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A CORRESPONDENCE BETWEEN HILBERT POLYNOMIALS AND CHERN POLYNOMIALS OVER PROJECTIVE SPACES

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Abstract. We construct a map ζ from $K_0(\mathbb{P}^d)$ to $(\mathbb{Z}[x]/x^{d+1})^\times \times \mathbb{Z}$, where $(\mathbb{Z}[x]/x^{d+1})^\times$ is a multiplicative Abelian group with identity 1, and show that ζ induces an isomorphism between $K_0(\mathbb{P}^d)$ and its image. This is inspired by a correspondence between Chern and Hilbert polynomials stated in Eisenbud [1, Exercise 19.18]. The equivalence of these two polynomials over $\mathbb{P}^d$ is discussed in this paper.

0. Introduction

Let $\mathbb{P}^d$ denote a projective space over an algebraically closed field. It is known that the Hilbert polynomial of a coherent sheaf over a projective scheme is closely related to the Chern polynomial of this sheaf by the Hirzebruch-Riemann-Roch theorem; in fact, as pointed out in Exercise 19.18 in Eisenbud [1], over $\mathbb{P}^d$ knowing the Hilbert polynomial is equivalent to knowing the Chern polynomial. Although the Chern and Hilbert polynomials are quite different in terms of degrees and coefficients, the Hirzebruch-Riemann-Roch theorem establishes a connection from one to the other. In the next section we will briefly describe the definitions and properties associated with a coherent sheaf.

Let $\mathcal{A}_0$ and $\mathcal{B}$ denote two Abelian groups which are generated by Chern polynomials and Hilbert polynomials, respectively. In Theorem 1 we show the existence of an isomorphism between the Grothendieck group $K_0(\mathbb{P}^d)$ and $\mathcal{A}_0 \times \mathbb{Z}$. This is analogous to the fact that $K_0(\mathbb{P}^d)$ and $\mathcal{B}$ are isomorphic, as shown in [1]. Let $P(t)$ and $C(x)$ denote the Hilbert and Chern polynomials of a coherent sheaf, respectively. In Theorem 2 we use this isomorphism to show the equivalence of the following three statements for any two coherent sheaves $\mathcal{M}$ and $\mathcal{N}$:

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(1) $\mathcal{M}$ and $\mathcal{N}$ represent the same class in $\mathrm{K}_0(\mathbb{P}^d)$.
(2) $P_{\mathcal{M}}(t) = P_{\mathcal{N}}(t)$.
(3) $C_{\mathcal{M}}(x) = C_{\mathcal{N}}(x)$ and $\operatorname{rank}\mathcal{M} = \operatorname{rank}\mathcal{N}$.

It is easy to see from the definitions that (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3). The implication (2) $\Longrightarrow$ (1) is proved in [1]. Here we give a proof for the remaining implication (3) $\Longrightarrow$ (1).

The paper is organized as follows. Section 1 contains some necessary background material. It is not possible to give complete definitions in this paper, but we state the main properties that we will use in our discussion. In Section 2 we prove Theorem 1, which describes two isomorphic group structures on $\mathrm{K}_0(\mathbb{P}^d)$. One is induced by a map $\eta : \mathrm{K}_0(\mathbb{P}^d) \longrightarrow \mathcal{B}$, $\mathcal{B} \subset (\mathbb{Q}[t]/(t^{d+1}))^{+}$ ([1, Exercises 19.16 and 19.17]), and the other is induced by a map $\zeta : \mathrm{K}_0(\mathbb{P}^d) \longrightarrow \mathcal{A}$, $\mathcal{A} \subset (\mathbb{Z}[x]/(x^{d+1}))^{\times} \times \mathbb{Z}$. The equivalence of the above statements on $P_{\mathcal{M}}(t)$ and $C_{\mathcal{M}}(x)$ then follows from this result. The isomorphism from $\mathcal{A}$ to $\mathcal{B}$ induced by ζ^{-1} and η recovers the Hirzebruch-Riemann-Roch theorem. This result together with the above-mentioned result in Eisenbud [1, Exercise 19.18] yields an algorithm for computing the Hilbert polynomial of a coherent sheaf from its Chern polynomial and vice versa without using an explicit sheaf structure. This one-to-one correspondence between the two polynomials is discussed in Section 3.

