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VECTOR-VALUED INVARIANT MEANS ON SPACES OF BOUNDED OPERATORS ASSOCIATED TO A LOCALLY COMPACT GROUP

CHING CHOU AND ANTHONY TO-MING LAU

ABSTRACT. The purpose of this paper is to introduce and study the notion of a vector-valued π-invariant mean associated to a unitary representation π of a locally compact group G on $\mathcal{S}$, a self-adjoint linear subspace containing I of $\mathcal{B}(H_\pi)$. We obtain, among other results, an extension theorem for π-invariant completely positive maps and π-invariant means which characterizes amenability of G. We also study vector-valued means on $\mathcal{S}$ of π-(weakly) almost periodic operators on H_π.

0. Introduction

Let G be a locally compact group, and let π be a continuous unitary representation of G on a Hilbert space H. Let $\mathcal{B}(H)$ be the space of bounded linear operators from H into H. Then π is called *amenable* if there is a state m on the C^*-algebra $\mathcal{B}(H)$ such that $m\big(\pi(x)T\pi(x^{-1})\big) = m(T)$ for all $T \in \mathcal{B}(H)$ and $x \in G$. The notion of amenable representation was first introduced and studied by M. Bekka [2]. This notion beautifully unifies the notions of amenable homogeneous spaces and of inner-amenable groups. In fact, a locally compact group G is amenable if and only if all their unitary representations are amenable.

The purpose of this paper is to introduce and study the notion of a vector-valued invariant mean Φ associated to a unitary representation π of G on $\mathcal{S}$, a self-adjoint linear subspace containing I in $\mathcal{B}(H_\pi)$.

Let $VN_\pi(G)$ be the von Neumann algebra generated by $\{\pi(x) : x \in G\}$ and $VN_\pi(G)'$ the commutant of $VN_\pi(G)$. A bounded linear operator $\Phi : \mathcal{S} \to \mathcal{B}(H_\pi)$ is called a *mean* if

$$\Phi(T) \in \overline{\mathrm{co}}^{\,\sigma}\{\pi(x)T\pi(x^{-1}) : x \in G\}$$

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for each $T \in \mathcal{S}$, where for a set $E \subset \mathcal{B}(H_\pi)$, $\overline{\text{co}}^{\,\sigma}(E)$ denotes the closed convex hull of E with respect to the ultraweak topology σ. Any such mean on $\mathcal{S}$ is completely positive (see Proposition 2.3). If $\mathcal{S}$ is invariant, i.e., if $\pi(x)T\pi(x^{-1}) \in \mathcal{S}$ whenever $T \in \mathcal{S}$ and $x \in G$, then a mean Φ is said to be *π-invariant* if for $T \in \mathcal{S}$ and $x \in G$, $\Phi(T) \in VN_\pi(G)'$ and $\Phi\big(\pi(x)T\pi(x^{-1})\big) = \Phi(T)$. If $\mathcal{B}(H_\pi)$ has a π-invariant mean, then π is said to be *strongly amenable* or simply *s-amenable.* When $\mathcal{S}$ is invariant, a state m on $\mathcal{S}$ (i.e., $m \in S^*$ and $\|m\| = m(I) = 1$) is called a π-invariant state if $m\big(\pi(x)T\pi(x^{-1})\big) = m(T)$, for $T \in \mathcal{S}$ and $x \in G$.

Let G be a finite group and (π, H_π) a unitary representation of G. For $T \in \mathcal{B}(H_\pi)$, let $\Phi(T)$ be the average of the operators $\{\pi(x)T\pi(x^{-1}) : x \in G\}$, i.e., $\Phi(T) = \frac{1}{n} \sum_{x \in G} \pi(x)T\pi(x^{-1})$, where n denotes the order of G. Then Φ is the unique π-invariant mean on $\mathcal{B}(H_\pi)$ (see Remark 4.3 (2)) and $\{\alpha \circ \Phi : \alpha$ is a state on $VN_\pi(G)'\}$ is the set of π-invariant states on $\mathcal{B}(H_\pi)$. If π is irreducible, then $\Phi(T) = \frac{1}{d}\,\mathrm{tr}(T)I$, where $d = \dim H_\pi$ and I denotes the identity operator on H_π.

Let λ be the left regular representation of a locally compact group G on $L^2(G)$ and let $\mathcal{M} = \{M_\varphi : \varphi \in L^\infty(G)\}$, where M_φ is the multiplication by φ operator on $L^2(G)$. Then $\mathcal{M}$ is an abelian von Neumann algebra and $\mathcal{M} \cap VN_\lambda(G)'$ only contains constant multiples of the identity operator I on $L^2(G)$. For a scalar-valued mean m on $L^\infty(G)$, let $\widetilde{m}$ be the state on $\mathcal{M}$ defined by $\widetilde{m}(M_\varphi) = m(\varphi)$, and Φ_m the bounded operator on $\mathcal{M}$ defined by $\Phi_m(M_\varphi) = m(\varphi)I$. Then $\widetilde{m}$ is a λ-invariant state on $\mathcal{M}$ if and only if m is a left invariant mean on $L^\infty(G)$ and Φ_m is a λ-invariant mean on $\mathcal{M}$ if and only if m is a two-sided invariant mean on $L^\infty(G)$ (see Section 2).

In Theorem 2.5 we show that if G is amenable then all unitary representations of G are s-amenable (strongly amenable). In Theorem 2.11 we obtain an extension theorem for π-invariant completely positive maps and π-invariant means which characterizes the amenability of G. This extension property is then applied in Proposition 3.2 to show that if G is a non-compact amenable group, then $\mathcal{B}\big(L^2(G)\big)$ has many invariant means with respect to the left regular representation.

It is easy to see that if π is s-amenable then π is amenable (in Bekka's sense), but the converse is not true. For example, if γ is the conjugation representation of F_2, the free group on two letters, then γ is amenable, but not s-amenable. In general, if a representation π has an amenable subrepresentation, then π is amenable [2, Theorem 2.3]; on the other hand, if π is s-amenable, then so are all its subrepresentations.

For a continuous unitary representation π of G on H_π, $T \in \mathcal{B}(H_\pi)$ is said to be *π-weakly almost periodic* (π-w.a.p., or just w.a.p.) if $O_\pi(T) = \{\pi(g)T\pi(g^{-1}) : g \in G\}$ is relatively compact in the $\sigma\big(\mathcal{B}(H_\pi), \mathcal{B}(H_\pi)^*\big)$-topology. T is said to be *π-almost periodic* (π-a.p.) if $O_\pi(T)$ is relatively compact in the norm topology. Let $\mathcal{WAP}_\pi(G)$ (resp. $\mathcal{AP}_\pi(G)$) be the space of all w.a.p. operators (resp. a.p. operators) in $\mathcal{B}(H_\pi)$.

Let $UC_\pi(G) = \{T \in \mathcal{B}(H_\pi) : x \mapsto \pi(x)T\pi(x^{-1})$ is norm continuous$\}$. Then $\mathcal{A}P_\pi(G) \subset \mathcal{WAP}_\pi(G) \subset UC_\pi(G)$, and if G is compact then $UC_\pi(G) = \mathcal{AP}_\pi(G)$. Note that for $\varphi \in L^\infty(G)$, $M_\varphi \in UC_\lambda(G)$ $\big(M_\varphi \in \mathcal{WAP}_\lambda(G)\big)$ $[M_\varphi \in \mathcal{AP}_\lambda(G)]$ if and only if φ is a right uniformly continuous (weakly almost periodic) [almost periodic] function on G. The space $\mathcal{WAP}_\pi(G)$ was defined and investigated by Q. Xu in his thesis [38], where he studied its π-invariant states. It is known that $\mathcal{WAP}_\pi(G)$ always has a unique π-invariant mean. We show that $\mathcal{WAP}_\pi(G)$ has a unique π-invariant state if and only if π is irreducible (Theorem 4.6). Our study of vector-valued means on $\mathcal{WAP}_\pi(G)$ and $\mathcal{AP}_\pi(G)$ is motivated by the work of K. DeLeeuw ([11], [12]) and U.B. Tewari and S. Somasundaram [32] (see also [31]) for locally compact abelian groups.

DeLeeuw [11] proved that $\mathcal{AP}_\lambda(\mathbb{T})$ $\big(= UC_\lambda(\mathbb{T})\big)$ is the norm closed linear span of $\{M_nU : n \in \mathbb{Z},\ U \in VN_\lambda(\mathbb{T})'\}$. Here $\mathbb{T}$ denotes the circle group, M_n the multiplication operator, defined by $(M_nf)(w) = w^n f(w)$ for $f \in L^2(\mathbb{T})$ and $w \in \mathbb{T}$, and U can be identified with a bounded function on $\mathbb{Z}$. Tewari and Somasundaram [32] extended this approximation theorem to $\mathcal{AP}_\lambda(G)$ if G is a locally compact abelian group. We show in this paper that, for a general locally compact group G, operators of the form $\sum\limits_{i=1}^{n} c_i M_{\gamma_i} T_i$ are norm dense in $\mathcal{AP}_\lambda(G)$, where γ_i are matrix coefficients of irreducible finite dimensional unitary representations of G, $T_i \in VN_\lambda(G)'$ and $c_i \in \mathbb{C}$. In fact, this approximation theorem holds more generally for some induced unitary representations of G (see Theorem 4.9). For example, it is well-known that if π is an infinite dimensional irreducible representation of $M(2)$, the motion group of the Euclidean plane, then π can be realized as an induced representation on $L^2(\mathbb{T})$; our approximation theorem implies that $\mathcal{AP}_\pi\big(M(2)\big) = \{M_h : h \in C(\mathbb{T})\}$, where M_h denotes the multiplication by h operator on $L^2(\mathbb{T})$.

We will also define the Fourier transform of operators in $UC_\lambda(G)$ if G is compact. This definition extends the usual notion of a Fourier transform of a continuous function φ on G if we identity φ with the multiplication operator M_φ.

