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ON HERZ'S PROJECTION THEOREM

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ABSTRACT. Let G be a locally compact group and H a discrete amenable subgroup. We prove the existence of a contractive projection $\mathcal{Q}$ of $CV_p(G)$ onto $CV_p(H)$ such that $\operatorname{supp} \mathcal{Q}(T) \subset \operatorname{supp} T$.

1. Introduction

Let G be a locally compact group and $1 < p < \infty$. We denote by $cv_p(G)$ the norm closure in $CV_p(G)$ of the set of all convolution operators with compact support. In [4, Corollaire 2] C. Herz proved, for G amenable and H a closed normal subgroup of G, the existence of a contractive projection of $cv_p(G)$ onto $cv_p(H)$. In [1] we were able to deal with non-amenable groups G, but we had to impose strong conditions on H, such as normality in G or compactness of H or $G \in [SIN]_H$. The example $\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \mid n \in \mathbb{Z}\}$ in $GL(2, \mathbb{R})$ was out of reach!

The main result of this work is the following theorem: *Suppose that G is an arbitrary locally compact group and H a discrete amenable subgroup. Then there is a contractive projection $\mathcal{Q}$ of $CV_p(G)$ onto $CV_p(H)$ such that* $\operatorname{supp} \mathcal{Q}(T) \subset \operatorname{supp} T$ *for every $T \in CV_p(G)$.*

2. Preliminaries

The case $H = G$ of the following result is due to V. Losert and H. Rindler [6, Theorem 3].

PROPOSITION 2.1. *Let G be a locally compact group and H a closed subgroup of G. Suppose that H is amenable. For every compact subset K of H, for every open neighborhood U of e in G and for every $\varepsilon > 0$ there is $k \in C_{00}^+(G)$ with $N_1(k) = 1$,* $\operatorname{supp} k \subset U$ *and $N_1({}_{s^{-1}}k_s \Delta_G(s) - k) < \varepsilon$ for $s \in K$.*

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Proof. Let U_1 be a compact neighborhood of e in G contained in U. There is $f \in C_{00}^+(H)$ with $N_1(f) = 1$ and $N_1({}_{s^{-1}}f - f) < \varepsilon$ for every $s \in K$. There is an open neighborhood V of e such that $hVh^{-1} \subset U_1$ for every $h \in \operatorname{supp} f$. Let $g \in C_{00}^+(G)$ with $N_1(g) = 1$ and $\operatorname{supp} g \subset V$. We define, for $x \in G$, $k(x) = \int_H f(h)g(h^{-1}xh)dh$. Then $\operatorname{supp} k \subset U_1$ and

$$\int_G k(x)dx = \int_H f(h)\left(\int_G g(h^{-1}xh)\Delta_G(h)dx\right)dh = 1.$$

For $s \in K$ we have

$$\begin{aligned}\int_G &|k(s^{-1}xs)\Delta_G(s) - k(x)|dx \\ &\leq \int_G \left(\int_H |f(s^{-1}h) - f(h)| g(h^{-1}xh)\Delta_G(h)dh\right)dx \\ &= \int_H |f(s^{-1}h) - f(h)|\left(\int_G g(h^{-1}xh)\Delta_G(h)dx\right)dh \\ &= N_1({}_{s^{-1}}f - f).\end{aligned}$$

□

Lemma 2.2. *Let G be a locally compact non-compact unimodular group, H a closed amenable subgroup of G, K a compact subset of H, $\varepsilon \in (0,\infty)$, $\delta \in (0,\infty)$ and U a neighborhood of e in G. Then there is an m_G-integrable subset V of G and an m_H-integrable subset N of H such that*

(i) $V = V^{-1}$,
(ii) $V \subset U$,
(iii) $m_G(V) > 0$,
(iv) $N \subset K$,
(v) $m_H(N) < \delta$,
(vi) *for every* $x \in K \setminus N$ *we have* $N_1(1_V - 1_{xVx^{-1}}) < \varepsilon\, m_G(V)$.

Proof. We suppose $m_H(K) > 0$. Let

$$\eta = \frac{\delta\varepsilon}{\delta\varepsilon + 3m_H(K)}.$$

A slight modification of Proposition 2.1 implies the existence of $f \in C_{00}^+(G)$ with $f = \check{f}$, $\operatorname{supp} f \subset U$, $N_1(f) = 1$ and $N_1({}_{x^{-1}}f_x - f) < \eta$ for every $x \in K$.

We can find

(1) $N \in \mathbb{N}$,
(2) m_G-integrable subsets $A_1, \dots, A_N$ of G,
(3) $\lambda_1, \dots, \lambda_N \in (0,\infty)$,

such that $A_N \subset \cdots \subset A_1$, $m_G(A_N) > 0$, $A_j^{-1} = A_j$ for every $1 \leq j \leq N$,

$$\sum_{j=1}^{N} \frac{\lambda_j}{m_G(A_j)} 1_{A_j} \leq f$$

and

$$N_1\left(f - \sum_{j=1}^{N} \frac{\lambda_j}{m_G(A_j)} 1_{A_j}\right) < \eta.$$