1. Background

We begin by introducing some notations that are used throughout the paper. Let $\mathbb{P}^d$ be a projective space over an algebraically closed field. For any coherent sheaf $\mathcal{M}$ over $\mathbb{P}^d$, $\mathcal{M}$ is associated with a graded finitely generated module $M = \oplus_n M_n$ over the polynomial ring $S = k[x_0, \ldots, x_d]$ with homogeneous components M_n. Conversely, a graded module also defines a coherent sheaf, but there are usually more than one graded module associated to a given coherent sheaf (cf. [5], [7]). For each $\mathcal{M}$, the *Hilbert polynomial* $P_{\mathcal{M}}(t)$ of $\mathcal{M}$ is a polynomial such that for any large enough integer $n \in \mathbb{N}$, $P_{\mathcal{M}}(n)$ coincides with the value of the Hilbert function of the module M, $H_M(n) = \operatorname{length}(M_n)$. For example, any twisted structure sheaf $\mathcal{O}(-m)$ with $m \in \mathbb{Z}$ is associated with the graded module $S[-m]$ and has Hilbert polynomial $P_{\mathcal{O}(-m)}(t) = \binom{t+d-m}{d}$. The Hilbert polynomials are additive on short exact sequences of sheaves.

The *Chern polynomial* $C_{\mathcal{M}}(x)$ (often called the *total Chern class*) of $\mathcal{M}$ is a formal sum of the Chern classes, which, in geometry, are usually viewed as cycles in the cohomology groups (cf. Griffiths and Harris [4]) or operators on the Chow groups (cf. Fulton [2] or Roberts [8]). The Chern classes considered in this paper are of the latter form. In general, their definitions are very complicated. We describe these notions for sheaves over $\mathbb{P}^d$ and recall a few properties that will be useful for our discussions. Complete details can be found in the above cited references.

The *Chow group* $\mathrm{A}_*(\mathbb{P}^d)$ of $\mathbb{P}^d$ is generated by the linear subspaces $\mathbb{P}^{d-\ell}, \ell = 0, \ldots, d$, so it has a simple structure, $\mathrm{A}_*(\mathbb{P}^d) \cong \mathbb{Z}^{d+1}$. Therefore, the Chern classes can be identified with integers. For a locally free sheaf $\mathcal{M}$ of finite rank r, there exist r Chern classes $c_1(\mathcal{M}), \ldots, c_r(M)$, and the *Chern polynomial* of $\mathcal{M}$ is defined to be a formal sum of the $c_i(\mathcal{M})$,

$$C_{\mathcal{M}}(x) = 1 + \sum_{i=1}^{r} c_i(\mathcal{M})x^i = 1 + c_1(\mathcal{M})x + \cdots + c_r(\mathcal{M})x^r \pmod{x^{d+1}}.$$

For instance, $c_1(\mathcal{O}(-m)) = -m$ for any twisted sheaf with $m \in \mathbb{Z}$ and there are no higher Chern classes. Thus, the Chern polynomial of $\mathcal{O}(-m)$ is $C_{\mathcal{O}(-m)}(x) = 1 - mx$. An important property, called the *Whitney sum formula*, states that the Chern polynomials are multiplicative on short exact sequences of sheaves.

Over a nonsingular variety, every coherent sheaf admits a unique minimal resolution of locally free sheaves up to quasi-isomorphisms,

(1)

$$0 \to \oplus_{j_d}\mathcal{O}(-j_d)^{\beta_{d,j_d}} \to \cdots \to \oplus_{j_1}\mathcal{O}(-j_1)^{\beta_{1,j_1}} \to \oplus_{j_0}\mathcal{O}(-j_0)^{\beta_{0,j_0}} \to \mathcal{M} \to 0.$$

Using the Whitney sum formula, the definition of Chern classes can be extended to coherent sheaves. By (1), the Chern polynomial of $\mathcal{M}$ over $\mathbb{P}^d$ is a polynomial modulo x^{d+1} with integer coefficients,

$$C_M(x) = \frac{\prod_{i:\text{even}} \prod_{j_i} (1 - j_i x)^{\beta_{i,j_i}}}{\prod_{i:\text{odd}} \prod_{j_i} (1 - j_i x)^{\beta_{i,j_i}}} \pmod{x^{d+1}}, \tag{2}$$

and the Hilbert polynomial is of degree at most d with rational coefficients,

$$P_M(t) \;=\; \textstyle\sum_{i,j_i} (-1)^{\beta_{i,j_i}} P_{S(-j_i)}(t).$$

Recall that $C_{\mathcal{O}(-m)}(x) = 1 - mx$ and $P_{\mathcal{O}(-m)}(t) = \binom{t+d-m}{d}$. The Chern polynomials of locally free sheaves always have integer coefficients and the degree varies, while the Hilbert polynomials of such sheaves have rational coefficients and the degree is fixed by $\dim X = d$. A consequence of the Hirzebruch-Riemann-Roch theorem is a connection between the Euler characteristic and Chern characters of coherent sheaves. This leads to a representation of Hilbert polynomials in terms of Chern classes in some special cases. We will recall this in Section 3.