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1. Preliminaries and some notations

Let E be a linear space and let ϕ be a linear functional on E; the value of ϕ at an element x in E will be written as $\phi(x)$ or $\langle \phi, x\rangle$. If E and F are Banach spaces, let $\mathcal{B}(E,F)$ be the space of bounded linear operators $E \to F$. We write $\mathcal{B}(E,F) = \mathcal{B}(E)$ when $E = F$.

Let (E,τ) be a linear topological space, and $A \subseteq E$. Then co A will denote the convex hull of A, and $\overline{\text{co}}^{\,\tau} A$ (or simply $\overline{\text{co}}\, A$) will denote the closed convex hull of E with respect to τ.

If M is a von Neumann algebra, the σ-topology or ultraweak topology on M is the weak*-topology on M.

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure μ. Integration with respect to μ will be denoted by $\int \dots dx$. The Banach spaces $L^p(G)$, $1 \leq p \leq \infty$, are defined as in [18]. Let $C(G)$ denote the C^*-algebra of bounded continuous complex-valued functions on G with the supremum norm, and $LUC(G)$ be the C^*-subalgebra of $C(G)$ consisting of all $f \in C(G)$ such that the mapping $g \mapsto \ell_g f$ from G into $C(G)$ is continuous, where $(\ell_g f)(t) = f(gt)$, $t \in G$. Then G is *amenable* if there is a *left invariant mean* (LIM) on $C(G)$ (or equivalently on $LUC(G)$), i.e., if $\phi \in C(G)^*$, $\phi \geq 0$, $\|\phi\| = 1$ and $\phi(\ell_a f) = \phi(f)$ for all $f \in C(G)$, $a \in G$. Amenable groups include all solvable groups and all compact groups. However the free group on two generators is not amenable. For more information on the subject, we refer the readers to the books of Pier [25] and Paterson [23].

Let π be a (continuous) unitary representation of G on a Hilbert space H_π (or simply H). Let $VN_\pi(G)$ be the von Neumann algebra generated by $\{\pi(g) : g \in G\} \subseteq \mathcal{B}(H)$, i.e., $VN_\pi(G) = \{\pi(g) : g \in G\}''$ is the ultra-weak closed linear span of $\{\pi(g) : g \in G\}$ by the von Neumann double commutation theorem, where $\mathcal{D}'$ denote the commutant of $\mathcal{D} \subseteq \mathcal{B}(H)$.

By *an operator system associated to* π we shall mean a self-adjoint linear subspace $\mathcal{S}$ of $\mathcal{B}(H)$ containing the identity operator I on H. For each positive integer $n = 1, 2, 3, \dots$, let $M_n(\mathcal{S})$ denote the $n \times n$ matrices with entries in $\mathcal{S}$. Then $M_n(\mathcal{S})$ may be regarded as an operator system associated to $\pi_n = \pi \oplus \cdots \oplus \pi$ (the n-fold direct sum of π) acting on $H^n = H \oplus \cdots \oplus H$ (the n-fold direct sum of H). We endow $M_n(\mathcal{S})$ with the norm and order structure inherited from $\mathcal{B}(H^n)$. If B is a C^*-algebra, and $\phi : \mathcal{S} \to B$ is a linear map, then we define $\phi_n : M_n(\mathcal{S}) \mapsto M_n(B)$ by $\phi_n\big((a_{ij})\big) = \big(\phi(a_{ij})\big)$. We call ϕ *completely positive* if ϕ_n is positive for each n.

The left regular representation of G on $L^2(G)$ will be denoted by λ, i.e., $\lambda(x)f(t) = f(x^{-1}t)$, for $x, t \in G$ and $f \in L^2(G)$.

2. Invariant vector-valued means

Let G be a locally compact group and let π be a unitary representation of G on a Hilbert space H. Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be an operator system. A bounded linear map $\Phi : \mathcal{S} \mapsto \mathcal{B}(H)$ is called a *vector-valued mean* (or simply a mean) *on* $\mathcal{S}$ *associated to* π if $\Phi(S) \in \overline{\mathrm{co}}^{\,\sigma}\{g \cdot S : g \in G\}$ for all $S \in \mathcal{S}$ and $g \in G$, where $g \cdot S = \pi(g) \cdot S \cdot \pi(g^{-1})$. The set of all means on $\mathcal{S}$ associated to π will be denoted by $\mathcal{M}_\pi(\mathcal{S})$.

Remark 2.1. Let $\ell^\infty = \ell^\infty\big(G, \mathcal{B}(H)\big)$ denote the set of all bounded functions from $G \mapsto \mathcal{B}(H)$. For $S \in \mathcal{S}$, let $f_S \in \ell^\infty$ be defined by $f_S(g) = g \cdot S$. Then $S \mapsto f_S$ is a linear isometry from $\mathcal{S}$ into $\ell^\infty\big(G, \mathcal{B}(H)\big)$ preserving involution, and $f_I(g) = I$ for all $g \in G$. Consequently $\mathcal{S}$ may be considered as a vector subspace of $\ell^\infty\big(G, \mathcal{B}(H)\big)$. Our notion of means on $\mathcal{S}$ coincides with the notion of weak*-means on $\mathcal{S}$ when considered as a subspace of ℓ^∞ as defined by [40]. Such vector-valued means were studied earlier by Husain and Wong [19].

Note that $E = \mathcal{B}\big(\mathcal{S}, \mathcal{B}(H)\big)$ is a dual Banach space. Indeed, $E \cong Z^*$, where Z is the closed linear span of linear functionals $x \otimes y$, $x \in \mathcal{S}$, $y \in \mathcal{T}(H)$, the trace class operators on H, where $\langle x \otimes y, T\rangle = \langle T(x), y\rangle$, $T \in E$. The $\sigma(E, Z)$ topology on E is referred to as the bounded weak topology (BW-topology) on E. Note that a *bounded* net T_λ converges to T in BW-topology if and only if for each $x \in \mathcal{S}$, $T_\lambda(x) \to T(x)$ in the weak*-topology of $\mathcal{B}(H)$ (W^*OT).

For each $g \in G$, let $\varepsilon(g)$ denote the element in $M_\pi(\mathcal{S})$ defined by $\varepsilon(g)S = \pi(g) \cdot S \cdot \pi(g^{-1})$.

Lemma 2.2. *For each π, $\big(M_\pi(\mathcal{S}), BW\big)$ is compact and convex. Also,* $\mathrm{co}\,\{\varepsilon(g);\, g \in G\}$ *is dense in $M_\pi(\mathcal{S})$.*

Proof. Clearly $M_\pi(\mathcal{S})$ is a BW-closed, convex subset of the unit ball of $\mathcal{B}\big(\mathcal{S}, \mathcal{B}(H)\big)$. Hence it must be compact in the BW-topology by the Banach Alaoglu Theorem. Since the BW-topology and W^*OT agree on $M_\pi(\mathcal{S})$, the last statement follows from [40, Theorem 3.5]. □

Proposition 2.3. *Each $\Phi \in M_\pi(\mathcal{S})$ is completely positive.*

Proof. Since the set $\{\Phi : \Phi \in \mathcal{B}\big(\mathcal{S}, \mathcal{B}(H)\big)$, Φ completely positive, and $\Phi(I) = I\}$ is BW-compact [24, Theorem 6.4], it suffices to show (in view of Lemma 2.2), that for each $g \in G$, $\Phi = \varepsilon(g)$ is completely positive. Let $\{n = 1, 2, \dots\}$ be fixed.

Suppose $(S_{ij}) \in M_n(\mathcal{S})$ and $(S_{ij}) \geq 0$. If $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in H^n$, then

$$\begin{aligned} \langle \Phi_n\big((S_{ij}\big)\alpha, \alpha\rangle &= \sum_{i,j=1}^{n} \langle \pi(g) S_{ij} \pi(g^{-1})\alpha_j, \alpha_i\rangle \\ &= \sum_{i,j=1}^{n} \langle S_{ij}\pi(g^{-1})\alpha_j,\, \pi(g^{-1})\alpha_i\rangle \geq 0. \end{aligned}$$ □

Proposition 2.4. *Let π be a unitary representation of G, $\mathcal{S}$ an operator system in $\mathcal{B}(H)$, and $\Phi : \mathcal{S} \mapsto \mathcal{B}(H)$. If Φ is a mean on $\mathcal{S}$ with respect to π, then for each $n = 1, 2, \dots$, Φ_n is a mean on $M_n(\mathcal{S})$ with respect to $\pi_n = \pi \oplus \cdots \oplus \pi$, the n-fold direct sum of π. Conversely, if Φ_n is a mean on $M_n(\mathcal{S})$ with respect to π_n for some n, then Φ is a mean on $\mathcal{S}$ with respect to π.*

Proof. If Φ is a mean, then by Lemma 2.2, there exists a net $\Phi_\alpha = \sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} \varepsilon(g_k^{(\alpha)})$, where $\lambda_k^{(\alpha)} \geq 0$, $\Sigma\lambda_k^{(\alpha)} = 1$, such that $\Phi_{\alpha n} \to \Phi$ in the BW topology.

Now clearly $\varepsilon(g)_n$ are means $M_n(\mathcal{S}) \mapsto \mathcal{B}(H^n)$ for $g \in G$, and so (Φ_α) are means on $M_n(\mathcal{S})$. Consequently, Φ_n is a mean on $M_n(\mathcal{S})$. Conversely, if Φ_n is a mean on $M_n(\mathcal{S})$ for some n, and $S \in \mathcal{S}$, let $A = (a_{ij}) \in M_n(\mathcal{S})$, such that $a_{11} = S$, and $a_{ij} = 0$ for $i \neq 1$, or $j \neq 1$. Now since $\Phi_n(A) \in \overline{\text{co}}^{\,\sigma}\{g \cdot A;\, g \in G\}$, it follows readily then $\Phi(S) \in \overline{\text{co}}^{\,\sigma}\{g \cdot S;\, g \in G\}$. So Φ is a mean also. □

Assume $\mathcal{S}$ is a π-invariant operator system in $\mathcal{B}(H_\pi)$. Then G acts on $\mathcal{B}(\mathcal{S}, \mathcal{B}(H_\pi))$ on the left and on the right by

$$(g\Phi)(T) = g \cdot \Phi(T), \qquad (\Phi g)(T) = \Phi(g \cdot T),$$

for $\Phi \in \mathcal{B}(\mathcal{S}, \mathcal{B}(H_\pi))$, $g \in G$ and $T \in \mathcal{S}$. A mean Φ on S is called *left π-invariant*, or simply *π_ℓ-invariant*, if $g\Phi = \Phi$ for all $g \in G$, i.e., if $\Phi(T) \in VN_\pi'$ for each $T \in \mathcal{S}$; Φ is called *right π-invariant*, or simply *π_r-invariant*, if $\Phi g = \Phi$ for all $g \in G$, i.e., if $\Phi(g \cdot T) = \Phi(T)$ for $T \in \mathcal{S}$. Φ is called *π-invariant* if it is both π_ℓ and π_r-invariant. If $\mathcal{B}(H_\pi)$ has a π-invariant mean then π is said to be s-amenable (strongly amenable).

