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CR EXTENSION FOR TUBE-LIKE CR MANIFOLDS OF CR DIMENSION 1

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ABSTRACT. Although previous research on CR extension has emphasized the concept of wedge extendability, wedges do not have some of the properties we expect of the regions described by a general theory. In particular, the regions we describe should be of roughly the same size and shape as the full regions of extendability, they should vary smoothly as one varies the base point or the size of the open neighborhood on the manifold, and they should satisfy a natural containment condition. We illustrate through an example the failure of wedges to satisfy these conditions. We then develop an alternative description of the sort outlined above for the class of tube-like CR submanifolds of $\mathbb{C}^n$ of CR dimension 1.

1. Introduction

The concept of wedge extendability, as described for example in the work of Tumanov [Tum88] and Baouendi and Rothschild [BR90], has been a major focus of research on CR extension. (See, for example, [BER99] for a general treatment of these and related topics.) We briefly describe what "wedge extendability" means. Consider a smooth, generic CR submanifold M of $\mathbb{C}^n$ with codimension d and CR dimension $n-d$. This means that if $p \in M$, there exists an open neighborhood $\mathcal{U}$ of p in $\mathbb{C}^n$ and a smooth function $\rho = (\rho_1, \dots, \rho_d) : \mathcal{U} \to \mathbb{R}^d$ so that

$$M = \{ z \in \mathcal{U} \mid \rho(z) = 0 \}$$

and $\partial\rho_1 \wedge \dots \wedge \partial\rho_d \neq 0$. Let Γ be an open, convex cone in $\mathbb{R}^d$ with vertex at the origin, and let $\mathcal{V} \subset \mathbb{C}^n$ be an open neighborhood of the point p. Define the *wedge* $\mathcal{W}(\Gamma, \mathcal{V}, p)$ with edge M centered at p to be

$$\{ z \in \mathcal{V} \mid \rho(z) \in \Gamma \}.$$

A number of additional hypotheses on M will guarantee the existence of a common wedge to which all CR functions on a fixed open subset of M extend

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holomorphically. The weakest such hypothesis is that of minimality. M is said to be minimal at p if there is no submanifold $S \subset M$ through p with strictly smaller real dimension but the same CR dimension as M. The following theorem holds:

THEOREM 1.1 (Tumanov; Baouendi and Rothschild). *If M is minimal at p, then for every open neighborhood U of p in M, there exists an open neighborhood $\mathcal{V}$ of p in $\mathbb{C}^n$ with $M \cap \mathcal{V} \subset U$ and a wedge $\mathcal{W} = \mathcal{W}(\Gamma, \mathcal{V}, p)$ such that every continuous CR function on U extends holomorphically to $\mathcal{W}$. Conversely, if M is not minimal at p, then there exists a continuous CR function defined in some neighborhood of p in M that does not extend holomorphically to any wedge with edge M centered at p.*

The sufficiency was established by Tumanov, and the necessity by Baouendi and Rothschild.

A major objective in the study of CR extension is to give a good description of the region Ω_U to which CR functions on $U \subset M$ extend. Wedges do not have some of the characteristics one might reasonably expect of the regions described by a general theory. In particular, we suggest four properties the regions we describe should have and indicate the progress the current work makes toward such a description.

1.1. Containment condition. If U and V are open sets in M and if $V \subseteq U$, then the region of extendability for CR functions on V should be contained in the region of extendability for CR functions on U. Boggess, Glenn, and Nagel [BGN98] show that wedges do not have this property. They give an example of a manifold for which, given any fixed wedge of extendability for CR functions on an open neighborhood U of the origin, one can find a sufficiently small open neighborhood V of the origin so that every wedge of extendability for V is disjoint from the fixed wedge for U. In Section 3, we present this example of Boggess, Glenn, and Nagel. Our approach not only illustrates the failure of wedges to satisfy the containment condition, but also describes the full region of extendability. It will be clear that the full regions have the containment property. We will also see in this case precisely how the regions change shape as the neighborhood of the origin shrinks.

1.2. Comparability. The containment property is one we expect the full region of extendability to have. Of course in general it is unreasonable to expect to be able to write down a set of inequalities that describe exactly the region of extendability. Moreover, even in those cases in which this is possible (as in the case of the model manifolds we consider), the inequalities may be too complicated to be of much use. We therefore aim instead to find a simple set of inequalities that describes regions *comparable* to the full regions of extendability. This notion of comparability will be described more precisely in the context of the theorem and example to follow. Roughly, two regions R and S are comparable if S contains and is contained in re-scalings of R.

The comparability condition can be thought of as essentially a sharpness condition. If a region is comparable to the full region of extendability for an open set, it means that the region is of roughly the right size and shape. The example in Section 3 will show that wedges do not satisfy the comparability condition. On the other hand, our main theorem, Theorem 2.5, will yield a description of regions that are comparable to the full region of extendability for open subsets of certain special classes of CR submanifolds.

1.3. Uniformity. Since the manifolds under consideration here are smooth, we expect that the regions we describe will vary smoothly as we vary the size of the open subset on the manifold or the base point. This is stronger than the requirement that the regions we describe be comparable to the full region at each point. For instance, in Section 3 we will consider a submanifold for which, for neighborhoods of the origin, the region of extendability is not comparable to a wedge, but the regions for neighborhoods of any point other than the origin are comparable to wedges. A description of this sort does not satisfy the uniformity property.

1.4. Homogeneity. When the manifold under consideration has some natural homogeneity (e.g., when it is invariant under a certain family of non-isotropic dilations, as is the case with many of the model manifolds, such as the Heisenberg group), we also expect that the regions of extendability will reflect this. The regions we describe associated with our model manifolds indeed have this property.

The paper is organized as follows: Section 2 includes basic definitions and notation, as well as the statement of the main theorem. In Section 3, we apply the main theorem to an example to illustrate how it can be used to obtain the sort of description of CR extension described above. Section 4 contains the necessary definitions and results on convex sets and moment sequences which will be needed for the proof of the main theorem. The complete proof of the main theorem is given in Section 5, and Section 6 extends the result to certain manifolds that make high-order contact with the models considered in the main theorem. Finally, in Section 7 we consider a manifold passing through the origin with the property that, for small neighborhoods of the origin on the manifold, the region to which CR functions extend lies entirely to one side of a hyperplane through the origin, but for sufficiently large neighborhoods, the region of extendability contains a full neighborhood of the origin.

2. CR manifolds and functions

Let M be a $(2n-d)$-dimensional smooth (C^∞) real submanifold of $\mathbb{C}^n = \mathbb{R}^{2n}$, and let $p \in M$. Denote by $T_p(M)$ the real tangent space to M at p, and by $H_p(M)$ the maximal complex subspace of $T_p(M)$. We say that M is a *CR submanifold* of $\mathbb{C}^n$ of *CR dimension* k if for all $p \in M$, $\dim_{\mathbb{R}} H_p(M) = 2k$.

Let $T^{\mathbb{C}}(M)$ denote the complexified tangent bundle and $H^{\mathbb{C}}(M)$ the complexified holomorphic tangent bundle. Then $H^{\mathbb{C}}(M) = H^{1,0}(M) \oplus H^{0,1}(M)$, where $H^{1,0}(M)$ is generated in a neighborhood U of p by

$$\left\{ L = \sum_{j=1}^{n} a_j(z) \frac{\partial}{\partial z_j} \,\middle|\, L\rho_k \equiv 0,\ 1 \le k \le d \right\}$$

and $H^{0,1}(M)$ is generated near p by

$$\left\{ \bar{L} = \sum_{j=1}^{n} b_j(z) \frac{\partial}{\partial \bar{z}_j} \,\middle|\, \bar{L}\rho_k \equiv 0,\ 1 \le k \le d \right\},$$

where $a_j, b_j \in C^\infty(U)$. We now define the class of functions we study.

Definition 2.1. Let f be a C^1 function on an open subset $U \subset M$. f is a CR function if $\bar{L}f \equiv 0$ on U for all $\bar{L} \in H^{0,1}(M)$.

Definition 2.2. Let f be a C^1 CR function on an open subset U of M. f extends holomorphically to an open subset Ω of $\mathbb{C}^n$ if $U \subset \overline{\Omega}$ and there exists a function F continuous on $\overline{\Omega}$ and holomorphic on Ω such that $F|_U = f$.

Under appropriate hypotheses on M, such as those in Tumanov's theorem, for $U \subset M$ there may exist an open set $\Omega_U \subset \mathbb{C}^n$ such that *every* C^1 CR function on U extends holomorphically to Ω_U. We will generally be interested in describing the largest such set, which we refer to simply as the region of extendability associated with U.

We restrict our attention to the following special class of manifolds:

Definition 2.3. M is a tube-like, generic CR submanifold of $\mathbb{C}^n$ of CR dimension 1 if there exist local coordinates in which p is the origin and M is given near the origin by

$$\{\, (x+iy, w_2, \dots, w_n) \in \mathbb{C}^n \mid \mathrm{Re}(w_j) = \phi_j(x),\ 2 \le j \le n \,\}. \tag{2.1}$$

Since each ϕ_j is C^∞,

$$\phi_j(x) = \sum_{\ell=2}^{m} a_{j,\ell} x^\ell + E_m^j(x), \tag{2.2}$$

where $E_m^j(x) = o(x^m)$ as $x \to 0$.

We assume further that $\{\, (a_{2,\ell}, \dots, a_{n,\ell}) \mid 2 \le \ell \le m \,\}$ spans $\mathbb{R}^{n-1}$. M is then said to be of *finite type* m at the origin.

We wish to describe the region of extendability associated with the open subset

$$M^\epsilon = \{ (z, w_2, \ldots, w_n) \in M \,|\, |\operatorname{Re}(z)| < \epsilon \}$$

The finite-type hypothesis implies minimality, and so Theorem 1.1 guarantees the existence of such a region. For *any* CR submanifold, the region of extendability for CR functions on an open set U must be contained in $\operatorname{ch}(U)$, the convex hull of U, though in general the region may be much smaller. (See Section 4 for definitions and results concerning convex sets and convex hulls.) However, the following special case of a theorem of Boivin and Dwilewicz states that for the subsets M^ϵ of the tube manifold under consideration here CR functions actually extend to the full convex hull.