Next, we define the Grothendieck group. The Grothendieck group of locally free sheaves, denoted by $\mathrm{K}_0(\mathbb{P}^d)$, is the Abelian group generated by all locally free sheaves $[\mathcal{M}]$ modulo the subgroup generated by $[\mathcal{M}] - [\mathcal{M}'] - [\mathcal{M}'']$ whenever $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ forms an exact sequence of sheaves. We also denote $[\mathcal{M}]$ by $[M]$ if the sheaf $\mathcal{M}$ is associated with the module M. The Grothendieck group of coherent sheaves $\mathrm{G}_0(\mathbb{P}^d)$ is defined in a similar way.

Since every coherent sheaf admits a locally free resolution in the form of (1), $\mathrm{K}_0(\mathbb{P}^d)$ is isomorphic to $\mathrm{G}_0(\mathbb{P}^d)$. Henceforth, we use $\mathrm{K}_0(\mathbb{P}^d)$ and refer to it as the Grothendieck group of $\mathbb{P}^d$ for simplicity. The generators of $\mathrm{K}_0(\mathbb{P}^d)$ can be described precisely as follows. The module $S/(x_0, \ldots, x_d)$ defines a zero sheaf since $(x_0, \ldots, x_d)$ is an irrelevant ideal. We take the Koszul resolution of $S/(x_0, \ldots, x_d)$ and obtain a long exact sequence on locally free sheaves,

$$0 \to \mathcal{O}(-d-1) \to (\mathcal{O}(-d))^{\binom{d+1}{d}} \to \cdots \to (\mathcal{O}(-1))^{\binom{d+1}{1}} \to \mathcal{O} \to 0. \tag{3}$$

This gives a relation for the twisted sheaves in $\mathrm{K}_0(\mathbb{P}^d)$ that expresses $[\mathcal{O}(-d-1)]$ as an alternating sum of $[\mathcal{O}], [\mathcal{O}(-1)], \ldots, [\mathcal{O}(-d)]$. If we twist the exact sequence (3) by a degree, say by degree one, then we get the following exact sequence:

$$0 \to \mathcal{O}(-d) \to (\mathcal{O}(-d+1))^{\binom{d+1}{d}} \to \cdots \to (\mathcal{O})^{\binom{d+1}{1}} \to \mathcal{O}(1) \to 0.$$

Thus, $[\mathcal{O}(1)]$ is also generated by the same set of twisted sheaves and similarly for other degrees. This implies that $\mathrm{K}_0(\mathbb{P}^d)$ is generated by $[\mathcal{O}(-m)]$ with $m = 0, \ldots, d$. Furthermore, these are free generators. A brief proof of this result will be given in the next section. On the other hand, let S_ℓ denote the graded module of S modulo ℓ variables,

$$S_\ell = k[x_0, \cdots, x_d]/(x_{d-\ell+1}, \cdots, x_d).$$

The Koszul complex on $x_{d-\ell+1}, \ldots, x_d$ provides a resolution of locally free sheaves for S_ℓ. By a standard argument, $[S_\ell]$, $\ell = 0, \ldots, d$, also generate $\mathrm{K}_0(\mathbb{P}^d)$.

It is known that different sheaves may have the same Hilbert polynomials and Chern polynomials. However, both polynomials are well-defined for the equivalence classes of coherent sheaves in the Grothendieck group. We have already seen in Eisenbud [1] that the Hilbert polynomials characterize the classes in $\mathrm{K}_0(\mathbb{P}^d)$. The main goal of this paper is to show that the Chern polynomials do the same job; namely, distinct classes have different pairs of Chern polynomial and rank.

2. Groups isomorphic to $\mathrm{K}_0(\mathbb{P}^d)$

If a polynomial with rational coefficients has integral values at large integers, then it can be written as a linear combination over $\mathbb{Z}$ of the binomial coefficient functions in t,

$$\binom{t}{0}, \binom{t}{1}, \binom{t}{2}, \ldots, \binom{t}{\ell}, \ldots.$$

These polynomials can be replaced by

$$\binom{t}{0}, \binom{t+1}{1}, \binom{t+2}{2}, \ldots, \binom{t+\ell}{\ell}, \ldots.$$

Indeed, if we let $a_\ell = \binom{t}{\ell}$ and $b_\ell = \binom{t+\ell}{\ell}$, then $b_\ell = \sum_{i=0}^{\ell} \binom{\ell}{i} a_i$ and $a_\ell = \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} b_i$.