Proposition 2.5. *For any unitary representation π of G, the set of π_ℓ-invariant means Σ on an operator system $\mathcal{S} \subseteq \mathcal{B}(H)$ is non-empty if and only if $K_S \cap VN_\pi(G)' \neq \emptyset$ for each $S \in \mathcal{S}$, where $K_S = \overline{\text{co}}^{\,\sigma}\{g \cdot S;\, g \in G\}$. In this case, $K_S \cap VN_\pi(G)' = \{\Phi(S);\, \Phi \in \Sigma\}$.*

Proof. This follows from Theorem 2.1 in [20]. □

Let $\lambda = \lambda_G$ be the left regular representation of G on $L^2(G)$. For $\varphi \in L^\infty(G)$, let M_φ be the multiplication by φ operator on $L^2(G)$. Then $x \cdot M_\varphi = \lambda(x) M_\varphi \lambda(x^{-1}) = M_{x\varphi}$, where ${}_x\varphi(y) = \varphi(x^{-1}y)$ for $y \in G$. Let $\mathcal{S} = M_{L^\infty(G)} = \{M_\varphi :\, \varphi \in L^\infty(G)\}$. Then $\mathcal{S}$ is a von Neumann subalgebra of $\mathcal{B}(L^2(G))$. For $\varphi \in L^\infty(G)$,

$$\overline{\text{co}}^{\,\sigma}\{x \cdot M_\varphi :\, x \in G\} = \big\{M_\psi :\, \psi \in \overline{\text{co}}^{\,\sigma}\{{}_x\varphi :\, x \in G\}\big\}.$$

Note that $M_\varphi \in VN_\lambda'$ if and only if φ is a constant function. Therefore, by Proposition 2.5, $\mathcal{S}$ has a λ_ℓ-invariant mean if and only if, for each $\varphi \in L^\infty(G)$, $\overline{\text{co}}^\sigma\{{}_x\varphi : x \in G\}$ contains a constant function, i.e., $L^\infty(G)$ is left stationary. It is known that $L^\infty(G)$ is left stationary if and only if G is amenable; see [37]. If Φ is a λ_ℓ-invariant mean on $\mathcal{S}$, write $\Phi(M_\varphi) = m(\varphi)I$. Then m is a right invariant mean on $L^\infty(G)$. It is now clear that Φ is a λ-invariant mean on $\mathcal{S}$ if and only if the corresponding m is a two-sided invariant mean on $L^\infty(G)$.

If π is a unitary representation of G, let UC_π denote the C^*-subalgebra of $\mathcal{B}(H_\pi)$ consisting of all operators $T \in \mathcal{B}(H_\pi)$ such that the map $g \mapsto g \cdot T$ from $G \mapsto \mathcal{B}(H_\pi)$ is continuous (see [2]).

THEOREM 2.6. *Let G be a locally compact group.*

(a) *If G is amenable, then each unitary representation π of G is s-amenable.*
(b) *If $M_{L^\infty(G)}$ has a λ_ℓ-invariant mean, then G is amenable.*
(c) *If $M_{L^\infty(G)}$ has a λ_r-invariant mean, then G is amenable.*

In particular, G is amenable if and only if each unitary representation of G is s-amenable.

Proof. (a) If G is amenable, then for each $S \in \mathcal{B}(H)$ consider the action on G of the weak*-compact convex set $K_S = \overline{\mathrm{co}}^\sigma\{g \cdot S;\ g \in G\}$ in $\mathcal{B}(H)$, defined by

$$G \times K_S \to K_S, \qquad (g, T) \mapsto g \cdot T, \tag{1}$$

where $g \in G$ and $T \in K_S$. It is easy to see that this action is affine and separately continuous on K_S in the weak operator topology (which coincides with the σ-topology on bounded sets) on K_S. Hence by Day's fixed point theorem (see [9] and [10]), there exists $T \in K_S$ such that $g \cdot T = T$ for all $g \in G$. Clearly $T \in VN_\pi(G)'$, since $VN_\pi(G)'$ is precisely the fixed point set in K_S of the action on G defined by (1). By Proposition 2.5, there is a π_ℓ-invariant mean Φ for $\mathcal{B}(H)$. We now show that Φ may be chosen to be also π_r-invariant. To this end, we let $\mathcal{S} = UC_\pi$.

By the above argument, the set Σ of π_ℓ-invariant means on UC_π is a non-empty, weak*-compact convex subset of $\mathcal{B}\big(UC_\pi, \mathcal{B}(H)\big)$. Furthermore, the action of G on Σ defined by $(g, \Phi) \to \Phi g$ is separately continuous when Σ has the weak*-topology. Recall that $\langle \Phi g, S\rangle = \langle \Phi, g \cdot S\rangle$, $g \in G$, $\Phi \in \Sigma$, $S \in UC_\pi$. Hence, by Day's fixed point theorem, again there is $\Phi \in \Sigma$ such that $\Phi g = \Phi$ for all $g \in \Phi$. Since $\mathcal{T}(H)$ is a Banach $L_1(G)$-module, defined by the strong integral, $f \cdot T = \int f(y)\pi(y)T\pi(y^{-1})dy$, $f \in L^1(G)$, $T \in \mathcal{T}(H)$, and $\mathcal{T}(H)^* = \mathcal{B}(H)$, $\mathcal{B}(H)$ is also a Banach $L_1(G)$-module. Let $f_0 \in L_1(G)$ be fixed with $f_0 \geq 0$ and $\|f_0\|_1 = 1$. By the Hahn-Banach theorem, there exists a net $\mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^{(\alpha)} \delta_{x_i}^{(\alpha)}$ of convex combinations of point evaluation δ_x on $C(G)$ such that $\int h(y)d\mu_\alpha \to \int h(y)f_0(y)dy$ for each $h \in C(G)$. Consequently, for each $S \in \mathcal{B}(H)$, $T \in \mathcal{T}(H)$,

$$\Big\langle \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \pi(x_i^\alpha) S \pi(x_i^{\alpha-1}), T \Big\rangle \to \langle f_0 \cdot S, T\rangle,$$

for each $T \in \mathcal{T}(H)$. Since $f_0 \cdot S \in UC_\pi$ for each $S \in \mathcal{B}(H)$ (see [2]), we may define $\widetilde{\Phi} : \mathcal{B}(H) \to \mathcal{B}(H)$ by $\widetilde{\Phi}(S) = \Phi(f_0 \cdot S)$, $S \in \mathcal{B}(H)$. Then the above argument shows that $\widetilde{\Phi}$ is a mean on $\mathcal{B}(H)$, and that $\widetilde{\Phi}$ is π_ℓ-invariant. Also an argument similar to the one used in the proof of Theorem 3.5 of [2] shows that $\widetilde{\Phi}$ is π_r-invariant. So $\widetilde{\Phi}$ is an invariant mean on $\mathcal{B}(H)$.

(b) This was proved in the remarks prior to this theorem.

(c) If Φ is a λ_r-invariant mean on $\mathcal{S} = M_{L^\infty(G)}$ and τ a state on $\mathcal{S}$, set $m(\varphi) = \tau\big(\Phi(M_\varphi)\big)$, $\varphi \in L^\infty(G)$. Then m is a scalar-valued left invariant mean on $L^\infty(G)$. So G is amenable. □

A von Neumann algebra $\mathcal{M}$ acting on a Hilbert space H is said to have property P if for each $T \in \mathcal{B}(H)$, $\overline{\mathrm{co}}^\sigma\{U^*TU;\, U \in \mathcal{M}^u\} \cap \mathcal{M}' \neq \emptyset$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ and $\mathcal{M}^u$ is the group of unitary elements in $\mathcal{M}$.

Remark 2.7. If π is a unitary representation of G, and $\mathcal{B}(H_\pi)$ has a π_ℓ-invariant mean, then $VN_\pi(G)$ has property P, but the converse is not true.

Proof. Indeed, if $G = SL(2, \mathbb{R})$, and λ is the left regular representation of G, then $VN_\lambda(G)$ has property P, but G is not amenable. Hence $\mathcal{B}\big(L^2(G)\big)$ cannot have a λ_ℓ-invariant mean by Theorem 2.6(b). □

It is known that if G is amenable, then, for each unitary representation π of G, VN_π has property P; see [23, p. 78]. This fact is also a direct consequence of Theorem 2.6 above.

Corollary 2.8. *If π is a unitary representation of G such that the von Neumann algebra $VN_\pi(G)$ is finite, and $\mathcal{B}(H_\pi)$ has a π_ℓ-invariant mean, then the representation π is amenable.*

Proof. This follows from Corollary 2.7 and Proposition 1 in [33]. □

Proposition 2.9. *Let π be a unitary representation of G.*

(a) *If $\mathcal{B}(H_\pi)$ has a π_r-invariant mean (in particular, if π is s-amenable), then π is amenable.*
(b) *If H_π is finite-dimensional, then π is s-amenable.*
(c) *Let τ be a subrepresentation of π. If $\mathcal{B}(H_\pi)$ has a π_ℓ-invariant (π_r-invariant) mean, then $\mathcal{B}(H_\tau)$ has a τ_ℓ-invariant (τ_r-invariant) mean. In particular, if π is s-amenable then so is τ.*

Proof. (a) Let Φ be a π_r-invariant mean on $\mathcal{B}(H_\pi)$ and τ a state on $\mathcal{B}(H_\pi)$. Then $\tau \circ \Phi$ is a π-invariant state $\mathcal{B}(H_\pi)$.

