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EXPONENTIAL ELLIPTICS GIVE DIMENSION TWO

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Abstract. Using the theory of infinite iterated function systems, we show that the Julia set of any function of the type $G = \lambda \exp \circ F$, $\lambda \in \mathbb{C} \setminus \{0\}$, with $F : \mathbb{C} \to \hat{\mathbb{C}}$ a non-constant elliptic function, has Hausdorff dimension two. However, there exist elliptic functions F such that the Julia sets of the maps $G = \exp \circ F$ are nowhere dense in $\mathbb{C}$.

1. Introduction

The aim of this paper is to prove the following result.

Theorem 1.1. *Let $F : \mathbb{C} \to \hat{\mathbb{C}}$ be any non-constant elliptic function and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then the Julia set $\mathcal{J}_G$ of the function $G = \lambda \exp \circ F$ has Hausdorff dimension two. Moreover, the hyperbolic dimension of G is also two.*

The main idea of the proof of this result is that we are able to associate with the dynamical system generated by the function G an infinitely generated iterated function system for which the Poincaré series turns out to converge if and only if its exponent is greater than or equal to two. This type of argument has been used in [KU] to show that the Hausdorff dimension $\mathrm{HD}(\mathcal{J}_F)$ is greater than $\frac{2q}{q+1}$, where q is the maximal multiplicity at all poles of F. Therefore, our result can be viewed as an "infinite multiplicity" version of the above estimate.

Our method of proof of Theorem 1.1 gives in fact more: the hyperbolic dimension, the dimension of the set of conical points of G (see [PU] for appropriate definitions), is equal to two. In contrast, in the case of hyperbolic exponential functions, this dimension is shown in [UZd] to be strictly less than two, whereas the dimension of the Julia set is still equal to two [MCM].

In the last section of this paper we provide a class of examples for which the Julia set is nowhere dense in the plane.

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2. Preliminaries on elliptic functions

An elliptic function is a meromorphic function $F : \mathbb{C} \to \hat{\mathbb{C}}$ which is doubly periodic: there is a lattice $\Lambda = \langle w_1, w_2 \rangle$, $w_1, w_2 \in \mathbb{C}$, with $\Im(w_1/w_2) \neq 0$, such that $F(z+\omega) = F(z)$ for every $\omega \in \Lambda$. We denote by $\mathcal{T} = \mathbb{C}/\Lambda$ the quotient torus and by $\pi : \mathbb{C} \to \mathcal{T}$ the canonical projection. The standard example is the Weierstrass elliptic function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) .$$

We make essential use of the fact that the set of poles

$$\mathcal{P} = F^{-1}(\infty) = \bigcup_{m,n \in \mathbb{Z}} \big(\mathcal{R} \cap F^{-1}(\infty) + mw_1 + nw_2 \big)$$

is non-empty. Let $b \in \mathcal{P}$ be such a pole of F and denote by $q \geq 1$ its multiplicity. Near this pole b the function F takes the form

$$F(z) = \frac{H(z)}{(z-b)^q},$$

where H is a holomorphic function defined near b and $H(b) \neq 0$. Note that a straightforward calculation gives that

$$|F'(z)| \asymp \frac{1}{|z-b|^{q+1}} \asymp |F(z)|^{\frac{q+1}{q}} \tag{2.1}$$

holds for all z in a sufficiently small neighborhood D of b. Here $A \asymp B$ means that the quotient A/B is bounded away from 0 and ∞, with bounds that are independent of the variables involved.

The objects in this paper are mappings of the form

$$G = \lambda \exp \circ F : \mathbb{C} \setminus \mathcal{P} \to \mathbb{C} ,$$

where λ is any non-zero complex number and F an elliptic function as described above. Since F is periodic with respect to Λ, the same is true for G. This allows us to project the map G onto the torus $\mathcal{T}$, given by semi-conjugation via the projection π:

$$\begin{array}{ccc} \mathbb{C} \setminus \mathcal{P} & \stackrel{G}{\longrightarrow} & \mathbb{C} \\ \big\downarrow \pi & & \big\downarrow \pi \\ \mathcal{T} \setminus \mathcal{P}_0 & \stackrel{g}{\longrightarrow} & \mathcal{T} \end{array} , \tag{2.2}$$

where $\mathcal{P}_0 = \pi(\mathcal{P})$.

As usual we denote by $\mathcal{F}_G$ the Fatou set of G, which is the set of points $z \in \mathbb{C}$ such that all the iterates of F are defined and form a normal family on a neighborhood of z. The Julia set $\mathcal{J}_G$ is the complement of $\mathcal{F}_G$ in $\hat{\mathbb{C}}$. An easy to verify fact is that the Julia set of the torus map g is $\mathcal{J}_g = \pi(\mathcal{J}_G)$.

3. Proof of the Theorem 1.1

Let $G = \lambda \exp \circ F : \mathbb{C} \to \hat{\mathbb{C}}$ be a map of the family we consider and let b be a pole of multiplicity $q \geq 1$ of the non-constant elliptic function F. As explained in the previous section, we associate to G the torus map $g : \mathcal{T} \setminus \mathcal{P}_0 \to \mathcal{T}$. We also set $a = \pi(b)$.