Let $\mathbb{P}^d = \mathrm{Proj}(S) = \mathrm{Proj}(k[x_0, \cdots, x_d])$ be as in the previous section. Then $\binom{t+d}{d}$ is exactly the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^d}$ and $P_{s_\ell}(t) = \binom{t+d-\ell}{d-\ell}$. Since Hilbert polynomials have integral values at large integers, for any graded module M, $P_M(t)$ can be written as a linear combination of $P_{S_\ell}(t), \ell \in \mathbb{N} \cup \{0\}$. Let $\mathcal{B}$ denote the Abelian group generated by all Hilbert polynomials of coherent sheaves over $\mathbb{P}^d$. The group $\mathcal{B}$ is a subgroup of the additive group $(\mathbb{Q}[t]/t^{d+1})^+$ with identity 0, and the polynomials $P_{S_\ell}(t)$, $\ell = 0, \ldots, d$, form a set of generators for $\mathcal{B}$. Moreover, these generators are linearly independent since $\deg P_{S_\ell}(t) = \ell$. Thus, $\mathcal{B}$ is a free Abelian group of rank $d+1$.

Let α be a class in $\mathrm{K}_0(\mathbb{P}^d)$ represented by some sheaf $\mathcal{M}$. The map

$$\eta : \mathrm{K}_0(\mathbb{P}^d) \longrightarrow \mathcal{B},$$

which takes α to the Hilbert polynomial $P_{\mathcal{M}}(t)$ of $\mathcal{M}$, induces an isomorphism. To see this, we note that η is surjective since P_{S_ℓ}, $\ell = 0, \ldots, d$, generate $\mathcal{B}$. The linear independence of these generators implies that $[S_\ell]$, $\ell = 0, \ldots, d$, are also linearly independent. Therefore $\mathrm{K}_0(\mathbb{P}^d)$ is generated freely by $[S_\ell]$, $\ell = 0, \ldots, d$, and the injectivity follows (cf. [1, Exercise 19.17]). Another proof of this fact, which uses $\{[\mathcal{O}(-m)] : m = 0, \ldots, d\}$ as a generating set and in which η is induced by a map taking $[\mathcal{O}(-m)]$ to its Hilbert series, can be found in [1, Exercise 19.16].

Alternatively, let $\mathcal{A}_0$ denote the Abelian group generated by all Chern polynomials of the coherent sheaves over $\mathbb{P}^d$. Similar to $\mathcal{B}$, $\mathcal{A}_0$ is a subgroup of the Abelian multiplicative group $(\mathbb{Z}[x]/x^{d+1})^\times$ with identity 1. Let $\mathcal{A}$ denote the subgroup $\mathcal{A}_0 \times \mathbb{Z}$ of $(\mathbb{Z}[t]/t^{d+1})^\times \times \mathbb{Z}$, endowed with the natural group structure, such that for any two elements $(f(x), r)$ and $(g(x), s)$, $(f(x), r) + (g(x), s) = (f(x)g(x), r+s)$, and $(1, 0)$ is the identity. For any α in $\mathrm{K}_0(\mathbb{P}^d)$ represented by a locally free sheaf $\mathcal{M}$, we define a map from $\mathrm{K}_0(\mathbb{P}^d)$ to $\mathcal{A} = \mathcal{A}_0 \times \mathbb{Z}$ by

$$\begin{array}{rccl} \zeta : & \mathrm{K}_0(\mathbb{P}^d) & \longrightarrow & \mathcal{A} = \mathcal{A}_0 \times \mathbb{Z} \\ & \alpha = [\mathcal{M}] & \longrightarrow & (C_{\mathcal{M}}(x), \operatorname{rank} \mathcal{M}). \end{array}$$

The map ζ is a well-defined group homomorphism by the Whitney sum formula. It should be noted that the component $\mathbb{Z}$ in $\mathcal{A}$ is necessary in order to distinguish different classes which have the same Chern polynomial. The simplest examples are $\alpha = [\mathcal{O}_{\mathbb{P}^d}]$ of rank one and $\beta = [\oplus_r \mathcal{O}_{\mathbb{P}^d}]$ of rank $r \neq 0, 1$. Both α and β have Chern polynomial equal to 1, while $\beta = r\alpha \neq \alpha$ in $\mathrm{K}_0(\mathbb{P}^d)$. Analogous to the isomorphism defined by η, we prove that ζ is also an isomorphism.

THEOREM 1. *$\zeta : \mathrm{K}_0(\mathbb{P}^d) \longrightarrow \mathcal{A}$ is an isomorphism of Abelian groups.*

The following lemma implies that $\mathcal{A}$ is free of rank $d+1$. The generators $(1-\ell x, 1)$ of $\mathcal{A}$ are the images of $[\mathcal{O}(-\ell)]$ for all ℓ, so η is surjective, and therefore an isomorphism since both groups are free of the same rank.