(b) See Remark 4.3(2).

(c) This follows directly from the definition of s-amenability. □

It is known that if a subrepresentation π_1 of a representation is amenable then so is π; see [2, Theorem 1.3]. So if π_1 is an amenable representation and π_2 is a non-amenable representation of G then $\pi = \pi_1 \oplus \pi_2$ is amenable, but not s-amenable. For example, let G be a non-amenable discrete group and let $\pi = 1_G \oplus \lambda_G$, where 1_G is the trivial one-dimensional representation of G. Then π is amenable but not s-amenable, since its subrepresentation λ_G is not s-amenable. In fact, since $VN_\lambda(G)$ does not have property P (see [27]), $VN_\pi(G)$ does not have a π_ℓ-invariant mean.

Bekka [2] showed that G is amenable if and only if each irreducible representation of G is amenable. This fact together with Theorem 2.6 and Proposition 2.9(a) imply the following result:

Corollary 2.10. *A locally compact group G is amenable if and only if all irreducible unitary representations of G are s-amenable.*

Let γ be the conjugation representation of a locally compact group G. It is known that G is inner amenable if and only if γ is amenable; see [2, Theorem 2.4]. But the inner amenability of G does not imply the s-amenability of γ, as the following example shows.

Example 2.11. Let $G = F_2$, the free group on two letters a, b. Consider the von Neumann subalgebra $M_{\ell^\infty(G)}$ of $\mathcal{B}\big(\ell^2(G)\big)$. Note that for $x \in G$, and $\varphi \in \ell^\infty(G)$, $x \cdot M_\varphi = \gamma(x^{-1}) M_\varphi \gamma(x) = M_{\varphi^x}$, where $\varphi^x(y) = \varphi(x^{-1}yx)$, $y \in G$. We claim that $M_{\ell^\infty(G)}$ does not have a γ_r-invariant mean (and hence γ is not s-amenable).

To prove the claim, we will identify $M_{\ell^\infty(G)}$ with $\ell^\infty(G)$. Assume $\ell^\infty(G)$ has a γ_r-invariant mean Φ, i.e., $\Phi \in \mathcal{B}\big(\ell^\infty(G)\big)$, $\Phi(\varphi) \in \overline{\mathrm{co}}^\sigma\{\varphi^x : x \in G\}$ for all $\varphi \in \ell^\infty(G)$ and $\Phi(\varphi^x) = \Phi(\varphi)$ for all $\varphi \in \ell^\infty(G)$ and $x \in G$. Let A be the subset of F_2 formed by all reduced words ending with a^k, $k \neq 0$. Then

$$F_2 = A \cup aAa^{-1} \cup \{e\}, \tag{1}$$

$$A,\ bAb^{-1} \quad \text{and} \quad b^{-1}Ab \quad \text{are mutually disjoint.} \tag{2}$$

Then $\Phi(\chi_G) = \chi_G \leq 2\Phi(\chi_A) + \Phi(\chi_{\{e\}})$, by (1). Note that $\Phi(\chi_{\{e\}}) = \chi_{\{e\}}$ and, by (2), $3\Phi(\chi_A) \leq \chi_G$. So $\Phi(\chi_A) \leq \chi_{\{e\}}$ and $\chi_G \leq 3\chi_{\{e\}}$. We have reached a contradiction.

Bekka [2] proved that γ has a subrepresentation γ_0 which is not amenable. This fact also implies that γ is not s-amenable.

A discrete group G is strongly inner amenable if the space of bounded functions on G has a non-trivial inner invariant mean; see [2]. If G is the direct product of the free group with two generators and a discrete amenable group H with at least two members then G is strongly inner amenable (see [16]), but, by following the proof of Example 2.11, we see that the conjugation representation of G is not s-amenable.

Example 2.12. Let H be an infinite dimensional Hilbert space and G the discrete group of all unitary operators on H. Let π be the identity representation of G on H. Then $VN_\pi = \mathcal{B}(H)$ and the unitary group of VN_π is G. Since $\mathcal{B}(H)$ has property P (see [23, p. 77]), $\mathcal{B}(H)$ has a π_ℓ-invariant mean.

However $\mathcal{B}(H)$ does not have a π_r-invariant mean, since $\mathcal{B}(H)$ is not a finite von Neumann algebra.

3. Extensions of invariant completely positive maps and means

As mentioned earlier (see Proposition 2.3), if Φ is a mean on an operator system $\mathcal{S}$ associated with a unitary representation π of a locally compact group G, then Φ is a completely positive map. In this section, we study the extensions of π_r-invariant completely positive maps and π-invariant means on $\mathcal{S}$.

THEOREM 3.1. *Let G be a locally compact group. The following are equivalent:*

(a) *G is amenable.*
(b) *Whenever $\{\pi, H\}$ is a representation of G, $\mathcal{A} \subseteq UC_\pi(G)$ is a C^*-algebra with unit, $\mathcal{S} \subseteq \mathcal{A}$ is a π-invariant operator system, and $\Phi : \mathcal{S} \mapsto \mathcal{B}(H)$ is a completely positive map [mean] such that $\Phi(g \cdot S) = \Phi(S)$ for each $g \in G$, $S \in \mathcal{S}$, then there exists a completely positive map [mean] $\Psi : \mathcal{A} \mapsto \mathcal{B}(H)$ such that Ψ extends Φ and $\Psi(g \cdot S) = \Psi(S)$ for each $g \in G$, $S \in \mathcal{S}$. Furthermore, if Φ is an invariant mean on $\mathcal{S}$, then Ψ may also be chosen to be an invariant mean on $\mathcal{A}$.*

Proof. (a) $\Longrightarrow$ (b): Without loss of generality we may assume that $\|\Phi\| = 1$. Let $\mathcal{K} = \{\Psi :\ \Psi : \mathcal{A} \mapsto \mathcal{B}(H),\ \Psi$ is completely positive and extends $\Phi\}$. Note that each $\Psi \in \mathcal{K}$ has norm one (since, by the complete positivity of Φ, $\|\Phi\| = \|\Phi(1)\| = 1$, so each $\Psi \in \mathcal{K}$, $\Psi(1) = \Phi(1)$, and so $\|\Psi\| = 1$), $\mathcal{K}$ is convex, and closed in the BW-topology. So $\mathcal{K}$ must be compact in the BW-topology of $\mathcal{B}(\mathcal{A}, \mathcal{B}(H))$. Consider the right action of G on $\mathcal{K}$ defined by

$$(g, \Psi) \mapsto \Psi g, \qquad (\Psi g)(S) = \Psi(g \cdot S).$$

This action is separately continuous when K has the BW-topology, and $\Psi g \in K$ for $\Psi \in K$ and $g \in G$.

(i) If $g_\alpha \to g$, and $\Psi \in \mathcal{K}$, then for $S \in \mathcal{A} \subset UC_\pi(G)$, $g_\alpha \cdot S \to g \cdot S$ in norm. So $(\Psi g_\alpha)(S) = \Psi(g_\alpha \cdot S) \to \Psi(g \cdot S) = (\Psi g)(S)$ in norm and in the w^*-topology of $B(H)$. Consequently, $\Psi g_\alpha \to \Psi g$ in the BW-topology.
(ii) If $\Psi_\alpha \to \Psi$ in the BW topology, then for each $g \in S$, $(\Psi_\alpha g)(S) = \Psi_\alpha(g \cdot S) \mapsto \Psi(g \cdot S)$ in the weak*-topology of $\mathcal{B}(H)$. So, by the boundedness of $\{\Psi g;\ \psi \in \mathcal{K}\}$, $\Psi_\alpha g \to \Psi g$ in the BW-topology.
(iii) If $g \in G$, $\Psi \in \mathcal{K}$, then for all $S \in \mathcal{S}$, $(\Psi g)(S) = \Psi(g \cdot S) = \Phi(g \cdot S) = \Phi(S)$. Hence $\Psi g \in \mathcal{K}$.

Also $\mathcal{K} \neq \emptyset$ by Arveson's extension theorem [1] (see also [24]). Hence, by Day's fixed point theorem, there exists $\Psi \in \mathcal{K}$ such that $\Psi g = \Psi$ for all $g \in G$.

(b) $\Longrightarrow$ (a): Consider the left regular representation λ of G. Let $\mathcal{S} = \{\alpha I : \alpha \in \mathbb{C}\}$ and $\mathcal{A} = \{M_f : f \in LUC(G)\}$. Note that $\mathcal{A} \subset UC_\lambda(G)$. Set $\Phi(\alpha I) = \alpha I$, $\alpha \in \mathbb{C}$. Clearly Φ is a completely positive map [mean] on $\mathcal{S}$ and $\Phi(g \cdot T) = \Phi(T)$ for $T \in \mathcal{S}$ and $g \in G$. Let $\Psi : \mathcal{A} \to \mathcal{B}\big(L^2(G)\big)$ be an extension of Φ such that $\Psi(M_g f) = \Psi(g \cdot M_f) = \Psi(M_f)$ for all $f \in LUC(G)$ and $g \in G$ and such that Ψ is completely positive. Let θ be a state on $\mathcal{A}$.

Set $m(f) = \theta\big(\Psi(M_f)\big)$. Then, for $f \in LUC(G)$ and $g \in G$,

$$m({}_g f) = \theta\big(\Psi(M_g f)\big) = \theta\big(\Psi(g \cdot M_f)\big) = \theta\big(\Psi(M_f)\big) = m(f).$$

So m is left invariant, $m \geq 0$, and $m(1_G) = \theta\big(\Psi(I)\big) = \theta(I) = 1$; i.e., m is a left invariant mean on $LUC(G)$ and hence G is amenable.