Let

$$k = \sum_{j=1}^{N} \frac{\lambda_j}{m_G(A_j)} 1_{A_j}.$$

Consider $l = k/N_1(k)$. For every $x \in K$ we have

$$N_1(l - {}_{x^{-1}}l_x) < \frac{2N_1(k-f) + N_1(f - {}_{x^{-1}}f_x)}{1-\eta} < \frac{3\eta}{1-\eta} = \frac{\delta\varepsilon}{m_H(K)}.$$

For $x \in G$ we have

$$N_1(l - {}_{x^{-1}}l_x) = \sum_{j=1}^{N} \frac{\lambda_j^{'}}{m_G(A_j)} N_1(1_{A_j} - 1_{xA_jx^{-1}})$$

with $\lambda_j^{'} = \lambda_j/N_1(k)$ for $1 \leq j \leq N$. We obtain

$$\int_K \left(\sum_{j=1}^{N} \frac{\lambda_j^{'}}{m_G(A_j)} N_1(1_{A_j} - 1_{hA_jh^{-1}})\right) dh < \delta\varepsilon$$

and therefore

$$\sum_{j=1}^{N} \frac{\lambda_j^{'}}{m_G(A_j)} \int_K N_1(1_{A_j} - 1_{hA_jh^{-1}}) dh < \delta\varepsilon.$$

Consequently there is $1 \leq j \leq N$ such that

$$\int_K \frac{N_1(1_{A_j} - 1_{hA_jh^{-1}})}{m_G(A_j)} dh < \delta\varepsilon.$$

Let $A = A_j$. We have $A = A^{-1}$, $A \subset U$. Let finally

$$N = \left\{h \mid h \in K, \frac{N_1(1_A - 1_{hAh^{-1}})}{m_G(A)} \geq \varepsilon\right\}.$$

Then N is a closed subset of H contained in K, and we have

$$\varepsilon m_H(N) \leq \int_N \frac{N_1(1_A - 1_{hAh^{-1}})}{m_G(A)} dh \leq \int_K \frac{N_1(1_A - 1_{hAh^{-1}})}{m_G(A)} dh.$$

This implies $m_H(N) < \delta$. For $x \in K \setminus N$ we get indeed

$$\frac{N_1(1_A - 1_{xAx^{-1}})}{m_G(A)} < \varepsilon. \qquad \square$$

REMARK 2.3. There are similarities between this proof and the method used by W. R. Emerson and F. P. Greenleaf to show that amenability implies Følner's condition (see [2, p. 374] or [7, p. 63]).

Proposition 2.4. *Let G be a locally compact, non-compact, non-discrete unimodular group, H a discrete amenable subgroup of G, F a finite subset of H, $\varepsilon \in (0,\infty)$ and U a neighborhood of e in G. Then there is an open neighborhood V of e in G such that V is relatively compact, $V \subset U$, $V^{-1} = V$ and $N_1(1_V - 1_{xVx^{-1}}) < \varepsilon\, m_G(V)$ for every $x \in F$.*

Proof. Let U_1 be an open relatively compact neighborhood of e in G with $U_1^{-1} = U_1$ and $U_1 \subset U$. According to the Lemma 2.2 there are sets $A \subset U_1$ and $N \subset F$ such that A is m_G-integrable, $A^{-1} = A$, $m_G(A) > 0$, $m_H(N) < 1$, and such that for every $x \in F\backslash N$ the inequality $N_1(1_A - 1_{xAx^{-1}}) < \frac{\varepsilon}{2} m_G(A)$ is satisfied. With m_H denoting the counting measure of H, we have $m_H(N) = 0$ and therefore $N = \emptyset$. Let $B = A \cup \{e\}$. Since the group G is non-discrete, we have $m_G(\{e\}) = 0$ and therefore $m_G(B) = m_G(A)$. We also have $B \subset U_1$ and $B^{-1} = B$. For $x \in F$ we have

$$\frac{N_1(1_B - 1_{xBx^{-1}})}{m_G(B)} \leq \frac{N_1(1_A - 1_{xAx^{-1}})}{m_G(A)}.$$

There is an open set W of G such that $B \subset W$ and $m_G(W) - m_G(B) < \frac{\varepsilon}{4} m_G(A)$. Consider now the set $V = W \cap W^{-1} \cap U_1$. For $x \in F$ we can write

$$\frac{N_1(1_V - 1_{xVx^{-1}})}{m_G(V))} \leq 2\frac{N_1(1_V - 1_B)}{m_G(B))} + \frac{N_1(1_B - 1_{xBx^{-1}})}{m_G(B))}.$$

We have

$$N_1(1_V - 1_B) = m_G(V) - m_G(B) < \frac{\epsilon}{4} m_G(A).$$

Hence we obtain, for every $x \in F$,

$$\frac{N_1(1_V - 1_{xVx^{-1}})}{m_G(V)} < \varepsilon. \qquad \square$$

Proposition 2.5. *Let G be a non-discrete, non-compact locally compact unimodular group, H a discrete amenable subgroup, U a neighborhood of e in G, K a compact subset of G and $\varepsilon \in (0,\infty)$. Then there is an open relatively compact neighborhood V of e in G, with $V^{-1} = V$, $V \subset U$ and*

$$\int_K |1_{HV}(x) - 1_{VH}(x)| dx < \varepsilon\, m_G(V).$$

Proof. We suppose that $e \in K$. There is a compact neighborhood U_0 of e in G with $U_0^{-1} = U_0$, $U_0 \subset U$ and $(U_0)^2 \cap H = \{e\}$. Let $F_0 = (KU_0 \cup U_0K) \cap H$. Then F_0 is a finite non-empty set. By Lemma 2.2 there is an open neighborhood V of e in G such that $V = V^{-1}$, $V \subset U_0$ and

$$N_1(1_V - 1_{xVx^{-1}}) < \frac{\varepsilon\, m_G(V)}{m_H(F_0)}$$

for every $x \in F_0$. Consider

$$I = \{h \in H \mid \; Vh \cap K \neq \emptyset \text{ or } hV \cap K \neq \emptyset\}.$$