LEMMA 1. *The group $\mathcal{A}_0$ is freely generated by $1-x, \dots, 1-dx$. Furthermore, $(1,1), (1-x,1), \dots, (1-dx,1)$ are free generators for $\mathcal{A}$.*

Proof. It is clear that $\mathcal{A}_0$ is generated by $1-x, \dots, 1-dx$ and that $\mathcal{A}$ is generated by $(1,1), (1-x,1), \dots, (1-dx,1)$, by the resolutions (1) and (3). If

$$r_0(1,1) + r_1(1-x,1) + \cdots + r_d(1-dx,1) = (1,0)$$

in $\mathcal{A}$ for some $r_0, \dots, r_d \in \mathbb{Z}$, then

$$(1-x)^{r_1} \cdots (1-dx)^{r_d} \equiv 1 \pmod{x^{d+1}}, \tag{4}$$

$$r_0 + r_1 + \cdots + r_d = 0. \tag{5}$$

It suffices to show that $1-x, \dots, 1-dx$ are linearly independent; that is, (4) implies $r_1 = \cdots = r_d = 0$. Then the linear independence of $(1,1), (1-x,1), \dots, (1-dx,1)$ follows from (5).

Without loss of generality, we may in the following argument assume that none of $r_1, \dots, r_d$ is zero. (If any of $r_0, \dots, r_d$ is zero, then a similar argument leads to the same contradiction.) We take the derivative of the equation in (4) and obtain

$$(1-x)^{r_1}(1-2x)^{r_2} \cdots (1-dx)^{r_d} \left(\frac{-r_1}{1-x} + \frac{-r_2}{1-2x} + \cdots + \frac{-r_d}{1-dx} \right) \equiv 0 \pmod{x^d}.$$

The above product is taken in the unique factorization domain $\mathbb{Z}[[x]]$ and a simple computation shows that

$$(1-x)^{r_1}(1-2x)^{r_2} \cdots (1-dx)^{r_d} = 1-(r_1+2r_2+\cdots+dr_d)x+\cdots \not\equiv 0 \pmod{x^d}.$$

Therefore,

$$\frac{r_1}{1-x} + \frac{2r_2}{1-2x} + \cdots + \frac{dr_d}{1-dx} \equiv 0 \pmod{x^d}.$$

Using Taylor expansions we obtain

$$\begin{aligned}(r_1 + 2r_2 + \cdots + dr_d) + (r_1 + 2^2 r_2 + \cdots + d^2 r_d)x + \cdots \\ + (r_1 + 2^d r_2 + \cdots + d^d r_d)x^{d-1} \equiv 0 \pmod{x^d}.\end{aligned} \tag{6}$$

The equivalence given by (6) provides a linear system in $r_1, \dots, r_d$ with Vandermonde coefficients if $r_1, \dots, r_d$ are all nonzero. This is a contradiction because a Vandermonde system has only trivial solutions. Therefore, $r_1 = r_2 = \cdots = r_d = 0$, and $r_0 = 0$ by (5). This completes the proof of both assertions in the lemma. $\square$

The isomorphisms η and ζ show that the three groups $\mathrm{K}_0(\mathbb{P}^d)$, $\mathcal{A}$ and $\mathcal{B}$ are isomorphic and give the following result.

THEOREM 2. *For any coherent sheaves $\mathcal{M}$ and $\mathcal{N}$ on $\mathbb{P}^d$, the following are equivalent:*

(1) $[\mathcal{M}] = [\mathcal{N}]$ *in* $\mathrm{K}_0(\mathbb{P}^d)$.
(2) $P_{\mathcal{M}}(t) = P_{\mathcal{N}}(t)$.
(3) $C_{\mathcal{M}}(x) = C_{\mathcal{N}}(x)$ *and* $\operatorname{rank}\mathcal{M} = \operatorname{rank}\mathcal{N}$.

3. The equivalence of Chern and Hilbert polynomials

In this section we establish the relationship between Chern and Hilbert polynomials which inspired the work presented in the previous section. The Hirzebruch-Riemann-Roch theorem relates the Euler characteristic to Chern characters. Not much is known about representing Hilbert polynomials in terms of Chern classes in general. We will discuss this and the converse problem over $\mathbb{P}^d$ in the current section. Proposition 1 provides a one-to-one correspondence between the two polynomials and an algorithm for computations.