THEOREM 2.4 ([BD98]). *Let*

$$\gamma_M^\epsilon = \{ (x, \phi_1(x), \ldots, \phi_{n-1}(x)) \,|\, |x| < \epsilon \}.$$

Then every continuous CR function on $M^\epsilon = \gamma_M^\epsilon + i\mathbb{R}^n$ can be continuously extended to a function on $\operatorname{Int}(\operatorname{ch}(\gamma_M^\epsilon)) \cup \gamma_M^\epsilon + i\mathbb{R}^n$ that is holomorphic on $\operatorname{Int}(\operatorname{ch}(\gamma_M^\epsilon)) + i\mathbb{R}^n$.

To develop an alternative to wedge extendability involving regions for CR extension having the properties described in Section 1, we begin by considering a model class for finite-type generic tube-like CR submanifolds of CR dimension 1. Following Boggess, Glenn, and Nagel [BGN98], we define

$$T_N = \{ (x + iy, u_2 + iv_2, \ldots, u_N + iv_N) \in \mathbb{C}^N \,|\, u_j = x^j,\ 2 \le j \le N \}.$$

CR functions on T_N^ϵ extend to functions holomorphic on $\Gamma_N^\epsilon + i\mathbb{R}^N$, where Γ_N^ϵ is the interior of the convex hull of the curve

$$\gamma_N^\epsilon = \{ (x, x^2, \ldots, x^N) \in \mathbb{R}^N \,|\, |x| < \epsilon \}.$$

We can now state our main theorem describing Γ_N^ϵ.

THEOREM 2.5.

(1) *The set Γ_{2n}^ϵ is comparable to the set S_{2n}^ϵ of points $(u_1, u_2, \ldots, u_{2n})$ satisfying*

$$\begin{gathered} 0 < u_{2p} < \sqrt{u_{2p-2}u_{2p+2}}, \quad 1 \le p \le n-1, \\ |u_{2p+1}| < \sqrt{u_{2p}u_{2p+2}}, \quad 0 \le p \le n-1, \\ u_{2n} < \epsilon^2 u_{2n-2}, \end{gathered} \tag{2.3}$$

(with $u_0 = 1$) in the sense that there exist two sets of positive constants, $\{ C_j \,|\, 0 \le j \le 2n \}$ and $\{ c_j \,|\, 0 \le j \le 2n \}$, depending only

on the dimension such that Γ^ϵ_{2n} is contained in the set of points $(u_1, u_2, \ldots, u_{2n})$ satisfying

$$\begin{aligned} 0 < u_{2p} < C_{2p}\sqrt{u_{2p-2}u_{2p+2}}, &\quad 1 \le p \le n-1, \\ |u_{2p+1}| < C_{2p+1}\sqrt{u_{2p}u_{2p+2}}, &\quad 0 \le p \le n-1, \\ u_{2n} < C_{2n}\epsilon^2 u_{2n-2}, & \end{aligned}$$

and the set of all points satisfying

$$\begin{aligned} 0 < u_{2p} < c_{2p}\sqrt{u_{2p-2}u_{2p+2}}, &\quad 1 \le p \le n-1, \\ |u_{2p+1}| < c_{2p+1}\sqrt{u_{2p}u_{2p+2}}, &\quad 0 \le p \le n-1, \\ u_{2n} < c_{2n}\epsilon^2 u_{2n-2}, & \end{aligned}$$

is contained in Γ^ϵ_{2n}.

(2) *Γ^ϵ_{2n+1} is comparable to the set S^ϵ_{2n+1} of points $(u_1, u_2, \ldots, u_{2n+1})$ satisfying the inequalities in* (2.3)*, together with*

$$|u_{2n+1}| < \epsilon u_{2n}.$$

We postpone the proof until Section 5. We show first how this theorem can be used in a specific example to give the kind of description of the regions for CR extension outlined in Section 1.

3. An example

Consider

$$M = \{(z = x+iy, w_2 = u_2 + iv_2, w_4 = u_4 + iv_4) \in \mathbb{C}^3 \mid u_2 = x^2,\ u_4 = x^4\}$$

and the open neighborhoods of the origin,

$$M^\epsilon = \{\, (z, w_2, w_4) \in M \mid |\operatorname{Re}(z)| < \epsilon \,\}.$$

Boggess, Glenn, and Nagel [BGN98] show that wedges to which CR functions on M^ϵ extend do not satisfy the containment condition described in Section 1. Specifically, they show that if $\epsilon_0 > 0$ is fixed and if $\mathcal{W}$ is a fixed wedge to which all CR functions on M^{ϵ_0} extend, then there exists $\epsilon_1 < \epsilon_0$ such that *every* wedge of extendability for M^{ϵ_1} is disjoint from $\mathcal{W}$. They summarize this phenomenon by saying that wedges *rotate* as $\epsilon \to 0$. We will use Theorem 2.5 to give an alternate proof of this fact and to describe the full region to which CR functions extend. We will then apply the theorem to describe the regions of extendability associated with neighborhoods of points on M other than the origin.

As above, let

$$\gamma^\epsilon_M = \{\, (x, x^2, x^4) \mid |x| < \epsilon \,\}$$

and

$$\gamma^\epsilon_4 = \{\, (x, x^2, x^3, x^4) \mid |x| < \epsilon \,\}.$$

If Π is the projection map

$$(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_4),$$

then since Π is linear and $\Pi(\gamma_4^\epsilon) = \gamma_M^\epsilon$,

$$\Pi\left(\mathrm{ch}\left(\gamma_4^\epsilon\right)\right) = \mathrm{ch}\left(\Pi(\gamma_4^\epsilon)\right) = \mathrm{ch}\left(\gamma_M^\epsilon\right).$$

We consider the portion of the convex hull in the space normal to the curve γ_M at the origin, that is, in $\{u_1 = 0\}$. Denote this set by $N_0\left(\gamma_M^\epsilon\right)$. Then

$$\begin{aligned} N_0(\gamma_M^\epsilon) &= \mathrm{ch}(\gamma_M^\epsilon) \cap \{\,(0, u_2, u_4) \in \mathbb{R}^3\,\} \\ &= \Pi(\mathrm{ch}(\gamma_4^\epsilon)) \cap \{\,(0, u_2, u_4) \in \mathbb{R}^3\,\} \\ &= \Pi\left(N_0(\gamma_4^\epsilon)\right). \end{aligned}$$

That is, we can obtain the slice of the convex hull of γ_M^ϵ in the space normal to the curve at the origin by projecting the slice of the convex hull of γ_4^ϵ in the normal space onto $\{\,(0, u_2, u_4) \in \mathbb{R}^3\,\}$.

By Theorem 2.5, Γ_4^ϵ is comparable to the set S_4^ϵ of points (u_1, u_2, u_3, u_4) satisfying

$$\begin{aligned} |u_1| &< \sqrt{u_2}, \qquad 0 < u_2 < \sqrt{u_4}, \\ |u_3| &< \sqrt{u_2 u_4}, \qquad 0 < u_4 < \epsilon^2 u_2. \end{aligned}$$

The slice of this set in the normal space is

$$\{\,(0, u_2, u_3, u_4) \,|\, u_2^2 < u_4 < \epsilon^2 u_2,\ |u_3| < \sqrt{u_2 u_4}\,\},$$

and the projection of this set under Π is clearly

$$\{\,(0, u_2, u_4) \,|\, u_2^2 < u_4 < \epsilon^2 u_2\,\}. \tag{3.1}$$

The set (3.1) is therefore comparable to $N_0\left(\gamma_M^\epsilon\right)$.

Let us now consider the intersection of wedges of extendability for M^ϵ with the normal space $\{\,(0, u_2, u_4) \in \mathbb{R}^3\,\}$. Thus let $\mathcal{W}$ be a fixed wedge to which CR functions on M^ϵ extend. There exist δ_1 and δ_2 positive such that if $(x + iy, u_2 + iv_2, u_4 + iv_4) \in \mathcal{W}$, then

$$\delta_1^2 u_2 < u_4 < \delta_2^2 u_2 \leq \epsilon^2 u_2. \tag{3.2}$$

Now, if $\widetilde{\mathcal{W}}$ is *any* wedge of extendability for CR functions on $M^{\delta_1/2}$, there exist η_1 and η_2 positive such that a point $(x + iy, u_2 + iv_2, u_4 + iv_4) \in \widetilde{\mathcal{W}}$ satisfies

$$\eta_1^2 u_2 < u_4 < \eta_2^2 u_2 \leq \frac{\delta_1^2}{4} u_2.$$

Therefore $\widetilde{\mathcal{W}}$ and $\mathcal{W}$ are disjoint. This example also shows that no wedge can be comparable to the region of extendability for CR functions on M^ϵ, for the smallest intersection a wedge containing the region of extendability can have with the u_2u_4 plane is the set of points satisfying

$$0 < u_4 < \epsilon^2 u_2, \quad u_2 < \epsilon^2.$$

This set is not comparable to a set of the kind given in (3.2).

We now consider regions of extendability associated with neighborhoods of points on M other than the origin. Let $p = (a + i0, a^2 + i0, a^4 + i0) \in M$ for $a > 0$. Consider

$$\{\, (z, w_2, w_4) \in \mathbb{C}^3 \mid u_2 = x^2,\ u_4 = x^4,\ |x - a| < \epsilon \,\}.$$

Expand the graphing functions in powers of $x - a$. Then perform the complex affine (hence biholomorphic) change of variables mapping the manifold to

$$M_a^\epsilon = \{\, (z, w_2, w_4) \mid u_2 = x^2,\ u_4 = x^4 + 4ax^3,\ |x| < \epsilon \,\}. \tag{3.3}$$

If $\zeta_j = t_j + is_j$ and

$$T_4 = \gamma_4 + i\mathbb{R}^4 = \{(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \mid t_j = t_1^j\},$$

and if A is the linear map

$$\begin{aligned} u_j &= t_j, \qquad && 1 \le j \le 3,\\ u_4 &= t_4 + 4at_3, && \end{aligned}$$

then $\gamma_{M_a}^\epsilon = \Pi\,(A\,(\gamma_4^\epsilon))$, where Π is the projection

$$(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_4).$$

Hence $N_0(\gamma_{M_a}^\epsilon) = \Pi\,(A\,(N_0(\gamma_4^\epsilon)))$. By Theorem 2.5, we know that $N_0(\gamma_4^\epsilon)$ is comparable to the set of points $(0, t_2, t_3, t_4)$ satisfying

$$\begin{gathered} t_2^2 < t_4 < \epsilon^2 t_2,\\ |t_3| < \sqrt{t_2 t_4}. \end{gathered} \tag{3.4}$$

Therefore $N_0(\gamma_{M_a}^\epsilon)$ is comparable to the set of points $(0, u_2, u_3, u_4)$ satisfying

$$\begin{gathered} u_2^2 + 4au_3 < u_4 < \epsilon^2 u_2 + 4au_3,\\ |u_3| < \sqrt{u_2(u_4 - 4au_3)}. \end{gathered} \tag{3.5}$$

We are interested in projecting the region in (3.5) onto the u_2u_4 plane. As above, we seek only to describe a region comparable to this projection, with scaling constants that are independent of a and ϵ. Since we are interested here in the behavior of the regions as $a \to 0$, we assume that $0 < a < \epsilon/4$.

