbb{R}$ denotes the norm squared of the shape operator, then $\int_M |A|^2 \geq \int_M (n-1)$. Moreover, if $\int_M |A|^2 = \int_M (n-1)$, then M must be a minimal Clifford hypersurface.*

For surfaces, the first part of Conjecture B follows from the Gauss-Bonnet theorem, the minimality of M and the fact that if M is a sphere immersed in S^3, then M must be totally geodesic [1]. The second part of this conjecture for surfaces is equivalent to Lawson's Conjecture: *The only embedded minimal torus in S^3 is the Clifford torus.* Another partial solution of Conjecture B states that if $M \subset S^n$ is a minimal immersed hypersurface with two principal curvatures at every point in M and $\int_M |A|^2 = \int_M (n-1)$, then M must be a Clifford hypersurface [9].

In this paper we will show that if M is an immersed minimal oriented hypersurface of S^n and the stability index of M is $n+2$, then $\int_M |A|^2 \leq \int_M (n-1)$. Moreover, we will show that if the stability index of M is $n+2$ and $\int_M |A|^2 = \int_M (n-1)$, then M must be a minimal Clifford hypersurface. In particular, Conjecture B implies Conjecture A for embedded hypersurfaces.

REMARK. For a minimal hypersurface M on S^n, the scalar curvature function $k : M \to \mathbb{R}$ satisfies $(n-1)(n-2)k(m) = 1 - |A|^2(m)$ for every $m \in M$ [11]. Therefore Conjecture B can be rewritten in terms of the average of the scalar curvature instead of the average $\int_M |A|^2$.

2. Preliminaries

Let $\phi : M \longrightarrow S^n$ be a minimal immersion of a compact oriented $(n-1)$-dimensional manifold into the unit sphere.

We identify M with the set $\phi(M) \subset \mathbb{R}^{n+1}$ and the space $T_m M$ with the linear subspace $d\phi_m(T_m M)$ of $\mathbb{R}^{n+1}$.

Let $w \in \mathbb{R}^{n+1}$ be fixed. We define functions $l_w : M \longrightarrow \mathbb{R}$ and $f_w : M \longrightarrow \mathbb{R}$ by

$$l_w(m) = \langle m, w \rangle, \quad f_w(m) = \langle \nu(m), w \rangle \quad \text{for all} \quad m \in M.$$

A direct computation using the minimality of M and the Codazzi equations gives the following result.

PROPOSITION 2.1. *The gradient and the Laplacian of the functions l_w and f_w are given by*

$$\nabla l_w = w^T, \qquad\qquad \nabla f_w = -A(w^T),$$
$$-\Delta l_w = (n-1)l_w, \qquad -\Delta f_w = \|A\|^2 f_w.$$

Here w^T denotes the tangential component of w on the tangent space $T_m M$.

The following lemma is based on the minimax characterization of eigenvalues of elliptic operators.

LEMMA 2.1. *Let $M \subset S^n$ be a non totally geodesic minimal compact oriented hypersurface of S^n and let ρ denote an eigenfunction of the stability*

operator J associated to the first eigenvalue of J. If $\mathrm{ind}(M) = n+2$, then for every smooth function $f : M \longrightarrow \mathbb{R}$ with $\int_M \rho f = 0$ we have

$$\int_M |\nabla f|^2 \geq \int_M \|A\|^2 f^2,$$

with equality only if $-\Delta f = \|A\|^2 f$.

Proof. Notice that Proposition 2.1 implies that the functions f_w satisfy $J(f_w) = -(n-1)f_w$. These functions span an $(n+1)$-dimensional space because M is non totally geodesic [7]. Therefore $-(n-1)$ is an eigenvalue of J with multiplicity at least $n+1$. Hence, if $\mathrm{ind}(M) = n+2$, then $-(n-1)$ is the second eigenvalue of J. Recall that, from the theory of elliptic operators, the multiplicity of the first eigenvalues must be 1. Since $-(n-1)$ is the second eigenvalue of J, the lemma follows using the minimax characterization of eigenvalues for elliptic operators.   □

The proof of the main result is based on a technique that uses the group of conformal applications from S^n to S^n, which was introduced by Li and Yau [5].