Then $I \subset F_0$, $K \cap VH = \bigsqcup_{h\in I} K \cap (Vh)$ and $K \cap HK = \bigsqcup_{h\in I} K \cap (hV)$. Consequently

$$1_K |1_{VH} - 1_{HV}| \leq \sum_{h\in I} 1_K |1_{Vh} - 1_{hV}|$$

and therefore

$$\begin{aligned}\int_G 1_K(x)|1_{VH}(x) - 1_{HV}(x)|dx &\leq \int_G \left(\sum_{h\in I} 1_K(x)|1_{Vh}(x) - 1_{hV}(x)|\right)dx \\ &= \sum_{h\in I}\int_G 1_K(x)|1_{Vh}(x) - 1_{hV}(x)|dx \leq \sum_{h\in I}\int_G |1_{Vh}(x) - 1_{hV}(x)|dx \\ &= \sum_{h\in I} N_1(1_V - 1_{hVh^{-1}}) < \frac{|I|\varepsilon m_G(V)}{m_H(F_0)}. \qquad \square\end{aligned}$$

Corollary 2.6. *Let G be a non-discrete, non-compact locally compact unimodular group, H a discrete amenable subgroup, U a neighborhood of e in G, K a compact subset of G and $\varepsilon \in (0,\infty)$. Then there is a relatively compact open neighborhood V of e in G, with $V^{-1} = V$, $V \subset U$ and*

$$\int_K |1_{HV}(x) - 1_{VH}(x)|dx < \varepsilon\; m_{G/H}(\omega(V)),$$

where ω is the canonical map of G onto G/H, m_H the counting measure of H, m_G a left invariant measure on G and $m_{G/H}$ the corresponding measure on G/H.

Proof. Let K_0 be a compact neighborhood of e in G and $f_0 \in C_{00}^+(G)$ with $f_0(x) = 1$ on K_0. By Proposition 2.5 there is an open neighborhood V of e in G with $V^{-1} = V$, $V \subset K_0 \cap U$ and

$$\int_K |1_{HV}(x) - 1_{VH}(x)|dx < \frac{\varepsilon\; m_G(V)}{\sup\{(T_H f_0)(\dot{x}) \mid \dot{x} \in G/H\}}.$$

The inequality $1_V \leq 1_{VH} f_0$ implies

$$\begin{aligned} m_G(V) &\leq \int_{G/H} 1_{\omega(V)}(\dot{x})\left(\int_H f_0(xh)dh\right)d\dot{x} \\ &\leq m_{G/H}(\omega(V))\sup\{(T_H f_0)(\dot{x}) \mid \dot{x} \in G/H\}. \qquad \square\end{aligned}$$

Lemma 2.7. *Let G be a locally compact group and H a closed subgroup of G and suppose that $\Delta_G(h) = \Delta_H(h)$ for $h \in H$. Let $1 < p < \infty$, $\varphi \in$*

$C_{00}(H,\mathbb{C})$, $k \in C_{00}(G,\mathbb{C})$ and U be a relatively compact open neighborhood of e in G. Then the following inequality holds:

$$N_p((\varphi *_H k)1_{UH}) \leq m_{G/H}(\omega(U))^{1/p} N_p(\varphi)\ \|T_H(|k|)\|_\infty^{1/p}\ \|T_H(|\check{k}|)\|_\infty^{1/p'}.$$

Proof. (1) $N_1((\varphi *_H k)1_{UH}) \leq m_{G/H}(\omega(U))\ N_1(\varphi)\ \|(T_H(|k|)\|_\infty$.
We have

$$\begin{aligned} N_1((\varphi *_H k)1_{UH}) &= \int_{G/H} 1_{\omega(U)}(\dot{x}) \left(\int_H |(\varphi *_H k)(xh)| dh \right) d\dot{x} \\ &\leq \int_{G/H} 1_{\omega(U)}(\dot{x}) N_1(\varphi)\ \|T_H(|k|)\|_\infty d\dot{x}. \end{aligned}$$

(2) $\|(\varphi *_H k)1_{UH}\|_\infty \leq \|\varphi\|_\infty\ \|T_H(|\check{k}|)\|_\infty$.
For every $x \in G$, we have

$$\begin{aligned} |1_{UH}(x)(\varphi *_H k)(x)| &\leq \left| \int_H \varphi(h) k(h^{-1}x) dh \right| \\ &\leq \|\varphi\|_\infty \int_H |\check{k}(x^{-1}h)| dh \leq \|\varphi\|_\infty\ \|T_H(|\check{k}|)\|_\infty. \end{aligned}$$

(3) It suffices to prove that for every step function $f \in \mathcal{L}^p_{\mathbb{C}}(H)$ with $N_p(f) = 1$ one has

$$N_p((f *_H k)1_{UH}) \leq (m_{G/H}(\omega(U)))^{1/p}\ \|T_H(|k|)\|_\infty^{1/p}\ \|T_H(|\check{k}|)\|_\infty^{1/p'}.$$