First we describe a region contained in the desired region. Since for any $\lambda \in (0, 1)$,

$$u_2^2 < \lambda u_2^2 + (1 - \lambda)\epsilon^2 u_2 < \epsilon^2 u_2,$$

by setting $1-\lambda = 4a^2/\epsilon^2$, we see that the set described in (3.5) contains the set of points satisfying

$$u_4 - 4au_3 = \lambda u_2^2 + 4a^2 u_2,$$
$$|u_3| < \sqrt{u_2(u_4 - 4au_3)}.$$

Then u_2 and u_4 satisfy

$$|u_4 - \lambda u_2^2 - 4a^2 u_2| < 4a\sqrt{\lambda u_2^3 + 4a^2 u_2^2}, \quad 0 < u_2 < \epsilon^2,$$

and this set in turn contains the set in the $u_2 u_4$ plane described by

$$\lambda u_2^2 - 4a^2 u_2 < u_4 < \lambda u_2^2 + 12a^2 u_2, \quad 0 < u_2 < \epsilon^2. \tag{3.6}$$

Next we consider the region we obtain by taking $\lambda = 1/2$. This choice yields a second region contained in (3.5), namely the set of points satisfying

$$\frac{1}{2}u_2^2 + \frac{1}{2}\epsilon^2 u_2 - ca\epsilon u_2 < u_4 < \frac{1}{2}u_2^2 + \frac{1}{2}\epsilon^2 u_2 + ca\epsilon u_2. \tag{3.7}$$

Observe that the point (ϵ^2, ϵ^4) satisfies the inequality (3.7). Since the origin is a point on the boundary of $N_0(\gamma^\epsilon_{M_a})$ and since $N_0(\gamma^\epsilon_{M_a})$ is convex, we conclude that it must contain all points satisfying

$$u_2^2 - 4a^2 u_2 < u_4 < \epsilon^2 u_2, \quad 0 < u_2 < \epsilon^2. \tag{3.8}$$

Next, we describe a region containing the region in (3.5). The inequalities imply $|u_3| < \epsilon u_2$, and so necessarily

$$u_4 < \epsilon^2 u_2 + 4a\epsilon u_2 < 2\epsilon^2 u_2.$$

We need a lower bound on u_4. The second inequality in (3.5) implies

$$u_3 > -2au_2 - \sqrt{4a^2 u_2 + u_2 u_4}.$$

If $u_4 < 0$, this implies

$$u_3 > -4au_2$$

and hence

$$u_4 > u_2^2 + 4au_3 > u_2^2 - 16a^2 u_2. \tag{3.9}$$

If $u_4 > 0$, then

$$\begin{aligned} u_4 &> u_2^2 + 4a(-2au_2 - \sqrt{4a^2u_2^2 + u_2u_4}) \\ &> u_2^2 - 8a^2u_2 - 4a(2au_2 + \sqrt{u_2u_4}) \\ &> u_2^2 - 16a^2u_2 - 4a^2u_2 - u_4. \end{aligned}$$

Therefore

$$u_4 > \frac{1}{2}u_2^2 - 10a^2 u_2. \tag{3.10}$$

Since

$$\frac{1}{2}u_2^2 - 10a^2u_2 > \frac{1}{2}u_2^2 - 16a^2u_2$$

and

$$u_2^2 - 16a^2u_2 > \frac{1}{2}u_2^2 - 16a^2u_2$$

for $u_2 > 0$, a region containing the region in (3.5) is

$$\frac{1}{2}u_2^2 - 16a^2u_2 < u_4 < 2\epsilon^2u_2, \quad 0 < u_2 < \epsilon^2. \tag{3.11}$$

We see that $N_0(\gamma^\epsilon_{M_a})$ is comparable to

$$\{\,(0, u_2, u_4) \,|\, u_2^2 - a^2u_2 < u_4 < \epsilon^2u_2\,\}. \tag{3.12}$$

We observe that for fixed non-zero a, it is possible to find a family $\{\,\mathcal{W}^\epsilon_a \,|\, \epsilon > 0\,\}$ of wedges such that $\mathcal{W}^\epsilon_a$ is comparable to the region of extendability for M^ϵ_a and $\mathcal{W}^{\epsilon_2}_a \subseteq \mathcal{W}^{\epsilon_1}_a$ if $\epsilon_2 < \epsilon_1$. For example, one can take $\mathcal{W}^\epsilon_a$ such that its intersection with $\{\,(0, u_2, u_4) \in \mathbb{R}^3\,\}$ is

$$\{\,(0, u_2, u_4) \,|\, -a^2u_2 < u_4 < \epsilon^2u_2,\ 0 < u_2 < \epsilon^2\,\}. \tag{3.13}$$

However, since the intersection of $N_0(\gamma^\epsilon_{M_a})$ with the u_2-axis is a segment of length roughly a^2, the largest scaled-down version of (3.13) that is contained in $N_0(\gamma^\epsilon_{M_a})$ consists of points for which $0 < u_2 < c\,a^2$. Therefore, the scaling constants depend on both a and ϵ. Furthermore, as $a \to 0$, the inner wedge shrinks. Thus although for points other than the origin wedges can give a description of CR extension satisfying the containment and comparability conditions, it is not a uniform description. On the other hand, the region (3.12) is comparable to $N_0(\gamma^\epsilon_{M_a})$ with scaling constants that are *independent* of a and ϵ, and as $a \to 0$, it deforms continuously into the region (3.1) observed for the origin.

4. Convex hulls and moment sequences

The first step in the proof of Theorem 2.5 is to use a result from the classical theory of moment sequences which characterizes points on Γ^ϵ_N as those for which two quadratic forms are positive-definite. We recall these results here. For complete proofs of these results and general background concerning moment problems, see [KN77]. Let $(\alpha_1, \ldots, \alpha_n)$ be a non-zero vector in $\mathbb{R}^n$ and $a \in \mathbb{R}$. The hyperplane

$$H = \left\{ (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \,\middle|\, \sum_{j=1}^{n} \alpha_j\xi_j = a \right\}$$

divides $\mathbb{R}^n$ into two open half spaces,

$$H^+ = \left\{ (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n} \alpha_j \xi_j > a \right\}$$

and

$$H^- = \left\{ (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n} \alpha_j \xi_j < a \right\}$$

whose common boundary is H. We denote the corresponding closed half spaces by $\overline{H^+}$ and $\overline{H^-}$.

Definition 4.1. H cuts $E \subset \mathbb{R}^n$ if there exist $x, y \in E$ such that $x \in H^+$ and $y \in H^-$. H supports E if either $E \subset \overline{H^+}$ or $E \subset \overline{H^-}$ and H contains a boundary point of E. If $E_1, E_2 \subset \mathbb{R}^n$, then H separates E_1 and E_2 if $E_1 \subset \overline{H^+}$ and $E_2 \subset H^-$, or vice versa.

Definition 4.2. $E \subset \mathbb{R}^n$ is convex if for all $x_1, x_2 \in E$, $\lambda x_1 + (1-\lambda) x_2 \in E$ for all $\lambda \in [0,1]$.

An easy induction argument shows that E is convex if and only if for all finite sets of points $\{x_1, x_2, \ldots, x_L\} \subset E$, if $\lambda_\ell \in [0,1]$, $1 \le \ell \le L$, with $\sum_{\ell=1}^{L} \lambda_\ell = 1$, $\sum_{\ell=1}^{L} \lambda_\ell x_\ell \in E$. The expression $\sum_{\ell=1}^{L} \lambda_\ell x_\ell$ is called a *convex combination* of the points $x_1, x_2, \ldots, x_L$.

Definition 4.3. The (closed) convex hull of a set $E \subset \mathbb{R}^n$ is the intersection of all (closed) convex sets containing E.

We denote the convex hull of E by $\mathrm{ch}(E)$. Clearly $\mathrm{ch}(E)$ is just the set of convex combinations of points in E, and the closed convex hull of E is just $\overline{\mathrm{ch}(E)}$. The next proposition follows easily from the definitions:

Proposition 4.4. *If $E \subset \mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $A(\mathrm{ch}(E)) = \mathrm{ch}(A(E))$.*

Lemma 4.5. *Suppose $E \subset \mathbb{R}^n$ is convex. If $c \in \mathrm{Int}(E^c)$, then there exists a hyperplane separating $\{c\}$ and E.*

Corollary 4.6. *A closed convex set is the intersection of all its closed support half spaces.*

The next theorem gives an important characterization of $\overline{\Gamma_N^\epsilon}$:

Theorem 4.7. *$\overline{\Gamma_N^\epsilon}$ is the set of all $(u_1, u_2, \ldots, u_N)$ admitting a representation*

$$u_j = \int_{-\epsilon}^{\epsilon} x^j \, d\sigma(x), \tag{4.1}$$

where σ is a finite, positive Borel measure on $[-\epsilon, \epsilon]$ satisfying

$$\int_{-\epsilon}^{\epsilon} d\sigma(x) = 1.$$

We call a sequence $(u_1, u_2, \ldots, u_N)$ of real numbers admitting a representation (4.1) a *moment sequence.* We obtain a second characterization of moment sequences. For each sequence $u = (u_1, u_2, \ldots, u_N)$ define a linear functional Λ_u on the set of polynomials of degree N as follows: If $P(x) = \sum_{j=0}^{N} \alpha_j x^j$, set

$$\Lambda_u(P(x)) = \sum_{j=0}^{N} \alpha_j u_j. \tag{4.2}$$

Definition 4.8. u is a positive sequence if $\Lambda_u(P(x)) \geq 0$ for all polynomials P non-negative on $[-\epsilon, \epsilon]$.