The Hirzebruch-Riemann-Roch theorem shows the existence of a certain map from the Grothendieck group of a scheme X to its Chow group that commutes with the maps induced by a projective map from X to a point. For projective spaces this theorem leads to a representation of Hilbert functions in terms of Chern classes. In order to make this precise, we need to introduce the Chern characters of $\mathcal{M}$. Suppose the Chern polynomial can be decomposed into

$$C_{\mathcal{M}}(x) = (1-\alpha_1 x)\cdots(1-\alpha_d x) \tag{7}$$

in $(\mathbb{Z}[x]/(x^{d+1}))$. In this case, $\alpha_1, \ldots, \alpha_d$ are called *Chern roots*. Then the *Chern character* of $\mathcal{M}$ is a power series defined by

$$\operatorname{ch}(\mathcal{M}) = e^{\alpha_1 x} + \cdots + e^{\alpha_d x}. \tag{8}$$

The coefficient of x^i in the Taylor expansion of (8) is called the *i-th Chern character* of $\mathcal{M}$ and denoted by $\operatorname{ch}_i(\mathcal{M})$. Since each $\operatorname{ch}_i(\mathcal{M})$ is a symmetric function of α_i and the Chern classes are elementary symmetric functions in α_i, the Chern characters $\operatorname{ch}_i(\mathcal{M})$ can be expressed as polynomials in the Chern classes. The first few terms are

$$\begin{aligned}\operatorname{ch}(\mathcal{M}) = r + c_1 x + \frac{1}{2!}(c_1^2 - 2c_2)x^2 + \frac{1}{3!}(c_1^3 - 3c_1c_2 + 3c_3)x^3 + \\ + \frac{1}{4!}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4)x^4 + \cdots ,\end{aligned} \tag{9}$$

where $c_i = c_i(\mathcal{M})$ and $r = \operatorname{rank}\mathcal{M}$ (see [2, Example 15.1.2] for the exact formulations). We should note that a factorization of the form (7) does not always exist over the current projective scheme. However, the representation (9) is independent of the existence of Chern roots.

For any power series $s(x)$ in x, let $\Phi(s(x))$ denote the *coefficient of* x^d *in the Taylor expansion of the expression* $s(x)\left(\frac{x}{1-e^{-x}}\right)^{d+1}$. The following theorem is the Hirzebruch Riemann-Roch theorem for $\mathbb{P}^d$; details can be found in the books by Fulton and Lang ([2, Example 15.1.4], [3]) and Hirzebruch ([6, Lemma 1.7.1]).

THEOREM 3 (Hirzebruch-Riemann-Roch). *Let* $X = \mathbb{P}^d$. *Then, for any locally free sheaf* $\mathcal{M}$ *on* X,

$$\Phi(\operatorname{ch}(\mathcal{M})) = \chi(\mathcal{M}), \tag{10}$$

where $\chi(\mathcal{M}) = \sum_{i\geq 0}(-1)^i \dim_k H^i(X,\mathcal{M})$*, the alternating sum of the cohomology groups, is the Euler characteristic of* $\mathcal{M}$.

Let $\mathcal{M}$ be a coherent sheaf defined by a finitely generated graded module $M = \oplus M_n$. By induction on the dimension of the support of $\mathcal{M}$, one can show that $H^i(X,\mathcal{M}(n)) = 0$ for all $i > 0$ and

$$\chi(\mathcal{M}(n)) = \sum_{i\geq 0}(-1)^i \dim H^i(X,\mathcal{M}(n)) = \dim H^0(X,\mathcal{M}(n)) = \dim M_n,$$

for $n \gg 0$ (cf. Hartshorne [5, Chapter III, Section 5]). In the case where $\mathcal{M}$ is locally free we have

$$\Phi(\operatorname{ch}(\mathcal{M}(n))) = \dim H^0(X,\mathcal{M}(n)) = \dim M_n \tag{11}$$

by the Hirzebruch-Riemann-Roch theorem. In particular, $\operatorname{ch}(\mathcal{M}(n)) = \operatorname{ch}(\mathcal{M}\otimes \mathcal{O}(n)) = \operatorname{ch}(\mathcal{M})\operatorname{ch}(\mathcal{O}(n)) = e^{nx}\operatorname{ch}(\mathcal{M})$. We replace n in (11) by an indeterminate t. The left hand side of (11), $\Phi(e^{tx}\operatorname{ch}(\mathcal{M}))$, becomes a polynomial in t whose values at large integers agree with the values of the Hilbert function of M. Therefore, this polynomial is the Hilbert polynomial (cf. Fulton [2, Example 15.2.7(a)]).

If $C_{\mathcal{M}}(x) = 1 + c_1(\mathcal{M})x + \cdots + c_r(\mathcal{M})x^r$ is the Chern polynomial of some locally free sheaf $\mathcal{M}$ of rank r, then $\operatorname{ch}(\mathcal{M})$ can be determined explicitly using (9). The Hilbert polynomial $P_{\mathcal{M}}(t)$ of $\mathcal{M}$ obtained by $\Phi(e^{tx}\operatorname{ch}(\mathcal{M}))$ is the coefficient of x^d in $e^{tx}\operatorname{ch}(\mathcal{M})\left(\frac{x}{1-e^{-x}}\right)^{d+1}$.