We will show that for every step function $g \in \mathcal{L}^{p'}_{\mathbb{C}}(G)$ with $N_{p'}(g) = 1$ one has

$$\left| \int_G f *_H k(x) 1_{UH}(x) g(x) dx \right| \leq (m_{G/H}(\omega(U)))^{1/p} \|T_H(|k|)\|_\infty^{1/p}\ \|T_H(|\check{k}|)\|_\infty^{1/p'}.$$

There exist $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{C}$, and disjoint integrable subsets $E_1, \ldots, E_m$ of H with $f = \sum_{j=1}^m a_j 1_{E_j}$ and $a_1 \ldots a_m \neq 0$. Let

$$B = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1\}.$$

For every $z \in B$, let $f_{(z)}$ denote the step function $\sum_{j=1}^m |a_j|^{(1-z)p} e^{i\vartheta_j} 1_{E_j}$, where $a_j = |a_j| e^{i\vartheta_j}$ with $0 \leq \vartheta_j < 2\pi$ for $1 \leq j \leq m$. Similarly, for the step function $g = \sum_{l=1}^n b_l 1_{F_l}$ with disjoint integrable subsets $F_1, \ldots, F_n$ of G and $b_1 \ldots b_n \neq 0$, define $g_{(z)} = \sum_{l=1}^n |b_l|^{(1-z)p'} e^{i\varphi_l} 1_{F_l}$.

For any step function $\varphi \in \mathcal{L}^p_{\mathbb{C}}(H)$ and $z \in B$ set

$$T_z \varphi = \frac{1_{UH}(\varphi *_H k)}{(m_{G/H}(\omega(U)))^{1-z}}.$$

For $z \in B$ let

$$F(z) = \int_G (T_z f_{(z)})(x) g_{(z)}(x) dx.$$

Then F is continuous on B, analytic on the interior of B and bounded on B. In fact, we have on B

$$|F(z)| \leq \frac{1}{\min\{m_{G/H}(\omega(G/H)),1\}} \sum_{j=1}^{m} \sum_{l=1}^{n} \max\{|a_j|^p,1\} \max\{|b_l|^{p'},1\} \cdot \left| \int_G 1_{E_j} *_H k(x) 1_{UH}(x) 1_{F_l}(x) dx \right|.$$

For $y \in \mathbb{R}$ we have $|F(iy)| \leq N_1(f_{(iy)})\ \|g_{(iy)}\|_\infty$ with

$$N_1(f_{(iy)}) = \int_G \frac{1_{UH}(x)|(f_{(iy)} *_H k)(x))|}{\left|m_{G/H}(\omega(H))^{1-iy}\right|} dx \leq \frac{N_1(f_{(iy)})\|T_H(|k|)\|_\infty}{m_{G/H}(\omega(U))}$$

according to (1). But $N_1(f_{(iy)}) = N_p(f)^p = 1$ and $|g_{(iy)}|_\infty = 1$, and consequently $|F(iy)| \leq \|T_H(|k|)\|_\infty$.

For $y \in \mathbb{R}$ we also have

$$|F(1+iy)| \leq \|T_{(1+iy)}\|_\infty\ N_1(g_{(1+iy)})$$

with $\|T_{(1+iy)}\|_\infty = \|1_{UH}(f_{(1+iy)} *_H k)\|_\infty$. Using (2) we get

$$\|T_{(1+iy)}\|_\infty \leq \|f_{(1+iy)}\|_\infty \|T_H(|\check{k}|)\|_\infty.$$

The relations $\|f_{(1+iy)}\|_\infty = 1$ and $N_1(g_{(1+iy)}) = N_{p'}(g)^{p'}$ then imply $|F(1+iy)| \leq \|T_H(|\check{k}|)\|_\infty$.

By the Phragmén-Lindelöf maximum principle, for every $t \in (0,1)$ we have $|F(t)| \leq \|T_H(|k|)\|_\infty^{1-t}\ \|T_H(|\check{k}|)\|_\infty^t$. We conclude from $f_{(1-1/p)} = f$, $g_{(1-1/p)} = g$ and

$$F\left(1-\frac{1}{p}\right) = \frac{\int_G 1_{UH}(x)(f *_H k)(x) g(x) dx}{(m_{G/H}(\omega(U)))^{1/p}}. \qquad \square$$

3. A projection theorem for cv_p

We use the notations and results of [1].

Proposition 3.1. *Let G be a non-discrete, non-compact locally compact unimodular group, H a discrete amenable subgroup, U a neighborhood of e in G, $\varepsilon \in (0,\infty)$, $m \in \mathbb{N}$, $p \in (1,\infty)$, and let m sequences $(r_n^{(j)})_{n=1}^\infty$, $j = 1,\dots,m$, of $\mathcal{L}_\mathbb{C}^p(H)$ and m sequences $(s_n^{(j)})_{n=1}^\infty$, $j = 1,\dots,m$, of $\mathcal{L}_\mathbb{C}^{p'}(H)$ be given. Suppose that $\sum_{n=1}^\infty N_p(r_n^{(j)}) N_{p'}(s_n^{(j)}) < \infty$ for every $1 \leq j \leq m$. Then there exist $k,l \in C_{00}^+(G)$ such that* $\operatorname{supp} k \subset U$, $\operatorname{supp} l \subset U$, $\|\Lambda_{k,l}\| \leq 1$ *and for every $1 \leq j \leq m$*

$$\sum_{n=1}^\infty \left| \langle \Lambda_{k,l}(i(S))[r_n^{(j)}],[s_n^{(j)}]\rangle_{L_\mathbb{C}^p(H),L_\mathbb{C}^{p'}(H)} - \langle S[r_n^{(j)}],[s_n^{(j)}]\rangle_{L_\mathbb{C}^p(H),L_\mathbb{C}^{p'}(H)} \right| \leq \varepsilon |||S|||_p$$

for every $S \in CV_p(H)$ [1].