Theorem 4.9. *u is a moment sequence if and only if it is positive.*

Combining Theorems 4.7 and 4.9, we see that $(u_1, \ldots, u_N) \in \overline{\Gamma_N^\epsilon}$ if and only if for all nonnegative polynomials $P(x) = \sum_{j=0}^{N} \alpha_j x^j$ on $[-\epsilon, \epsilon]$, $\sum_{j=0}^{N} \alpha_j u_j \geq 0$ (where, as above, we set $u_0 = 1$).

The close connection between points in $\overline{\Gamma_N^\epsilon}$ and nonnegative polynomials on the finite interval $[-\epsilon, \epsilon]$ motivates the study of convenient characterizations of the latter. The following theorem of Markov and Lukacs generalizes better-known results concerning nonnegative polynomials on the real line.

Theorem 4.10 (Markov and Lukacs). *Let $P(x)$ be an algebraic polynomial of degree $\leq N$ nonnegative on the (finite) interval $[a, b]$. Then P admits the representation*

$$P(x) = \left(\sum_{j=0}^{n} \xi_j x^j\right)^2 + (b-x)(x-a)\left(\sum_{j=0}^{n-1} \eta_j x^j\right)^2 \quad \textit{if } N = 2n \tag{4.3}$$

or

$$P(x) = (b-x)\left(\sum_{j=0}^{n} \xi_j x^j\right)^2 + (x-a)\left(\sum_{j=0}^{n} \eta_j x^j\right)^2 \quad \textit{if } N = 2n+1. \tag{4.4}$$

Equation (4.3) can be rewritten

$$\begin{aligned} P(x) &= \sum_{j,k=0}^{n} \xi_j \xi_k x^{j+k} + (b-x)(x-a) \sum_{j,k=0}^{n-1} \eta_j \eta_k x^{j+k} \\ &= \sum_{j,k=0}^{n} \xi_j \xi_k x^{j+k} + \sum_{j,k=0}^{n-1} \eta_j \eta_k (-abx^{j+k} + (a+b)x^{j+k+1} - x^{j+k+2}). \end{aligned}$$

Similarly, (4.4) can be rewritten

$$\begin{aligned} P(x) &= (b-x) \sum_{j,k=0}^{n} \xi_j \xi_k x^{j+k} + (x-a) \sum_{j,k=0}^{n} \eta_j \eta_k x^{j+k} \\ &= \sum_{j,k=0}^{n} \xi_j \xi_k (bx^{j+k} - x^{j+k+1}) + \sum_{j,k=0}^{n} \eta_j \eta_k (x^{j+k+1} - ax^{j+k}). \end{aligned}$$

Now take $a = -\epsilon$ and $b = \epsilon$. We have thus established:

THEOREM 4.11 ([KN77], Chapter III, Theorem 2.3).

(i) $(u_1, \ldots, u_{2n}) \in \Gamma^\epsilon_{2n}$ *if and only if the quadratic forms*

$$f = \sum_{j,k=0}^{n} u_{j+k} \xi_j \xi_k \quad \textit{and} \quad F = \sum_{j,k=0}^{n-1} (\epsilon^2 u_{j+k} - u_{j+k+2}) \xi_j \xi_k$$

(with $u_0 = 1$) are positive-definite.

(ii) $(u_1, \ldots, u_{2n+1}) \in \Gamma^\epsilon_{2n+1}$ *if and only if the quadratic forms*

$$g = \sum_{j,k=0}^{n} (\epsilon u_{j+k} + u_{j+k+1}) \xi_j \xi_k \quad \textit{and} \quad G = \sum_{j,k=0}^{n} (\epsilon u_{j+k} - u_{j+k+1}) \xi_j \xi_k$$

(with $u_0 = 1$) are positive-definite.

We will use this theorem in conjunction with the following special case of the signature rule of Jacobi:

PROPOSITION 4.12. *A Hermitian matrix $[a_{j,k}]^s_{j,k=0}$ is positive-definite if and only if the successive principal minors $\Delta_r = \det[a_{j,k}]^r_{j,k=0}$, $0 \le r \le s$, are positive.*

5. Proof of Theorem 2.5

To prove Theorem 2.5, we prove first in Lemma 5.1 that Γ^ϵ_N is comparable to the set of points satisfying a larger set of inequalities which includes those defining S^ϵ_N. We then show in Lemma 5.4 that these two sets of inequalities actually define the same set.

LEMMA 5.1.

(a) Γ^{ϵ}_{2n} *is comparable to the set* E^{ϵ}_{2n} *of points* $(u_1, u_2, \dots, u_{2n})$ *satisfying*

$$\begin{aligned} |u_{j+k}| &< \sqrt{u_{2j}u_{2k}}, \quad 0 \le j \ne k \le n, \\ 0 < u_{2j+2} &< \epsilon^2 u_{2j}, \quad 0 \le j \le n-1, \end{aligned} \tag{5.1}$$

in the sense that Γ^{ϵ}_{2n} *is contained in* E^{ϵ}_{2n} *and the set* $\mathcal{E}^{\epsilon}_{2n}$ *of points satisfying*

$$\begin{aligned} |u_{j+k}| &< \frac{1}{(2n+2)!}\sqrt{u_{2j}u_{2k}}, \quad 0 \le j \ne k \le n, \\ 0 < u_{2j+2} &< \frac{\epsilon^2}{(2n+2)!}u_{2j}, \quad 0 \le j \le n-1, \end{aligned}$$

is contained in Γ^{ϵ}_{2n}.

(b) Γ^{ϵ}_{2n+1} *is comparable to the set* E^{ϵ}_{2n+1} *of points* $(u_1, u_2, \dots, u_{2n+1})$ *satisfying*

$$\begin{aligned} u_{2j} &> 0, \\ |u_{j+k}| &< \sqrt{u_{2j}u_{2k}}, \quad 0 \le j \ne k \le n, \\ |u_{j+k+1}| &< \epsilon\sqrt{u_{2j}u_{2k}}, \quad 0 \le j, k \le n. \end{aligned} \tag{5.2}$$

Proof of (a). We show first that $\Gamma^{\epsilon}_{2n} \subseteq E^{\epsilon}_{2n}$.

Let $(u_1, u_2, \dots, u_{2n}) \in \Gamma^{\epsilon}_{2n}$, so that the forms $f = \sum_{j,k=0}^{n} u_{j+k}\xi_j\xi_k$ and $F = \sum_{j,k=0}^{n-1}(\epsilon^2 u_{j+k} - u_{j+k+2})\xi_j\xi_k$ are positive-definite. Let j be a positive integer with $0 \le j \le n$ and suppose that in the form f, $\xi_k = 0$ if $k \ne j$. By the positivity of f, we must have $u_{2j}\xi_j^2 > 0$, and hence $u_{2j} > 0$. Next, if both ξ_j and ξ_k are non-zero for integers j and k with $0 \le j \ne k \le n$ and all other ξ_ℓ are zero, then the positivity of f yields

$$u_{2j}\xi_j^2 + 2u_{j+k}\xi_j\xi_k + u_{2k}\xi_k^2 > 0,$$

which implies that

$$u_{j+k}^2 - u_{2j}u_{2k} < 0,$$

or $|u_{j+k}| < \sqrt{u_{2j}u_{2k}}$ for $0 \le j \ne k \le n$. Similarly, the positivity of F implies that $u_{2j+2} < \epsilon^2 u_{2j}$. This proves that $\Gamma^{\epsilon}_{2n} \subseteq E^{\epsilon}_{2n}$.

Next, we show that $\mathcal{E}^{\epsilon}_{2n} \subseteq \Gamma^{\epsilon}_{2n}$. We proceed by induction on n. If $n = 1$, then the point (u_1, u_2) is in $\mathcal{E}^{\epsilon}_2$ if

$$\begin{aligned} |u_1| &< \frac{1}{4!}\sqrt{u_2}, \\ 0 < u_2 &< \frac{\epsilon^2}{4!}. \end{aligned}$$

On the other hand, $(u_1, u_2) \in \Gamma^{\epsilon}_2$ if and only if

$$u_1^2 < u_2 \quad \text{and} \quad u_2 < \epsilon^2.$$

Thus if $(u_1, u_2) \in \mathcal{E}^{\epsilon}_2$, it is also in Γ^{ϵ}_2, and the result holds for $n = 1$.

Suppose, then, that the result holds for $n-1 \geq 1$. That is, suppose $\mathcal{E}^{\epsilon}_{2(n-1)} \subseteq \Gamma^{\epsilon}_{2(n-1)}$. We claim $\mathcal{E}^{\epsilon}_{2n} \subseteq \Gamma^{\epsilon}_{2n}$. To show this, we must show that for $(u_1, u_2, \dots, u_{2n}) \in \mathcal{E}^{\epsilon}_{2n}$ the quadratic forms f and F are positive-definite. By the signature rule of Jacobi, $f = \sum_{j,k=0}^{n} u_{j+k}\xi_j\xi_k$ is positive-definite if and only if the determinants

$$\det[u_{j+k}]_{j,k=0}^{m}$$

are positive for all $0 \leq m \leq n$. Similarly, F is positive-definite if and only if the determinants

$$\det[\epsilon^2 u_{j+k} - u_{j+k+2}]_{j,k=0}^{m}$$

are positive for $1 \leq m \leq n$. By the induction hypothesis, $(u_1, u_2, \dots, u_{2n-2})$ is in Γ^{ϵ}_{2n-2}, and hence all of the above determinants for $m \leq n-1$ are positive. Therefore, we need only show that the full determinants

$$\det[u_{j+k}]_{j,k=0}^{n} \quad \text{and} \quad \det[\epsilon^2 u_{j+k} - u_{j+k+2}]_{j,k=0}^{n-1}$$

are positive. Now,

$$\det[u_{j+k}]_{j,k=0}^{n} = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=0}^{n} u_{j+\sigma(j)},$$

where σ is a permutation on $\{0, 1, \dots, n\}$, and $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ. Then,

$$\det[u_{j+k}]_{j,k=0}^{n} \geq \prod_{j=0}^{n} u_{2j} - \sum_{\sigma \neq id} \prod_{j=0}^{n} |u_{j+\sigma(j)}|.$$

If $\sigma \neq id$, for at least two integers j, j is not equal to $\sigma(j)$ and we have

$$|u_{j+\sigma(j)}| < \frac{\sqrt{u_{2j}u_{2\sigma(j)}}}{(2n+2)!}.$$

Observe that if $j = \sigma(j)$, the estimate $|u_{j+\sigma(j)}| \leq \sqrt{u_{2j}u_{2\sigma(j)}}$ holds trivially. Thus for *all* j, $|u_{j+\sigma(j)}| \leq \sqrt{u_{2j}u_{2\sigma(j)}}$. It follows that

$$\begin{aligned}
\det[u_{j+k}]_{j,k=0}^{n} &> \prod_{j=0}^{n} u_{2j} - \sum_{\sigma \neq id} \frac{1}{[(2n+2)!]^2} \left(\prod_{j=0}^{n} \sqrt{u_{2j}u_{2\sigma(j)}} \right) \\
&= \prod_{j=0}^{n} u_{2j} - \sum_{\sigma \neq id} \frac{1}{[(2n+2)!]^2} \prod_{j=0}^{n} u_{2j} \\
&> \left(1 - \frac{(n+1)!}{[(2n+2)!]^2}\right) \prod_{j=0}^{n} u_{2j} \\
&> 0,
\end{aligned}$$

since each $u_{2j} > 0$.