Let $r > 0$ be such that the restriction of π to $\mathbb{D}(b, 2r)$ is injective and set $\Omega = \pi(D)$ with $D = \mathbb{D}(b, r)$ and $\Omega' = \pi(\mathbb{D}(b, 2r))$. We suppose that $r > 0$ has been chosen sufficiently small so that the estimation (2.1) holds on D. Clearly $a \in \Omega$. We will define an infinite iterated function sytem $S = \{\Phi_\omega\}_{\omega \in \Lambda^+}$ (where Λ^+ is a subset of Λ to be specified later in the course of the proof) on the domain Ω whose limit set J_S is contained in the Julia set $\mathcal{J}_g$. Theorem 1.1 then follows from the fact that the Poincaré series associated to the system S,

$$\psi(t) = \sum_{\omega \in \Lambda^+} \|\Phi'\|^t \asymp \sum_{\omega \in \Lambda^+} |\Phi'(a)|^t,$$

will converge if and only if $t \geq 2$. Indeed, in the terminology of [MU] this means that the critical exponent of the series ψ is $\theta(S) = 2$. Therefore (see [MU] again) $\mathrm{HD}(J_S) \geq \theta(S) = 2$. Since it will follow from the construction that the limit set J_S is contained in the closure of the repelling periodic points of g, we have $\mathcal{J}_g \supset J_S$. We are thus left to show that $\theta(S) = 2$.

The generators Φ_ω of the system are appropriately chosen inverse branches of the map g that are defined on Ω' in order to have the Koebe distortion property for the mappings of the system S. They are defined as follows: let

$$\Lambda^+ = \{\omega \in \Lambda \; ; \; \mathrm{Re}(b + \omega) - 2r > 1\} ,$$

and, for $\omega \in \Lambda^+$, let $\pi_\omega^{-1} : \Omega' \to D'_\omega = \mathbb{D}(b + \omega, 2r)$ be the inverse branch of π sending a to $b + \omega$. The map $z \mapsto \lambda \exp(z)$ has well-defined all holomorphic inverse branches on the domain $U = \{\mathrm{Re}\, z > 1\}$. We denote by $\log_*$ one such inverse branch, selected so that $\log_*(U)$ is contained in the neighborhood of infinity $F(D)$ and so that no critical values of F (there are only finitely many of them) belong to $\log_*(U)$. We finally denote by F_b^{-1} the inverse branch of F defined on $\log_*(U)$, given by $F_b^{-1}(\infty) = b$. The generators of the system we look for are

$$\Phi_\omega = \pi \circ F_b^{-1} \circ \log_* \circ \pi_\omega^{-1} \quad , \; \omega \in \Lambda^+ .$$

We now evaluate the size of $|\Phi'_\omega(a)|$, $\omega \in \Lambda^+$. Indeed, first notice that

$$|\Phi'_\omega(a)| = |G'(F_b^{-1} \circ \log_k(b + \omega))|^{-1} .$$

Next, since $G'(z) = G(z)F'(z)$, we have that

$$\left|G'(F_b^{-1} \circ \log_*(b + \omega))\right| = |b + \omega| \left|F'(F_b^{-1} \circ \log_*(b + \omega))\right| .$$

Notice that $F_b^{-1}\circ\log_*(\mathbb{D}(b+\omega,r))\subset D$ and that the estimation (2.1) is true on D. Therefore,

$$|\Phi'_\omega(a)| \asymp |b+\omega|^{-1}|\log_k(b+\omega)|^{-\frac{q+1}{q}} \ .$$

It follows that the series

$$\psi(t)=\sum_{\omega\in\Lambda^+}|\Phi'_\omega|^t \asymp \sum_{\omega\in\Lambda^+}|b+\omega|^{-t}|\log_k(b+\omega)|^{-\frac{q+1}{q}t}$$

converges if and only if $t\geq 2$. This means that the critical exponent is $\theta(S)=2$ and we are done.

4. Nowhere dense examples

Our aim in this section is to describe a large class of elliptic functions $F:\mathbb{C}\to\hat{\mathbb{C}}$ such that the Julia set of the map $G=\exp\circ F$ is nowhere dense in $\mathbb{C}$. Indeed, start of with an arbitrary non-constant elliptic function H such that $b=H'(1)\neq 0$. Put $a=H(1)$. Then

$$F(z)=\frac{1}{2b}H(z)-\frac{a}{2b}$$

is again an elliptic function with respect to the same lattice as H. Immediate calculations give that $F(1)=0$ and $F'(1)=1/2$. Then $G(1)=\exp(F(1))=e^0=1$ and $G'(1)=G(1)F'(1)=1/2$. This means that 1 is an attracting fixed point of G. Since its basin of attraction is a non-empty open subset of the Fatou set of G, we are done.

References

[KU] J. Kotus and M. Urbański, *Hausdorff dimension and Hausdorff measures of Julia sets of elliptic functions*, Bull. London Math. Soc. **35** (2003), 269–275. MR 1952406 (2003j:37067)

[MU] R. D. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. (3) **73** (1996), 105–154. MR 1387085 (97c:28020)

[MCM] C. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. **300** (1987), 329–342. MR 871679 (88a:30057)

[PU] F. Przytycki and M. Urbański, *Fractals in the plane—ergodic theoretic methods*, to be published by Cambridge Univ. Press; available at Urbański's website.

[UZd] M. Urbański and A. Zdunik, *The finer geometry and dynamics of the hyperbolic exponential family*, Michigan Math. J. **51** (2003), 227–250. MR 1992945 (2004d:37068)

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