Proof. We suppose that $\varepsilon < 1$. For every $1 \leq j \leq m$ there are sequences $(\varphi_n^{(j)})_{n=1}^\infty, (\psi_n^{(j)})_{n=1}^\infty$ of $C_{00}(H, \mathbb{C})$ such that

$$N_p(r_n^{(j)} - \varphi_n^{(j)}) < \frac{\varepsilon}{3 \cdot 2^{n+1}\big(1 + N_{p'}\big(s_n^{(j)}\big)\big)}$$

and

$$N_{p'}(s_n^{(j)} - \psi_n^{(j)}) < \frac{\varepsilon}{3 \cdot 2^{n+1}\big(1 + N_p\big(r_n^{(j)}\big)\big)}$$

for every $n \in \mathbb{N}$.

For every $1 \leq j \leq m$ and $n \in \mathbb{N}$ we have

$$N_p\big(\varphi_n^{(j)}\big)N_{p'}\big(\psi_n^{(j)}\big) < \frac{1}{9 \cdot 2^{2n+2}} + \frac{2}{3 \cdot 2^{n+1}} + N_p\big(r_n^{(j)}\big)N_{p'}\big(s_n^{(j)}\big)$$

and therefore $\sum_{n=1}^\infty N_p(\varphi_n^{(j)})N_{p'}(\psi_n^{(j)}) < \infty$. Consequently there is $N \in \mathbb{N}$ such that

$$\sum_{n=1+N}^\infty N_p(\varphi_n^{(j)})N_{p'}(\psi_n^{(j)}) < \frac{\varepsilon}{2^5}$$

for every $1 \leq j \leq m$.

Let U_0 be a compact neighborhood of e in G with $U_0^{-1} = U_0$ and $U_0 \subset U$. According to Lemma 1 of [1] there is $k' \in C_{00}^+(G)$ with $\operatorname{supp} k' \subset U_0$, $(\operatorname{supp} k') \cap H = \{e\}$, $\sum_{h\in H} k'(h) = 1$, and $\sum_{h \in H} k'(xh) \leq 1$ for all $x \in G$.

For every $n \in \mathbb{N}$ and $1 \leq j \leq m$ we have $\varphi_n^{(j)} = \operatorname{Res}_H(\varphi_n^{(j)} *_H k')$ and $\psi_n^{(j)} = \operatorname{Res}_H(\psi_n^{(j)} *_H k')$.

Let

$$0 < \varepsilon_1 < \min\left\{ \frac{\varepsilon}{3 \cdot 2^{n+2}\big(1 + N_p\big(\varphi_n^{(j)}\big) + N_p\big(\psi_n^{(j)}\big)\big)} \;\middle|\; 1 \leq n \leq N, 1 \leq j \leq m \right\}.$$

There is a relatively compact open neighborhood U_1 of e in G such that for $1 \leq n \leq N, 1 \leq j \leq m$ and $x \in U_1$ we have

$$N_p\big((\varphi_n^{(j)} *_H k')_{x,H} - (\varphi_n^{(j)} *_H k')_H\big) < \varepsilon_1$$

and

$$N_{p'}\big((\psi_n^{(j)} *_H k')_{x,H} - (\psi_n^{(j)} *_H k')_H\big) < \varepsilon_1.$$

This implies

$$N_p\big((\varphi_n^{(j)} *_H k')_{x,H} - \varphi_n^{(j)}\big) < \varepsilon_1$$

and

$$N_{p'}\big((\psi_n^{(j)} *_H k')_{x,H} - \psi_n^{(j)}\big) < \varepsilon_1.$$

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[1] For $f \in F^G$, where F is a set, $[f]$ denotes the set all $g \in F^G$ with $g = f$ a.e.

Let A be an open neighborhood of e in G with $A \subset U_1$. Using a Bruhat function for H, G (as in [1, p. 1430]), we obtain for every $1 \le n \le N, 1 \le j \le m$ and $S \in CV_p(H)$ the following inequality:

$$\left| \frac{\langle i(S)[1_{AH}(\varphi_n^{(j)} *_H k')], [1_{AH}(\psi_n^{(j)} *_H k')]\rangle_{L^p_{\mathbb{C}}(G), L^{p'}_{\mathbb{C}}(G)}}{m_{G/H}(\omega(A))} - \langle S[\varphi_n^{(j)}], [\psi_n^{(j)}]\rangle_{L^p_{\mathbb{C}}(H), L^{p'}_{\mathbb{C}}(H)} \right| \le |||S|||_p \varepsilon_1 \big(1 + N_p(\varphi_n^{(j)}) + N_{p'}(\psi_n^{(j)})\big).$$

Let K be a finite subset of H containing $\operatorname{supp} \varphi_n^{(j)}$ and $\operatorname{supp} \psi_n^{(j)}$ for $1 \le n \le N$ and $1 \le j \le m$. Then $\operatorname{supp}(\varphi_n^{(j)} *_H k') \subset KU_0$ and $\operatorname{supp}(\psi_n^{(j)} *_H k') \subset KU_0$ for $1 \le n \le N$ and $1 \le j \le m$.