Similarly,

$$
\begin{aligned}
&\det[\epsilon^2 u_{j+k} - u_{j+k+2}]_{j,k=0}^{n-1} \\
&\quad \geq \prod_{j=0}^{n-1} (\epsilon^2 u_{2j} - u_{2j+2}) - \sum_{\sigma \neq id} \prod_{j=0}^{n-1} |\epsilon^2 u_{j+\sigma(j)} - u_{j+\sigma(j)+2}| \\
&\quad > \prod_{j=0}^{n-1} \left(\epsilon^2 u_{2j} - \frac{\epsilon^2 u_{2j}}{(2n+2)!} \right) - \sum_{\sigma \neq id} \prod_{j=0}^{n-1} (\epsilon^2 |u_{j+\sigma(j)}| + |u_{j+\sigma(j)+2}|) \\
&\quad > \epsilon^{2n} \left(1 - \frac{1}{(2n+2)!} \right)^n \prod_{j=0}^{n-1} u_{2j} - \sum_{\sigma \neq id} \prod_{j=0}^{n-1} (\epsilon^2 |u_{j+\sigma(j)}| + \sqrt{u_{2j+2} u_{2\sigma(j)+2}}) \\
&\quad > \epsilon^{2n} \left(1 - \frac{1}{(2n+2)!} \right)^n \prod_{j=0}^{n-1} u_{2j} - \sum_{\sigma \neq id} \prod_{j=0}^{n-1} \left(\epsilon^2 |u_{j+\sigma(j)}| + \frac{\epsilon^2 \sqrt{u_{2j} u_{2\sigma(j)}}}{(2n+2)!} \right).
\end{aligned}
$$

As above, for all $\sigma \neq id$, $|u_{j+\sigma(j)}| \leq \sqrt{u_{2j} u_{2\sigma(j)}}$ for all j, and the stronger estimate

$$
|u_{j+\sigma(j)}| < \frac{\sqrt{u_{2j} u_{2\sigma(j)}}}{(2n+2)!}
$$

holds for at least two j. Thus,

$$
\begin{aligned}
&\det[\epsilon^2 u_{j+k} - u_{j+k+2}]_{j,k=0}^{n-1} \\
&\quad > \epsilon^{2n} \left(1 - \frac{1}{(2n+2)!} \right)^n \prod_{j=0}^{n-1} u_{2j} - \sum_{\sigma \neq id} \epsilon^{2n} \frac{2^n}{[(2n+2)!]^2} \prod_{j=0}^{n-1} \sqrt{u_{2j} u_{2\sigma(j)}} \\
&\quad > \epsilon^{2n} \left\{ \left(1 - \frac{1}{(2n+2)!} \right)^n - \frac{2^n}{(2n+2)!} \right\} \prod_{j=0}^{n-1} u_{2j}.
\end{aligned}
$$

The constant $1/(2n+2)!$ has been chosen so that this last expression is strictly positive. This shows that $\mathcal{E}_{2n}^{\epsilon} \subseteq \Gamma_{2n}^{\epsilon}$ and completes the proof of part (a). $\square$

Proof of (b). We show first that Γ_{2n+1}^{ϵ} is contained in a set comparable to E_{2n+1}^{ϵ}. We begin with an easy lemma:

LEMMA 5.2. *Let Γ_N denote the interior of the convex hull of the curve*

$$
\gamma_N = \{ (x, x^2, \ldots, x^N) \mid x \in \mathbb{R} \}.
$$

Then $\overline{\Gamma_{2n+1}} = \overline{\Gamma_{2n}} \times \mathbb{R}$. Thus the closure of the convex hull of γ_{2n+1} is the tube over $\overline{\Gamma_{2n}}$, the closure of the convex hull of γ_{2n}.

Proof of Lemma 5.2. Let $(u_1, u_2, \ldots, u_{2n+1}) \in \Gamma_{2n+1}$ and suppose that $S = \{ (\xi_1, \xi_2, \ldots \xi_{2n+1}) \mid \sum_{j=1}^{2n+1} \alpha_j \xi_j \geq a \}$ is a closed half-space containing

γ_{2n+1}. Then for all $x \in \mathbb{R}$,

$$\sum_{j=1}^{2n+1} \alpha_j x^j \geq a.$$

This forces $\alpha_{2n+1} = 0$, and hence S naturally gives rise to a half space $S' = \{ (\xi_1, \xi_2, \dots, \xi_{2n}) \mid \sum_{j=1}^{2n} \alpha_j \xi_j \geq a \}$ containing γ_{2n}. Conversely, every half-space S' containing γ_{2n} gives rise to a closed half-space $S' \times \mathbb{R}$ containing γ_{2n+1}. Since the closure of the convex hull of a set E is the intersection of all the closed half-spaces containing E, the lemma is established. □

In light of the lemma, it remains only to show that if $(u_1, u_2, \dots, u_{2n+1}) \in \Gamma^{\epsilon}_{2n+1}$, then

$$|u_{j+k+1}| < C_{j,k} \epsilon \sqrt{u_{2j} u_{2k}}$$

for positive constants $C_{j,k}$. As in the proof of part (a), it follows easily that if the forms g and G are positive-definite,

$$|u_{2j+1}| < \epsilon u_{2j}.$$

Also,

$$(\epsilon u_{j+k} \pm u_{j+k+1})^2 < (\epsilon u_{2j} \pm u_{2j+1})(\epsilon u_{2k} \pm u_{2k+1}).$$

Using the inequality $|u_{2j+1}| < \epsilon u_{2j}$ established above and the estimate $|u_{j+k}| < \sqrt{u_{2j} u_{2k}}$ when $j \neq k$, we conclude that

$$\begin{aligned} |u_{j+k+1}| &< \sqrt{(\epsilon u_{2j} + |u_{2j+1}|)(\epsilon u_{2k} + |u_{2k+1}|)} + \epsilon |u_{j+k}| \\ &< \sqrt{(2\epsilon u_{2j})(2\epsilon u_{2k})} + \epsilon \sqrt{u_{2j} u_{2k}} \\ &= 3\epsilon \sqrt{u_{2j} u_{2k}}. \end{aligned}$$

This proves that Γ^{ϵ}_{2n+1} is contained in a region comparable to E^{ϵ}_{2n+1}.

Next, we claim that if $(u_1, u_2, \dots, u_{2n+1})$ is in the set $\mathcal{E}^{\epsilon}_{2n+1}$ of points satisfying

$$\begin{gathered} u_{2j} > 0, \quad 1 \leq j \leq n, \\ |u_{j+k}| < \frac{\sqrt{u_{2j} u_{2k}}}{(2n+3)!}, \quad 0 \leq j \neq k \leq n, \\ |u_{j+k+1}| < \epsilon \frac{\sqrt{u_{2j} u_{2k}}}{(2n+3)!}, \quad 0 \leq j, k \leq n, \end{gathered}$$

then it is in Γ^{ϵ}_{2n+1}. As in part (a), the proof is by induction on n.

If $n = 1$, $(u_1, u_2, u_3) \in \mathcal{E}_3^\epsilon$ if

$$\begin{aligned}
|u_1| &< \frac{\sqrt{u_2}}{5!}, \\
|u_1| &< \frac{\epsilon}{5!}, \\
0 < u_2 &< \epsilon \frac{\sqrt{u_2}}{5!}, \\
|u_3| &< \epsilon \frac{u_2}{5!}.
\end{aligned}$$

We must show that for such a point, g and G are positive-definite. Thus consider

$$\begin{pmatrix} \epsilon + u_1 & \epsilon u_1 + u_2 \\ \epsilon u_1 + u_2 & \epsilon u_2 + u_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \epsilon - u_1 & \epsilon u_1 - u_2 \\ \epsilon u_1 - u_2 & \epsilon u_2 - u_3 \end{pmatrix}.$$

Both matrices will be positive-definite if

$$\begin{aligned}
\epsilon \pm u_1 &> 0, \\
(\epsilon + u_1)(\epsilon u_2 + u_3) - (\epsilon u_1 + u_2)^2 &> 0, \\
(\epsilon - u_1)(\epsilon u_2 - u_3) - (\epsilon u_1 - u_2)^2 &> 0.
\end{aligned}$$

The first inequality, which says that $|u_1| < \epsilon$, is clearly satisfied by each point in $\mathcal{E}_3^\epsilon$. Also, for such a point,

$$\begin{aligned}
(\epsilon \pm u_1)(\epsilon u_2 \pm u_3) - (\epsilon u_1 \pm u_2)^2 &> \frac{\epsilon}{2} \cdot \frac{\epsilon u_2}{2} - \left(\epsilon \frac{\sqrt{u_2}}{5!} + \frac{\epsilon \sqrt{u_2}}{5!} \right)^2 \\
&> \epsilon^2 u_2 \left(\frac{1}{4} - \frac{4}{(5!)^2} \right) \\
&> 0.
\end{aligned}$$

The result is thus established for $n = 1$.