If Φ is a mean, then by Lemma 2.2 there exists a net $\Phi_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \varepsilon(g_i^\alpha)$ of convex combinations of $\varepsilon(g)$'s such that $\Phi_\alpha \to \Phi$ in the BW topology of $\mathcal{B}\big(\mathcal{S}, \mathcal{B}(H)\big)$. Clearly each Φ_α may be regarded as a mean $\mathcal{A} \to \mathcal{B}(H)$. Let Ψ be a cluster point of $\{\Phi_\alpha\}$ in the BW-topology of $\mathcal{B}\big(\mathcal{A}, \mathcal{B}(H)\big)$. Then Ψ is a mean, and Ψ extends Φ. This shows that the set $\mathcal{K} = \{\Psi :\ \Psi$ is a mean in $\mathcal{B}\big(\mathcal{A}, \mathcal{B}(H)\big)$ and Ψ extends $\Phi\}$ is non-empty. Following the proof of the completely positive case for the implication (a) $\Longrightarrow$ (b), we conclude that $\mathcal{K}$ contains a π_r-invariant mean Ψ.

If Φ is a π-invariant mean on $\mathcal{S}$, let $\mathcal{K}' = \{\Psi :\ \Psi\ \pi_r$-invariant mean on $\mathcal{A}$ extending $\Phi\}$. Then $\mathcal{K}'$ is a non-empty, BW-compact convex subset of $\mathcal{B}\big(\mathcal{A}, \mathcal{B}(H)\big)$. Consider the left action of G on $\mathcal{K}'$ defined by $(g, \Psi) \mapsto g\Psi$, where $(g\Psi)(A) = g \cdot \Psi(A) = \pi(g)\Psi(A)\pi(g^{-1})$. Then, as is readily checked, $g\Psi \in \mathcal{K}'$ for any $g \in G$ and $\Psi \in \mathcal{K}'$, and the map $G \times \mathcal{K}' \to \mathcal{K}'$ is separately continuous when $\mathcal{K}'$ has the BW-topology. So an application of Day's fixed point theorem shows that there exists $\Psi \in \mathcal{K}'$ such that $g\Psi = \Psi$ for all $g \in G$. Such a function Ψ is an invariant mean on $\mathcal{A}$. $\square$

Let G be a locally compact amenable group. Let $\mathcal{S} = \{M_f : f \in LUC(G)\} \subset UC_\lambda(G) \subset \mathcal{B}\big(L^2(G)\big)$. For a given scalar-valued (two-sided) invariant mean m on $LUC(G)$, let $\Phi_0(M_f) = m(f)I$, for $f \in LUC(G)$. Then Φ_0 is a λ-invariant mean on $\mathcal{S}$. By Theorem 3.1, Φ_0 can be extended to an invariant mean Φ on $UC_\lambda(G)$. Fix $h \in L^1(G)$ with $h \geq 0$ and $\|h\|_1 = 1$. Define a mean Ψ on $\mathcal{B}\big(L^2(G)\big)$ by $\Psi(T) = \Phi(h \cdot T)$. Then Φ is a λ-invariant mean and Ψ extends Φ (and hence Φ_0); see [2, p. 390]. So $\mathcal{B}\big(L^2(G)\big)$ has at least as many λ-invariant means as (two-sided) scalar-valued invariant means on $LUC(G)$.

It is known that if G is a noncompact amenable group and d is the smallest possible cardinality for a covering of G by compact sets, then $LUC(G)$ has 2^{2^d} invariant means (see [39]), so Theorem 3.1 has the following consequence.

Proposition 3.2. *Let G be a noncompact locally compact amenable group. Then $\mathcal{B}\big(L^2(G)\big)$ has at least 2^{2^d} λ-invariant means.*

Remark 3.3. If a locally compact group G is amenable when regarded as a discrete group, then part (b) of Theorem 3.1 holds for any π-invariant C^*-algebra $\mathcal{A} \subset \mathcal{B}(H_\pi)$ (without having to assume $\mathcal{A} \subset UC_\pi(G)$).

For a compact group G the weak integral

$$\Lambda(T) = \int_G (g \cdot T) dg, \quad T \in \mathcal{B}(H_\pi)$$

defines a π-invariant mean on $\mathcal{B}(H_\pi)$. For $G = \mathbb{T}$ and $\pi = \lambda$, the mean Λ was first considered by DeLeeuw (see [11] and [12]) in his study of the operators in $\mathcal{B}\big(L^2(\mathbb{T})\big)$.

Note that the restriction of Λ to $\mathcal{S} = \{M_f : f \in L^\infty(G)\}$ is just $\Lambda(M_f) = m_0(f) \cdot I$, where $m_0(f) = \int_G f(y)dy$. When an infinite compact group G is amenable as a discrete group, $L^\infty(G)$ has an invariant mean m different from m_0; see [26] and [17]. By Remark 3.3, there exists a λ-invariant mean Φ on $\mathcal{B}\big(L^2(G)\big)$ such that $\Phi(M_f) = m(f)I$, $f \in L^\infty(G)$. Then $\Phi \neq \Lambda$. Thus we have the following result.

Proposition 3.4. *Assume G is an infinite compact group which is amenable as a discrete group. Then $\mathcal{B}\big(L^2(G)\big)$ has more than one λ-invariant mean.*

Remark 3.5. Let G be a connected compact simple Lie group, e.g., $G = SU(n)$, $n \geq 2$. Then $L^\infty(G)$ has a unique invariant mean, namely m_0, as defined above; see [29], [22], [15], [7]. It would be interesting to know if Λ is the only λ-invariant mean on $\mathcal{B}\big(L^2(G)\big)$.

4. Almost periodic operators

Let G be a locally compact group and $\{\pi, H\}$ a continuous unitary representation of G. Recall that $T \in \mathcal{B}(H)$ is said to be w.a.p. (or π-w.a.p.) if $\{g \cdot T : g \in G\}$ is relatively $\sigma\big(\mathcal{B}(H), \mathcal{B}(H)^*\big)$ compact in $\mathcal{B}(H)$, and T is π-a.p. if $\{g \cdot T : g \in G\}$ is relatively compact in $\mathcal{B}(H)$.

Let $\mathcal{AP}_\pi(\mathcal{WAP}_\pi)$ be the space of π-a.p. (π-w.a.p.) operators in $\mathcal{B}(H)$.

Proposition 4.1 ([38]).

(1) *$\mathcal{WAP}_\pi$ is a π-invariant operator system and $\mathcal{WAP}_\pi \subset UC_\pi$.*
(2) *$\mathcal{AP}_\pi$ is a C^*-subalgebra of $\mathcal{WAP}_\pi$.*

It is known that $\mathcal{WAP}_\pi$ has a unique π-invariant mean; see [30, p. 35]. For completeness, we outline a proof of this result.

Proposition 4.2. *$\mathcal{WAP}_\pi$ has a unique π-invariant mean.*

Proof. For $T \in \mathcal{WAP}_\pi$, let $K(T) = \overline{\text{co}}\{g \cdot T : g \in G\}$. Then $K(T)$ is weakly compact and by the Ryll-Nardzewski fixed point theorem (see [8]), $K(T) \cap VN'_\pi \neq \phi$. For a bounded linear functional θ on WAP_π, $x \mapsto f(x) = \langle \theta, x \cdot T\rangle$ defines a continuous weakly almost periodic function on G, and if $S \in K(T) \cap VN'_\pi$ then $\langle \theta, S\rangle$ is the unique constant function in $\overline{\text{co}}\{f_t : t \in G\}$, where $f_t(x) = f(xt)$ for $t, x \in G$. So by the Hahn-Banach theorem, $K(T) \cap VN'_\pi$ is a singleton.

For $T \in \mathcal{WAP}_\pi$, let $\Phi(T)$ be the unique member of $K(T) \cap VN'_\pi$. By Proposition 2.5, Φ is linear and π_ℓ-invariant. Since, for $T \in \mathcal{WAP}_\pi$ and $x \in G$, $K(T) = K(x \cdot T)$, we have $\Phi(x \cdot T) = \Phi(T)$. So Φ is also π_r-invariant. $\square$

REMARK 4.3. (1) The restriction of the π-invariant mean of $\mathcal{WAP}_\pi$ to $\mathcal{AP}_\pi$ is the unique π-invariant mean of $\mathcal{AP}_\pi$. Here is a direct construction of the π-invariant mean of $\mathcal{AP}_\pi$: For $g \in G$, let $\widetilde{g} \in \mathcal{B}(\mathcal{AP}_\pi)$ be the operator which sends T to $g \cdot T$. Then the strong operator closure of $\{\widetilde{g} : g \in G\}$ in $\mathcal{B}(\mathcal{AP}_\pi)$ is a compact topological group G^π and $g \mapsto \widetilde{g}$ is strong operator continuous. (Thus G^π is an almost periodic compactification of G.) For $s \in G^\pi \subset \mathcal{B}(\mathcal{AP}_\pi)$, we denote $s(T)$ by $s \cdot T$. So for $g \in G$, $\widetilde{g} \cdot T$ is just $g \cdot T$. Let μ be the unique normalized Haar measure of the compact group G^π. For $T \in \mathcal{AP}_\pi$, let $\Phi(T) = \int_{G^\pi} (s \cdot T) d\mu(s)$. Then Φ is the π-invariant mean on $\mathcal{AP}_\pi$. Note that Φ is π_ℓ-invariant since μ is left invariant, and Φ is π_r-invariant since μ is right invariant.

(2) Note that if π is a finite dimensional unitary representation of G, then $\mathcal{AP}_\pi = \mathcal{B}(H_\pi)$; hence, by (1), $\mathcal{B}(H_\pi)$ has a unique π-invariant mean and π is s-amenable. Bekka [2] proved that finite dimensional representations are amenable.

(3) When G is abelian, Tawari and Somasundaram [32] showed that $\mathcal{AP}_\lambda(G)$ has a λ-invariant mean.