Let B^{n+1} be the open unit ball in $\mathbb{R}^{n+1}$. For each point $g \in B^{n+1}$ we consider the map

$$F_g(p) = \frac{p + (\mu\langle p, g\rangle + \lambda)g}{\lambda(\langle p, g\rangle + 1)}$$

for all $p \in S^n$, where $\lambda = (1 - |g|^2)^{-1/2}$ and $\mu = (\lambda - 1)|g|^{-2}$. A direct verification (see [6]) shows that F_g is a conformal transformation from S^n to S^n. Moreover, for every $v, w \in T_p S^n$, its differential dF_g satisfies

$$\langle dF_g(v), dF_g(w)\rangle = \frac{1 - |g|^2}{(\langle p, g\rangle + 1)^2}\langle v, w\rangle.$$

In [5], Li and Yau proved the following lemma:

LEMMA 2.2.   *If h is a Riemannian metric on M and $\phi : (M, h) \longrightarrow S^n$ is a conformal immersion, then there exists $g \in B^{n+1}$ such that $\int_M (F_g \circ \phi)dv = (0, \ldots, 0)$. Here the differential of volume dv is taken with respect to the metric h.*

3. Proof of the main result

We start this section with the following lemma, which gives an expression for the volume of M, $|M|$, in terms of one of the coordinate functions.

LEMMA 3.1.   *Let M^k be a k-dimensional compact minimal manifold immersed in S^n. If $w \in \mathbb{R}^{n+1}$ is a fixed vector such that the function $1 + l_w(m)$*

is always positive on M, then

$$|M| = \int_M 1 = \int_M \frac{1 - |w|^2}{(1 + l_w)^2} + \int_M \frac{|w|^2 - l_w^2 - \frac{2}{k}|\nabla l_w|^2}{(1 + l_w)^2}.$$

Proof. Let us define $f : M^k \longrightarrow \mathbb{R}$ by $f = \ln(1 + l_w)$. A direct verification shows that $\nabla f = \nabla l_w/(1 + l_w)$ and

$$
\begin{aligned}
\Delta f = \operatorname{div} \nabla f &= \frac{-k l_w}{1 + l_w} - \frac{|\nabla l_w|^2}{(1 + l_w)^2} \\
&= -\frac{k}{2} \left(\frac{2 l_w (l_w + 1) + \frac{2}{k}|\nabla l_w|^2}{(1 + l_w)^2} \right) \\
&= -\frac{k}{2} \left(\frac{2 l_w + l_w^2 + l_w^2 + 1 - 1 + \frac{2}{k}|\nabla l_w|^2}{(1 + l_w)^2} \right) \\
&= -\frac{k}{2} \left(1 + \frac{-1 + l_w^2 + |w|^2 - |w|^2 + \frac{2}{k}|\nabla l_w|^2}{(1 + l_w)^2} \right) \\
&= -\frac{k}{2} \left(1 - \frac{1 - |w|^2}{(1 + l_w)^2} - \frac{|w|^2 - l_w^2 - \frac{2}{k}|\nabla l_w|^2}{(1 + l_w)^2} \right).
\end{aligned}
$$

Since $\int_M \Delta f = 0$, the lemma follows. $\qquad\square$

LEMMA 3.2.   *Let $\phi : M \longrightarrow S^n$ be a smooth map, $g \in B^{n+1}$ and $\{e_i\}_{i=1}^{n+1}$ be an orthonormal basis of $\mathbb{R}^{n+1}$. If we define $h_i : M \longrightarrow \mathbb{R}$ by $h_i(m) = \langle F_g(\phi(m)), e_i \rangle$ and $s_i : M \longrightarrow \mathbb{R}$ by $s_i(m) = \langle \phi(m), e_i \rangle$, then*

$$\sum_{i=1}^{n+1} |\nabla h_i|^2(m) = \frac{1 - |g|^2}{(1 + \langle \phi(m), g \rangle)^2} \sum_{i=1}^{n+1} |\nabla s_i|^2(m).$$

Proof. Let $\{v_i\}_{i=1}^{n-1}$ be an orthonormal basis of $T_m M$. We have

$$
\begin{aligned}
|\nabla h_i|^2(m) &= \sum_{j=1}^{n-1} \left(v_j(h_i) \right)^2 \\
&= \sum_{j=1}^{n-1} \left(\langle (dF_g)_{\phi(m)}(d\phi(v_j)), e_i \rangle \right)^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{i=1}^{n+1} |\nabla h_i|^2(m) &= \sum_{i=1}^{n+1} \sum_{j=1}^{n-1} \left(\langle (dF_g)_{\phi(m)}(d\phi(v_j)), e_i \rangle \right)^2 \\
&= \sum_{j=1}^{n-1} \| (dF_g)_{\phi(m)}(d\phi(v_j)) \|^2
\end{aligned}
$$

$$= \sum_{j=1}^{n-1} \frac{1 - |g|^2}{(1 + \langle \phi(m), g \rangle)^2} \|d\phi(v_j)\|^2$$

$$= \sum_{j=1}^{n-1} \frac{1 - |g|^2}{(1 + \langle \phi(m), g \rangle)^2} \sum_{i=1}^{n+1} \left(v_j(s_i) \right)^2$$

$$= \frac{1 - |g|^2}{(1 + \langle \phi(m), g \rangle)^2} \sum_{i=1}^{n+1} |\nabla s_i|^2 (m). \qquad \square$$