Suppose then that the result holds for $n - 1$, so that $\mathcal{E}_{2n-1}^\epsilon \subseteq \Gamma_{2n-1}^\epsilon$. We claim that $\mathcal{E}_{2n+1}^\epsilon \subseteq \Gamma_{2n+1}^\epsilon$. As in the proof of part (a), if $(u_1, u_2, \ldots, u_{2n+1}) \in \mathcal{E}_{2n+1}^\epsilon$, the induction hypothesis yields that the determinants

$$\det[\epsilon u_{j+k} \pm u_{j+k+1}]_{j,k=0}^m$$

are positive for $1 \le m \le n - 1$, and hence we need only consider the two full determinants:

$$
\begin{aligned}
&\det[\epsilon u_{j+k} \pm u_{j+k+1}]_{j,k=0}^{n} \\
&\quad = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=0}^{n} (\epsilon u_{j+\sigma(j)} \pm u_{j+\sigma(j)+1}) \\
&\quad \geq \prod_{j=0}^{n} (\epsilon u_{2j} \pm u_{2j+1}) - \sum_{\sigma \neq id} \prod_{j=0}^{n} |\epsilon u_{j+\sigma(j)} \pm u_{j+\sigma(j)+1}| \\
&\quad \geq \prod_{j=0}^{n} (\epsilon u_{2j} - |u_{2j+1}|) - \sum_{\sigma \neq id} \prod_{j=0}^{n} (\epsilon |u_{j+\sigma(j)}| + |u_{j+\sigma(j)+1}|) \\
&\quad > \prod_{j=0}^{n} \left(\epsilon u_{2j} - \frac{\epsilon u_{2j}}{(2n+3)!} \right) - \sum_{\sigma \neq id} \prod_{j=0}^{n} \left(\epsilon |u_{j+\sigma(j)}| + \frac{\epsilon \sqrt{u_{2j} u_{2\sigma(j)}}}{(2n+3)!} \right) \\
&\quad > \epsilon^{n+1} \left\{ \left(1 - \frac{1}{(2n+3)!}\right)^{n+1} \prod_{j=0}^{n} u_{2j} \right. \\
&\qquad\qquad \left. - \left(1 + \frac{1}{(2n+3)!}\right)^{n-1} \left(\frac{4}{[(2n+3)!]^2}\right) \sum_{\sigma \neq id} \prod_{j=0}^{n} u_{2j} \right\} \\
&\quad > \epsilon^{n+1} \left\{ 2^{-(n+1)} - 2^{n-1} \frac{2^2}{(2n+3)!} \right\} \prod_{j=0}^{n} u_{2j} \\
&\quad > 0.
\end{aligned}
$$

Hence $\mathcal{E}^{\epsilon}_{2n+1} \subseteq \Gamma^{\epsilon}_{2n+1}$. This completes the proof of part (b). $\square$

To complete the proof of Theorem 2.5, we must show that $E^{\epsilon}_{N} = S^{\epsilon}_{N}$. Since the inequalities defining S^{ϵ}_{N} are a subset of those defining E^{ϵ}_{N}, it suffices to prove that the inequalities defining S^{ϵ}_{N} imply those defining E^{ϵ}_{N}. This is accomplished in Lemma 5.4. In the course of the proof, it will be helpful to know an even larger set of inequalities than those for E^{ϵ}_{N} that still define a region comparable to Γ^{ϵ}_{N}. This is achieved in Lemma 5.3.

LEMMA 5.3 (Larger set of inequalities). *E^{ϵ}_{N} is equal to the set R^{ϵ}_{N} of points satisfying*

$$
\begin{aligned}
&u_{2j} > 0, \\
&|u_{j+k+p}| < \epsilon^{p} \sqrt{u_{2j} u_{2k}}, \quad 1 \leq j+k+p \leq N,\ 0 \leq j,k \leq n,\ j \neq k \text{ if } p = 0,
\end{aligned}
$$

where $n = \lfloor N \rfloor$. Thus R^{ϵ}_{N} is comparable to Γ^{ϵ}_{N}.

Proof. Clearly, $R^{\epsilon}_{N} \subseteq E^{\epsilon}_{N}$. For the reverse containment, we show that if $(u_1, u_2, \ldots, u_N) \in E^{\epsilon}_{N}$, then

$$
|u_{j+k+p}| < \epsilon^{p} \sqrt{u_{2j} u_{2k}} \quad 0 \leq j,k \leq n \text{ and } j \neq k \text{ if } p = 0.
$$

Case 1: $N = 2n$. The proof is by induction on p. When $p = 0$, the inequality reduces to one of those defining E^ϵ_{2n}, and the claim holds trivially.

Suppose then that the claim holds for some p with $0 \le p \le N-1$. That is, suppose that for all integers j and k with $0 \le j, k \le n$, $1 \le j+k+p \le 2n$, and $j \ne k$ if $p = 0$,

$$|u_{j+k+p}| < \epsilon^p \sqrt{u_{2j}u_{2k}}.$$

Consider the case for $p+1$. Thus suppose j and k are integers with $0 \le j, k \le n$ and $1 \le j+k+(p+1) \le 2n$. Then either $j \le n-1$ or $k \le n-1$. Assume without loss of generality that it is j. Then $j+1 \le n$ and by the induction hypothesis,

$$\begin{aligned} |u_{j+k+p+1}| &= |u_{(j+1)+k+p}| \\ &< \epsilon^p \sqrt{u_{2j+2}u_{2k}} \\ &< \epsilon^p \sqrt{\epsilon^2 u_{2j}u_{2k}} \\ &= \epsilon^{p+1} \sqrt{u_{2j}u_{2k}}. \end{aligned}$$

Therefore the claim holds for $p+1$ and the lemma follows in this case.

Case 2: $N = 2n+1$. The argument used to prove Case 1 goes through here if we show that the inequalities defining E^ϵ_{2n+1} imply

$$u_{2j+2} < \epsilon^2 u_{2j}.$$

Indeed,

$$\begin{aligned} u_{2j+2} &= u_{(j+1)+j+1} \\ &< \epsilon\sqrt{u_{2j+2}u_{2j}}. \end{aligned}$$

Dividing by $\sqrt{u_{2j+2}}$ and squaring (both legitimate since $u_{2j+2} > 0$) gives the inequality. □

Lemma 5.4.

(a) *Suppose $N = 2n$. The inequalities*

$$\begin{aligned} u_{2j} &> 0, \quad 0 \le j \le n, \\ |u_{j+k}| &< \sqrt{u_{2j}u_{2k}}, \quad 0 \le j \ne k \le n, \\ u_{2n} &< \epsilon^2 u_{2n-2}, \end{aligned} \tag{5.3}$$

imply

$$u_{2j} < \epsilon^2 u_{2j-2}, \quad 1 \le j \le n-1.$$

(b) *The inequalities*

$$u_{2p} < \sqrt{u_{2p-2}u_{2p+2}}, \quad 1 \le p \le n-1, \tag{5.4}$$

imply

$$|u_{j+k}| < \sqrt{u_{2j}u_{2k}}$$

for all integers j *and* k *such that* $0 \le j \ne k \le n$ *and* $j+k = 2q$ *for some positive integer* q.

(c) *The inequalities* (5.4) *together with*

$$|u_{2p+1}| < \sqrt{u_{2p}u_{2p+2}}, \quad 0 \le p \le n-1, \tag{5.5}$$

imply

$$|u_{j+k}| < \sqrt{u_{2j}u_{2k}}$$

for all integers j *and* k *such that* $0 \le j \ne k \le n$ *and* $j+k = 2q+1$ *for some positive integer* q.

Proof of (a). The proof is by induction on ℓ, where $j = n - \ell$. If $\ell = 0$, $j = n$ and the desired inequality

$$u_{2n} < \epsilon^2 u_{2n-2}$$

is just one of the inequalities in (5.3).

Suppose then that the result holds for some integer L with $0 \le L \le n-2$. That is, if $2 \le J = n - L \le n$, the inequalities in (5.3) imply

$$u_{2J} < \epsilon^2 u_{2J-2}.$$

Consider the case for $L+1$. We must estimate $u_{2(J-1)} = u_{2J-2}$.

$$\begin{aligned} u_{2J-2} &= u_{J+(J-2)} \\ &< \sqrt{u_{2J}u_{2J-4}} \\ &< \sqrt{\epsilon^2 u_{2J-2}u_{2J-4}}. \end{aligned}$$

Since $u_{2J-2} > 0$, we may square both sides and divide by u_{2J-2} to obtain

$$u_{2j-2} < \epsilon^2 u_{2J-4}.$$

The result therefore holds for $J - 1 = n - (L+1)$. □

Proof of (b). It suffices to show that the $n-1$ inequalities given in (5.4), namely

$$u_{2p} < \sqrt{u_{2p-2}u_{2p+2}}, \quad 1 \le p \le n-1,$$

imply

$$u_{2p} < \sqrt{u_{2p-2\ell}u_{2p+2\ell}}$$

whenever $1 \le \ell \le \ell_p = \min\{p, n-p\}$. The proof is by induction on ℓ. If $\ell = 1$, the result holds trivially.

Suppose then that for some integer L with $1 \le L \le n-2$, for all ℓ with $1 \le \ell \le L$, for all q for which $\ell \le \ell_q$, the inequalities (5.4) imply

$$u_{2q} < \sqrt{u_{2q-2\ell}u_{2q+2\ell}}. \tag{5.6}$$

Consider the case for $L+1$. We consider two subcases, depending on the parity of $L+1$.

Subcase 1: $L+1$ is even. Then $\frac{L+1}{2}$ is an integer $\leq L$, and hence applying the inductive hypothesis twice for all those integers q for which $L+1 \leq \ell_q$, we obtain

$$\begin{aligned} u_{2q}^2 &< u_{2q-2\frac{L+1}{2}} u_{2q+2\frac{L+1}{2}} \\ &< \sqrt{u_{2q-2(L+1)} u_{2q}} \sqrt{u_{2q} u_{2q+2(L+1)}}. \end{aligned}$$

Dividing by u_{2q} gives

$$u_{2q} < \sqrt{u_{2q-2(L+1)} u_{2q+2(L+1)}},$$

and the result holds for $L+1$. This completes the proof if $L+1$ is even.

Subcase 2: $L+1$ is odd. Write $L+1 = 2R+1$ for some integer R. We want to show that the inequalities (5.4) imply

$$\begin{aligned} u_{2q} &< \sqrt{u_{2q-2(L+1)} u_{2q+2(L+1)}} \\ &= \sqrt{u_{2q-2-4R} u_{2q+2+4R}}. \end{aligned}$$

This will follow if we show that for all non-negative r with $2r+1 \leq \ell_q - 2$,

$$u_{2q-2-4r} u_{2q+2+4r} \leq u_{2q-2-4(r+1)} u_{2q+2+4(r+1)}, \tag{5.7}$$

for then we will have

$$u_{2q} < \sqrt{u_{2q-2} u_{2q+2}} \leq \sqrt{u_{2q-2-4R} u_{2q+2+4R}}.$$

The proof of (5.7) is by induction on r.