REMARK 4.4. Let $\mathcal{W}_{\pi,0} = \big\{T \in \mathcal{WAP}_\pi : 0$ is in the weak closure of $\{g \cdot T : g \in G\}\big\}$. DeLeeuw and Glicksberg [14] have shown that $\mathcal{W}_{\pi,0}$ is a G-invariant closed subspace of $\mathcal{WAP}_\pi$ and

$$\mathcal{WAP}_\pi = \mathcal{AP}_\pi \oplus \mathcal{W}_{\pi,0}.$$

We outlined a construction of a π-invariant mean Φ of $\mathcal{AP}_\pi$ in Remark 4.3 (1) above. For $T \in \mathcal{WAP}_\pi$, let $T = T_a + T_0$ be the unique decomposition of T with $T_a \in \mathcal{AP}_\pi$ and $T_0 \in \mathcal{WAP}_{\pi,0}$. Set $\Phi(T) = \Phi(T_a)$. It can be shown directly that Φ is the unique π-invariant mean of $\mathcal{WAP}_\pi$.

LEMMA 4.5. *If Φ is the unique invariant mean of $\mathcal{WAP}_\pi$ and θ a state on VN'_π then $\theta \circ \pi$ is a π-invariant state on $\mathcal{WAP}_\pi$. Conversely, if m is a π-invariant state on $\mathcal{WAP}_\pi$ then there exists a state θ on VN'_π such that $m = \theta \circ \Phi$.*

Proof. Let m be a π-invariant state of $\mathcal{WAP}_\pi$. Let $\theta = m|VN'_\pi$, and let Φ be the unique mean on $\mathcal{WAP}_\pi$. Then $m = \theta \circ \Phi$. Indeed, for $T \in \mathcal{WAP}_\pi$ and $\varepsilon > 0$ there exists $\sum_{i=1}^{n} c_i(g_i \cdot T) \in \text{co}\{g \cdot T : g \in G\}$ such that $\|\Phi(T) - \sum_{i=1}^{n} c_i(g_i \cdot T)\| < \varepsilon$. Then

$$\Big|m\Big(\Phi(T) - \sum_{i=1}^{n} c_i(g_i \cdot T)\Big)\Big| = |\theta\big(\Phi(T)\big) - m(T)| < \varepsilon,$$

and so $m(T) = \theta\big(\Phi(T)\big)$. □

THEOREM 4.6. *$\mathcal{WAP}_\pi$ has a unique π-invariant state if and only if π is irreducible.*

This theorem answers a question in Q. Xu's thesis [38].

REMARK 4.7. For a locally compact group G, let $WAP(G)$ $\big(AP(G)\big)$ be the C^*-algebra of weakly almost periodic (almost periodic) functions on G; cf. [4]. For $f \in L^\infty(G)$ we have $f \in WAP(G)$ $\big(f \in AP(G)\big)$ if and only if M_f, the multiplication by f operator on $L^2(G)$, is λ-w.a.p. (λ-a.p.). It is well-known that $WAP(G)$ has a unique invariant mean $m = m_G$, and $WAP(G) = AP(G) \oplus WAP_0(G)$, where $WAP_0(G) = \{f \in WAP(G) : m_G(|f|) = 0\}$; see [4]. So Proposition 4.2 and Theorem 4.6 show that the unique vector-valued mean on WAP_π is the proper generalization of the scalar-valued invariant mean on $WAP(G)$.

LEMMA 4.8. *Let $\mathcal{F}$ be the set of $T \in \mathcal{B}(H)$ such that the linear span of $\{g \cdot T : g \in G\}$ is finite dimensional. Then $\mathcal{F}$ is dense in $\mathcal{AP}_\pi$.*

Proof. This is a known result, but we outline a proof here. Let $\varphi \in C(G^\pi)$ (as defined in Remark 4.3 (1)) and $T \in \mathcal{AP}_\pi$. Let $\varphi \cdot T \in \mathcal{AP}_\pi$ be defined by $\varphi \cdot T = \int_{G^\pi} \varphi(s)(s \cdot T) d\mu(s)$, where μ is the Haar measure on G^π.

For $T \in \mathcal{AP}_\pi$ and $\varepsilon > 0$ choose a neighborhood V of e in G^π such that if $s \in G^\pi$ then $\|s \cdot T - T\| < \varepsilon$. Choose $\varphi \in C(G^\pi)$ such that $\varphi \geq 0$, $\|\varphi\|_1 = 1$ and $\varphi = 0$ off V. Then $\|\varphi \cdot T - T\| < \varepsilon$. Since G^π is compact, by the Peter-Weyl theorem, there exists a function ψ which is a linear combination of matrix coefficients of finite dimensional irreducible representations such that $\|\varphi - \psi\|_1 < \varepsilon$. Then $\|T - \psi \cdot T\| < \varepsilon + \varepsilon\|T\|$. It is easy to check that $\psi \cdot T \in \mathcal{F}$. □

Let G^a be the universal almost periodic compactification of G and ε the canonical homomorphism of G into G^a; see [4]. Then, for a unitary representation π of G, the compact group G^π is a continuous homomorphic image of G^a. Let α_π be the unique continuous homomorphism of G^a onto G^π such that $\alpha_\pi\big(\varepsilon(g)\big) = \widetilde{g}$, for $g \in G$. We write $\alpha_\pi(s) \cdot T$ as $s \cdot T$ if $s \in G^a$

and $T \in \mathcal{AP}_\pi(G)$. Note that the unique π-invariant mean Φ on $\mathcal{AP}_\pi(G)$ is also given by $\Phi(T) = \int_{G^a}(s \cdot T)ds$, where $T \in \mathcal{AP}_\pi(G)$ and ds denotes the normalized Haar measure on G^a. Let $\varphi \in AP(G)$. We identify φ with a continuous function on G^a. For $T \in \mathcal{AP}_\pi(G)$, let $\varphi \cdot T = \int_{G^a} \varphi(s)(s \cdot T)ds$. Then $\varphi \cdot T \in \mathcal{AP}_\pi(G)$.

Let $\widehat{G}$ be the dual of G, i.e., the set of irreducible unitary representations of G, with its equivalent members identified. Let $\widehat{G}_f = \{\sigma \in \widehat{G} : \sigma \text{ is finite dimensional}\}$. For $\sigma \in \widehat{G}_f$ and $x \in G$, we consider $\sigma(x)$ as a $d_\sigma \times d_\sigma$ unitary matrix $\sigma(x) = \big(\sigma_{ij}(x)\big)$, where d_σ is the dimension of σ and the numbers σ_{ij} are the matrix coefficients of σ. We consider each σ_{ij} as a continuous function on G^a. For $T \in \mathcal{AP}_\pi(G)$, let $T^\sigma = (\sigma_{ij} \cdot T)$ be a member of $M_{d_\sigma}\big(\mathcal{B}(H_\pi)\big)$, the algebra of $d_\sigma \times d_\sigma$ matrices with entries in $\mathcal{B}(H_\pi)$. Note that

$$x \cdot (\sigma_{ij} \cdot T) = \sum_{k=1}^{d_\sigma} \sigma_{ik}(x^{-1})(\sigma_{kj} \cdot T). \tag{1}$$

Let (θ, H_θ) be a unitary representation of a closed normal subgroup N of G. Let $\pi = \mathrm{ind}_N^G \theta$ be the representation of G induced by θ. Then H_π can be realized as a space of H_θ-valued functions f on G satisfying $f(xn) = \theta(n)^{-1} f(x) for\ x \in G$ and $n \in N$, and for $f \in H_\pi$ we have $\pi(x)f(y) = f(x^{-1}y)$, for all $x, y \in G$; see [36, p. 374]. For $\varphi \in L^\infty(G/N)$, there is a multiplication operator M_φ on H_π, defined by $(M_\varphi f)(x) = \varphi(xN)f(x)$ for $x \in G$. Note that

$$g \cdot M_\varphi\big(= \pi(g) M_\varphi \pi(g^{-1})\big) = M_{g}\varphi,$$

where $({}_g\varphi)(xN) = \varphi(g^{-1}xN)$.

Theorem 4.9. *Let G be a locally compact group and N a closed normal subgroup of G such that the only irreducible finite dimensional unitary representations of G are pull-backs of representations of G/N. Assume $\pi = \mathrm{ind}_N^G \theta$, where θ is a unitary representation of N. Then the operators of the form $\sum_{k=1}^{n} M_{\varphi_k} U_k$, where φ_k are matrix coefficients of finite dimensional irreducible unitary representations of G/N, and $U_k \in VN_\pi(G)'$, are norm dense in $\mathcal{AP}_\pi(G)$. If π is further assumed to be irreducible, then $\mathcal{AP}_\pi(G) = \{M_\varphi : \varphi \in \mathcal{AP}(G/N)\}$.*

Proof. Let $\sigma \in \widehat{G}_f = (G/N)\widehat{\ }_f$. Assume $\dim \sigma = n$. Then

$$x \cdot M_{\sigma_{ij}} = M_{x\sigma_{ij}} = \sum_{k=1}^{n} \sigma_{ik}(x^{-1}) M_{\sigma_{kj}}. \tag{2}$$

Let M^σ and $M^{\check\sigma}$ be the members of $M_n\big(\mathcal{B}(H_\pi)\big)$ with their (i,j)-th entries equal to $M_{\sigma_{ij}}$ and $M_{\check\sigma_{ij}}$ respectively. (For a function h on G, the function $\check h$

is defined by $\check{h}(x) = h(x^{-1})$, $x \in G$.) Note that

$$M^\sigma M^{\check{\sigma}} = (M_{\sigma_{ij}})(M_{\check{\sigma}_{ij}}) = \Big(M_{\sum_{k=1}^n \sigma_{ik}\check{\sigma}_{kj}}\Big) = I_n,$$

where $(I_n)_{ij} = \delta_{ij} I$, with I denoting the identity operator on H_π.

For $A = (A_{ij}) \in M_n\big(\mathcal{B}(H_\pi)\big)$ and $x \in G$, let $x \cdot A = (x \cdot A_{ij})$. Note that

$$\begin{aligned} x \cdot (M^{\check{\sigma}} T^\sigma) &= (x \cdot M^{\check{\sigma}})(x \cdot T^\sigma) \\ &= M^{\check{\sigma}}\big(\sigma_{ij}(x)\big)\big(\sigma_{ij}(x^{-1})\big)T^\sigma \quad \text{(by (1) and (2))} \\ &= M^{\check{\sigma}} T^\sigma. \end{aligned}$$

Therefore, if $M^{\check{\sigma}} T^\sigma = (S_{ij})$, then $S_{ij} \in VN_\pi(G)'$. So,

$$\begin{aligned} T^\sigma &= (M^\sigma M^{\check{\sigma}})T^\sigma = M^\sigma (M^{\check{\sigma}} T^\sigma) \\ &= \Big(\sum_{k=1}^n M_{\sigma_{ik}} S_{kj}\Big). \end{aligned} \tag{3}$$

That is to say that, for each (i,j), we have $\sigma_{ij} T = \sum_{k=1}^n M_{\sigma_{ik}} S_{kj}$.