Let

$$0 < \varepsilon_2 < \min\left\{ \left(\frac{\varepsilon_1}{2^{n+3}\big(1 + \|\varphi_n^{(j)} *_H k'\|_\infty\big)\big(1 + N_{p'}\big(\psi_n^{(j)}\big)\big)} \right)^p, \left(\frac{\varepsilon_1}{2^{n+3}\big(1 + \|\psi_n^{(j)} *_H k'\|_\infty\big)\big(1 + N_p\big(\varphi_n^{(j)}\big)\big)} \right)^{p'} \middle| 1 \le n \le N, 1 \le j \le m \right\}.$$

Corollary 2.6 implies the existence of an open neighborhood U_2 of e in G with $U_2^{-1} = U_2$, $U_2 \subset U_1$ and

$$\int_{KU_0} |1_{HU_2}(x) - 1_{U_2H}(x)| dx < \varepsilon_2 \ m_{G/H}(\omega(U_2)).$$

(1) For $1 \le n \le N$, $1 \le j \le m$ and $S \in CV_p(H)$ we have

$$\left| \frac{\langle i(S)[1_{HU_2}(\varphi_n^{(j)} *_H k')], [1_{HU_2}(\psi_n^{(j)} *_H k')]\rangle_{L^p_{\mathbb{C}}(G), L^{p'}_{\mathbb{C}}(G)}}{m_{G/H}((\omega(U_2))} - \frac{\langle i(S)[1_{U_2H}(\varphi_n^{(j)} *_H k')], [1_{U_2H}(\psi_n^{(j)} *_H k')]\rangle_{L^p_{\mathbb{C}}(G), L^{p'}_{\mathbb{C}}(G)}}{m_{G/H}((\omega(U_2))} \right| \le \frac{|||S|||_p \varepsilon_1}{2^{n+2}}.$$

We first show that

$$\frac{N_p(1_{HU_2}(\varphi_n^{(j)} *_H k'))}{m_{G/H}((\omega(U_2))^{1/p}} \le N_p(\varphi_n^{(j)}).$$

We have indeed

$$\int_G 1_{HU_2}(x)|(\varphi_n^{(j)} *_H k')(x)|^p dx = \int_G 1_{U_2H}(x)|((k')\check{\ } *_H (\varphi_n^{(j)})\check{\ })(x)|^p dx$$
$$= \int_{G/H} 1_{\omega(U_2)}(\dot{x}) \left(\int_H |((k')\check{\ } *_H (\varphi_n^{(j)})\check{\ })(xh)|^p dh \right) d\dot{x},$$

and for every $x \in G$ we have

$$\int_H |((k')\check{\ } *_H (\varphi_n^{(j)})\check{\ })(xh)|^p dh \leq N_p(\varphi_n^{(j)})^p.$$

We claim that

$$\frac{N_{p'}((1_{HU_2} - 1_{U_2H})(\psi_n^{(j)} *_H k'))}{m_{G/H}((\omega(U_2))^{1/p'}} < \frac{\varepsilon_1}{2^{n+3}\big(1 + N_p\big(\varphi_n^{(j)}\big)\big)}.$$

Since $\operatorname{supp}(\psi_n^{(j)} *_H k') \subset KU_0$ we have

$$\begin{aligned} &N_{p'}((1_{HU_2} - 1_{U_2H})(\psi_n^{(j)} *_H k'))^{p'} \\ &\qquad = \int_{KU_0} |1_{HU_2}(x) - 1_{U_2H}(x)|^{p'} |(\psi_n^{(j)} *_H k')(x)|^{p'} dx \\ &\qquad \leq \|\psi_n^{(j)} *_H k'\|_\infty^{p'} \int_{KU_0} |1_{HU_2}(x) - 1_{U_2H}(x)| dx \\ &\qquad \leq \|\psi_n^{(j)} *_H k'\|_\infty^{p'} \, \varepsilon_2 \, m_{G/H}(\omega(U_2)). \end{aligned}$$

Similarly,

$$\frac{N_p((1_{HU_2} - 1_{U_2H})(\varphi_n^{(j)} *_H k'))}{m_{G/H}((\omega(U_2))^{1/p}} < \frac{\varepsilon_1}{2^{n+3}\big(1 + N_{p'}\big(\psi_n^{(j)}\big)\big)}.$$

Lemma 2.7 implies

$$\frac{N_{p'}(1_{U_2H}(\psi_n^{(j)} *_H k'))}{m_{G/H}((\omega(U_2))^{1/p'}} \leq N_{p'}(\psi_n^{(j)}) \|T_H(k')\|_\infty^{1/p'} \|T_H(\check{k}')\|_\infty^{1/p}.$$

But $\|T_H(k')\|_\infty \leq 1$ and $\|T_H(\check{k}')\|_\infty \leq 1$. This justifies Step (1).