THEOREM 3.1.   *Let M be a compact oriented $(n-1)$-dimensional manifold and let $\phi : M \to S^n$ be a minimal immersion of M in S^n. If the stability index of M is $n+2$, then $\int_M \|A\|^2 \leq \int_M (n-1)$. Moreover, if $\int_M \|A\|^2 = \int_M (n-1)$, then M must be isometric to a minimal Clifford hypersurface.*

Proof. Let ρ be a first eigenfunction of the stability operator J. We can assume ρ is always positive. Let us consider the Riemannian manifold (M, h), where h is the metric $\rho^{2/(n-1)}$ times the metric induced by ϕ. Since $\phi(M, h) \to S^n$ is a conformal map, by Lemma 2.2 we can find $g \in B^{n+1}$ such that

$$\int_{(M,h)} F_g \circ \phi = \int_M \rho F_g \circ \phi = (0, \ldots, 0).$$

The above equality implies that the functions $h_i = \langle F_g(\phi(m)), e_i \rangle$ with $e_1 = (1, 0, \ldots, 0), \ldots, e_{n+1} = (0, \ldots, 0, 1)$, are perpendicular to the function ρ, i.e., $\int_M \rho h_i = 0$. By Lemma 2.1 we have

$$(1) \qquad \sum_{i=1}^{n+1} \int_M |\nabla h_i|^2 \geq \sum_{i=1}^{n+1} \int_M \|A\|^2 h_i^2 = \int_M \|A\|^2,$$

with equality only if $-\Delta h_i = \|A\|^2 h_i$. On the other hand, by Lemma 3.2 we have

$$(2) \qquad \sum_{i=1}^{n+1} |\nabla h_i|^2 = \frac{1 - |g|^2}{(1 + \langle \phi(m), g \rangle)^2} \sum_{i=1}^{n+1} |\nabla l_{e_i}|^2 = (n-1) \frac{1 - |g|^2}{(1 + \langle \phi(m), g \rangle)^2}.$$

Therefore, integrating the equations (1) and (2) and using Lemma 3.1, we get

$$\int_M \|A\|^2 \leq (n-1) \int_M \frac{1 - |g|^2}{(1 + \langle \phi(m), g \rangle)^2}$$

$$= (n-1)|M| - (n-1) \int_M \frac{|g|^2 - l_g^2 - \frac{2}{n-1}|\nabla l_g|^2}{(1 + l_g)^2},$$

with equality only if $-\Delta h_i = \|A\|^2 h_i$. Since the expression

$$\int_M \frac{|g|^2 - l_g^2 - \frac{2}{n-1}|\nabla l_g|^2}{(1 + l_g)^2}$$

is positive unless $g = 0$, we have $\int_M \|A\|^2 \leq (n-1)|M|$. Moreover, if $\int_M \|A\|^2 = (n-1)|M|$, then $g = 0$. Therefore, for $i = 1, \ldots, n+1$ we have $h_i = l_{e_i}$ and

$$\|A\|^2 h_i = \|A\|^2 l_{e_i} = -\Delta h_i = -\Delta l_{e_i} = (n-1)l_{e_i}.$$

The above equality implies that $\|A\|^2 \equiv n-1$, and hence that M is isometric to a Clifford hypersurface [2]. $\square$

COROLLARY 3.1.   *If $M \subset S^3$ is a minimal surface with index 5, then M is a Clifford torus.*

Proof. Since the index of M is 5, M is not totally geodesic and therefore is not topologically a sphere [1]. Hence, by the Gauss-Bonnet theorem and the minimality of M, we have $\int_M \|A\| \geq 2|M|$ (see, e.g., [8]). On the other hand, Theorem 3.1 implies that $\int_M \|A\|^2 \leq 2|M|$. Therefore $\int_M \|A\|^2 = 2|M|$. Applying again Theorem 3.1, we obtain that M is isometric to the Clifford torus. $\square$

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