Let $T \in \mathcal{AP}_\pi(G)$ and $\varepsilon > 0$. Then, as explained in the proof of Lemma 4.8, there exists ψ, a linear combination of matrix coefficients of members of $\widehat{G}_f = (G/N)\widehat{\ }_f$, such that $\|\psi \cdot T - T\| < \varepsilon$. By the argument in the above paragraph, $\psi \cdot T$ is a linear combination of operators of the form $M_\varphi U$, where φ is a matrix coefficient of some $\sigma \in \widehat{G}_f$ and $U \in VN_\pi(G)'$. $\square$

Corollary 4.10 [1]. *Assume G is a locally compact group. Then the operators of the form*

$$\sum_{i=1}^n M_{\gamma_i} T_i,$$

where γ_i are matrix coefficients of finite dimensional unitary representations of G and $T_i \in VN'_\lambda$, are dense in $\mathcal{AP}_\lambda$.

Remark 4.11. When G is abelian, the above corollary is Theorem 3.3 of Tawari and Somusundaram [32]. Let $\widehat{G}$ be the dual group of G. Then $VN'_\lambda(G) = VN_\lambda(G)$, which can be identified with $L^\infty(\widehat{G})$: each $T \in VN_\lambda(G)$ equals T_φ for some $\varphi \in L^\infty(\widehat{G})$ where, for $\xi \in L^2(G)$, $T_\varphi \xi = (\varphi \widehat{\xi})^\wedge$. (For $\xi \in L^2(G)$, $\widehat{\xi}$ denotes the Fourier-Plancherel transform of ξ.) So $\mathcal{AP}_\lambda$ is the closed linear span of operators of the form $M_f T_\varphi$, $f \in AP(G)$ and $\varphi \in L^\infty(\widehat{G})$.

We now give more examples of almost periodic operators.

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[1] We would like to thank Professor Steven Schanuel for showing us that this result holds for finite groups.

Example 4.12. Let G be a compact group and π a unitary representation of G. Then $UC_\pi(G) = \mathcal{WAP}_\pi = \mathcal{AP}_\pi$. For $T \in UC_\pi(G)$,

$$\Phi(T) = \int_G \pi(g)T\pi(g^{-1})dg$$

is the unique invariant mean of T.

Example 4.13. A locally compact group G is said to be minimally almost periodic if $AP(G)$ only contains constant functions. For example, it is known that $SL(n, \mathbb{R})$, $n \geq 2$, is minimally a.p.; see [35] or [28]. If G is minimally a.p., then $\mathcal{AP}_\pi = VN_\pi'$ for any unitary representation π of G. If π is, in addition, irreducible, then $\mathcal{AP}_\pi = \mathbb{C} \cdot I$.

Example 4.14. Let G be the "$ax+b$" group. More explicitly, $G = \{(a,b) : a, b \in \mathbb{R},\ a > 0\}$ with multiplication $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1 + a_1b_2)$. Then $H = \{(1,b) : b \in \mathbb{R}\}$ is a closed normal subgroup of G and $K = \{(a, 0) : a > 0\}$ is a closed subgroup of G; G is a semidirect product of H and K and $G/H \simeq K$. All the finite dimensional irreducible representations of G are one-dimensional and are defined by continuous characters of K. Furthermore, G has exactly two inequivalent irreducible infinite dimensional unitary representations; these are induced by characters of H. Consider one of the two irreducible representations π, defined by

$$\pi(a,b)f(t) = e^{ie^t b} f(t + \log a),$$

for $(a,b) \in G$, $f \in L^2(\mathbb{R})$. (See [36, p. 441] for all of the above.) Let $h \in L^\infty(\mathbb{R})$ and let $T = M_h$ be the corresponding multiplication operator on $L^2(\mathbb{R})$. Then, as can be calculated directly, $\big(\pi(a,b)M_h\pi(a,b)^{-1}\big)f(t) = h(t + \log a)f(t)$. So $\{(a,b) \cdot M_h : (a,b) \in G\} = \{M_{h_a} : a \in \mathbb{R}, a > 0\}$, where $h_a(t) = h(t + \log a)$. So M_h belongs to $UC_\pi(\mathcal{AP}_\pi)$ [$\mathcal{WAP}_\pi$] if and only if h is uniformly continuous (almost periodic) [weakly almost periodic] on $\mathbb{R}$. Denote the unique invariant mean on $\mathcal{WAP}_\pi$ by Φ. Then for $h \in WAP(\mathbb{R})$, $\Phi(M_h) = m(h) \cdot I$, where m is the unique invariant mean on $WAP(\mathbb{R})$. By Theorem 4.9, $\mathcal{AP}_\pi = \{M_h : h \in AP(\mathbb{R})\}$.

Example 4.15. Let $G = M(n) = SO(n) \times \mathbb{R}^n$ be the Euclidean motion group of $\mathbb{R}^n$. For $t \in SO(n)$ and $x \in \mathbb{R}^n$, let $t(x)$ denote the evaluation of t at x. The multiplication in G is given by $(t,x)(t',x') = \big(tt', x + t(x')\big)$. So $\mathbb{R}^n$ is a closed normal subgroup of G and $G/\mathbb{R}^n \simeq SO(n)$. It is known that each finite dimensional irreducible unitary representation of G is lifted from an irreducible unitary representation of the compact group $SO(n)$; see [36, p. 442]. Let S^{n-1} be the $(n-1)$-sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure. For $\rho > 0$, let π_ρ be the irreducible unitary representation of G on $L^2(S^{n-1})$ given by $\pi_\rho(t,x)f(\xi) = e^{i\rho\langle x,\xi\rangle} f\big(t^{-1}(\xi)\big)$, where $f \in L^2(S^{n-1})$, $(t,x) \in M(n)$, $\xi \in S^{n-1}$ and $\langle x, \xi\rangle$ denotes the inner product of x and ξ in $\mathbb{R}^n$.

Note that π_ρ is induced by a character of $SO(n-1)\times\mathbb{R}^n$ (see [36, p. 442]). For $h\in L^\infty(S^{n-1})$, let M_h be the multiplication operator on $L^2(S^{n-1})$ defined by h. Then $(t,x)\cdot M_h \bigl(=\pi_\rho(t,x)M_h\pi_\rho(t,x)^{-1}\bigr)=M_{h_t}$, where $h_t(\xi)=h\bigl(t^{-1}(\xi)\bigr)$. Note that $M_h\in UC_{\pi_\rho}\iff M_h\in\mathcal{WAP}_{\pi_\rho}\iff M_h\in\mathcal{AP}_{\pi_\rho}\iff h\in C(S^{n-1})$. Denote the unique π_ρ-invariant mean on $\mathcal{WAP}_{\pi_\rho}$ by Φ. Then for $h\in C(S^{n-1})$, $\Phi(M_h)=\bigl(\int_{S^{n-1}}h(s)ds\bigr)\cdot I$, since the Lebesgue measure on S^{n-1} is $SO(n)$-invariant.

If $n=2$, then π_ρ is induced by a character of $\mathbb{R}^2$. Note that $\mathbb{R}^2$ is normal in $M(2)$ and all finite dimensional irreducible representations of $M(2)$ are lifted from characters of $M(2)/\mathbb{R}^2\approx\mathbb{T}$. So by Theorem 4.9, $\mathcal{AP}_{\pi_\rho}=\{M_h : h\in C(\mathbb{T})\}$. We do not know if $\mathcal{AP}_{\pi_\rho}=\{M_h : h\in C(S^{n-1})\}$ if $n>2$.

Given a Hilbert space H, we denote the algebra of compact operators on H by $\mathcal{K}(H)$. Tewari and Somasundaram [32] proved that, for an abelian group G, if G is compact then $\mathcal{K}\bigl(L^2(G)\bigr)\subset\mathcal{AP}_\lambda(G)$, and if G is noncompact then $\mathcal{K}\bigl(L^2(G)\bigr)\cap\mathcal{AP}_\lambda(G)=(0)$. The following two propositions generalize their results.

Proposition 4.16. *Let (π,H) be a unitary representation of G and $\mathcal{K}(H)$ the space of compact operators on H. Then $\mathcal{K}(H)\subset\mathcal{WAP}_\pi$.*

Proof. Recall that $\mathcal{K}(H)^*=\mathcal{T}(H)$ and $\mathcal{T}(H)^*=\mathcal{B}(H)$. For $T\in\mathcal{K}(H)$, it is easy to check that $\{g\cdot T : g\in G\}\subset\mathcal{K}(H)$. Therefore $\{g\cdot T : g\in G\}$ is relatively weak compact in $\mathcal{B}(H)$ if and only if it is relatively weak* compact in $\mathcal{B}(H)$, and $\{g\cdot T : g\in G\}$ is relatively weak* compact since it is bounded. □

Proposition 4.17. *If G is noncompact then $\mathcal{K}\bigl(L^2(G)\bigr)\cap\mathcal{AP}_\lambda=(0)$.*

Proof. Let $T\in\mathcal{K}\bigl(L^2(G)\bigr)$ and let $\{g_\alpha\}$ be a net in G with $g_\alpha\to\infty$. We claim that $g_\alpha\cdot T\to 0$ weakly. As explained above, it suffices to show that $g_\alpha\cdot T\to 0$ in the σ-weak topology. Since on bounded subsets of $\mathcal{B}\bigl(L^2(G)\bigr)$, the σ-weak topology is equal to the weak operator topology, we only have to show $g_\alpha\cdot T\to 0$ in the weak operator-topology.