PROPOSITION 1. *Let* $X = \mathbb{P}^d$ *and let* $\mathcal{M}$ *be a coherent sheaf on* X *of rank* r.

A. *Let* $C_{\mathcal{M}}(x)$ *be the Chern polynomial of* $\mathcal{M}$. *Then the Hilbert polynomial of* $\mathcal{M}$ *is*

$$P_{\mathcal{M}}(t) = \Phi(e^{tx}\operatorname{ch}(\mathcal{M})). \tag{12}$$

B. *If*

$$P_{\mathcal{M}}(t) = \sum_{\ell=0}^{d} a_\ell \binom{t+d-\ell}{d-\ell},$$

then

$$C_{\mathcal{M}}(x) \equiv \prod_{\ell=0}^{d} [C_{S_\ell}(x)]^{a_\ell} \pmod{x^{d+1}}, \tag{13}$$

where $C_{S_\ell}(x)$ *is the Chern polynomial of* S_ℓ *as a module over* S.

Proof. Part A is an immediate consequence of Theorem 3, as explained above, so it remains to prove Part B. We recall from Section 1 that S_ℓ defines the linear subspace $\mathbb{P}^{d-\ell}$ in $\mathbb{P}^d$. Since the Hilbert polynomial of S_ℓ is $\binom{t+d-\ell}{d-\ell}$, by the assumption and Theorem 2, we have

$$P_{\mathcal{M}}(t) = \sum_{\ell=0}^{d} a_\ell \binom{t+d-\ell}{d-\ell} = \sum_{\ell=0}^{d} a_\ell P_{S_\ell}(t)$$

in $\mathcal{B}$ if and only if $[\mathcal{M}] = \sum_{\ell=0}^{d} a_\ell [S_\ell]$ in $\mathrm{K}_0(\mathbb{P}^d)$. Therefore,

$$C_{\mathcal{M}}(x) \equiv \prod_{\ell=0}^{d} (C_{S_\ell}(x))^{a_\ell} \pmod{x^{d+1}}. \qquad \square$$

We note that the representation of $P_{\mathcal{M}}(t)$ given in the hypothesis of Part B is always possible since $\binom{t+d-\ell}{d-\ell}$, $\ell = 0, \dots, d$, form a basis for the group $\mathcal{B}$ of all Hilbert polynomials. The correspondence between the two polynomials can be made explicit as follows. Let σ denote the group homomorphism $\eta \circ \zeta^{-1}$,

$$\sigma = \eta \circ \zeta^{-1} : \mathcal{A} \longrightarrow \mathcal{B}.$$

Any element in the group $\mathcal{A}$ of the Chern polynomials can be written as $(\prod_{m=1}^{d} (1-mx)^{r_m}, s)$ for some r_m and s in $\mathbb{Z}$. It is not hard to see that such an element has a preimage via ζ^{-1} in $\mathrm{K}_0(\mathbb{P}^d)$ of the form

$$\sum_{m=1}^{d} a_m [\mathcal{O}(-m)] + (s-r)[\mathcal{O}],$$

where $a = a_1 + \cdots + a_m$. This implies

$$\begin{aligned}\sigma\left(\left(\prod (1-mx)^{a_m}, s\right)\right) &= \sum_{m=1}^{d} a_m \binom{t+d-m}{d} + (s-a)\binom{t+d}{d} \\ &= \sum_{m=1}^{d} a_m P_{\mathcal{O}(-m)}(t) + (s-a) P_{\mathcal{O}}(t).\end{aligned}$$

An element $(f(x), s)$ in $\mathcal{A}$ is said to be *represented by* $\mathcal{M}$ (or $\mathcal{M}$ *represents* $(f(x), s)$) if there exists a sheaf $\mathcal{M}$ such that $f(x)$ is the Chern polynomial of $\mathcal{M}$ and $\operatorname{rank}\mathcal{M} = s$. Thus, for any representative $(f(x), s)$ in $\mathcal{A}$, σ takes $(f(x), s)$ to the Hilbert polynomial of M. The computation can be carried out by means of (12). Conversely, the preimage of the Hilbert polynomial of $\mathcal{M}$ is the pair consisting of the Chern polynomial and the rank of $\mathcal{M}$. This preimage is uniquely determined since $\sigma = \eta \circ \zeta^{-1}$ is an isomorphism. More precisely, (13) gives the Chern polynomial and $a_0 + \cdots + a_d$ gives the rank, which is independent from the choices of a representing sheaf.