(2) Let

$$k'' = \frac{(1_{HU_2}k')\check{\ }}{m_{G/H}((\omega(U_2))^{1/p}}$$

and

$$l'' = \frac{(1_{HU_2}k')\check{\ }}{m_{G/H}((\omega(U_2))^{1/p'}}.$$

Then $N_p(T_H(k'')) \leq 1$, $N_{p'}(T_H(l'')) \leq 1$ and

$$\sum_{n=1}^{\infty}\left|\langle\Lambda_{k'',l''}(i(S))[\varphi_n^{(j)}],[\psi_n^{(j)}]\rangle_{L^p_{\mathbb{C}}(H),L^{p'}_{\mathbb{C}}(H)}-\langle S[\varphi_n^{(j)}],[\psi_n^{(j)}]\rangle_{L^p_{\mathbb{C}}(H),L^{p'}_{\mathbb{C}}(H)}\right|$$
$$\leq\frac{5\;\varepsilon}{2^5}|||S|||_p$$

for every $1\leq j\leq m,\ S\in CV_p(H)$.

We have $N_p(T_H(k''))^p=\int_{G/H}(T_H(k''))^p d\dot{x}$, but

$$T_H(k'')(\omega(x))=\frac{1_{\omega(U_2)}(\dot{x})}{m_{G/H}((\omega(U_2))^{1/p}}\sum_{h\in H}k'(hx^{-1}).$$

Hence $N_p(T_H(k''))\leq 1$. Similarly we obtain $N_{p'}(T_H(l''))\leq 1$.

For $1\leq n\leq N$ we get, using (1),

$$\left|\langle\Lambda_{k'',l''}(i(S))[\varphi_n^{(j)}],[\psi_n^{(j)}]\rangle_{L^p_{\mathbb{C}}(H),L^{p'}_{\mathbb{C}}(H)}-\langle S[\varphi_n^{(j)}],[\psi_n^{(j)}]\rangle_{L^p_{\mathbb{C}}(H),L^{p'}_{\mathbb{C}}(H)}\right|$$
$$\leq\frac{\varepsilon_1}{2^{n+2}}|||S|||_p+\left|\frac{\langle i(S)[1_{U_2H}(\varphi_n^{(j)}*_H k')],[1_{U_2H}(\psi_n^{(j)}*_H k')]\rangle_{L^p_{\mathbb{C}}(G),L^{p'}_{\mathbb{C}}(G)}}{m_{G/H}((\omega(U_2))}\right.$$
$$\left.-\langle S[\varphi_n^{(j)}],[\psi_n^{(j)}]\rangle_{L^p_{\mathbb{C}}(H),L^{p'}_{\mathbb{C}}(H)}\right|$$
$$\leq\frac{3\varepsilon}{2^{n+5}}|||S|||_p.$$

The estimate

$$\sum_{n=1+N}^{\infty}\left|\langle\Lambda_{k'',l''}(i(S))[\varphi_n^{(j)}],[\psi_n^{(j)}]\rangle_{L^p_{\mathbb{C}}(H),L^{p'}_{\mathbb{C}}(H)}-\langle S[\varphi_n^{(j)}],[\psi_n^{(j)}]\rangle_{L^p_{\mathbb{C}}(H),L^{p'}_{\mathbb{C}}(H)}\right|$$
$$\leq 2|||S|||_p\sum_{n=1+N}^{\infty}N_p(\varphi_n^{(j)})N_{p'}(\psi_n^{(j)})\leq\frac{2\varepsilon}{2^5}|||S|||_p$$

gives (2).

(3) Let

$$0<\varepsilon_3<\min\left\{\frac{\varepsilon}{2^6\big(1+\sum_{n=1}^{\infty}N_p(\varphi_n^{(j)})N_{p'}(\psi_n^{(j)})\big)}\;\middle|\;1\leq j\leq m\right\}$$

and let $f,g\in C^+_{00}(G/H)$ with

$$N_p\bigg(f-\frac{1_{\omega(U_2)}}{m_{G/H}((\omega(U_2))^{1/p}}\bigg)<\varepsilon_3$$

and

$$N_{p'}\left(g - \frac{1_{\omega(U_2)}}{m_{G/H}((\omega(U_2))^{1/p'}}\right) < \varepsilon_3.$$

Then, setting $k''' = f \circ \omega\check{k}'$, $l''' = g \circ \omega\check{k}'$, we have $k''', l''' \in C_{00}^+(G)$, $N_p(T_H(k''')) \leq 1 + \varepsilon_3$, $N_{p'}(T_H(l'')) \leq 1 + \varepsilon_3$ and

$$\sum_{n=1}^{\infty} \left| \langle \Lambda_{k''',l'''}(i(S))[\varphi_n^{(j)}], [\psi_n^{(j)}] \rangle_{L^p_{\mathbb{C}}(H), L^{p'}_{\mathbb{C}}(H)} - \langle S[\varphi_n^{(j)}], [\psi_n^{(j)}] \rangle_{L^p_{\mathbb{C}}(H), L^{p'}_{\mathbb{C}}(H)} \right| \leq \frac{7\varepsilon |||S|||_p}{2^5}$$

for every $1 \leq j \leq m$, $S \in CV_p(H)$.