First assume that T is a rank one operator, i.e., there exists $\eta,\zeta\in L^2(G)$ such that $T(\xi)=\langle\xi,\eta\rangle\zeta$, $\xi\in L^2(G)$. Let ξ,ω belong to $L^2(G)$. Then

$$\langle(g_\alpha\cdot T)(\xi),\omega\rangle=\langle\xi,\lambda(g_\alpha)\eta\rangle\ \langle\omega,\lambda(g_\alpha)\zeta\rangle\to 0,$$

since $\langle\xi,\lambda(g_\alpha)\eta\rangle\to 0$ and $\langle\omega,\lambda(g_\alpha)\eta\rangle\to 0$. So $g_\alpha\cdot T\to 0$ weakly. Since finite rank operators are dense in $\mathcal{K}\bigl(L^2(G)\bigr)$, it follows then $g_\alpha\cdot T\to 0$ weakly, for each $T\in\mathcal{K}\bigl(L^2(G)\bigr)$. □

Remark 4.18. Let π_ρ be the irreducible representation of $M(2)$ given in Example 4.15. Then $\mathcal{AP}_{\pi_\rho}=\{M_f : f\in C(\mathbb{T})\}$. So $\mathcal{AP}_{\pi_\rho}\cap\mathcal{K}\bigl(L^2(\mathbb{T})\bigr)=(0)$. On the other hand, if $\lambda=\lambda_{\mathbb{T}}$ is the regular representation of $\mathbb{T}$ then both $\mathcal{K}\bigl(L^2(\mathbb{T})\bigr)$ and $\{M_f : f\in C(\mathbb{T})\}$ are contained in $\mathcal{AP}_\lambda(\mathbb{T})$.

Let π be a unitary representation of G. Let

$$UC_{\pi,0} = \{T \in UC_\pi : g_\alpha \cdot T \to 0 \quad \text{weakly whenever} \quad g_\alpha \to \infty\}.$$

Note that if G is noncompact then $UC_{\pi,0} \subset \mathcal{W}_{\pi,0}$.

Remark 4.19. Let G be a non-compact group and $\varphi \in L^\infty(G)$. Then $\varphi \in C_0(G) \Longleftrightarrow M_\varphi \in UC_{\lambda,0}$. Also $\mathcal{K}(L^2(G)) \subset UC_{\lambda,0}$; see the proof of Proposition 4.17. Hence $UC_{\pi,0}$ may be considered as a generalization of $C_0(G)$, in our setting.

For a locally compact group G, let G^w be the universal w.a.p. compactification of G; see [3]. Then, for any unitary representation π of G, the action of G on $\mathcal{WAP}_\pi$, which sends (g,T) to $g \cdot T = \pi(g)T\pi(g^{-1})$ for $g \in G$ and $T \in \mathcal{WAP}_\pi$, extends to an action of G^w on $\mathcal{WAP}_\pi$.

Recall that a noncompact locally compact group is said to be minimally w.a.p. if $WAP_0(G) = C_0(G)$; see [6]. It is known that noncompact simple analytic groups with finite center and the motion group $M(n)$ are minimally w.a.p.; see [34] and [6].

Proposition 4.20. *If G is a non-compact minimally weakly almost periodic group then $\mathcal{W}_{\pi,0} = UC_{\pi,0}$, and therefore $\mathcal{WAP}_\pi = \mathcal{AP}_\pi \oplus UC_{\pi,0}$.*

Proof. Note that, by assumption, $G^w = G \cup \mathcal{K}(G^w)$. Here we consider G as a subgroup of G^w and $\mathcal{K}(G^w)$ denotes the minimal ideal of G^w. As is well-known, $\mathcal{K}(G^w)$ is a compact group. Assume $T \in \mathcal{W}_{\pi,0}$ and $T \neq 0$. Then there exists a net (g_α) in G such that $g_\alpha \cdot T \to 0$. Since $T \neq 0$, $g_\alpha \to \infty$. By taking a subnet if needed, we may assume $t = \lim g_\alpha \in \mathcal{K}(S^w)$. Then $t \cdot T = 0$. So $s \cdot T = 0$ for each $s \in \mathcal{K}(S^w)$. So $T \in UC_{\pi,0}$. □

Remark 4.21. (1) Let G be a non-compact, simple, analytic group with finite center, and let π be a unitary representation of G. Then $\mathcal{AP}_\pi = VN'_\pi$ and $\mathcal{W}_{\pi,0} = UC_{\pi,0}$. In particular, if π is irreducible then $\mathcal{WAP}_{\pi,0} = \mathbb{C} \cdot I \oplus UC_{\pi,0}$.

(2) Let π_ρ be the irreducible representation of $M(2)$ as defined in Example 4.15. Then $\mathcal{WAP}_{\pi_\rho} = M_{C(\mathbb{T})} \oplus UC_{\pi_\rho,0}$.

In the remainder of this paper we assume that G is a compact group. For $T \in UC_\lambda(G)$ $(= \mathcal{AP}_\lambda(G))$ and $\sigma \in \widehat{G}$, recall that $T^\sigma = (\sigma_{ij} \cdot T) \in M_{d_\sigma}\Big(\mathcal{B}(L^2(G))\Big)$. If G is abelian then each $\sigma \in \widehat{G}$ is one dimensional and hence $T^\sigma \in \mathcal{B}(L^2(G))$. In [11], DeLeeuw called the formal series $\sum_{\sigma \in \widehat{G}} T^\sigma$ the Fourier series of T; see also [32]. He proved that if $T = M_\varphi$, $\varphi \in C(G)$, then $(M_\varphi)^\sigma = \widehat{\varphi}(\sigma) M_\sigma$, where $\widehat{\varphi}(\sigma)$ denotes the Fourier transform of φ at σ. So the

Fourier series of M_φ is $\sum_\sigma \widehat{\varphi}(\sigma)M_\sigma$, or, if we identify φ with M_φ, the Fourier series of φ is $\sum_\sigma \widehat{\varphi}(\sigma)\sigma$, the Fourier series of φ in the classical sense.

For a general compact group G, we have shown that, for $T \in UC_\lambda(G)$, $T^\sigma = M^\sigma(M^{\check{\sigma}}T^\sigma)$, where $M^{\check{\sigma}}T^\sigma \in M_{d_\sigma}\big(VN_\lambda(G)'\big)$. We call $\widehat{T}(\sigma) = M^{\check{\sigma}}T^\sigma$ the Fourier transform of T at σ.

If $T = M_\varphi$, $\varphi \in C(G)$, then it is easy to check that $(M_\varphi)\widehat{\ }(\sigma) = \widehat{\varphi}(\sigma)$, where $\widehat{\varphi}(\sigma) = \int_G \varphi(x)\sigma(x^{-1})dx$ denotes the Fourier transform of φ at $\sigma \in \widehat{G}$; see Hewitt and Ross [18]. A study of the Fourier transform of operators will be carried out elsewhere, but we include here two results concerning the Fourier transform.

PROPOSITION 4.22. *Let G be a compact group and $T \in UC_\lambda(G)$. If $\widehat{T}(\sigma) = 0$ for all $\sigma \in \widehat{G}$ then $T = 0$.*

Proof. If $\widehat{T}(\sigma) = 0$ then $T^\sigma = 0$, or $\sigma_{ij}\cdot T = 0$ for all (i,j). But, given $\varepsilon > 0$, the proof of Theorem 4.9 shows that there exists φ, a linear combination of matrix coefficients of members of $\widehat{G}$, such that $\|\varphi\cdot T - T\| < \varepsilon$. Since $\varphi\cdot T = 0$, we have $\|T\| < \varepsilon$. So $T = 0$. □

PROPOSITION 4.23. *Let G be a compact group and $T \in \mathcal{B}\big(L^2(G)\big)$. Then the following statements are equivalent:*

(1) $T \in \mathcal{K}\big(L^2(G)\big)$;
(2) $T \in \mathcal{K}\big(L^2(G)\big) \cap UC_\lambda(G)$;
(3) *$T \in UC_\lambda(G)$ and, for each $\sigma \in \widehat{G}$, the entries of $\widehat{T}(\sigma)$ are compact operators;*
(4) *T belongs to the closed linear span of operators of the form $M_\gamma r(\chi)$, where γ and χ are matrix coefficients of irreducible representations of G and $r(\chi)$ denotes the convolution on the right by χ on $L^2(G)$.*

Proof. (1) $\Longrightarrow$ (2): This is just Proposition 4.16.

(2) $\Longrightarrow$ (3): If $T \in \mathcal{K}\big(L^2(G)\big) \cap UC_\lambda(G)$, then, for $\sigma \in \widehat{G}$, $\sigma_{ij}T$, the $(ij)^{\text{th}}$ entry of T^σ is compact. But $\widehat{T}(\sigma) = M^{\check{\sigma}}T^\sigma$, so each entry of $\widehat{T}(\sigma)$ is also compact.

(3) $\Longrightarrow$ (4): We have shown in formula (3) of the proof of Theorem 4.9 that, for each $\sigma \in \widehat{G}$,

$$\sigma_{ij}\cdot T = \sum_{k=1}^{d_\sigma} M_{\sigma_{ik}} S_{kj},$$

where $(S_{kj}) = \widehat{T}(\sigma)$. By assumption, $S_{kj} \in VN_\lambda(G)' \cap \mathcal{K}\big(L^2(G)\big)$. Note that $VN_\lambda(G)'$ equals $VN_r(G)$, the von Neumann algebra generated by the right translation operators on $L^2(G)$. It is well-known that $VN_\lambda(G) \cap \mathcal{K}\big(L^2(G)\big)$ is the C^*-algebra generated by the right convolution operators $\{r(f) : f \in$

$L^1(G)\}$; see [7, Proposition 2.2]. Since T can be approximated in norm by a linear combination of operators of the form $\sigma_{ij} \cdot T$, $\sigma \in \widehat{G}$, $1 \leq i, j \leq d_\sigma$, (4) holds.

The implication (4) $\Longrightarrow$ (1) is trivial. $\square$

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Ching Chou, Department of Mathematics, SUNY Buffalo, Buffalo, NY 14214-3093, USA

E-mail address: chouc@acsu.buffalo.edu

Anthony To-Ming Lau, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

E-mail address: tlau@math.ualberta.ca