We end the paper with the following two remarks which are often useful in this context.

Remark 1. We like to point out the special case when $\mathcal{M}$ is a twisted structure sheaf $\mathcal{O}(-m)$, which has drawn the attention of those who attempted to solve Exercise 19.18 in [1]. For any $m \in \mathbb{Z}$ we obtain

$$P_{\mathcal{O}(-m)}(t) = \sum_{\ell=0}^{m} (-1)^{\ell} \binom{m}{\ell} P_{S_\ell}(t) \tag{14}$$

by an inductive computation of the binomial coefficient functions

$$\binom{t+d-m}{d} = \binom{t+d-(m-1)}{d} - \binom{t+(d-1)-(m-1)}{d-1}. \tag{15}$$

(We use the convention that $\binom{a}{b} = 0$ if $a < b$.) Then the congruence in Part B in Proposition 1 can be reduced to the congruence

$$C_{\mathcal{O}(-m)}(x) = 1 - mx \equiv \prod_{\ell=0}^{m} (C_{S_\ell})^{(-1)^{\ell}\binom{m}{\ell}} \pmod{x^{d+1}}. \tag{16}$$

Since C_{S_ℓ} can be computed by the Koszul complex as shown in (2), a naive attempt to prove the above congruence (16) would be to compute the coefficient of each x^i on the right hand side and to show that the coefficients of higher terms are zero. However, the computation of the coefficient of a general term x^i in terms of the binomial coefficients is rather complicated. It is not clear how these terms vanish for $i \geq 2$ if they are treated as combinatorial formulae.

The result in the previous section provides an alternative approach: (16) follows from the fact that σ is a group isomorphism; more precisely, we have

$$\begin{aligned}(C_{\mathcal{O}(-m)}(x), 1) = (1 - mx, 1) &= \sigma^{-1}(P_{\mathcal{O}(-m)}(t)) \\ &= \sigma^{-1}\left(\sum_{\ell=0}^{m} (-1)^{\ell} \binom{m}{\ell} P_{S_\ell}(t)\right) \\ &= \left(\prod_{\ell=0}^{m} (C_{S_\ell}(x))^{(-1)^{\ell}\binom{m}{\ell}}, 1\right).\end{aligned}$$

Hence, $C_{\mathcal{O}(-m)}(x)$ and $\prod_{\ell=0}^{m}(C_{S_\ell}(x))^{(-1)^\ell\binom{m}{\ell}}$ are equal in the groups $\mathcal{A}_0$. Although (15) is a combinatorial property, it also represents the relation between Hilbert polynomials of the sheaves in the short exact sequence

$$0 \longrightarrow \mathcal{O}(-m) \xrightarrow{\mathcal{H}} \mathcal{O}(-m+1) \longrightarrow \mathcal{O}_{\mathcal{H}}(-m+1) \longrightarrow 0, \tag{17}$$

where $\mathcal{H}$ is a hyperplane in $\mathbb{P}^d$. Since the Chern polynomials depend on the ambient scheme, a similar inductive decomposition as for $P_{\mathcal{M}}(t)$ in (14) does not hold for Chern polynomials. However, (17) induces an identity on the cycles $\alpha = [\mathcal{O}(-m)] = \sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}[S_\ell]$ in $\mathrm{K}_0(\mathbb{P}^d)$, which implies the corresponding decomposition (16) in $\mathcal{A}$.

Remark 2. A more fundamental correspondence between Chern and Hilbert polynomials should be pointed out. Let a_i denote the coefficient of x^i in the Taylor expansion of $(\frac{x}{1-e^{-x}})^{d+1}$. (12) can be written explicitly as

$$\begin{aligned} P_M(t) = \frac{1}{d!}a_0 r t^d + \frac{1}{(d-1)!}(a_0 \operatorname{ch}_1 + a_1 r)t^{d-1} + \cdots \\ \cdots + (a_0 \operatorname{ch}_d + a_1 \operatorname{ch}_{d-1} + \cdots + a_{d-1}\operatorname{ch}_1 + a_d r), \end{aligned} \tag{18}$$

in which we abbreviated $\mathrm{ch}_i(\mathcal{M})$ by ch_i. Replacing the above ch_i by the appropriate terms in (9), the coefficients of $P_{\mathcal{M}}$ can be expressed in terms of Chern classes. Conversely, if $P_{\mathcal{M}}$ is known, that is, if the coefficients of $P_{\mathcal{M}}(t)$ are determined, then the Chern classes can be found inductively using (18). However, the computation of a_i is tedious. Part B of Proposition 1 avoids such lengthy computation.

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