Suppose $r=0$. By Subcase 1,

$$\begin{aligned} u_{2(q-1)} &< \sqrt{u_{2(q-1)-4} u_{2(q-1)+4}} = \sqrt{u_{2q-6} u_{2q+2}}, \\ u_{2(q+1)} &< \sqrt{u_{2(q+1)-4} u_{2(q+1)+4}} = \sqrt{u_{2q-2} u_{2q+6}}. \end{aligned}$$

Hence

$$u_{2q-2} u_{2q+2} < \sqrt{u_{2q-6} u_{2q+6}} \sqrt{u_{2q-2} u_{2q+2}}.$$

Dividing by $\sqrt{u_{2q-2} u_{2q+2}}$ then establishes the result for $r=0$.

Suppose then that (5.7) holds for some integer $R \geq 0$, and consider the case for $R+1$. Again by Subcase 1,

$$\begin{aligned} u_{2q-2-4(R+1)} &< \sqrt{u_{2q-2-4(R+1)-4} u_{2q-2-4(R+1)+4}} \\ &= \sqrt{u_{2q-2-4(R+2)} u_{2q+-2-4R}}, \\ u_{2q+2+4(R+1)} &< \sqrt{u_{2q+2+4(R+1)-4} u_{2q+2+4(R+1)+4}} \\ &= \sqrt{u_{2q+2+4R} u_{2q++2+4(R+2)}}. \end{aligned}$$

Hence, multiplying these inequalities and applying the inductive hypothesis,

$$\begin{aligned}
&u_{2q-2-4(R+1)}u_{2q+2+4(R+1)} \\
&\quad < \sqrt{u_{2q-2-4(R+2)}u_{2q++2+4(R+2)}}\sqrt{u_{2q-2-4R}u_{2q+2+4R}} \\
&\quad < \sqrt{u_{2q-2-4(R+2)}u_{2q++2+4(R+2)}}\sqrt{u_{2q-2-4(R+1)}u_{2q+2+4(R+1)}}.
\end{aligned}$$

Dividing by $\sqrt{u_{2q-2-4(R+1)}u_{2q+2+4(R+1)}}$ establishes the claim for $R+1$ and completes the proof of part (b). □

Proof of (c). We prove that the inequalities (5.4) and (5.5) imply that for any q,

$$|u_{2q+1}| < \sqrt{u_{2q-2\ell}u_{2q+2+2\ell}}$$

whenever $\ell \leq \tilde{\ell}_q = \min\{\frac{N}{2} - q - 1, q\}$. This will follow if for all ℓ,

$$u_{2q-2\ell}u_{2q+2+2\ell} < u_{2q-2(\ell+1)}u_{2q+2+2(\ell+1)}. \tag{5.8}$$

The proof of (5.8) is by induction on ℓ.

If $\ell = 0$, by (5.4),

$$\begin{aligned}
u_{2q} &< \sqrt{u_{2q-2}u_{2q+2}}, \\
u_{2q+2} &< \sqrt{u_{2q}u_{2q+4}},
\end{aligned}$$

and hence

$$u_{2q}u_{2q+2} < \sqrt{u_{2q-2}u_{2q+4}}\sqrt{u_{2q}u_{2q+2}},$$

and

$$u_{2q}u_{2q+2} < u_{2q-2}u_{2q+4},$$

establishing the claim in this case.

Suppose then that the claim holds for some $L \geq 0$ and consider the claim for $L+1$. Then

$$\begin{aligned}
u_{2q-2(L+1)} &< \sqrt{u_{2q-2(L+2)}u_{2q-2L}}, \\
u_{2q+2+2(L+1)} &< \sqrt{u_{2q+2+2L}u_{2q+2+2(L+2)}},
\end{aligned}$$

and hence, applying the inductive hypothesis,

$$\begin{aligned}
&u_{2q-2(L+1)}u_{2q+2+2(L+1)} \\
&\quad < \sqrt{u_{2q-2(L+2)}u_{2q+2+2(L+2)}}\sqrt{u_{2q-2L}u_{2q+2+2L}} \\
&\quad < \sqrt{u_{2q-2(L+2)}u_{2q+2+2(L+2)}}\sqrt{u_{2q-2(L+1)}u_{2q+2+2(L+1)}}.
\end{aligned}$$

Dividing by $\sqrt{u_{2q-2(L+1)}u_{2q+2+2(L+1)}}$ gives the result for $L+1$ and completes the proof of part (c). □

6. Approximation by the models

Our knowledge of the model T_N can be used to understand the region of extendability for CR functions on more general CR submanifolds. We begin with a lemma.

LEMMA 6.1. *Let*

$$\begin{aligned} M &= \gamma_M + i\mathbb{R}^m \\ &= \{ (x+iy, u_2+iv_2, \dots, u_m+iv_m) \,|\, u_j = x^j + \mathcal{E}_j^{j+1}(x),\ 2 \le j \le m \}, \end{aligned}$$

where $\mathcal{E}_j^{j+1}(x)$ is $o(x^j)$ as $x \to 0$. Then there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, $\mathrm{ch}(\gamma_M^\epsilon)$ is contained in a set comparable to E_m^ϵ.

Proof. If $(u_1, u_2, \dots, u_m) \in \mathrm{ch}(\gamma_M^\epsilon)$, there exist a positive integer L, coefficients $\lambda_\ell \in [0,1]$, $1 \le \ell \le L$, with $\sum_{\ell=1}^L \lambda_\ell = 1$, and real numbers $x_\ell \in (-\epsilon, \epsilon)$ such that

$$u_j = \sum_{\ell=1}^L \lambda_\ell (x_\ell^j + \mathcal{E}_j^{j+1}(x_\ell)). \tag{6.1}$$

Observe that (6.1) also holds for $j=1$ if we take $\mathcal{E}_0^1(x) \equiv 0$. Fix a number η with $0 < \eta < 1$. Then there exists an $\epsilon_0 > 0$ such that if $|x| < \epsilon_0$, then $|\mathcal{E}_j^{j+1}(x)| < \eta |x|^j$ for all $1 \le j \le m$.

For positive integers j, k, and p with $0 \le j,k \le \lfloor \frac{m}{2} \rfloor$, $1 \le j+k+p \le m$, and $j \ne k$ if $p = 0$, if $|x| < \epsilon_0$,

$$\begin{aligned} |u_{j+k+p}| &= \left| \sum_{\ell=1}^L \lambda_\ell \left(x_\ell^{j+k+p} + \mathcal{E}_{j+k+p}^{j+k+p+1}(x_\ell) \right) \right| \\ &< \sum_{\ell=1}^L \lambda_\ell \left(|x_\ell|^{j+k+p} + \eta |x_\ell|^{j+k+p} \right) \\ &= (1+\eta) \sum_{\ell=1}^L \lambda_\ell |x_\ell|^{j+k+p}. \end{aligned}$$

The expression $\sum_{\ell=1}^L \lambda_\ell |x_\ell|^{j+k+p}$ is the $(j+k+p)$th coordinate of a convex linear combination of the L points $(|x_\ell|, |x_\ell|^2, \dots, |x_\ell|^m)$ on the curve $\gamma_m^{\epsilon_0}$ associated with the model $T_m^{\epsilon_0}$. Since $\Gamma_m^{\epsilon_0} \subseteq E_m^{\epsilon_0}$, it follows that

$$
\begin{aligned}
|u_{j+k+p}| &< (1+\eta)\epsilon^p \sqrt{\left(\sum_{\ell=1}^{L} \lambda_\ell x_\ell^{2j}\right)\left(\sum_{\ell=1}^{L} \lambda_\ell x_\ell^{2k}\right)} \\
&< \epsilon^p \frac{1+\eta}{1-\eta} \sqrt{\left(\sum_{\ell=1}^{L} \lambda_\ell (1-\eta) x_\ell^{2j}\right)\left(\sum_{\ell=1}^{L} \lambda_\ell (1-\eta) x_\ell^{2k}\right)} \\
&< \epsilon^p \frac{1+\eta}{1-\eta} \sqrt{\left(\sum_{\ell=1}^{L} \lambda_\ell (x_\ell^{2j} + \mathcal{E}_{2j}^{2j+1}(x_\ell))\right)\left(\sum_{\ell=1}^{L} \lambda_\ell (x_\ell^{2k} + \mathcal{E}_{2k}^{2k+1}(x_\ell))\right)} \\
&= \frac{1+\eta}{1-\eta}\left(\epsilon^p \sqrt{u_{2j} u_{2k}}\right).
\end{aligned}
$$

This proves that if $0 < \epsilon < \epsilon_0$, then $\mathrm{ch}(\gamma_M^\epsilon)$ is contained in a region comparable to E_m^ϵ. □

The next theorem says roughly that if M makes mth order contact with the model T_m at the origin, then the region of extendability for CR functions in a sufficiently small neighborhood of the origin on M is comparable to that for the model.

Theorem 6.2. *Suppose*

$$
\begin{aligned}
M &= \{\, (x+iy, u_2+iv_2, \ldots, u_m+iv_m) \in \mathbb{C}^m \mid u_j = x^j + P_j^{m+1}(x) \,\} \\
&= \gamma_M + i\mathbb{R}^m,
\end{aligned}
$$

where $P_j^{m+1}(x) = \sum_{\ell=m+1}^{N} a_{j,\ell} x^\ell$ for some integer $N \geq m+1$. Then there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then $\mathrm{ch}(\gamma_M^\epsilon)$ is comparable to E_m^ϵ.

Proof. By Lemma 6.1, there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ then $\mathrm{ch}(\gamma_M^\epsilon)$ is contained in a region comparable to E_m^ϵ. Hence we need only establish the reverse containment.