We finally set $k = k'''/(1+\varepsilon_3)$ and $l = l'''/(1+\varepsilon_3)$. Then $\|\Lambda_{k,l}\| \leq N_p(T_H(k))\ N_p(T_H(l)) \leq 1$, $\operatorname{supp} k \subset U$, $\operatorname{supp} l \subset U$ and

$$\sum_{n=1}^{\infty} \left| \langle \Lambda_{k,l}(i(S))[\varphi_n^{(j)}], [\psi_n^{(j)}] \rangle_{L^p_{\mathbb{C}}(H), L^{p'}_{\mathbb{C}}(H)} - \langle S[\varphi_n^{(j)}], [\psi_n^{(j)}] \rangle_{L^p_{\mathbb{C}}(H), L^{p'}_{\mathbb{C}}(H)} \right| \leq \frac{\varepsilon |||S|||_p}{3}.$$

Consequently,

$$\sum_{n=1}^{\infty} \left| \langle \Lambda_{k,l}(i(S))[r_n^{(j)}], [s_n^{(j)}] \rangle_{L^p_{\mathbb{C}}(H), L^{p'}_{\mathbb{C}}(H)} - \langle S[r_n^{(j)}], [s_n^{(j)}] \rangle_{L^p_{\mathbb{C}}(H), L^{p'}_{\mathbb{C}}(H)} \right| \leq \varepsilon |||S|||_p.$$

□

We can now state our main result.

THEOREM 3.2. *Let G be a locally compact group and H a discrete amenable subgroup. Then there is a linear contraction $\mathcal{Q}$ from $\mathcal{L}(L^p_{\mathbb{C}}(G))$ into $\mathcal{L}(L^p_{\mathbb{C}}(H))$ such that*

1. *$\mathcal{Q}(T) \in CV_p(H)$ for every $T \in CV_p(G)$,*
2. *$\operatorname{supp} \mathcal{Q}(T) \subset \operatorname{supp} T$ for every $T \in CV_p(G)$,*
3. *$\mathcal{Q}(i(S)) = S$ for every $S \in CV_p(H)$.*

Proof. Theorem 3 of [1] permits us to assume that G is non-compact, non-discrete and unimodular. The preceding proposition then allows us to repeat step by step the proof of Theorem 3 of [1]. □

COROLLARY 3.3. *Let G be a locally compact group and H a discrete amenable subgroup. Then there is a contractive projection of $cv_p(G)$ onto $cv_p(H)$.*

Proof. Let $\mathcal{Q}$ be the map of Theorem 3.2. Claim (2) of this result implies that $\mathcal{Q}(T) \in cv_p(H)$ for $T \in cv_p(G)$. Let $S \in cv_p(H)$. Then $i(S) \in cv_p(G)$ and consequently $\mathcal{Q}(i(S)) = S$. □

COROLLARY 3.4. *Let G be a locally compact group and H a discrete amenable subgroup. Then, via i, the Banach algebra $cv_p(H)$ is isometrically isomorphic to $\{T \mid T \in cv_p(G), \operatorname{supp} T \subset H\}$.*

Proof. We have $i(cv_p(H)) \subset cv_p(G)$. Let $T \in cv_p(G)$ with $\operatorname{supp} T \subset H$. There is $S \in CV_p(H)$ with $i(S) = T$. Let $\mathcal{Q}$ be the projection of Theorem 3.2. We then have $\mathcal{Q}(T) \in cv_p(H)$ and therefore $S \in cv_p(H)$. □

REMARKS 3.5. (1) For G abelian, the Banach algebra $cv_2(G)$ is canonically isomorphic to $C_u^b(\widehat{G})$. In this case, for an arbitrary closed subgroup of G and $p = 2$, Corollary 3.4 is due to H. Reiter [8, Theorem 2].

(2) If G is an amenable group and H an arbitrary closed subgroup, Corollary 3.4 also holds. Indeed, let $T \in cv_p(G)$ with $\operatorname{supp} T \subset H$. By the Cohen-Hewitt factorization theorem, there exist $u \in A_p(G)$, $R \in CV_p(G)$ and a sequence $(u_n)_{n=1}^{\infty}$ of $A_p(G)$ such that $T = uR$ and $\lim_{n\to\infty} |||R - u_nT|||_p = 0$. There is also $S \in CV_p(H)$ with $i(S) = T$. For $m, n \in N$ we have $|||u_mT - u_nT|||_p = |||\operatorname{Res}_H(u_mS) - \operatorname{Res}_H(u_nS)|||_p$. There exists $S' \in CV_p(H)$ with $\lim_{n\to\infty} |||\operatorname{Res}_H(u_nS) - S'|||_p = 0$. We then have $T = uR = i(\operatorname{Res}_H uS')$, but $\operatorname{Res}_H uS' \in cv_p(H)$.

(3) In the case when $p = 2$ and H is a discrete amenable subgroup of G.[2] Corollary 3.4 is precisely part (ii) of Lemma 3.2 of [5].

(4) In Corollaries 3.3 and 3.4 it is possible to replace cv_p by the norm closure in $\mathcal{L}(L^p)$ of the finitely supported convolution operators. This Banach algebra was considered by E. Granirer [3].

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[2] In Lemma 3.2(ii) replace $r^*(UC(\hat{G})) = UC(\hat{G}) \cap VN_H(G)$ by $r^*(UC(\hat{H})) = UC(\hat{G}) \cap VN_H(G)$.

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