Let

$$
\gamma_{\widetilde{M}} = \{\, (x, x^2 + P_2^{m+1}(x), \ldots, x^m + P_m^{m+1}(x), x^{m+1}, \ldots, x^N) \mid x \in \mathbb{R} \,\}.
$$

Then if $\Pi : \mathbb{R}^N \to \mathbb{R}^m$ is the projection of $\mathbb{R}^N$ onto its first m coordinates, $\gamma_M = \Pi(\gamma_{\widetilde{M}})$. Let $\gamma_N = \{\, (x, x^2, \ldots, x^N) \mid x \in \mathbb{R} \,\}$. Then $\gamma_{\widetilde{M}}$ is the image of the model curve γ_N under the (nonsingular) linear map A given by

$$
\begin{aligned}
u_1 &= t_1, \\
u_j &= t_j + \sum_{\ell=m+1}^{N} a_{j,\ell} t_\ell, \quad 2 \leq j \leq m, \\
u_j &= t_j \qquad m+1 \leq j \leq N,
\end{aligned}
$$

with inverse

$$\begin{aligned} t_1 &= u_1, \\ t_j &= u_j - \sum_{\ell=m+1}^{N} a_{j,\ell} u_\ell, \quad 2 \le j \le m, \\ t_j &= u_j, \qquad\quad m+1 \le j \le N. \end{aligned}$$

We obtain upper and lower bounds on $|t_j|$ in terms of the u's. We consider two cases, depending on the parity of j. For $2 \le j = 2p \le m$,

$$\begin{aligned} \sum_{\ell=m+1}^{N} |a_{2p,\ell}||u_\ell| &= \sum_{\ell=m+1}^{N} |a_{2p,\ell}||t_\ell| \\ &\le \sum_{\ell=m+1}^{N} |a_{2p,\ell}|\epsilon^{\ell-2p} t_{2p}, \end{aligned} \tag{6.2}$$

where we have used Lemma 5.3. Since $\ell - 2p \ge 1$, there exists $\epsilon_1 > 0$ such that if $\epsilon \le \epsilon_1$ the last expression is $\le \frac{1}{4} t_{2p}$ for all $2 \le 2p \le m$. Thus if $\epsilon \le \epsilon_1$,

$$u_{2p} - \frac{1}{4} t_{2p} \le t_{2p},$$

or

$$\frac{4}{5} u_{2p} \le t_{2p}.$$

On the other hand,

$$t_{2p} \le u_{2p} + \frac{1}{4} t_{2p},$$

and so

$$t_{2p} \le \frac{4}{3} u_{2p}.$$

Hence

$$\frac{4}{5} u_{2p} \le t_{2p} \le \frac{4}{3} u_{2p}. \tag{6.3}$$

Observe that (6.3) holds trivially for $m < 2p \le N$.

Suppose next that $3 \le j = 2p+1 \le m$. Then

$$\begin{aligned} \sum_{\ell=m+1}^{N} |a_{2p+1,\ell}||u_\ell| &= \sum_{\ell=m+1}^{N} |a_{2p+1,\ell}||t_\ell| \\ &\le \sum_{\ell=m+1}^{N} |a_{2p+1,\ell}|\epsilon^{\ell-(2p+1)} \sqrt{t_{2p} t_{2p+2}}. \end{aligned}$$

Since $\ell - (2p+1) \geq 1$, there exists $\epsilon_2 > 0$ such that if $\epsilon \leq \epsilon_2$, then

$$|u_{2p+1}| - \frac{1}{4} c_N \sqrt{t_{2p} t_{2p+2}} \leq |t_{2p+1}| \leq |u_{2p+1}| + \frac{1}{4} c_N \sqrt{t_{2p} t_{2p+2}}.$$

If $N = 2n$, a region contained in Γ_N^ϵ is

$$\begin{gathered} 0 < t_{2p} < c_{2n} \sqrt{t_{2p-2} t_{2p+2}}, \quad 1 \leq p \leq n-1, \\ |t_{2p+1}| < c_{2n} \sqrt{t_{2p} t_{2p+2}}, \quad 0 \leq p \leq n-1, \\ t_{2n} < c_{2n} \epsilon^2 t_{2n-2}. \end{gathered}$$

These inequalities are satisfied if the u_j's satisfy

$$\begin{gathered} 0 < \frac{4}{3} u_{2p} < c_{2n} \sqrt{\frac{16}{25} u_{2p-2} u_{2p+2}}, \quad 1 \leq p \leq n-1, \\ |u_{2p+1}| + \frac{1}{4} c_{2n} \sqrt{\frac{16}{9} u_{2p} u_{2p+2}} < c_{2n} \sqrt{\frac{16}{25} u_{2p} u_{2p+2}}, \quad 0 \leq p \leq n-1, \\ \frac{4}{3} u_{2n} < \frac{4}{5} c_{2n} \epsilon^2 u_{2n-2}. \end{gathered}$$

That is, if

$$\begin{gathered} 0 < u_{2p} < \frac{3}{5} c_{2n} \sqrt{u_{2p-2} u_{2p+2}}, \quad 1 \leq p \leq n-1, \\ |u_{2p+1}| < \left(\frac{4}{5} c_{2n} - \frac{4}{3} \cdot \frac{1}{4} c_{2n} \right) \sqrt{u_{2p} u_{2p+2}}, \quad 0 \leq p \leq n-1, \\ u_{2n} < \frac{3}{5} c_{2n} \epsilon^2 u_{2n-2}. \end{gathered}$$

This shows that $\mathrm{ch}(\gamma_{\widetilde{M}}^\epsilon)$ contains a region comparable to S_N^ϵ for $\epsilon < \min\{\epsilon_1, \epsilon_2\}$.

On the other hand, if $N = 2n+1$, then a region contained in Γ_N^ϵ is

$$\begin{gathered} 0 < t_{2p} < c_{2n+1} \sqrt{t_{2p-2} t_{2p+2}}, \quad 1 \leq p \leq n-1, \\ |t_{2p+1}| < c_{2n+1} \sqrt{t_{2p} t_{2p+2}}, \quad 0 \leq p \leq n-1, \\ t_{2n} < c_{2n+1} \epsilon^2 t_{2n-2}, \\ |t_{2n+1}| < c_{2n+1} \epsilon t_{2n}. \end{gathered}$$

These inequalities are satisfied if the u_j's satisfy

$$\begin{gathered} 0 < \frac{4}{3} u_{2p} < c_{2n+1} \sqrt{\frac{4}{5} u_{2p-2} \cdot \frac{4}{5} u_{2p+2}}, \quad 1 \leq p \leq n-1, \\ |u_{2p+1}| + \frac{1}{4} c_{2n+1} \sqrt{\frac{4}{3} u_{2p} \cdot \frac{4}{3} u_{2p+2}} < c_{2n+1} \sqrt{\frac{4}{5} u_{2p} \cdot \frac{4}{5} u_{2p+2}}, \quad 0 \leq p \leq n-1, \\ \frac{4}{3} u_{2n} < \frac{4}{5} c_{2n+1} \epsilon^2 u_{2n-2}, \\ |u_{2n+1}| < \frac{4}{5} c_{2n+1} u_{2n}, \end{gathered}$$

where we have used the fact that since $N > m$, $t_N = u_N$. That is,

$$0 < u_{2p} < \frac{3}{5} c_{2n+1} \sqrt{u_{2p-2} u_{2p+2}}, \quad 1 \le p \le n-1,$$
$$|u_{2p+1}| < \left(\frac{4}{5} c_{2n+1} - \frac{4}{3} \cdot \frac{1}{4} c_{2n+1} \right) \sqrt{u_{2p} u_{2p+2}}, \quad 0 \le p \le n-1,$$
$$u_{2n} < \frac{3}{5} c_{2n+1} \epsilon^2 u_{2n-2},$$
$$|u_{2n+1}| < \frac{4}{5} c_{2n+1} \epsilon u_{2n}.$$

This shows that $\mathrm{ch}(\gamma^\epsilon_{\widetilde{M}})$ contains a region comparable to S^ϵ_N for $\epsilon < \min\{\epsilon_1, \epsilon_2\}$. Thus for both the even and the odd case, the theorem will be proved if we can show that $\Pi(S^\epsilon_N)$ contains a region comparable to S^ϵ_m. This follows immediately from Theorem 5.1 and the fact that since Π is linear, $\Pi(\Gamma^\epsilon_N) = \Gamma^\epsilon_m$. $\square$

One can naturally ask if an analogous theorem holds if $P^{m+1}_j(x)$ is replaced by the more general error $\mathcal{E}^{j+1}_j(x)$ of Lemma 6.1 since the lemma established the containment in one direction. The reverse containment in this more general case is not yet clear.

7. Behavior for large ϵ: criteria for a full neighborhood

One may also consider how the regions of extendability behave as the neighborhood of the origin expands. An interesting phenomenon can occur. Consider

$$\widetilde{M} = \{ (x+iy, u_2+iv_2, u_3+iv_3) \in \mathbb{C}^3 \mid u_2 = x^2 + \alpha x^4,\ u_3 = x^3 + \beta x^4 \}.$$

By Theorem 6.2, for small ϵ, $N_0\left(\gamma^\epsilon_{\widetilde{M}}\right)$ lies to one side of a hyperplane through the origin. But if $\beta^2 + \alpha < 0$, for large ϵ, it contains a full neighborhood of the origin.

To see this, observe that $\gamma^\epsilon_{\widetilde{M}} = \Pi_4(\widetilde{A}(\gamma^\epsilon_4))$, where $\widetilde{A}$ is the linear map

$$\begin{aligned}(u_1, u_2, u_3, u_4) &= A(t_1, t_2, t_3, t_4)\\ &= (t_1, t_2 + \alpha t_4, t_3 + \beta t_4, t_4)\end{aligned}$$

and Π_4 is the projection $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_3)$. Then $\widetilde{A}(N_0(\gamma^\epsilon_4))$ is comparable to the set of $(0, u_2, u_3, u_4)$ satisfying

$$\begin{aligned}(u_2 - \alpha u_4)^2 &< u_4 < \epsilon^2 (u_2 - \alpha u_4),\\ |u_3 - \beta u_4| &< \sqrt{(u_2 - \alpha u_4) u_4}.\end{aligned} \tag{7.1}$$

The origin is a point of a region comparable to $N_0(\gamma^{\epsilon}_{\widetilde{M}})$ if there is some u_4 such that $(0,0,0,u_4)$ satisfies (7.1). That is, we need a real number u_4 satisfying

$$\alpha^2 u_4^2 < u_4 < -\alpha\epsilon^2 u_4,$$
$$|\beta| u_4 < \sqrt{-\alpha} u_4.$$

Thus we require

$$\epsilon^2 > \frac{1}{-\alpha} \quad \text{and} \quad \beta^2 + \alpha < 0. \tag{7.2}$$

In other words, although for small ϵ $N_0(\gamma^{\epsilon}_{\widetilde{M}})$ is entirely to one side of a hyperplane through the origin, whenever $\beta^2+\alpha<0$, for ϵ sufficiently large (greater than $1/\sqrt{-\alpha}$), $N_0(\gamma^{\epsilon}_{\widetilde{M}})$ contains a full neighborhood of the origin.

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